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Directed cycles and multi-stability of coherent dynamics in systems of coupled nonlinear oscillators

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Abstract: We analyse the dynamics of networks of coupled nonlinear systems in terms of both topology of interconnections as well as the dynamics of individual nodes. Here we focus on two basic and extremal components of any network: chains and cycles. In particular, we investigate the effect of adding a directed feedback from the last element in a directed chain to the first. Our analysis shows that, depending on the network size and internal dynamics of isolated nodes, multiple coherent and orderly dynamic regimes co-exist in the state space of the system. In addition to the fully synchronous state an attracting rotating wave solution occurs. The basin of attraction of this solution apparently grows with the number of nodes in the loop. The effect is observed in networks exceeding a certain critical size. Emergence of the attracting rotating wave solution can be viewed as a "topological bifurcation" of network dynamics in which removal or addition of a single connection results in dramatic change of the overall coherent dynamics of the system.

Keywords: Coupled systems, reaction network, eigenvalue, synchronization, wave solutions

1. INTRODUCTION

Networks of interconnected and interacting systems have been in the focus of attention for many decades. Substantial progress has been made towards understanding of some of the facets of their collective behavior such as e.g. synchronization (see e.g. (Pikovsky et al., 2001), (Strogatz, 2003) and references therein). Various authors reported that under certain conditions, a large class of nonlinear dynamical systems may exhibit globally asymptotically stable synchronization (Pogromsky, 1998), (Scardovi et al., 2010) as well as the partial one (Pogromsky et al., 2002), Belykh et al. (2000), (Belykh et al., 2004). Emergence and asymptotic properties of these collective behaviors has been shown to depend crucially on the particular network topologies (Chandrasekar et al., 2014). Yet, the link between network topology, dynamics of individual network nodes, and the overall collective network dynamics is not completely understood.

Here we look at two basic and extremal topological ingredients of complex networks: a chain and a cycle. Our motivation to focus on these two very basic configurations is that despite simplicity of these configurations, many complex physical networks can be viewed as an interconnection of the two (Gorban et al., 2010). Furthermore, recent computational studies revealed that cycles on their own could be important for explaining sustained coherent oscillatory network activity (Garcia et al., 2014).

We begin our investigation by analysing the dynamics of a system of coupled neutrally stable linear equations. The dynamics is essentially governed by a coupling matrix that corresponds to directed interconnections in the system. The equations can also be viewed as a model describing dynamics of damped oscillations in kinetic systems, and thus in what follows we refer to it as such. The results are provided in Section 2. In Section 3 we consider a more generalized setting, in which the dynamics of each individual node is governed by a nonlinear, albeit semi-passive (Pogromsky, 1998), oscillator. The equations describing oscillators in each node are of the FizHugh-Nagumo (FHN) type (FitzHugh, 1961). These oscillators are, in turn, an adaptation of the van der Pol oscillator. We found that

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for sufficiently long directed chains, adding feedback from the last node to the first one gives rise to the existence of rotating waves. Section 4 contains a discussion of our findings, and Section 5 concludes the paper.

2. COUPLED NEUTRALLY STABLE SYSTEMS

Consider the following system of linear first-order differential equations:

$$\dot{P} = KP, \tag{1}$$

where $P = \mathrm{col}(p_1, p_2, \dots, p_n) \in \mathbb{R}^n$, and matrix $K = (k_{ij})$ is defined as follows:

$$k_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j; \\ -\sum\limits_{m,\, m\neq i} q_{mi} & \text{if } i = j. \end{cases} \tag{2}$$

Note that K is a Metzler matrix [1] with zero column sums. Off-diagonal elements k_{ij}, $i \neq j$, of the matrix K can be viewed as the connection weights between the i-th and the j-th nodes in the network. The matrix K can be related to the Laplacian matrix L (see e.g. (Bollobas, 1998)) of an associated directed network in which the overall connectivity pattern is the same except for that the direction of all connections is altered. The Laplacian for the latter network is thus $L = -K^T$. Note, however, that this relation does not necessarily hold for the original network.

