Uniform asymptotic formulae for Green's tensors in elastic singularly perturbed domains with multiple voids

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Structure of the talk

1. Domain and notations
2. Green’s tensor for 2D elasticity, asymptotic approximation
3. Example: Asymptotic formulae versus numerical solution
4. Anti-plane shear: Green’s function
5. Green’s tensor for 3D elasticity, asymptotic approximation
6. Example: 3D Green’s function for the Laplacian
7. Conclusions
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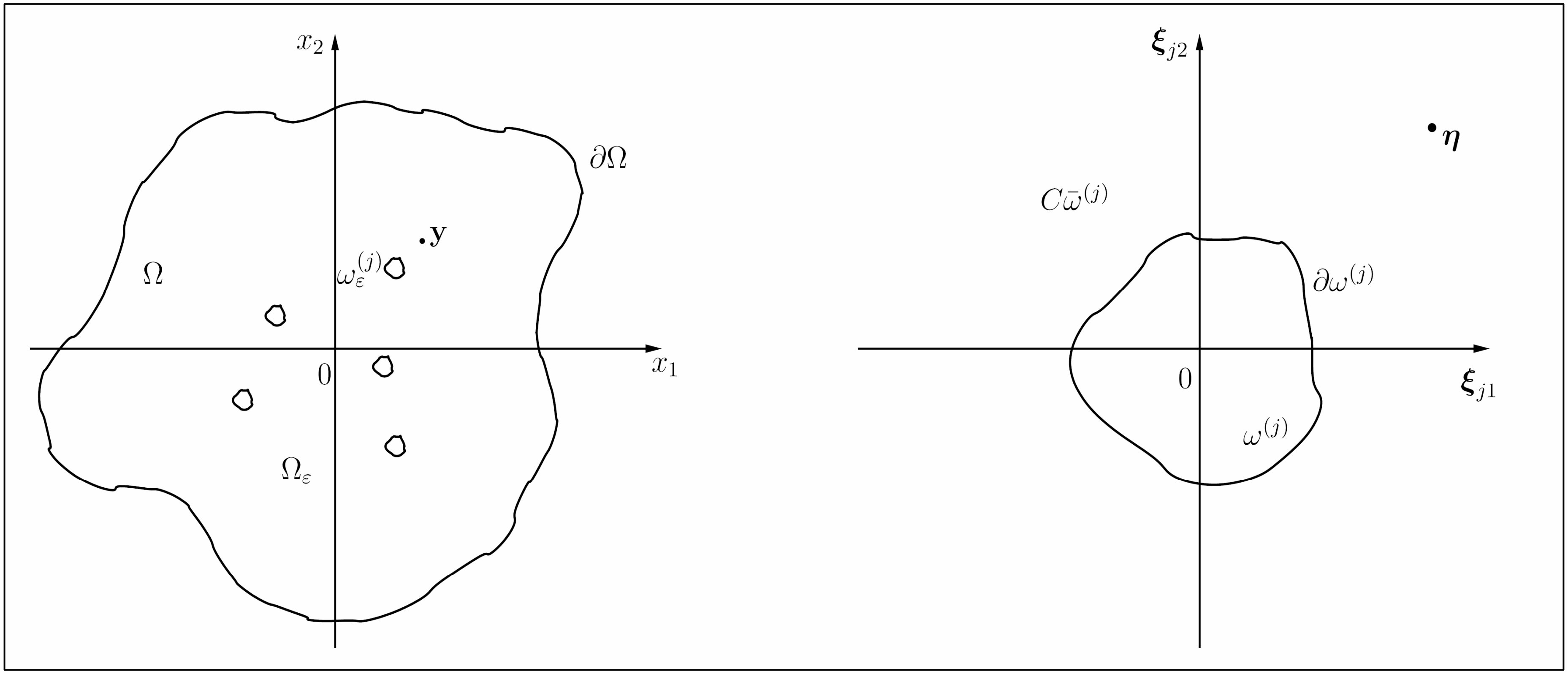
Domain and notations



Scaled variables:

$$\boldsymbol{\xi}_j = \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)})$$
$$\boldsymbol{\eta}_j = \varepsilon^{-1}(\mathbf{y} - \mathbf{O}^{(j)})$$

for $j = 1, \ldots, N$

$$\omega_\varepsilon^{(j)} = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)}) \in \omega^{(j)}\}$$

Two-dimensional elasticity, Green's tensors

Now we proceed with the formulation of the problem of Green's tensor for 2D elasticity with the Dirichlet boundary conditions. In this case we consider the isotropic Lamé operator

$$L(\partial/\partial\mathbf{x}) := \mu\Delta_{\mathbf{x}} + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot\ )\ .$$

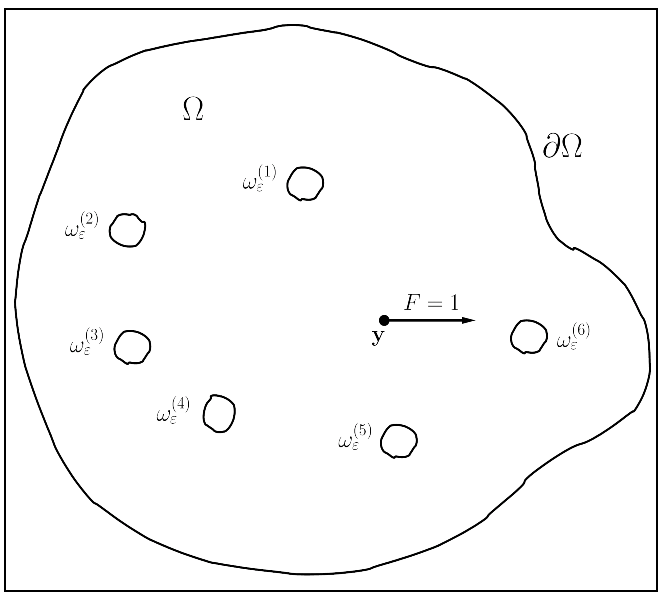
Here λ and μ are elastic moduli.

We also have the fundamental solution $\gamma=[\gamma_{pq}]_{p,q=1}^{2}$ for the Lamé operator whose entries are given by

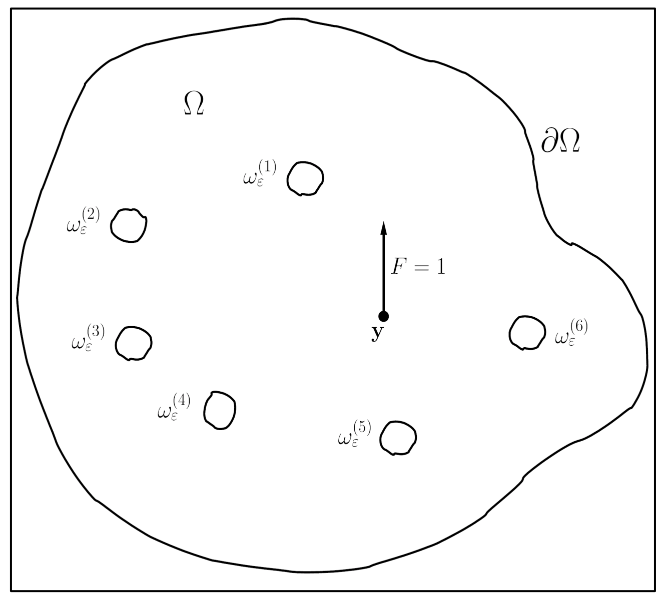
$$\begin{aligned}\gamma_{pq}(\mathbf{x},\mathbf{y}) \quad = \quad & (\lambda+3\mu)(4\pi\mu(\lambda+2\mu))^{-1}(-\log|\mathbf{x}-\mathbf{y}|\delta_{pq} \\ & +(\lambda+\mu)(\lambda+3\mu)^{-1}(x_p-y_p)(x_q-y_q)|\mathbf{x}-\mathbf{y}|^{-2})\ .\end{aligned}$$

