On the choice of coarse variables for dynamics

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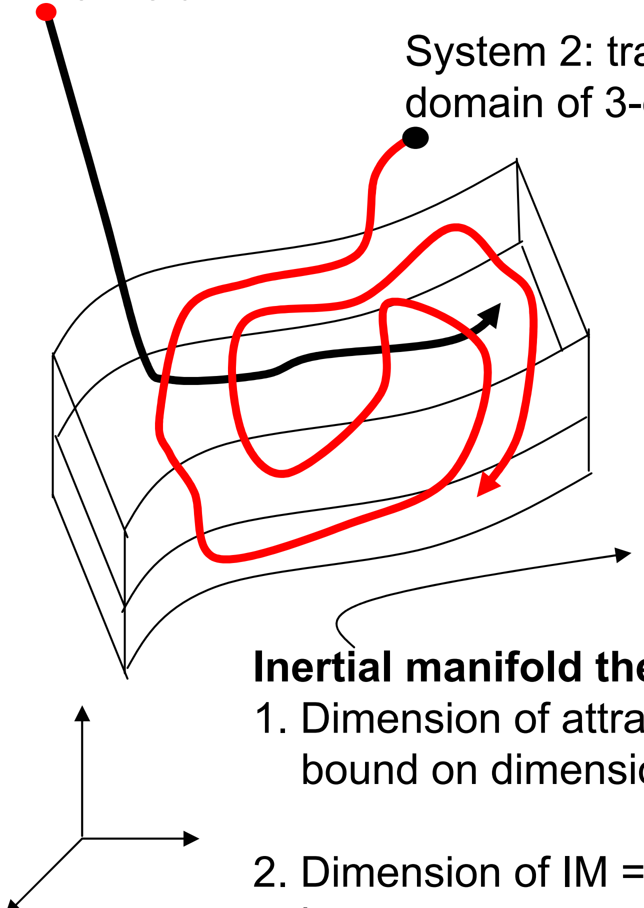
Workshop on Mathematics of Model Reduction
University of Leicester, Aug. 28-30, 2007

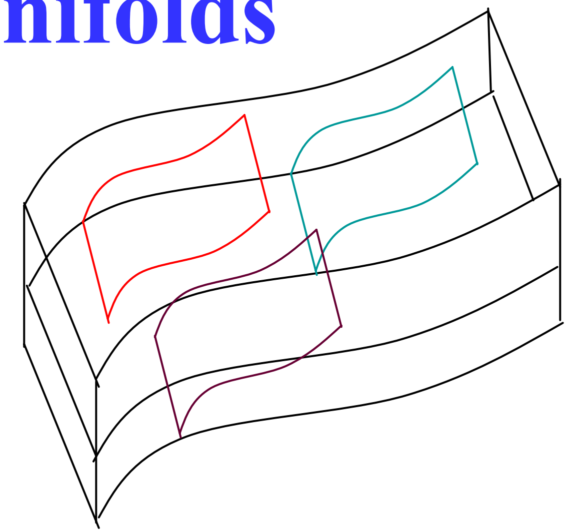
Acknowledgment: US DOE

Locally Invariant Manifolds

System 1: trajectories end up in 2-d manifold

System 2: trajectories end up in thin domain of 3-d phase space





3-d phase space

Temam et al.

Inertial manifold theory =>

1. Dimension of attractor = lower bound on dimension of IM

2. Dimension of IM = # of coarse variables in ***autonomous*** reduced dynamics

Present Idea: (Parametrized LIM)

1. Fill phase space region with many low dimensional LI manifolds (choose dim, say **d**)
2. Reduced dynamics is **d-**dimensional
3. For IM problems, reduction not linked to dimension of IM
4. If coarse var. chosen arbitrarily, need knowledge of fine ic for consistency.

Coarse Variables

- Want to avoid having knowledge of fine initial conditions
- Ideally
  - want to work with a small set of user-defined coarse variables +
  - a small augmentation of it for autonomous coarse dynamics
- Main user-defined coarse variables of interest
  - Time averages

Scheme

$$\frac{df}{dt}(t) = H\left(f\left(t\right)\right)$$
$$f\left(0\right) = f_*$$

← Original system

$$c\left(t\right) = \frac{1}{\tau}\int_t^{t+\tau} \Lambda\left(f\left(s\right)\right) ds$$
$$p\left(t\right) = \Pi\left(f\left(t\right)\right)$$

← Coarse variables

Prescribed functions

Prescribed Time interval

Introduce new fine delay variables

$$f_f\left(t\right) := f\left(t+\tau\right)$$
$$\frac{df_f}{dt}(t) = H\left(f_f\left(t\right)\right)$$

$$\frac{dc}{dt}(t) = \frac{1}{\tau}\left[\Lambda\left(f\left(t+\tau\right)\right) - \Lambda\left(f\left(t\right)\right)\right]$$
$$\frac{dp}{dt}(t) = \sum_{J=1}^{N} \frac{\partial \Pi}{\partial f^J}\left(f\left(t\right)\right) H^J\left(f\left(t\right)\right)$$