Consider the simple simplex

$$\Delta_n = \left\{ P | p_i \geq 0, \sum\nolimits_i p_i = 1 \right\}.$$

Δ_n is clearly forward invariant under the dynamics (1) since it preserves non-negativity and obeys the "conservation law" $\sum_i p_i = \text{const}$. (The latter follows immediately from the fact that K has zero column sums.) Thus any solution $P(\cdot; t_0, P_0)$ of (1) starting from $P_0 = P(t_0) \in \Delta_n$ remains in Δ_n for all $t \geq t_0$.

The invariance of Δ_n under (1) can be used to prove certain important properties of K and its associated system (1). Two examples are presented below.

- **Equilibria.** The Perron-Frobenius theorem implies the existence of a non-negative vector P^* such that $KP^* = 0$, i.e. P^* defines an equilibrium of system (1). The existence of this vector P^* can also be deduced from the forward invariant of Δ_n. Indeed, as any continuous map $\Phi : \Delta_n \to \Delta_n$ has a fixed point (Brouwer fixed point theorem), $\Phi = \exp(Kt)$ has a fixed point in Δ_n for any $t \geq t_0$. If $\exp(Kt)P^* = P^*$ for some $P^* \in \Delta_n$ and sufficiently small $t > t_0$, then $KP^* = 0$ because
$$\exp(Kt)P = P + tKP + o(t^2).$$
- **Eigenvalues of K.** It is clear that K has a zero eigenvalue. In fact, Gerschgorin's theorem states that all eigenvalues of K are in the union of closed discs
$$D_i = \{\lambda \in \mathbb{C} | \|\lambda - k_{ii}\| \leq |k_{ii}|\}.$$
Thus K does not have purely imaginary eigenvalues. This fact can also be deduced from the forward invariance of Δ_n in combination with the assumption of a positive equilibrium P^*. We exclude the eigenvector corresponding to the zero eigenvalue and consider K on the invariant hyperplane where $\sum_i p_i = 0$. If K has a purely imaginary eigenvalue λ, then there exists a $2D$ K-invariant subspace U, where K has two conjugated imaginary eigenvalues, λ and $\overline{\lambda} = -\lambda$. Restriction of $\exp(Kt)$ on U is one-parametrer group of rotations. For the positive equilibrium P^* the intersection $(U+P^*) \cap \Delta_n$ is a convex polygon. It is forward invariant with respect to (1) because U is invariant, P^* is equilibrium and Δ_n is forward invariant. But a polygon on a plane cannot be invariant with respect to one-parametrer semigroup of rotations $\exp(Kt)$ $(t \geq 0)$. This contradiction proves the absence of purely imaginary eigenvalues.

The main result of this section is the following theorem:

Theorem 1. For every nonzero eigenvalue λ of matrix K

$$\frac{|\Im\lambda|}{|\Re\lambda|} \leq \cot\frac{\pi}{n}. \tag{3}$$

The the proof of this theorem can be extracted from the general Dmitriev–Dynkin–Karlelevich theorems (Dmitriev and Dynkin, 1946; Karpelevich, 1951). Due to space limitation, however, we do not provide it here and refer the reader to (Gorban et al., 2015).

Remark 1. It is important to note that the bound given in Theorem 1 is sharp. Indeed, let K define a directed ring with uniform weights q, e.g.

$$K = \begin{pmatrix} -q & 0 & 0 \cdots & 0 & q \\ q & -q & 0 \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & \ddots & q & -q & 0 \\ 0 & 0 \cdots & 0 & q & -q \end{pmatrix}.$$

The eigenvalues of K are

$$\lambda_k = -q + q\exp\left(\frac{2\pi k \mathrm{i}}{n}\right), \quad k = 0, 1, \dots, n-1,$$

cf. (Davis, 1994), with $\mathrm{i} = \sqrt{-1}$ the imaginary unit. Thus $\frac{|\Im\lambda_1|}{|\Re\lambda_1|} = \cot\frac{\pi}{n}$. Note that for large n,

$$\cot\frac{\pi}{n} \approx \frac{n}{\pi},$$

which means that oscillations in a directed ring with a large number of systems decay very slowly.

An important consequence of this extremal property of a directed ring is that not only transients in the cycle decay very slowly but also that the overall behavior of transients becomes extremely sensitive to perturbations. This, as we show in the next sections, gives rise to resonances and bistabilities if neutrally stable nodes in (1) are replaced with the ones exhibiting oscillatory dynamics.