Green’s tensor in a domain with several voids

For Green’s tensor, the first column gives displacements corresponding to a force acting parallel to horizontal axis and the second column gives displacements for the case of the force acting parallel to the vertical axis at an arbitrary fixed point in an elastic body.



a)



b)

a) Configuration for first column   b) Configuration for second column

Theorem 1: Green's tensor for the two-dimensional solid with several voids

Green's tensor for the Lamé operator in Ω_ε admits the representation

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - N\gamma(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})$$
$$+\sum_{j=1}^{N}\{P_\varepsilon^{(j)}(\mathbf{x})A^{(j)}P_\varepsilon^{(j)T}(\mathbf{y}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)}\}$$
$$-\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} P_\varepsilon^{(j)}(\mathbf{x})G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P_\varepsilon^{(k)T}(\mathbf{y}) + O(\varepsilon)\,,$$

uniformly with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$, where

$$A^{(j)} = (\lambda+3\mu)(4\pi\mu(\lambda+2\mu))^{-1}\log\varepsilon I_2 + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}\,.$$

Example 1: The regular part of Green's tensor. An elastic half-plane with five circular voids

We consider the right half-plane with five circular voids. Here the point force acts at (250, 50) and $\lambda = \mu = 5.6 \times 10^{10}$ (Cast Iron).

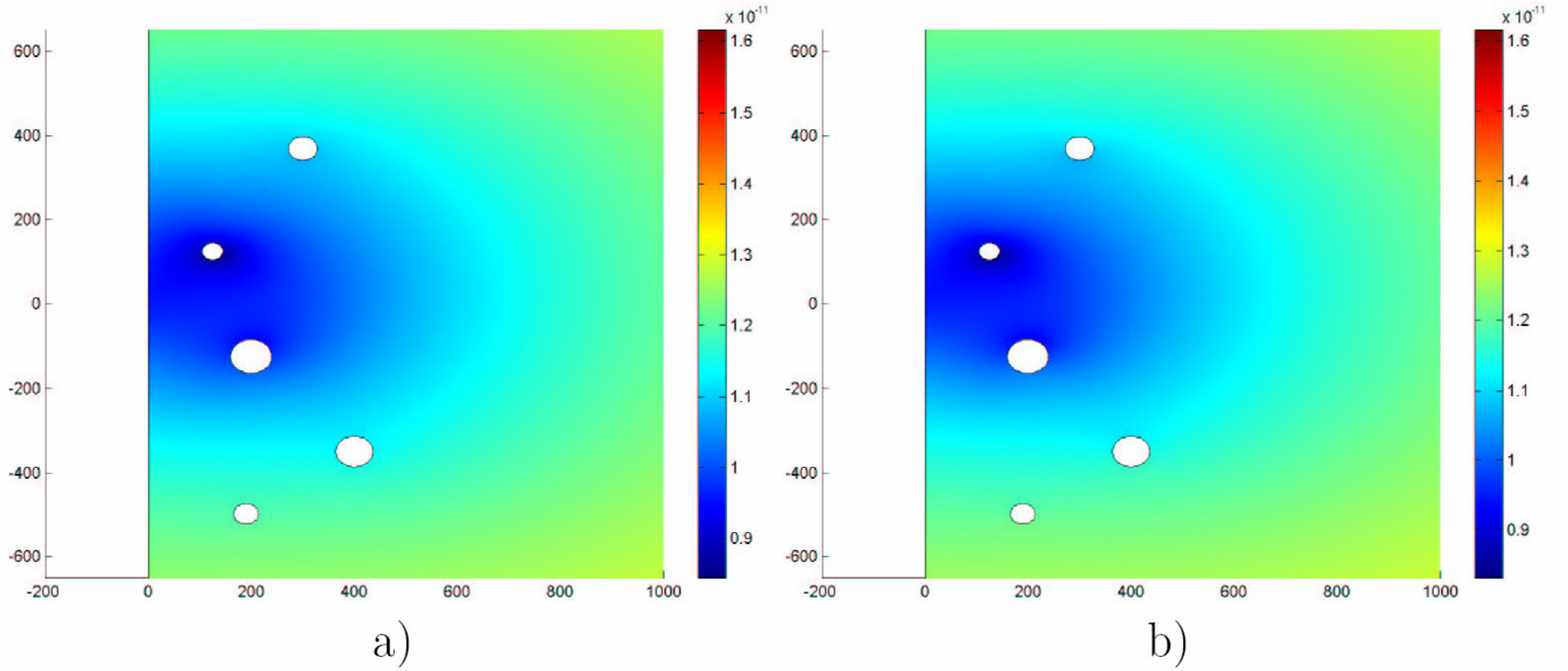
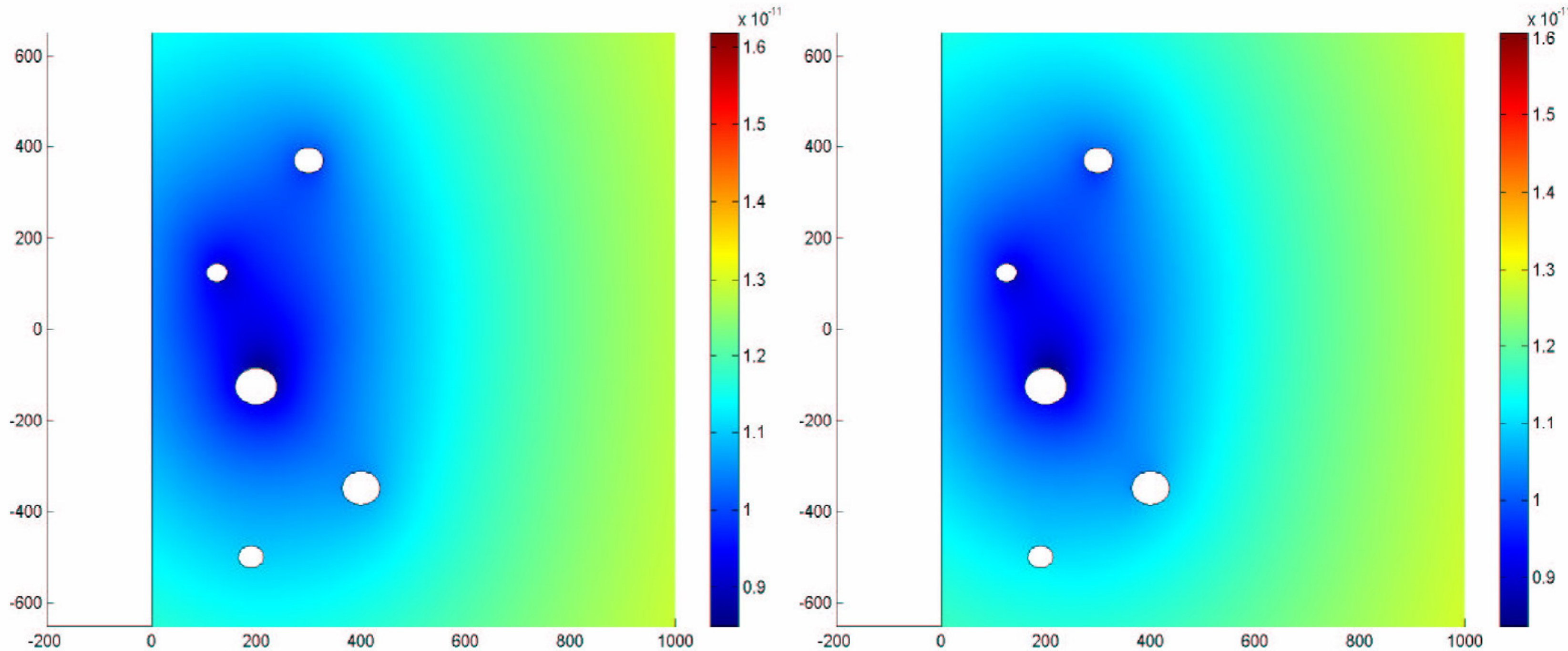


Figure 1 a) Numerical solution produced in FEMLAB on a mesh containing 66480 elements, b) Computations based on the asymptotic formula for the first column of $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.32$.