Augmented system with singular perturbation structure

$$\frac{df_f}{dt}(t) = H\left(f_f(t)\right)$$

$$\frac{df}{dt}(t) = H\left(f(t)\right)$$

$$\frac{dp}{dt}(t) = \sum_{J=1}^{N} \frac{\partial \Pi}{\partial f^J}\left(f(t)\right) H^J\left(f(t)\right)$$

$$\frac{dc}{dt}(t) = \frac{1}{\tau}\left[\Lambda\left(f_f(t)\right) - \Lambda\left(f(t)\right)\right] \quad ; \quad \tau \gg 1$$

Coarse variables

Instantaneous

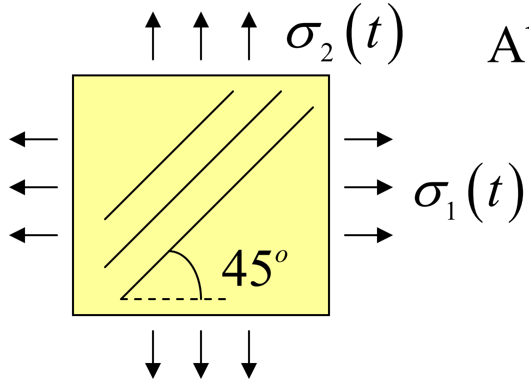
Time averaged

Special structure of invariance equation for time averaged coarse variables

$$\left.\begin{aligned}
&\sum_{k=1}^{m}\frac{\partial G_f^I}{\partial c^k}\left(\frac{1}{\tau}\left[\Lambda^k\left(G_f\right)-\Lambda^k\left(G\right)\right]\right)+\sum_{l=1}^{n}\sum_{K=1}^{N}\frac{\partial G_f^I}{\partial p^l}\frac{\partial \Pi^l}{\partial f^K}\left(G\right)H^K\left(G\right)=H^I\left(G_f\right)\\
&\sum_{k=1}^{m}\frac{\partial G^I}{\partial c^k}\left(\frac{1}{\tau}\left[\Lambda^k\left(G_f\right)-\Lambda^k\left(G\right)\right]\right)+\sum_{l=1}^{n}\sum_{K=1}^{N}\frac{\partial G_f^I}{\partial p^l}\frac{\partial \Pi^l}{\partial f^K}\left(G\right)H^K\left(G\right)=H^I\left(G\right)
\end{aligned}\right\}\quad I=1 \text{ to } N$$

(possibly oscillatory) fine vector field does not appear!

Kinetics of Material With Wiggly Energy (joint with Aarti Sawant, *2005*)



Abeyaratne, Chu & James (1996)

$$\dot{\lambda}(t) = -\mu \frac{dW\left(\lambda(t), \sigma_1(t), \sigma_2(t)\right)}{d\lambda}$$

Gradient-Flow Type Kinetic Law

Evolution of volume fraction of Martensite variant under different loading programs

Energy Consideration:

$$W_{load} = -\begin{Bmatrix} \lambda^2\left(\sigma_1^2+\sigma_2^2\right)\left(\alpha^2-\gamma^2\right)^2 \Big/ \left(\alpha^2+\gamma^2\right) \\ +2\lambda\left(\sigma_1^2\gamma^2-\sigma_2^2\alpha^2\right)\left(\alpha^2-\gamma^2\right)/\left(\alpha^2+\gamma^2\right) \\ +\left(\sigma_1\gamma+\sigma_2\alpha\right)^2 \end{Bmatrix}^{\frac{1}{2}}$$

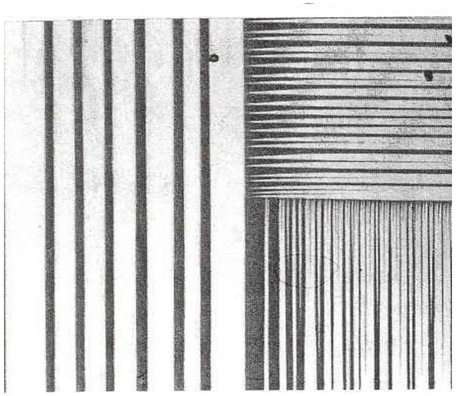
$$W_{tr.layer} = c_1\lambda^2 + c_2\left(1-\lambda^2\right)$$

$$W_o = W_{load} + W_{trn.layers}$$

$$W_{pert} = a.\varepsilon.Cos\left(\frac{\lambda}{\varepsilon}\right)\left(\varepsilon \to 0\right)$$

$$W = W_o + W_{pertb}$$

modification



microstructure showing several laminates

Carnegie Mellon

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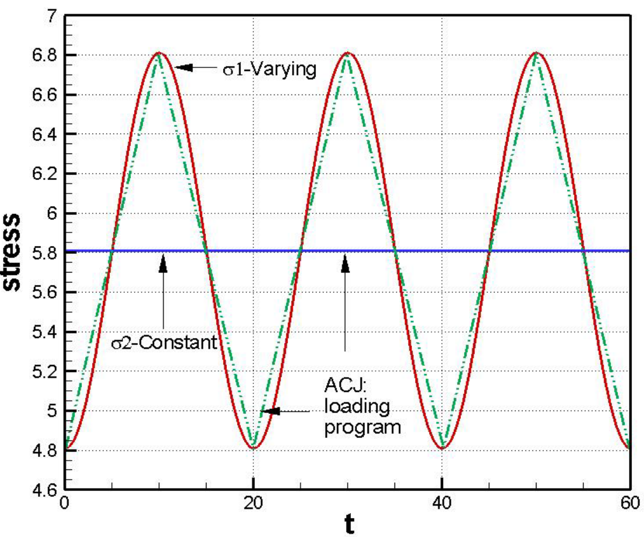
Comparison of Volume Fraction for Different Energy Considerations