3. COUPLED NONLINEAR NEURAL OSCILLATORS

Consider now a network of FitzHugh-Nagumo (FHN) neurons

$$\begin{cases} \dot{z}_j = \alpha\,(y_j - \beta z_j) \\ \dot{y}_j = y_j - \gamma y_j^3 - z_j + u_j, \end{cases} \tag{4}$$

$j = 1, 2, \dots, n$ with parameters α, β, γ chosen as

$$\alpha = \tfrac{8}{100},\ \beta = \tfrac{8}{10},\ \gamma = \tfrac{1}{3}$$

[1] A Metzler matrix is a matrix with non-negative off-diagonal entries

The FHN neurons interact via diffusive coupling

$$u_j = \sigma \sum_{l=1}^{n} q_{jl}(y_l - y_j) \tag{5}$$

with constant $\sigma \in \mathbb{R}$, $\sigma > 0$, being the coupling strength. For convenience, let

$$y = (y_1, \ldots, y_n),\ u = (u_1, \ldots, u_n),\ x = (y, z),$$

and $x(\cdot; x_0, \sigma)$ denote a solution of the coupled system with the coupling strength σ and satisfying the initial condition $x(0) = x_0$. The topology of network connections in (5) is characterized by the adjacency matrix Q with zeros on the main diagonal and entries identical to the values of q_{ij} for $i \neq j$, $i, j \in \{1, \ldots, n\}$. The matrix Q is now assumed to be a circulant matrix

$$Q = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Thus besides assuming the network structure to be a directed ring we have also assumed the interaction weights q_{jl} to be identical and, without loss of generality, we have set these weights of interaction to 1.

At the first glance, the connectivity pattern specified by Q differs from that specified by matrix K in (2). Yet, if coupling (5) is rewritten in the vector-matrix notation then the following identity holds

$$u = \sigma(Q - I_n)y \triangleq -\sigma L y. \tag{6}$$

As remarked before, the network Laplacian matrix $L = \mathrm{Diag}(\sum_{j \neq l} q_{jl}) - Q = I_n - Q$ in this case can be related to K as $L = -K^T$.

In what follows we will employ the notions of *semi-passivity* and *strict semi-passivity* that have been introduced first in (Pogromsky, 1998). For consistency, we recall these notions below

Definition 1. Consider a system of first-order nonlinear ordinary differential equations

$$\dot{x} = f(t, x, u(t)), y = h(x)$$

were $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous and locally Lipschitz function, $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $u : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $x(\cdot; t_0, x_0, [u])$ be a solution of the Cauchy problem $x(t_0; t_0, x_0, [u]) = x_0$, and let $\mathcal{U} \subset \mathcal{C}^0$ be the set of inputs u for which the solution $x(\cdot; t_0, x_0, [u])$ is defined in forward time.

The system is called semi-passive if there is a non-negative function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (a storage function) and a function $H : \mathbb{R}^n \to \mathbb{R}$ such that for each $x(\cdot; t_0, x_0, [u])$ the following holds for all $t \geq t_0$ in the domain of this solution definition:

$$S(x(t; t_0, x_0, [u])) - S(x_0) \leq \int_{t_0}^{t} y(\tau)u(\tau) - H(x(\tau, t_0, x_0, [u]))d\tau,$$

where the function H is non-negative outside a ball in $\mathbb{R}^n$.

The system is called strictly semi-passive if the function H is strictly positive outside a ball in $\mathbb{R}^n$.

In what follows we demonstrate existence and boundedness of solutions of the coupled system and specify some bounds on the values of coupling strength σ that guarantee global asymptotic synchronization in the systems. Due to space limitations proofs of the corresponding theoretical results are omitted; they can be found, however, in (Gorban et al., 2015).

3.1 Boundedness of solutions in the coupled system

We begin the analysis with demonstrating that solutions of the networks being investigated are defined for all $t \geq 0$ and are bounded in forward time. In particular, the following statement holds:

Lemma 2. The solutions of the ring network of FHN neurons are ultimately bounded uniformly in x_0, $\sigma \in \mathbb{R}_{\geq 0}$. That is, there is a compact set $\Omega \in \mathbb{R}^{2n}$ such that for all $x_0 \in \mathbb{R}^{2n}$, $\sigma \in \mathbb{R}_{\geq 0}$

$$\lim_{t \to \infty} \mathrm{dist}\,(x(t, x_0, \sigma), \Omega) = 0.$$

The proof is based on the fact that FHN neuron is strictly semi-passive (see also (Steur et al., 2009)), and we refer the reader to (Gorban et al., 2015) for further details. Note also that similar statement can be formulated for the chain of FHN neurons.