Example 1: An elastic half-plane with five circular voids (continued)



a) b)

Figure 2.: a) Numerical solution produced in FEMLAB on a mesh containing 66480 elements, b) Computations based on the asymptotic formula for the second column of $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.32$.

Example 1: An elastic half-plane with five circular voids (continued)

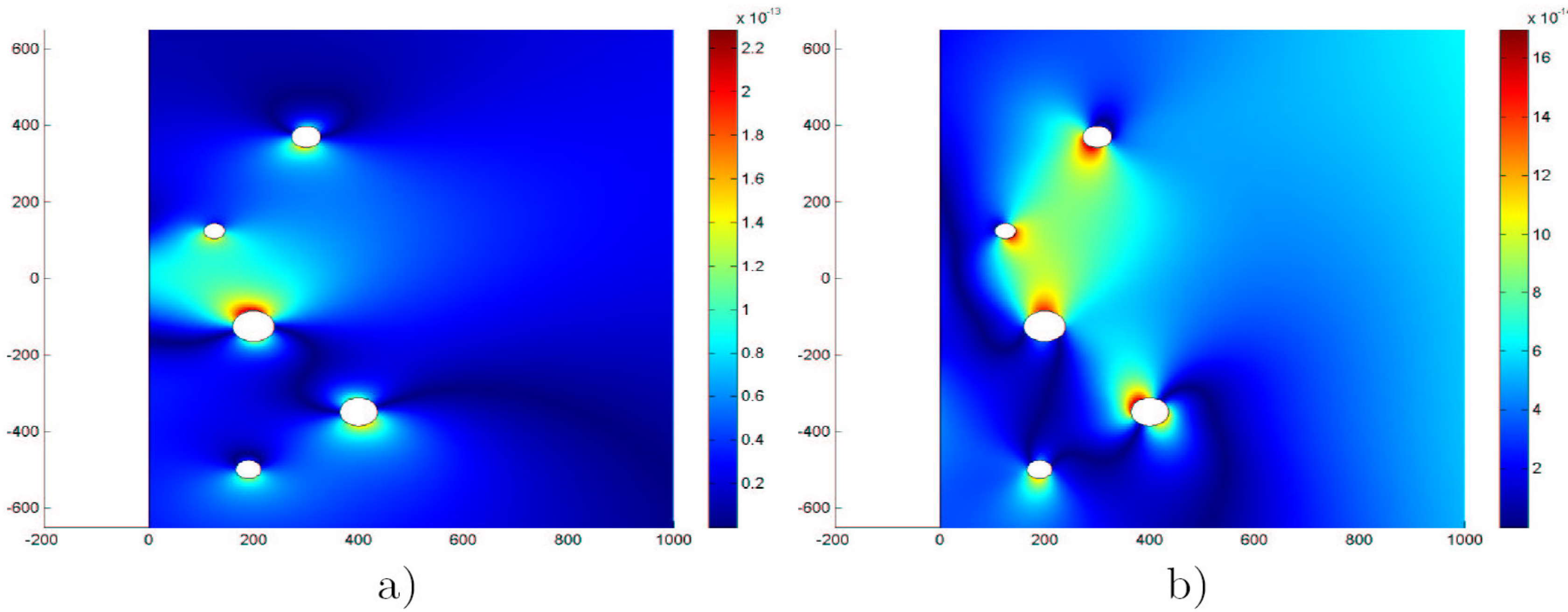


Figure 3: a) Absolute error between computations given in FEMLAB and those by the asymptotic formula for a) the first column and b) the second column of $\mathcal{H}_\varepsilon$ when $\varepsilon = 0.32$ in the vicinity of the inclusions. In a) the maximum absolute error is 2.285×10^{-13}, which occurs on boundary of the inclusion with centre (200, -125), and in b) the maximum absolute error is 1.697×10^{-13}, which occurs on boundary of the inclusion with centre (400, -350).

Green’s function for the case of anti-plane shear

(sketch of technical derivations)

Green's function in Ω_ε for the operator $-\Delta$

We first consider Green's function G_ε for the Laplacian in the domain Ω_ε . The function G_ε is a solution of

$$
\boxed{\begin{gathered}
-\Delta_{\mathbf{x}} G_\varepsilon(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \\
G_\varepsilon(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x}\in\partial\Omega_\varepsilon, \mathbf{y}\in\Omega_\varepsilon\ .
\end{gathered}}
$$

Green's functions in Ω and $C\overline{\varpi}^{(j)}$

Let G and $g^{(j)}$ denote Green's functions for the Laplacian in the domains Ω and $C\overline{\varpi}^{(j)}$, respectively. The function G solves the following

$$
\begin{array}{c}
-\Delta_{\mathrm{x}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Omega, \\
G(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega,
\end{array}
$$

and the functions $g^{(j)}$ are a solution of

$$
\begin{array}{c}
-\Delta_{\boldsymbol{\xi}} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \delta(\boldsymbol{\xi}_j - \boldsymbol{\eta}_j), \quad \boldsymbol{\xi}_j, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}, \\
g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = 0, \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)},
\end{array}
$$

also this formulation is also supplied with the following condition at infinity

$$
g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \text{ is bounded as } |\boldsymbol{\xi}_j| \to \infty, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}.
$$

Green's functions in Ω and $C\overline{\omega}^{(j)}$ (continued)

We represent G and $g^{(j)}$ as

$$G(\mathbf{x},\mathbf{y}) = \underbrace{-(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|}_{\text{fundamental solution for } -\Delta} - H(\mathbf{x},\mathbf{y})\,, \qquad g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = -(2\pi)^{-1}\log|\boldsymbol{\xi}_j-\boldsymbol{\eta}_j| - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\,,$$

where H and $h^{(j)}$ are the *regular parts* of G and $g^{(j)}$, respectively. For the asymptotic algorithm we need the following Lemma

Lemma 1 *For $|\boldsymbol{\xi}_j| > 2$ and $\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}$ the following estimate holds*

$$h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = -(2\pi)^{-1}\log|\boldsymbol{\xi}_j| - \zeta^{(j)}(\boldsymbol{\eta}_j) + O(|\boldsymbol{\xi}_j|^{-1})\,,$$

for $j = 1,\dots,N$.

Here $\zeta^{(j)}$ is the limit of Green's function $g^{(j)}$ at infinity.

Some auxiliary functions

For the asymptotic formula for G_ε , we also introduce the function $\zeta^{(j)}$ and the constant $\zeta^{(j)}_\infty$

$$\zeta^{(j)}(\boldsymbol{\eta}_j) = \lim_{|\boldsymbol{\xi}_j|\to\infty} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \, , \qquad \zeta^{(j)}_\infty = \lim_{|\boldsymbol{\eta}_j|\to\infty} \{\zeta^{(j)}(\boldsymbol{\eta}_j) - (2\pi)^{-1} \log |\boldsymbol{\eta}_j|\} \, .$$

We also have the following estimate for $\zeta^{(j)}$

Lemma 2 *For $|\boldsymbol{\xi}_j| > 2$, the following representation for $\zeta^{(j)}$ holds*

$$\zeta^{(j)}(\boldsymbol{\xi}_j) = (2\pi)^{-1} \log |\boldsymbol{\xi}_j| + \zeta^{(j)}_\infty + O(|\boldsymbol{\xi}_j|^{-1}) \, ,$$

for $j = 1, \ldots, N$.