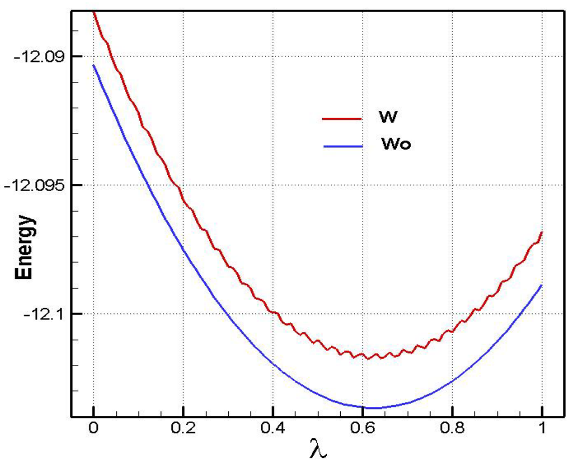
W_0 & W

Loading Program

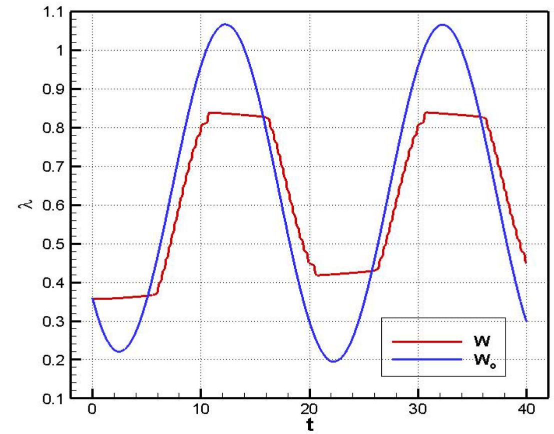


σ vs. t

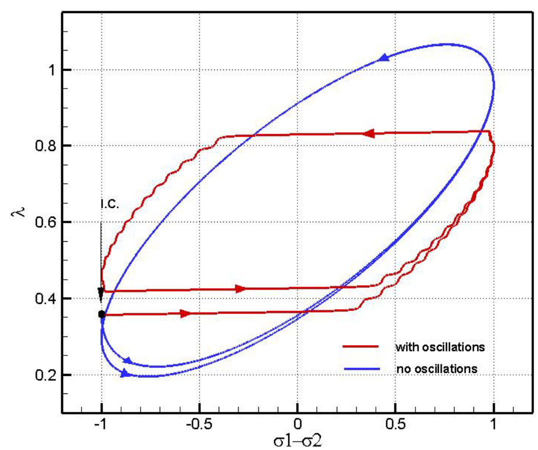
$$\dot{\sigma}_1(t) = \eta.\sigma_3(t),$$
$$\dot{\sigma}_3(t) = \eta.(\sigma_0 - \sigma_1(t))$$