3.2 Sufficient conditions for synchronization

Directed chain First we consider the dynamics of two coupled systems in the leader-follower configuration:

$$\begin{cases} \dot{z}_1 = \alpha\,(y_1 - \beta z_1) \\ \dot{y}_1 = y_1 - \gamma y_1^3 - z_1 \end{cases} \tag{7}$$

$$\begin{cases} \dot{z}_2 = \alpha\,(y_2 - \beta z_2) \\ \dot{y}_2 = y_2 - \gamma y_2^3 - z_2 + \sigma(y_1 - y_2) \end{cases} \tag{8}$$

Theorem 3. Consider the system of coupled FHN oscillators (7) in which the parameter σ is chosen so that

$$\sigma > 1.$$

Then solutions of the system asymptotically synchronize for all values of initial conditions.

Generalizing two coupled systems to a directed chain of n oscillators, we observe that the Laplacian matrix of this configuration is

$$L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

The matrix L has only real eigenvalues; a simple zero eigenvalue and $n - 1$ eigenvalues equal to 1. The only type of stable correlated oscillations we can find in the chain are the completely synchronous oscillations. These synchronous oscillations will emerge for values of the coupling strength σ for which the chain of 2 FHN oscillators synchronize. Thus the conditions for synchronization are independent of the size of the network (i.e. the length of the chain).

Directed ring Suppose now that the n-th oscillator is feeding back its output to the input of 1st, that is the network topology is that of the directed ring. As we shall see later the presence of such an extra connection has a drastic effect on the system's performance with respect to the coupling strength needed to maintain stable full-state synchrony. This is reflected in the statement of the theorem below.

Theorem 4. Consider the system of coupled FHN oscillators (4), (5), and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the symmetrized Laplacian of the network $\frac{1}{2}(L + L^T)$. Then solutions of the coupled system asymptotically synchronize providing that

$$\sigma\lambda_2 > 1.$$

Corollary 5. For the network of n coupled FHN oscillators, solutions globally asymptotically synchronize if the following inequality holds:

$$\sigma\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) > 1.$$

3.3 Synchronization and rotating waves

The results in the previous sections show that, on the one hand, when a system has a directed ring topology and the number of systems in the ring grows then their relative dynamics becomes more and more under-damped (Theorem 1). On the other hand, in accordance with Corollary 5, estimates of attraction rates of the diagonal synchronization manifold rapidly diminish to zero for increasing number of systems. The latter results are, however, sufficient and may be conservative. To get a clearer view of the network dynamics we performed an exhaustive numerical exploration of the system dynamics for various values of coupling strengths σ as well as the network sizes n.

We construct a grid (n, σ) for number of systems $n = 2, \ldots, 20$ and coupling strengths $\sigma = \{0.05, 0.1, 0.15, \ldots, 10\}$. For each (n, σ) 100 sets of initial conditions are drawn uniformly randomly from the domain $|y_i(0)| \leq \frac{3}{2}\sqrt{3}$ and $|z_i(0)| \leq \frac{15}{8}\sqrt{3}$, which can be shown to be positively invariant for both connectivity configurations (i.e. the directed simple cycle and the directed chain). The *MATLAB* numerical solver *ode45* was used with relative and absolute error tolerances of order 10^{-5} to integrate dynamics for a maximum of $20,000$ time steps. At regular intervals of $1000, 2000, \ldots, 20,000$ time steps we interrupt integration to check for synchronization or rotating wave solutions. After $20,000$ time steps, if neither synchronization nor a rotating wave solution is detected, we register 'no solution'. Note that coupling is non-invasive for the case of synchronization but invasive for the case of the rotating wave.