Two-dimensional equilibrium potential

Let $P_\varepsilon^{(j)}$ be the *equilibrium potential* corresponding to the jth void. The function $P_\varepsilon^{(j)}$ is defined as a solution of

$$\Delta P_\varepsilon^{(j)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon \ ,$$
$$P_\varepsilon^{(j)}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in \partial\Omega \ ,$$
$$P_\varepsilon^{(j)}(\mathbf{x}) = \delta_{ij} \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(i)} \ , \quad i = 1, \ldots, N.$$

Approximation of the equilibrium potential

We shall also make use of the approximation of $P_\varepsilon^{(j)}$. We set

$$P_\varepsilon(\mathbf{x}) = \{P_\varepsilon^{(j)}(\mathbf{x})\}_{j=1}^{N} \ .$$

Approximation of the equilibrium potential (continued)

Theorem 2 *The asymptotic approximation of $P_\varepsilon(\mathbf{x})$ is given by the formula,*

$$P_\varepsilon(\mathbf{x}) = \left(\operatorname*{diag}_{1\le j\le N} \{\alpha_\varepsilon^{(j)}\} - \mathfrak{M} \right)^{-1} \mathcal{S}(\mathbf{x}) + p_\varepsilon(\mathbf{x})$$

where

$$\alpha_\varepsilon^{(j)} = (2\pi)^{-1} \log \varepsilon + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta_\infty^{(j)} \,,$$

$$\mathfrak{M} = \{(1-\delta_{kj}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\}_{k,j=1}^{N} \,,$$

$$\mathcal{S}(\mathbf{x}) = \{-G(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1} \log |\boldsymbol{\xi}_j| - \zeta_\infty^{(j)}\}_{j=1}^{N} \,,$$

and the vector $p_\varepsilon(\mathbf{x})$ is the remainder term such that

$$|p_\varepsilon(\mathbf{x})| \le \mathrm{const}\, \varepsilon (\log \varepsilon)^{-2},$$

uniformly with respect to $\mathbf{x} \in \Omega_\varepsilon$.

Approximation of the equilibrium potential (continued)

Associated with the preceding result is the asymptotic identity (which will be used in the algorithm)

$$P_\varepsilon^{(j)}(\mathbf{x}) = \Big(- G(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1} \log|\boldsymbol{\xi}_j| - \zeta_\infty^{(j)} + \sum_{\substack{k \neq j \\ 1 \le k \le N}}^{N} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) \Big) (\alpha_\varepsilon^{(j)})^{-1} + O(\varepsilon (\log \varepsilon)^{-1}) , \quad (1)$$

Theorem 3: A uniform asymptotic formula for G_ε for the operator $-\Delta$ in two-dimensions

Green's function for the operator $-\Delta$ in Ω_ε admits the representation

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + N(2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\ &+ \sum_{j=1}^{N}\left\{\alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{x}) P_\varepsilon^{(j)}(\mathbf{y}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)}\right\} \\ &- \sum_{j=1}^{N} \sum_{\substack{k\neq j \\ 1\leq k\leq N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) P_\varepsilon^{(j)}(\mathbf{x}) P_\varepsilon^{(k)}(\mathbf{y}) + O(\varepsilon) \end{aligned}$$

uniformly with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$, where

$$\alpha_\varepsilon^{(j)} = (2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}\ .$$

Proof: The Algorithm

For the asymptotic algorithm we propose that G_ε be written as follows

$$G_\varepsilon(\mathrm{x},\mathrm{y}) = -(2\pi)^{-1}\log|\mathrm{x}-\mathrm{y}| - H_\varepsilon(\mathrm{x},\mathrm{y}) - \sum_{j=1}^{N} h_\varepsilon^{(j)}(\mathrm{x},\mathrm{y})\ ,$$

where it is sufficient to seek the approximation the functions $H_\varepsilon(\mathrm{x},\mathrm{y})$ and $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ which solve

$$\begin{gathered}
\Delta_{\mathbf{x}} H_\varepsilon(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \\
H_\varepsilon(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \\
H_\varepsilon(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon, 1\le j\le N\ ,
\end{gathered}$$

and

$$\begin{gathered}
\Delta_{\mathbf{x}} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \\
h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \\
h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon\ , \\
h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon\ , 1\le k\le N\ , k\ne j\ .
\end{gathered}$$

Some Remarks

The same algorithm can be applied to the case of two dimensional elasticity. We use for the remainder estimates the result:

Let $\mathbf{u}$ be the displacement vector which satisfies the Dirichlet boundary value problem in the domain $\Omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$

$$L(\partial_{\mathbf{x}})\,\mathbf{u}(\mathbf{x}) = \mathbf{O}\,, \quad \mathbf{x} \in \Omega_\varepsilon\,, \tag{1}$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x})\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{2}$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\varphi}_j(\varepsilon^{-1}\mathbf{x})\,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}\,, 1 \le j \le N\,, \tag{3}$$

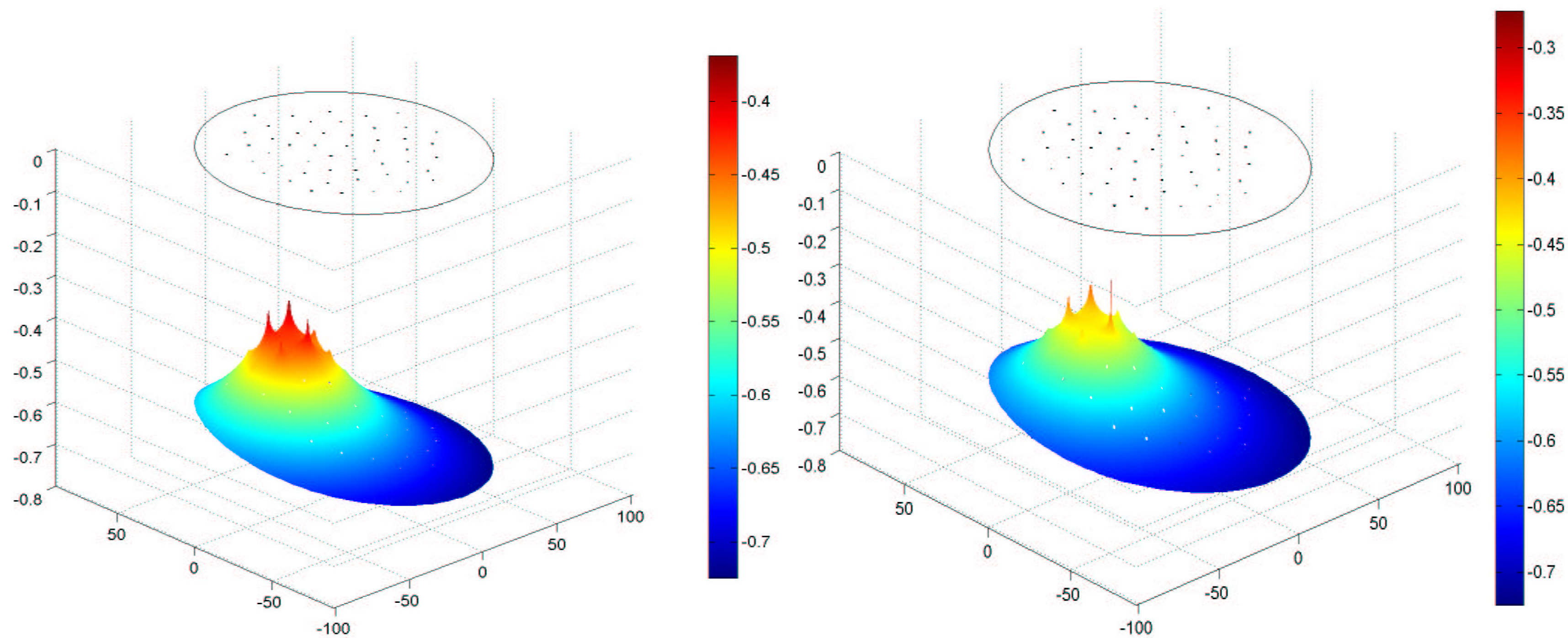
where $\mathbf{O}$ is the zero vector, and we assume that $\boldsymbol{\varphi}_j$ and $\boldsymbol{\psi}$ are continuous vector functions.