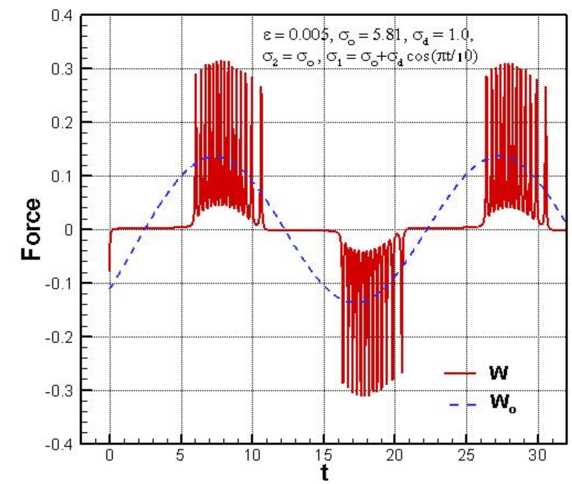
W vs. λ



λ vs. t

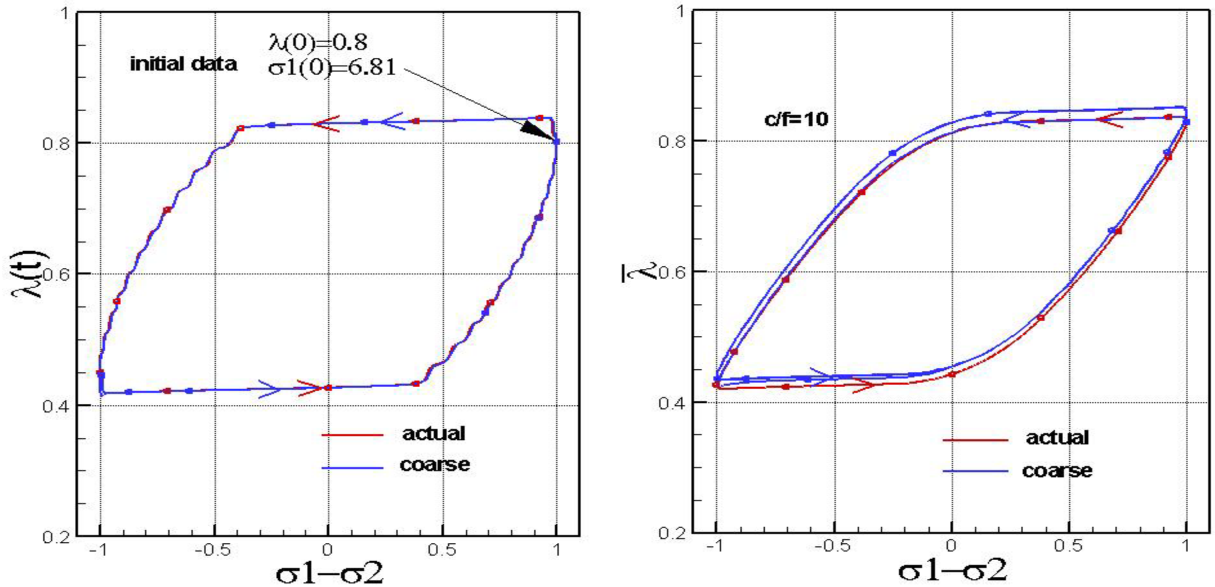


λ vs. $\sigma_1 - \sigma_2$

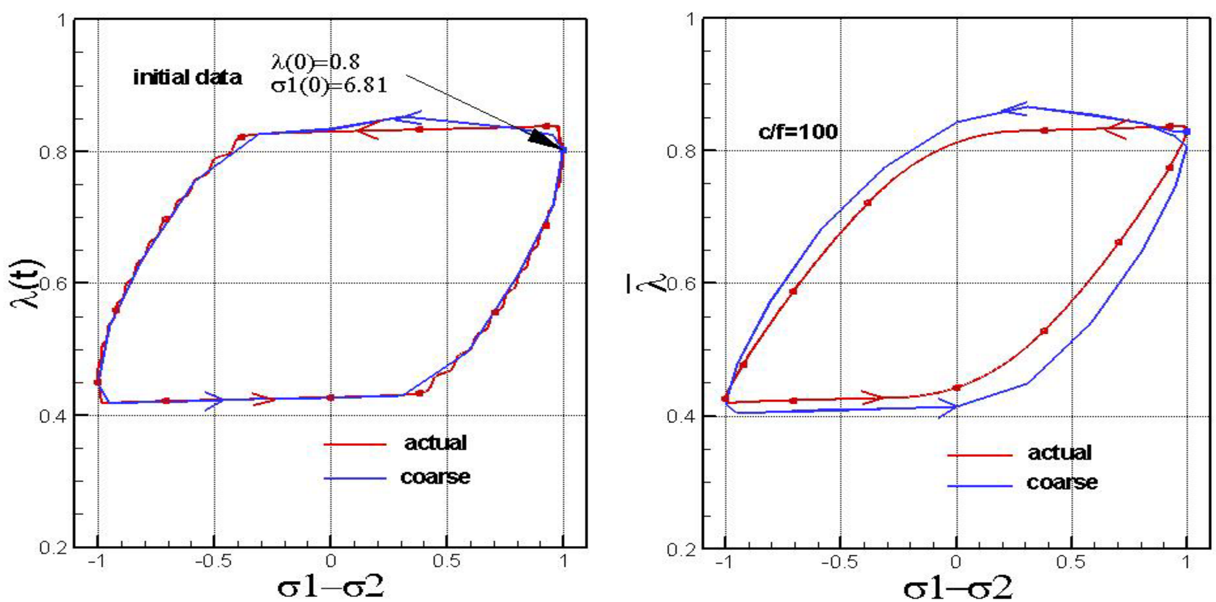


$\dot{\lambda}$ vs. t

Hysteresis Plots for different ratios coarse-to-fine time stepping



$c/f = 10$



$c/f = 100$

Choice of coarse variables for autonomous coarse dynamics

- *Theorem* (Takens, 1981): Let M be a compact manifold of dimension m. For pairs (φ, y), $\varphi : M \to M$ a smooth diffeomorphism and $y : M \to R$ (reals) a smooth function, it is a generic property that the map $\Phi_{(\varphi,y)} : M \to R^{2m+1}$, defined by

$$\Phi_{(\varphi,y)}(x) = \left(y(x), y \circ \varphi(x), \cdots, y \circ \varphi^{2m}(x)\right)$$

is an embedding; by 'smooth' we mean at least C^2.

Delay Reconstruction Technique:

- **Take aperiodic, dense (on attractor), fine trajectory**
- **Take *any* observable (y)**
- **Since original traj. is aperiodic, Takens guarantees that with >= (2m+1) delay coords.,**
  - **- have a DR image trajectory without self-intersection**
  - **- a one-to-one DR image of original dense set of points**
- **In practice, *m* is unknown, so**
- **Delay Reconstruct as above, increasing delay coordinates progressively**
- **By some algorithm, judge when there is no self-intersection in image trajectory; then**
  - **- the set of obtained delay variables follows autonomous dynamics**

Ruelle, Abarbanel, Yorke, etc.

Trial Idea

- **Consider the possibility of similar DR for coarse dim << *m***

IS THIS POSSIBLE?