Synchronization is identified in terms of the absolute error between the states of neighbouring systems averaged over a 1000 time step window being less than 2×10^{-5}. In case of no synchronization, we investigate the existence of rotating waves of Mode Type 1. Rotating waves are defined as periodic solutions where all systems take identical orbits with constant non-zero and equal phase shifts between neighbouring systems. The mode type describes the group velocity of the wave; for a periodic wave, Mode Type 1 describes the case where the period of a rotating wave having non-zero wave velocity equals the period of individual oscillators. Mode types other than 1 correspond to integer multiples of the period. Identical orbits are identified if the absolute difference between the time shifted orbits - so that orbits are in-phase - of neighbouring systems averaged over the period of the orbit is less than 10^{-4}. Constant and equal phase shifts (for a Mode Type 1 rotating wave) are identified if the maximum from all absolute differences between n times the phase shifts between pairwise neighbouring systems and period T is less than a tolerance of 10^{-2}.

The results of this exploration are summarized in Fig. 1.

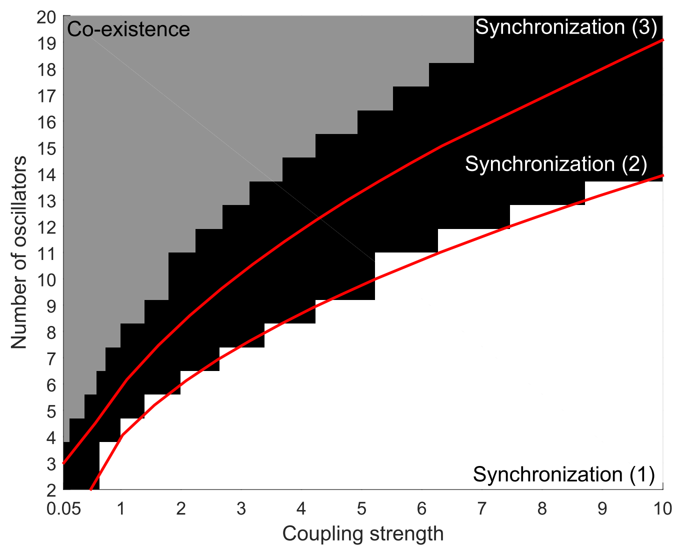

Fig. 1. Bifurcation diagram for directed rings of FHN oscillators. *Synchronization (1)*, global asymptotic synchronization that is guaranteed by the semi-passivity argument; *Synchronization (2)*, synchronization registered for every set of random initial conditions during numerical simulations; *Synchronization (3)* synchronization registered for every set of random initial conditions during numerical simulations, but Floquet stability analysis of solutions of the auxiliary system indicated existence of a locally asymptotically stable rotating wave solution; *Co-existence*, both the fully synchronous and rotating wave solutions were registered during numerical simulation.

Fig. 2 shows for each σ and n the proportion of initial conditions that yield a rotating wave solution of Mode Type 1. For low coupling σ and for increasing number of systems n, rotating wave solutions are more often found. This suggests a larger basin of attraction for the rotating wave than that for synchronization. Fig. 3 shows for each σ and n the proportion of initial conditions that produced a rotating wave solution for all mode types, i.e. rotating waves that resonate with individual systems period of oscillation.

Local stability analysis of the rotating wave Throughout this section we consider only the rotating wave having Mode Type 1. Similar analysis can also be performed for other mode types.

Suppose that n identical coupled systems have a non-constant, T-periodic solution $x_j = (z_j, y_j)$ for constant

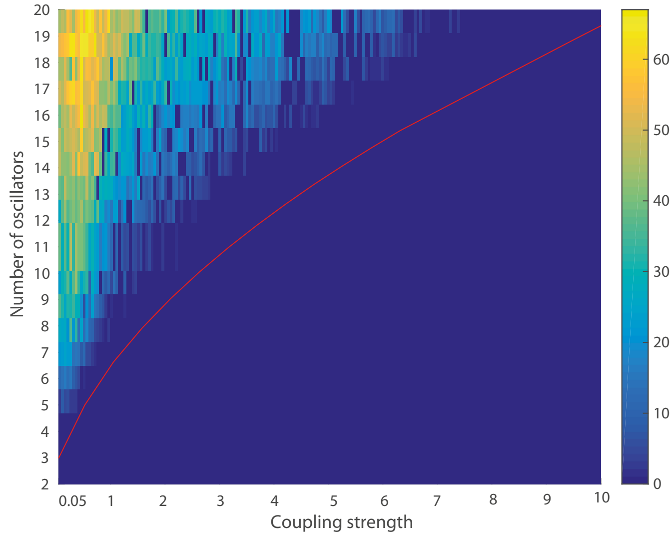

Fig. 2. Proportion of samples that yield a rotating wave solution of Mode Type 1. Red curve shows the boundary the rotating wave is stable determined by Floquet multipliers.