Lemma 3 *There exists a unique solution* $\mathbf{u} \in C(\bar{\Omega}_\varepsilon)$ *of problem* (1) – (3) *which satisfies the estimate*

$$\max_{\bar{\Omega}_\varepsilon} |\mathbf{u}(\mathbf{x})| \le \text{const}\max\{\max_{1\le j\le N}\{\|\boldsymbol{\varphi}_j\|_{C(\partial\omega_\varepsilon^{(j)})}\}\,, \|\boldsymbol{\psi}\|_{C(\partial\Omega)}\}\,. \tag{4}$$

Example 2: The regular part of Green's function. The case of a large number of holes

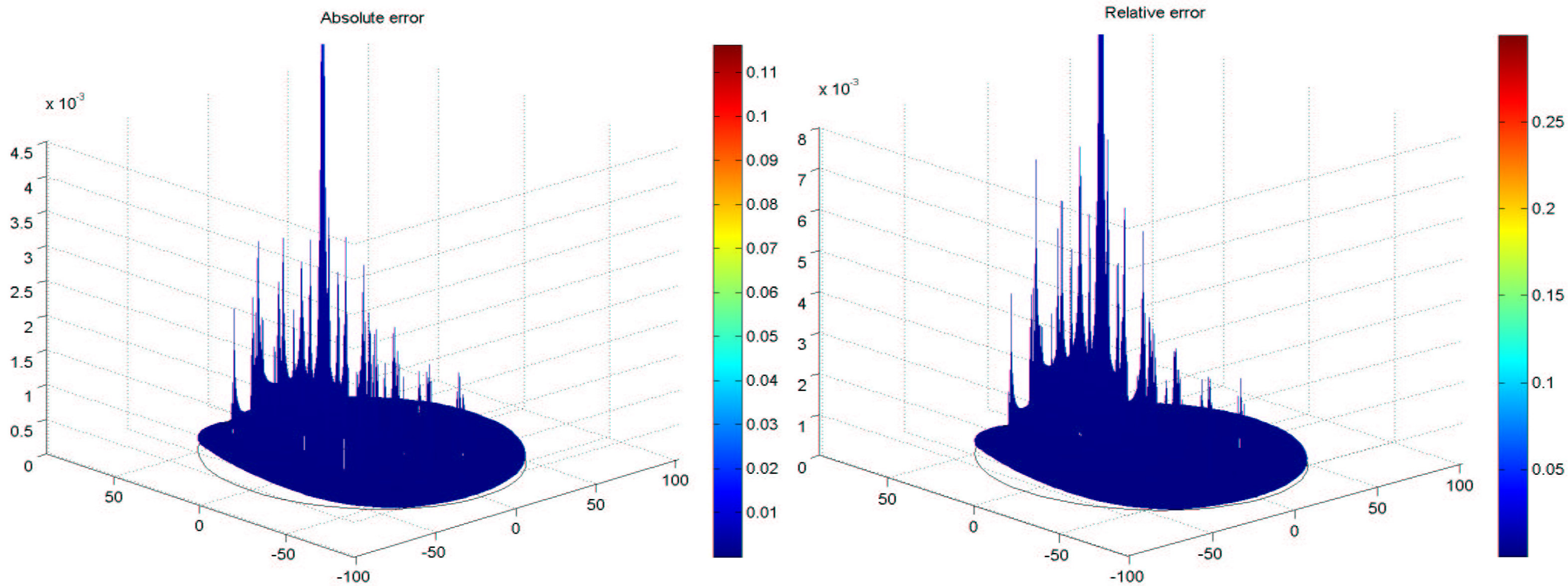
We take Ω to be a disk of radius 70 centred at the origin containing 50 small disks whose radii not exceed 0.5. The force is located at the point (-20, 15).



a) b)

Figure 4 a) Numerical solution produced in FEMLAB on a mesh containing 188112 elements, b) Computation based on the asymptotic formula for $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.0498$.

Example 2: The case of a large number of holes (continued)



a) b)

Figure 5 a) Absolute error and b) relative error between numerical solution and the computations produced by the asymptotic formula for $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.0498$ and the mesh contains 188112 elements. All the spikes occur on the boundaries of the holes. Maximum absolute error is 0.1162, maximum relative error is 0.2995, which is attained on the boundary of the inclusion with centre (-20, 4), near the point (-20, 15) where the force is applied.

Example 2: The case of a large number of holes (continued)

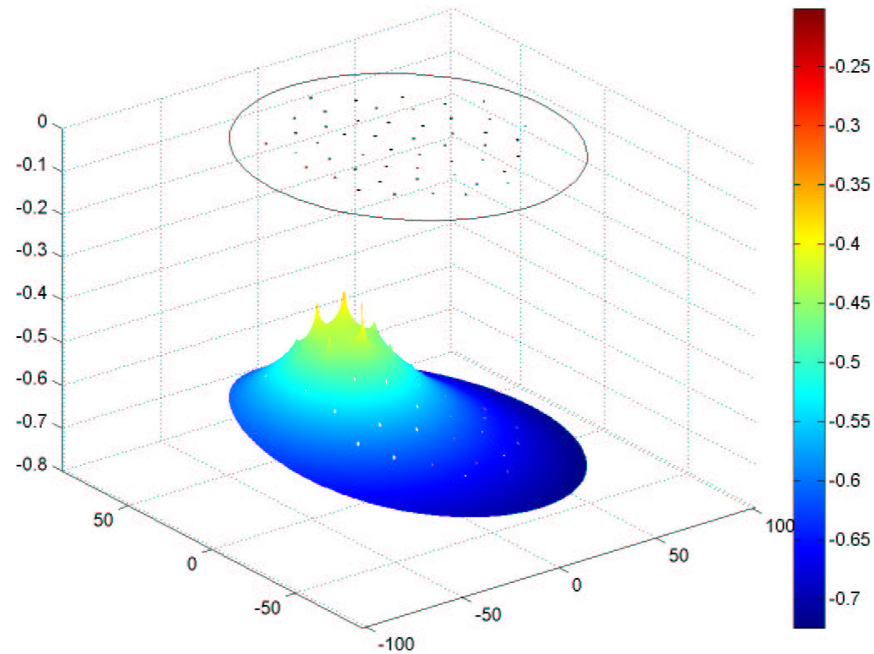


Figure 6: The computation based on the asymptotic formula for the regular part $\mathcal{H}_\varepsilon$ of Green's function on the refined mesh, when $\varepsilon = 0.0498$ and the mesh contains 752448 elements.