Coarse variables for autonomous dynamics

- While Takens's result seems only like a sufficient condition, have to consider Ding et al. result
  - Plateau in correlation dimension of DR image of attractor occurs at
    - Ceiling [c.d. of attractor]
- Eckmann and Ruelle show similar result for embedding dimension

Non-generic nature of observables for ‘good’ coarse variables

Consider systems for which one has ergodicity on a set of non zero Lebesgue measure containing the attractor.

Consider iterated map of original dynamics as the observable

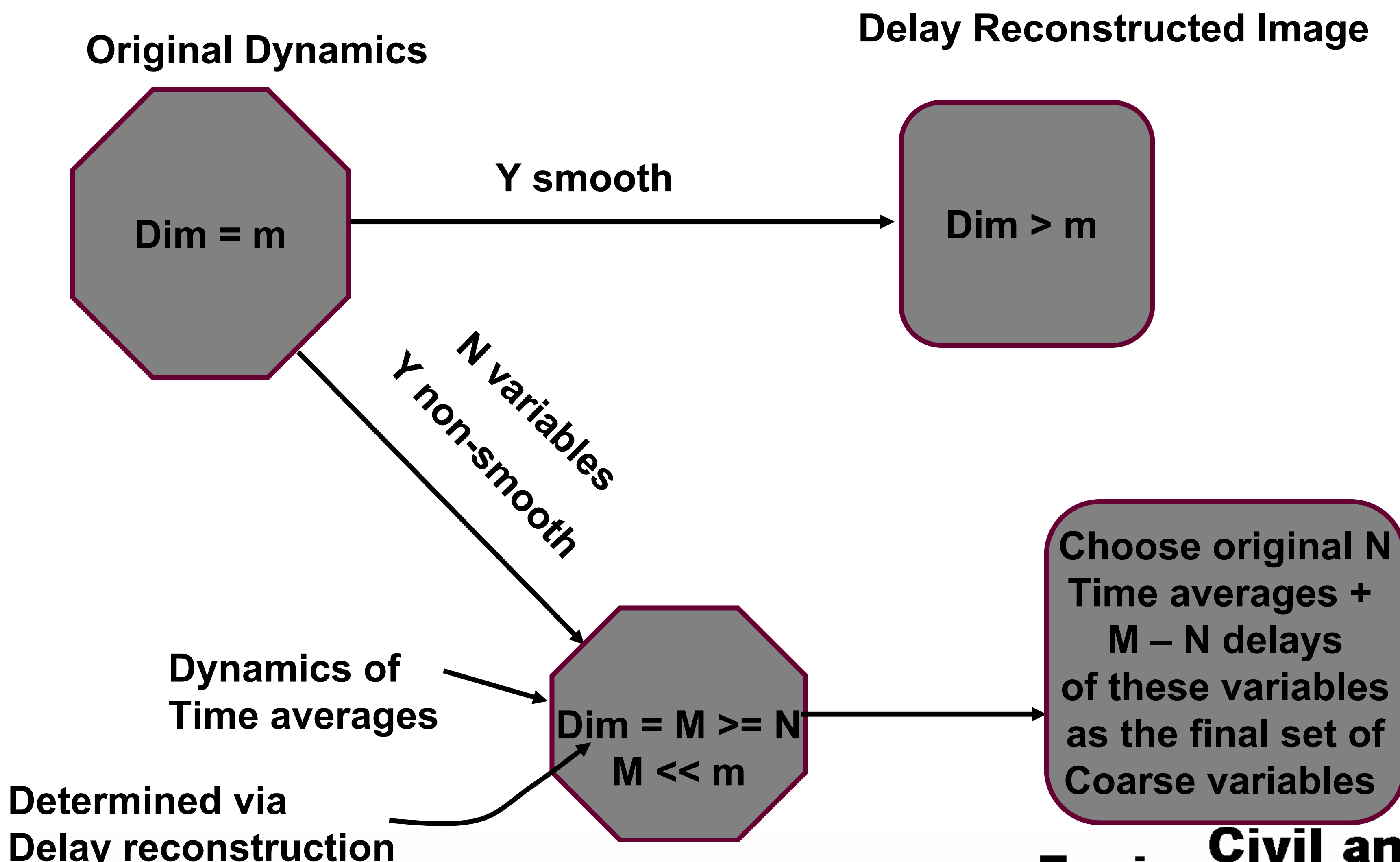
$$y(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{p=0}^{N} \varphi^{p}(x)$$

IN THE VICINITY OF FIXED POINTS SUCH A MAP CANNOT BE SMOOTH!!