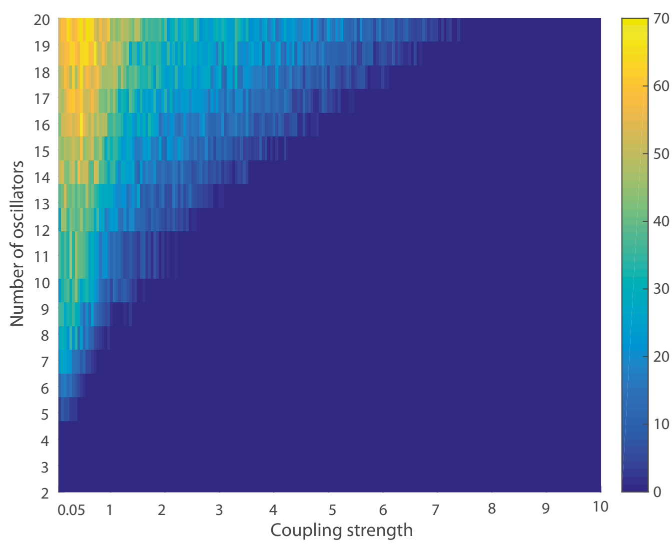

Fig. 3. Proportion of samples that yield a rotating wave solution for all mode types.

$T > 0$, and for which the orbit of each system is identical and time shifted by some constant $\tau = \frac{T}{n}$:

$$\begin{aligned} x_1(t) &= x_2(t+\tau) = x_3(t+2\tau) = \cdots = x_n(t+(n-1)\tau) \\ &= x_1(t+n\tau) = x_1(t+T). \end{aligned} \tag{9}$$

We refer to this as the *rotating wave solution*. Recall (4) with $x_j = (z_j, y_j)$. If we restrict the coupled dynamics of the FHN oscillators to the rotating wave manifold, then using the periodicity of the rotating wave solution, substitution of Equation (9) into the dynamics of each coupled FHN oscillator (4) yields n identical uncoupled delay differential equations (DDEs) of the form

$$\dot{x}_1(t) = f(x_1(t)) + \sigma BC\left(x_1(t - \frac{(n-1)}{n}T) - x_1(t)\right)$$

$$\vdots$$

$$\dot{x}_n(t) = f(x_n(t)) + \sigma BC\left(x_n(t - \frac{(n-1)}{n}T) - x_n(t)\right).$$

Thus the rotating wave solution can only exist if the auxiliary system

$$\begin{aligned} \dot{s}(t) =& f(s(t)) - \sigma BC[s(t) - s(t-\tau^*)], \\ \tau^* :=& T - \tau = \frac{n-1}{n}T, \end{aligned} \tag{10}$$

has a non-constant, T-periodic solution:

$$s_t = s_{t+T} \in \mathcal{C} = \mathcal{C}([0,T], \mathbb{R}^2),$$

for which the set $\mathcal{C}$ is the set of continuous functions that map the interval $[0,T]$ into $\mathbb{R}^2$, and $s_t(\theta) := s(t+\theta)$, $\theta \in [0,T]$.

For the local stability analysis we computed periodic solutions of the auxiliary system (10). The periodic solutions are determined using continuation methods that are available in the numerical software package *DDE-Biftool* (Engelborghs et al., 2001). Stability of rotating wave solutions was assessed by applying Floquet theory to the dynamics of the ring of FHN neurons (eq. (4), (5)) linearized around the rotating wave solution (again with *DDE-Biftool*). Recall that if all Floquet multipliers except one (at 1) have modulus strictly smaller than 1, then the zero solution of the linearized error system is asymptotically stable, which implies the rotating wave solution to be locally asymptotically stable. The red line in Fig. 2 is defined by the crossing of (at least) one multiplier with the boundary of the unit disc in $\mathbb{C}$.