Example 3: The configuration with holes of relatively large size

Now we take Ω to be a disk of radius 150, again centred at the origin with 5 circular holes whose radii where varied throughout this example. The force now acts at the point (-25, 70)

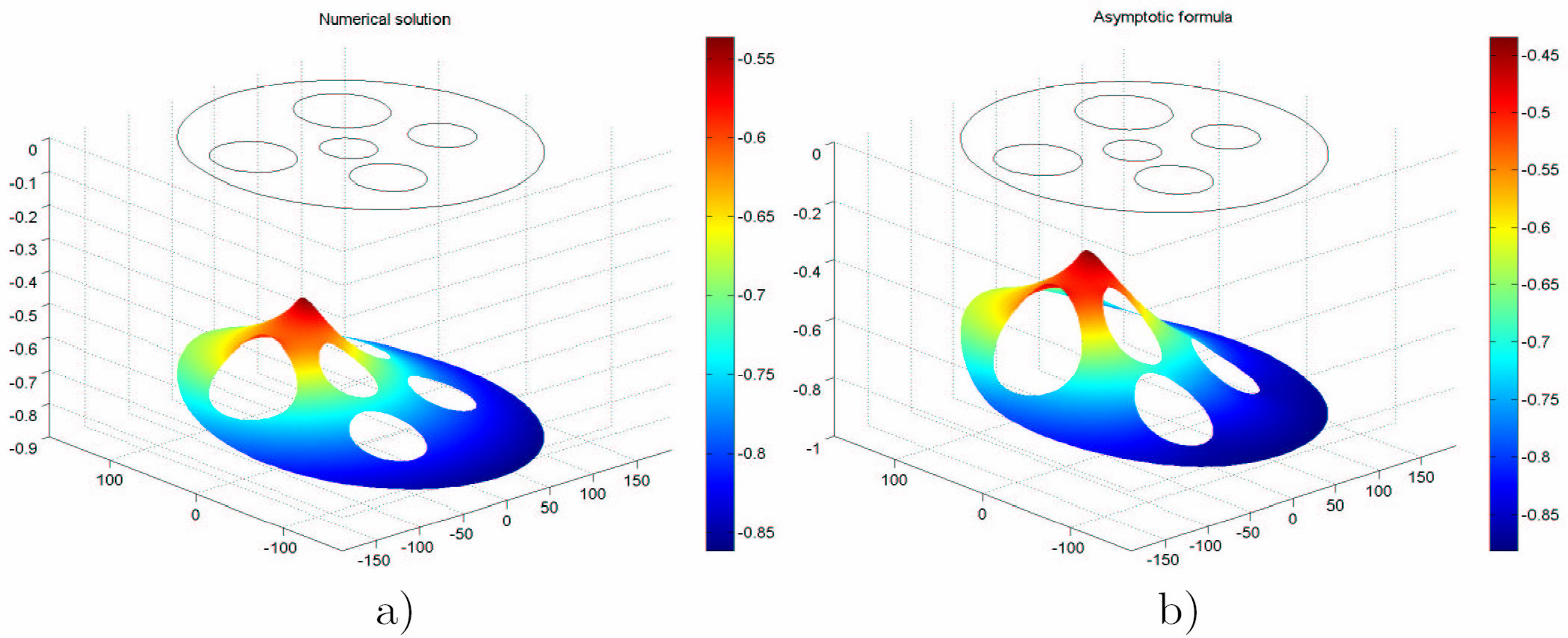


Figure 7 a) Computations produced by FEMLAB, b) Computations produced by the asymptotic formula for $\mathcal{H}_\varepsilon$ for the case $\varepsilon = 0.7436$.

Example 3: The configuration with holes of relatively large size (continued)

For the numerical experiments we define $\varepsilon = m/d$, as a non-dimensional parameter, where m is the maximum radius of all the holes and

$$d = \min\{ \min_{1 \le j \le N} \{\text{dist}(\mathbf{O}^{(j)}, \partial\Omega)\}, \min_{1 \le i,k \le N} \{\text{dist}(\mathbf{O}^{(i)}, \mathbf{O}^{(k)})\}\}$$

Example 3: The configuration with holes of relatively large size (continued)

m	ε	A_{max}	R_{max}
40	0.7436	0.1219	0.1991
36	0.6692	0.09741	0.157
32	0.5949	0.07637	0.1216
28	0.5205	0.05845	0.09204
24	0.4462	0.04335	0.06752
20	0.3718	0.0308	0.04749
16	0.2974	0.0206	0.03156
12	0.2231	0.01298	0.02
8	0.1487	0.007266	0.0111
4	0.0744	0.001395	0.004503
2	0.0372	0.0006608	0.001991
1	0.0186	0.002993	0.0009269
0.5	0.0093	0.0003156	0.0004448
0.25	0.0046	0.0001515	0.0002171

Table 1: Maximum absolute and relative error corresponding to various values of ε.

Example 3: The configuration with holes of relatively large size (continued)

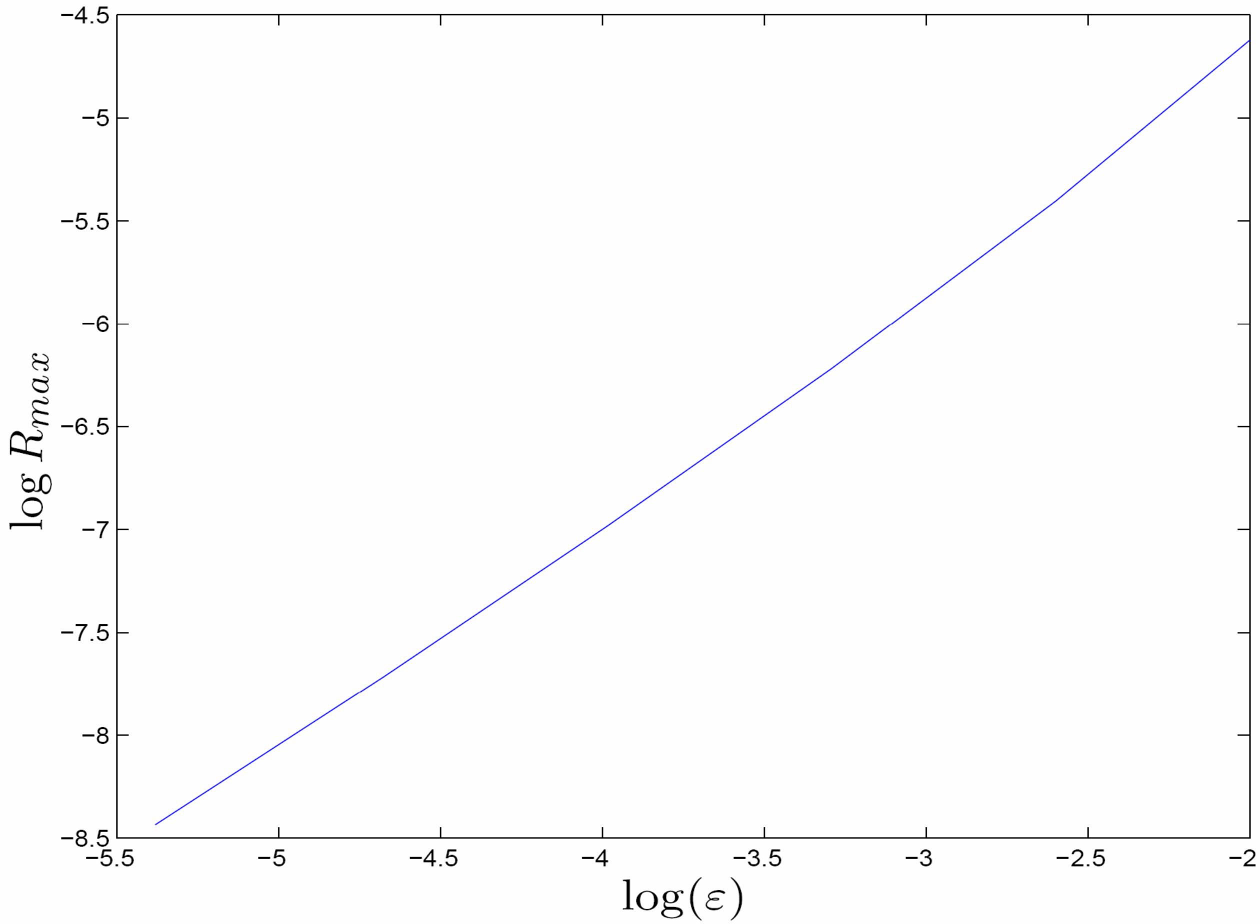


Figure 6: Plot of $\log(\varepsilon)$ against $\log R_{max}$.