Backing off from infinite sum to finite, but long, sum, theoretically smooth map
but practically with huge gradients

Takens does not apply ; but such variables + DR remain useful

Key Idea


Original Dynamics
Delay Reconstructed Image
Dim = m
Y smooth
Dim > m
N variables
Y non-smooth
Dynamics of
Time averages
Determined via
Delay reconstruction
Dim = M >= N
M << m
Choose original N
Time averages +
M – N delays
of these variables
as the final set of
Coarse variables

Dynamics of Coarse Variables

- Choose time-averages of fine state functions and their delays, chosen by DR, as coarse variable set
- Define dynamics by PLIM
  - Due to a-priori knowledge about autonomous nature of coarse dynamics
    - only one LIM need be calculated for each local region of coarse space (??)
    - (??) – i.e. can do with mathematical help of the theorem-proving type to understand precise implications of what is and is not possible)
- Dealing with 1-1 map between sets of different dim. - Typically not smooth in one direction
- For time-averaged variables, PLIM requires smoothness in coarse-to-fine direction; this is OK

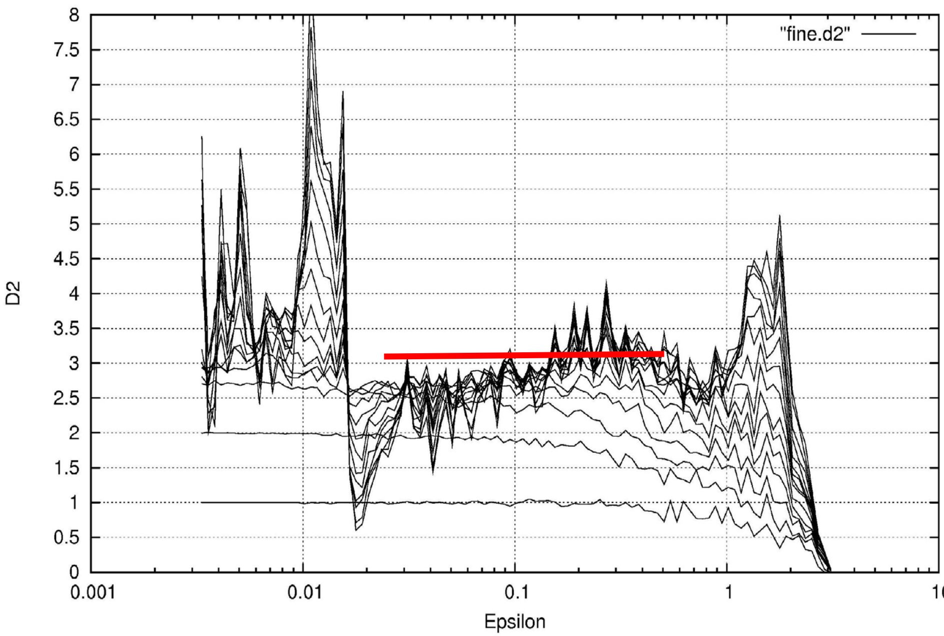
Example : Hamiltonian system (joint work with Will Kotterman)

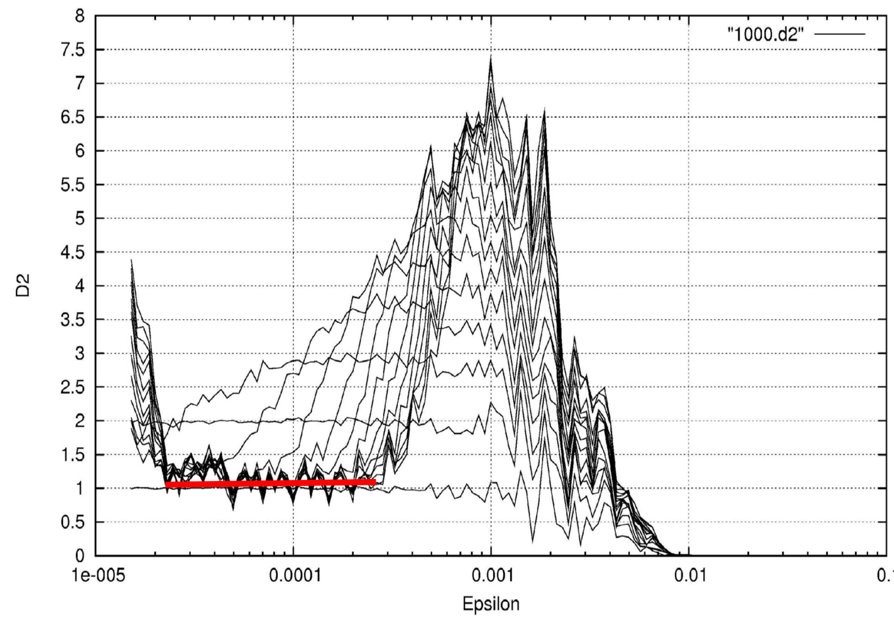
$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_3 = x_4$$
$$\dot{x}_2 = -x_1\left(1 + x_3^2\right)$$
$$\dot{x}_4 = -x_3\left(1 + x_1^2\right)$$