4. DISCUSSION

So far we have been focusing only on extremal properties of a single directed ring. The question, however, is that if new phenomena may emerge when two directed rings are diffusively coupled via an undirected link between an oscillator in each cycle? For a total of $2n$ coupled systems, we constructed two cycles as follows: systems $1, \ldots, n$ formed the first directed ring and systems $n+1, \ldots, 2n$ formed the second. The cycles were diffusively coupled via systems x_1 and x_{n+1} by a undirected link. Clearly the synchronization manifold exists, as does the rotating wave solution in the form of two synchronized rotating waves, $x_1(t) = x_{n+1}(t) = x_2(t+\tau) = x_{n+2}(t+\tau) = \ldots = x_{n-1}(t+(n-2)\tau) = x_{2n-1}(t+(n-2)\tau) = x_n(t+(n-1)\tau) = x_{2n}(t+(n-1)\tau)$. Full description of the phenomena in a system comprised of two coupled cycles is beyond the scope of this work, however, as an illustration of new emergent phenomenon, we present the following brief example. We take $(n, \sigma) = (10, 0.75)$, which for a directed ring lies in the region of co-existence of synchronization and rotating wave solutions. We observe in Figure (4) a stable state in which the trajectories of all systems in the first cycle (in red) are attracted to the synchronization manifold, whilst all trajectories of systems in the second cycle (in green) are attracted to the rotating wave solution. There is a clear competition of each cycle to attract the other to its own dynamical regime. The two diffusively coupled oscillators from each cycle periodically perturb

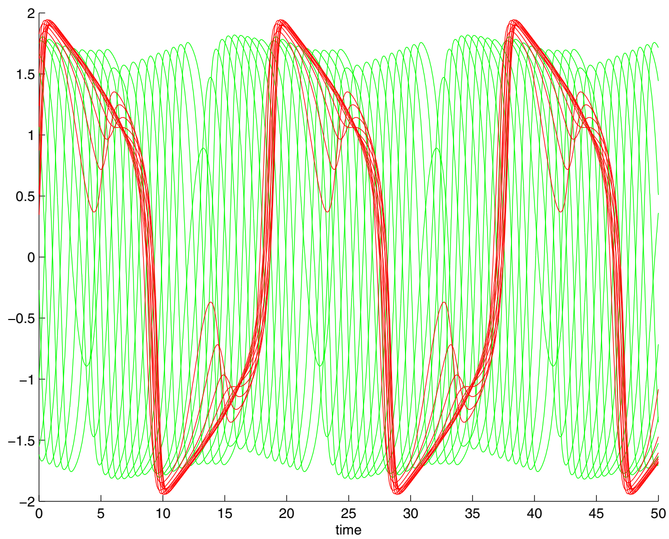

Fig. 4. Two coupled cycles and their y-dynamics; in red the y-dynamics of the first cycle, and in green the y-dynamics of the second cycle.

each other, which prevents asymptotic convergence of systems to either the synchronization manifold or the rotating wave solution. Clearly, the extremal properties of the directed ring can give rise to multiple regimes of complex patterns of dynamics when embedded into larger network structures.

5. CONCLUSION

We considered the problem of how network topology influences the dynamics of collective behavior in the system. We approached the problem by studying how "closing" a chain of interconnected systems with directed coupling by a directed feedback from the last element in the chain to the first may affect the dynamics of the system. Two general settings have been investigated. In the first we analyzed the behavior of a simple linear system. We showed that the simple cycle with equal interaction weights has the slowest decay of the oscillations among all linear systems with the same number of states. In the second setting we considered directed rings and chains of identical nonlinear oscillators. For directed rings of nonlinear oscillators, a lower bound for the connection strengths that guarantee asymptotic synchronization in the network is found to follow a pattern similar to that of a directed ring of linear systems. Furthermore, numerical analysis revealed that, depending on the network size, multiple dynamic regimes co-exist in the system's state space. In addition to the fully synchronous state, for sufficiently large networks or sufficiently low coupling an asymptotically stable rotating wave solution emerges. The phenomenon persists over a broad range of coupling strengths and network sizes, and can be viewed as a form of extreme sensitivity of the network dynamics to removal or addition of a single connection.

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