Green's tensors for three dimensional elasticity (the sketch of technical derivations)

Green's tensor for a 3D elastic solid with several voids

Now G_ε is a 3x3 matrix, and is defined as a solution of

$$\mu \Delta_{\mathbf{x}} G_\varepsilon(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot G_\varepsilon(\mathbf{x},\mathbf{y})) + \delta(\mathbf{x}-\mathbf{y}) I_3 = 0 I_3 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ ,$$

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = 0 I_3 \ , \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ ,$$

where Ω_ε is a 3-dimensional elastic perturbed body (with multiple voids). As before, we shall use the notation L for the Lamé operator.

Model problems of 3D elasticity

We once again use the model tensors G and $g^{(j)}$ defined in Ω and $C\bar{\omega}^{(j)} = \mathbb{R}^3 \backslash \bar{\omega}^{(j)}$, respectively. The tensor G is a solution of

$$L(\partial_{\mathbf{x}})G(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y})I_3 = 0I_3 \;, \quad \mathbf{x},\mathbf{y} \in \Omega \;,$$
$$G(\mathbf{x},\mathbf{y}) = 0I_3 \;, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega \;,$$

and the tensors $g^{(j)}$ solve the following problem

$$L(\partial_{\boldsymbol{\xi}_j})g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + \delta(\boldsymbol{\xi}_j-\boldsymbol{\eta}_j)I_3 = 0I_3 \;, \quad \boldsymbol{\xi}_j,\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;,$$
$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = 0I_3 \;, \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;,$$
$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}_j| \to \infty, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;.$$

Model problems of 3D elasticity (continued)

We also represent G as

$$G(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \ ,$$

where $\Gamma(\mathbf{x},\mathbf{y}) = [\Gamma_{ij}(\mathbf{x},\mathbf{y})]_{i,j=1}^{3}$ is the fundamental solution of the Lamé operator in 3-dimensions, whose entries are given by

$$\begin{aligned}\Gamma_{ij}(\mathbf{x},\mathbf{y}) \quad &= \quad (8\pi\mu(\lambda+2\mu)|\mathbf{x}-\mathbf{y}|)^{-1}((\lambda+\mu)(x_i-y_i)(x_j-y_j)|\mathbf{x}-\mathbf{y}|^{-2} \\ &\quad +(\lambda+3\mu)\delta_{ij}) \ ,\end{aligned}$$

and H is the regular part of Green's tensor G .

Model problems of 3D elasticity (continued)

Also let $h^{(j)}$ be the regular part of $g^{(j)}$, then this function solves

$$L(\partial_{\boldsymbol{\xi}_j})h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = 0I_3\ , \quad \boldsymbol{\xi}_j,\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}\ ,$$

$$h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = \Gamma(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}\ ,$$

$$h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}_j| \to \infty, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}\ ,$$

and we have the estimate

Lemma 2 *For all* $\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}$ *and* $\boldsymbol{\xi}_j$ *with* $|\boldsymbol{\xi}_j| > 2$, *the following estimate for* $h^{(j)}$ *holds*

$$h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = \Gamma(\boldsymbol{\xi}_j,\mathbf{O})P^{(j)T}(\boldsymbol{\eta}_j) + O(|\boldsymbol{\xi}_j|^{-2}|\boldsymbol{\eta}_j|^{-1})\ ,$$

where $j = 1,\ldots,N$.

The elastic capacitary potential matrix

Let $P^{(j)}(\boldsymbol{\xi}_j)$ be the elastic capacitary potential matrix for the set $C\bar{\omega}^{(j)}$, defined as a solution of the following problem

$$L(\partial_{\boldsymbol{\xi}_j})P^{(j)}(\boldsymbol{\xi}_j) = 0I_3 \ , \quad \boldsymbol{\xi}_j \in C\bar{\omega}^{(j)} \ ,$$

$$P^{(j)}(\boldsymbol{\xi}_j) = I_3 \ , \quad \boldsymbol{\xi}_j \in \partial\omega^{(j)} \ ,$$

$$P^{(j)}(\boldsymbol{\xi}_j) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}_j| \to \infty \ ,$$

We also introduce the elastic capacity matrix $B^{(j)}$ of the set $C\bar{\omega}^{(j)}$. This is a constant symmetric matrix.

The elastic capacitary potential matrix

We also need the following result related to the elastic capacitary potential, for the asymptotic algorithm in 3D

Lemma 1 *i) If* $|\boldsymbol{\xi}_j| \geq 2$*, then for* $P^{(j)}$*, the following estimate holds*

$$P^{(j)}(\boldsymbol{\xi}_j) = \Gamma(\boldsymbol{\xi}_j, \mathbf{O})B^{(j)} + O(|\boldsymbol{\xi}_j|^{-2}) ,$$

where $B^{(j)}$ *is the symmetric elastic capacity matrix of the set* $\omega^{(j)}$*.*

ii) The columns $P^{(j,i)}$*,* $i = 1, 2$ *or* 3*, of the elastic capacitary potential of the set* $\omega^{(j)}$*,* $j = 1, \dots, N$*, satisfy the inequality*

$$\sup_{\boldsymbol{\xi}_j \in C\overline{\omega}^{(j)}} \{|\boldsymbol{\xi}_j||P^{(j,i)}(\boldsymbol{\xi}_j)|\} \leq \text{const} , \quad j = 1, \dots, N .$$

Theorem 4: A uniform asymptotic formula for Green's tensor for the Lamé operator in 3-dimensions

Green's tensor for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^3$ admits the representation

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}\sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - N\Gamma(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N}\Big\{P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y})$$

$$+H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) - P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) - \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\Big\}$$

$$+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\leq k\leq N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O\left(\sum_{j=1}^{N}\varepsilon^2(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|\,,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right)\,,$$

which is uniform with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

The Algorithm for 3D elasticity

In a similar way to the case of anti-plane shear we represent G_ε as

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) - H_\varepsilon(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ ,$$

where H_ε and $h_\varepsilon^{(j)}$ are matrices which solve the problems

$$L(\partial_\mathbf{x})H_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ ,$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ ,$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon, 1 \le j \le N \ ,$$

and

$$L(\partial_\mathbf{x})h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ ,$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ ,$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \ ,$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \ , 1 \le k \le N \ , k \ne j \ .$$

Theorem 5: A uniform asymptotic formula for Green's function for the Laplacian in 3D

Green's function for the operator $-\Delta$ in $\Omega_\varepsilon \subset \mathbb{R}^3$ admits the representation

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}\sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - N(4\pi|\mathbf{x}-\mathbf{y}|)^{-1} + \sum_{j=1}^{N}\Big\{H(\mathbf{O}^{(j)},\mathbf{y})P^{(j)}(\boldsymbol{\xi}_j)$$