Chorin, Hald Kupferman

Use TISEAN for data analysis

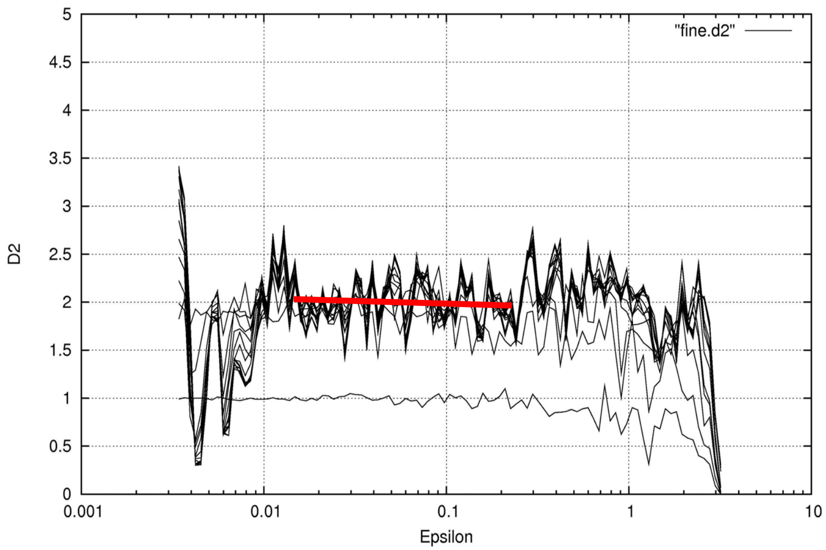
Choose delay from First minimum of Mutual information Statistic (Abarbanel)