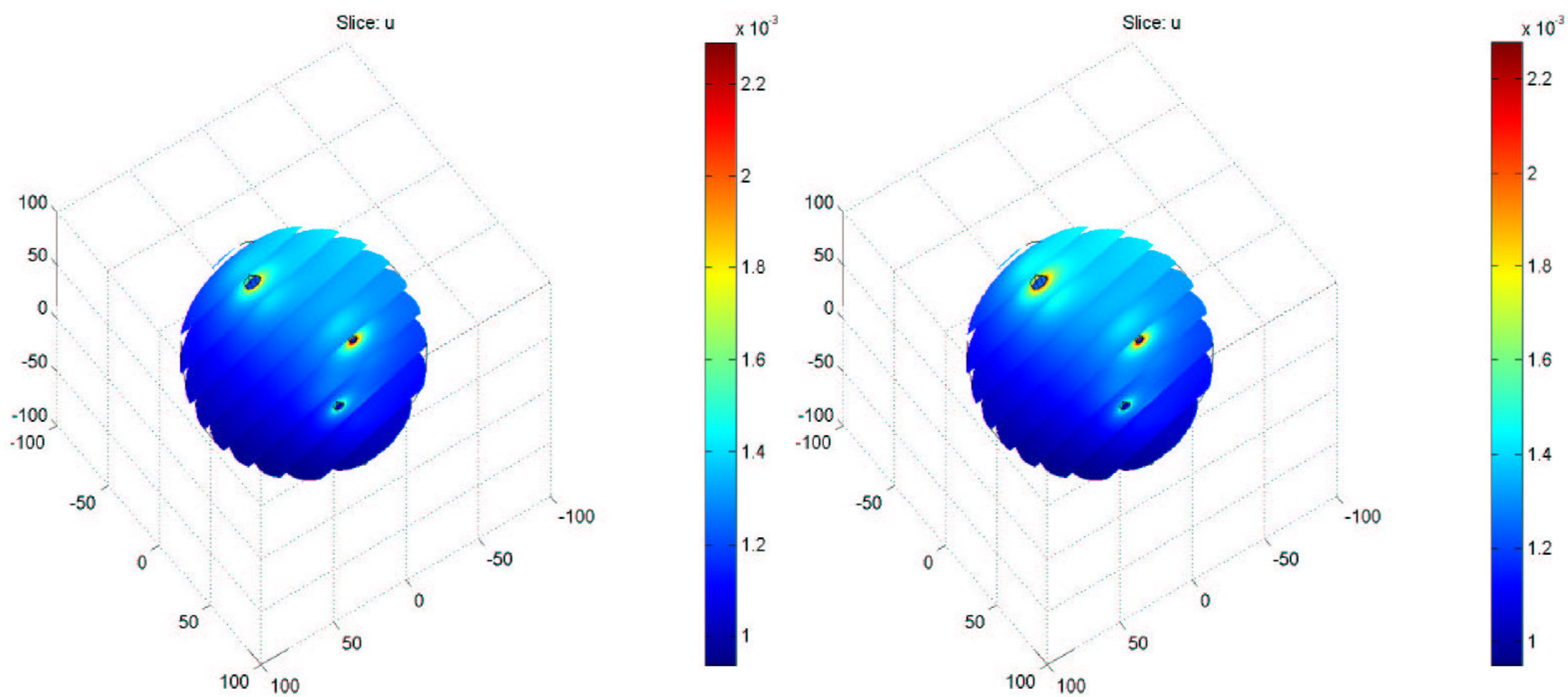
$$+H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)}(\boldsymbol{\xi}_j)P^{(j)}(\boldsymbol{\eta}_j) - \varepsilon \operatorname{cap} \bar{\omega}^{(j)} H(\mathbf{x},\mathbf{O}^{(j)})H(\mathbf{O}^{(j)},\mathbf{y})\Big\}$$

$$+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(k)}(\boldsymbol{\xi}_k)P^{(j)}(\boldsymbol{\eta}_j) + O\left(\sum_{j=1}^{N}\varepsilon^2(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|\,,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right),$$

which is uniform with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

Example: The regular part of Green’s function in 3D. A spherical body with five spherical voids

We consider a spherical body with five spherical voids. Here the point force acts at $\mathbf{y} = (-10, -10, 5)$.



a) b)

Figure 1: a) Numerical solution produced in FEMLAB, b) Computations based on the asymptotic formula for the regular part of Green’s function.

Example: The regular part of Green's function in 3D. A spherical body with five spherical voids (continued)

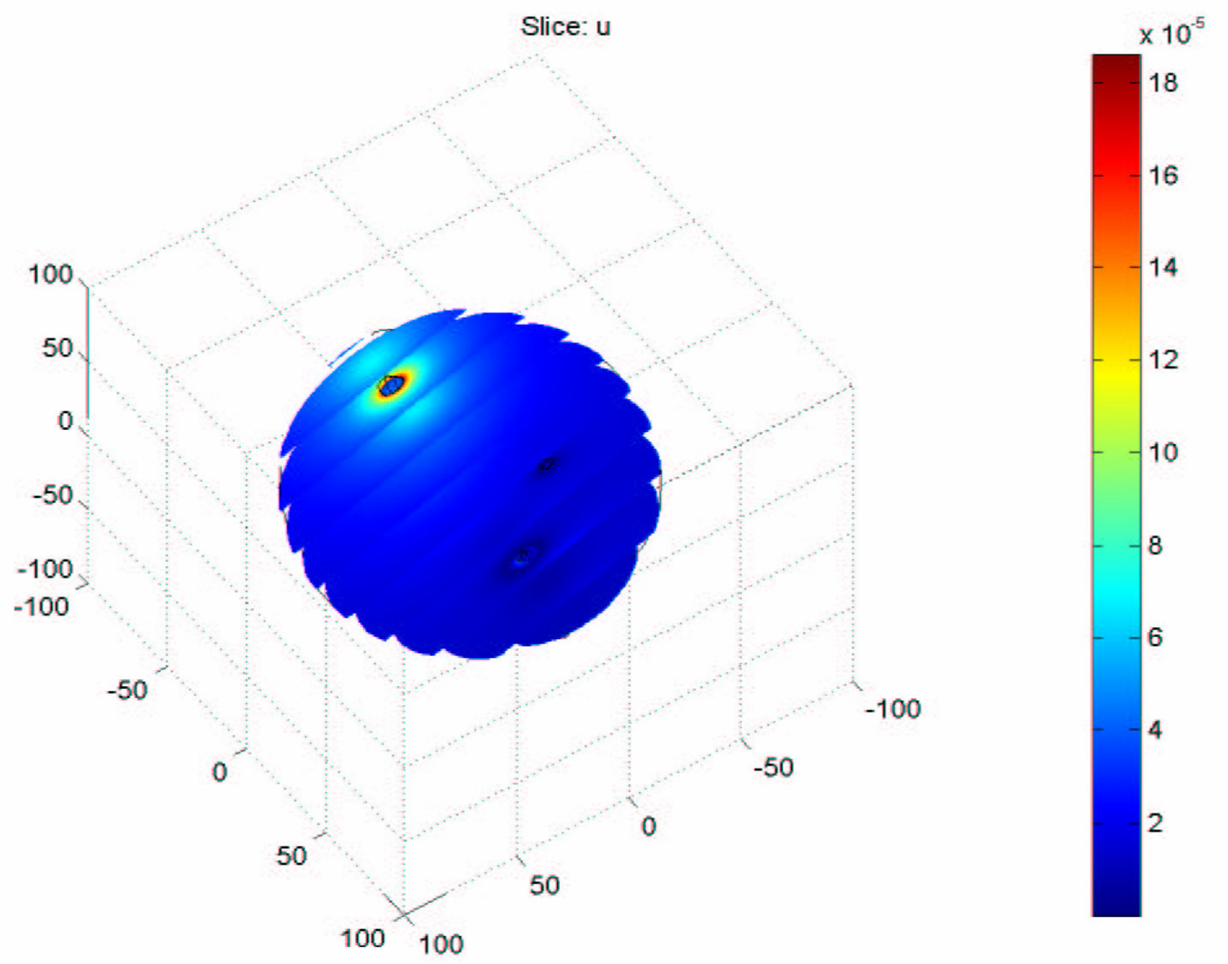


Figure 2: Absolute error between computations given in FEMLAB and those by the asymptotic formula for the regular part of Green's function.

Conclusions

For the asymptotic formulae, for Green's tensors in the perturbed domain, we may draw these conclusions:

- The new feature of the asymptotic formulae is their uniformity with respect to the independent spatial variables
- The asymptotic algorithm produces formulae within a theoretical good degree of accuracy
- The asymptotic formulae give a good approximation to the benchmark numerical computations, even in the extreme cases considered (and in some cases are more efficient)
- Numerical results show that the error produced by the approximation is in agreement with the theoretical prediction

Further Work

Next, we aim to extend this theory to the mixed boundary value problem for Green's tensors in elasticity, for the case of when Neumann conditions are prescribed on the boundaries of the small holes and with have the Dirichlet condition on the exterior boundary.

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