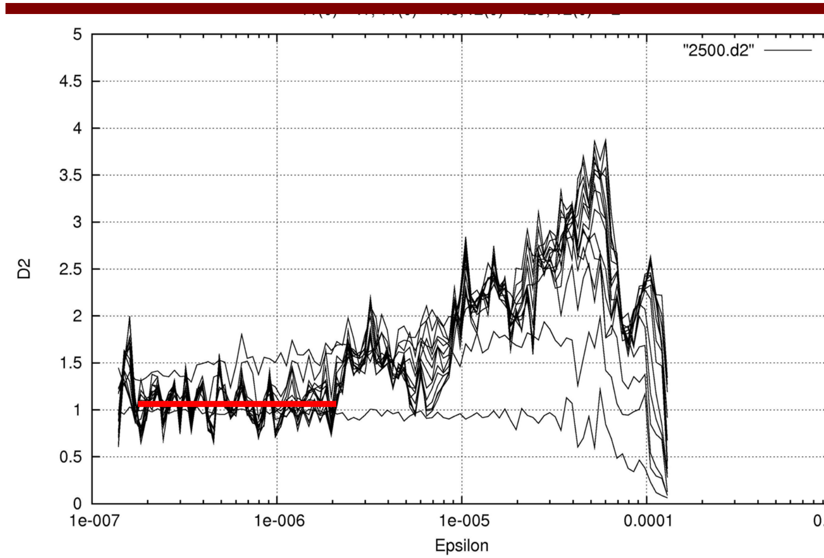


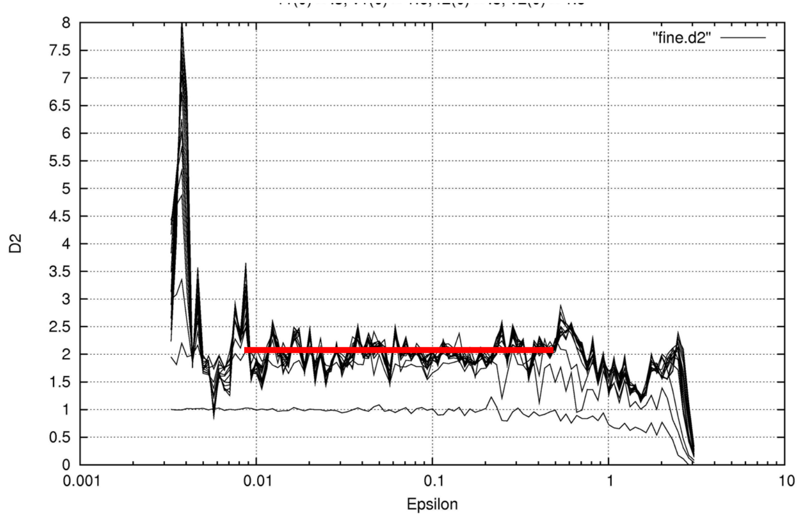


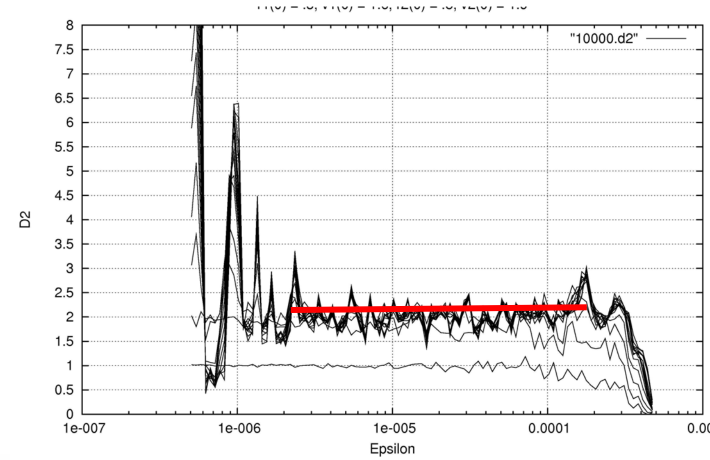
Example Hamiltonian system

Fine | **Coarse**

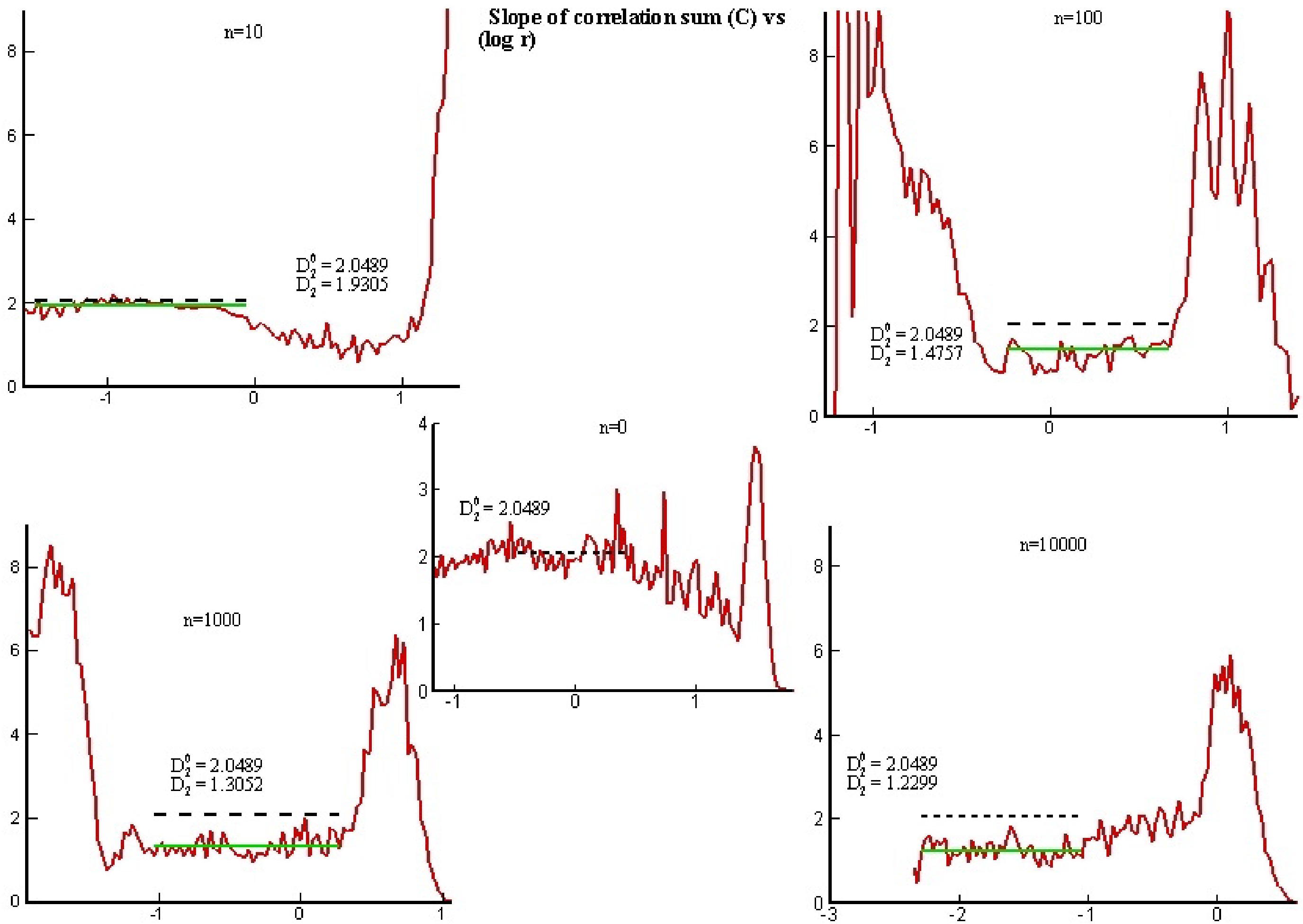








Example: Lorenz
(joint work with Rajarshi Singh)



n = number of time steps used to average original signal
D_2^0 = correlation dimension of original dynamics
D_2 = correlation dimension of averaged dynamics

Dimension Reduction on Delay Reconstruction in the literature

- Broomhead, Huke and Muldoon (1992)
  - Construct explicit example of non-recursive, inverse all pole filter that reduces dimension of a particular signal
  - Example of 'non-generic' observable.
- Observables for homogenization should be 'non-generic.'

Scheme 2: Adapted Projections

Want to define Coarse projection for autonomous coarse dynamics

$$c(t) := \Pi\left(f(t)\right) \quad \Rightarrow \quad \dot{c}(t) = \frac{\partial \Pi}{\partial f}\left(f(t)\right) H\left(f(t)\right)$$

$$W_c := \left\{ f : \Pi(f) = c \right\}$$

require ⟶ $\frac{\partial \Pi}{\partial f}(f) H(f) = A_c$ **on** W_c

or, locally $\lambda \frac{\partial}{\partial f}(\Pi) = \frac{\partial}{\partial f}\left(\frac{\partial \Pi}{\partial f} H\right)$ for λ scalar

Choose arbitrary φ s.t.

$$\lambda := \partial \varphi / \partial \Pi \qquad \frac{\partial}{\partial f}\left(\varphi(\Pi) - \frac{\partial \Pi}{\partial f} H\right) = 0$$

$$\varphi(\Pi) = \frac{\partial \Pi}{\partial f} H$$

⟵ governing eqn. for Π

For every choice of φ, $\exists \; \Pi$!!!