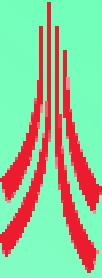


Large Fluctuations in Chaotic Systems



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Outline

- Large Fluctuational Approach and Model Reduction
- Escape in quasi-hyperbolic systems
- Escape in non-hyperbolic systems
- Conclusions



Environment induces

Dissipation and Fluctuations

$$H = H_S + H_B + H_{SB}$$

H_S System Hamiltonian

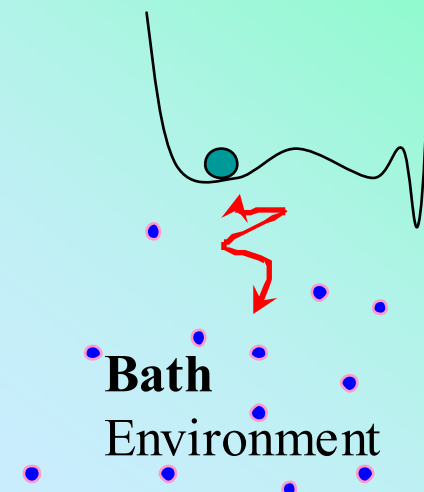
H_B Bath (Environmental) Hamiltonian

H_{SB} Hamiltonian of interaction

Elimination of the environmental degrees of freedom leads to

- Dissipation and
- Fluctuations

Note: Elimination is, as a rule, a challenge task and it is often phenomenological



(*external* or *internal* degrees of freedom)



Deterministic Chaos and Noise: Environment

Archetypical Example: Environment as a Collection of Linear Oscillators

$$H = H_S + H_B + H_{SB}$$

$$H_S = \frac{p^2}{2m} + V(q;t) \quad \text{The system is a model of a particle in potential}$$

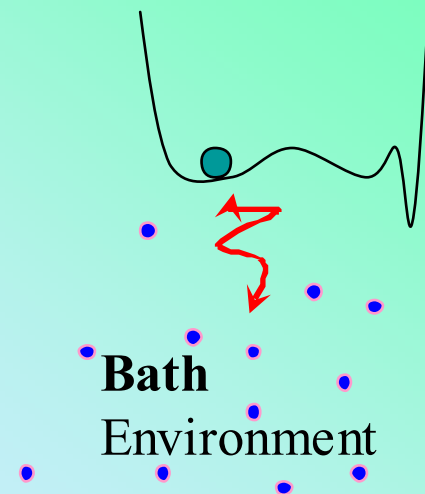
$$H_B = \sum_{n=1}^N \left(\frac{p_n^2}{2m_n} + \frac{m_n}{2} \omega_n^2 x_n^2 \right) \quad \text{The collection of harmonic oscillators}$$

$$H_{SB} = -q \sum_{n=1}^N c_n x_n + q^2 \sum_{n=1}^N \frac{c_n^2}{2m_n \omega_n^2} \quad \text{Linear coupling between system and bath}$$

Elimination leads to

$$m\ddot{q} + 2\gamma m\dot{q} + \frac{\partial V}{\partial q} = \xi(t)$$

Damping (dissipation)
Fluctuations noise



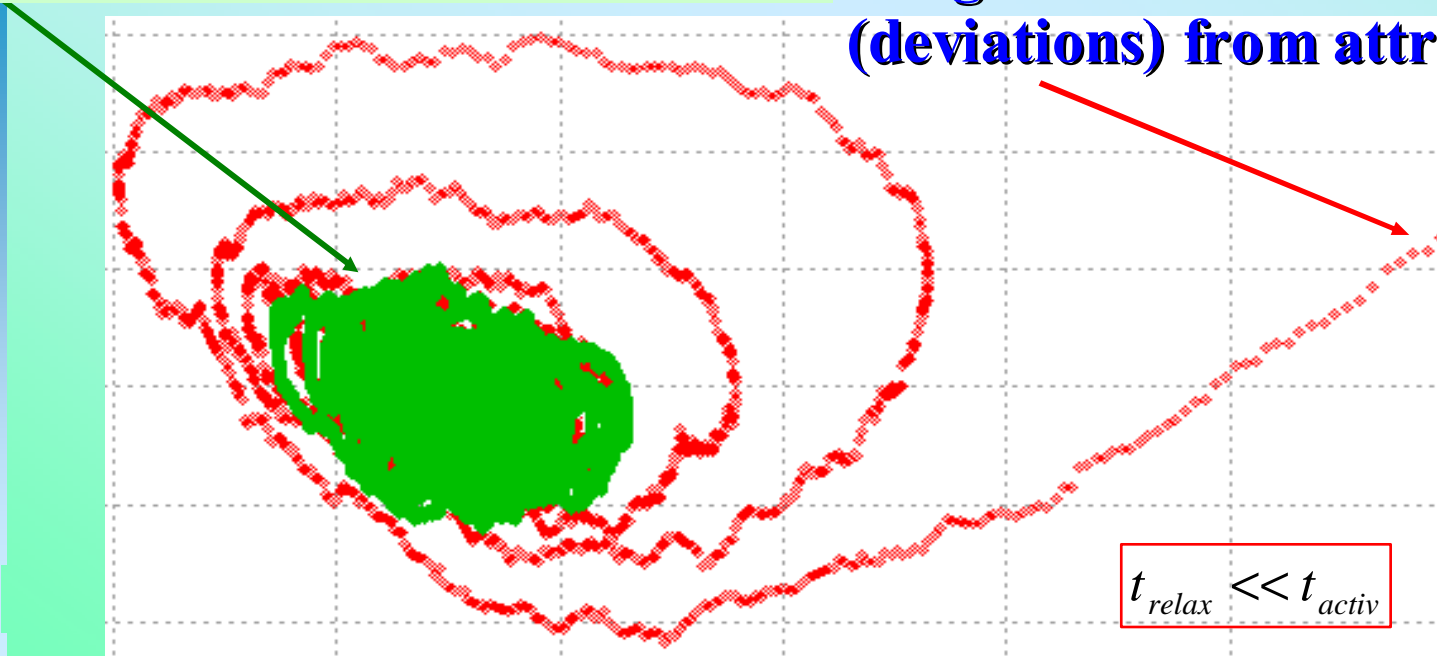
Dissipation and Fluctuations have the same origin

The simplification of dynamics: considering dynamics related to Large Fluctuations

Different manifestations of fluctuations:

Diffusion in a vicinity of attractor

Large fluctuations
(deviations) from attractors



Chaos and Noise: Large Fluctuation Approach

The system described by Langevin equations:

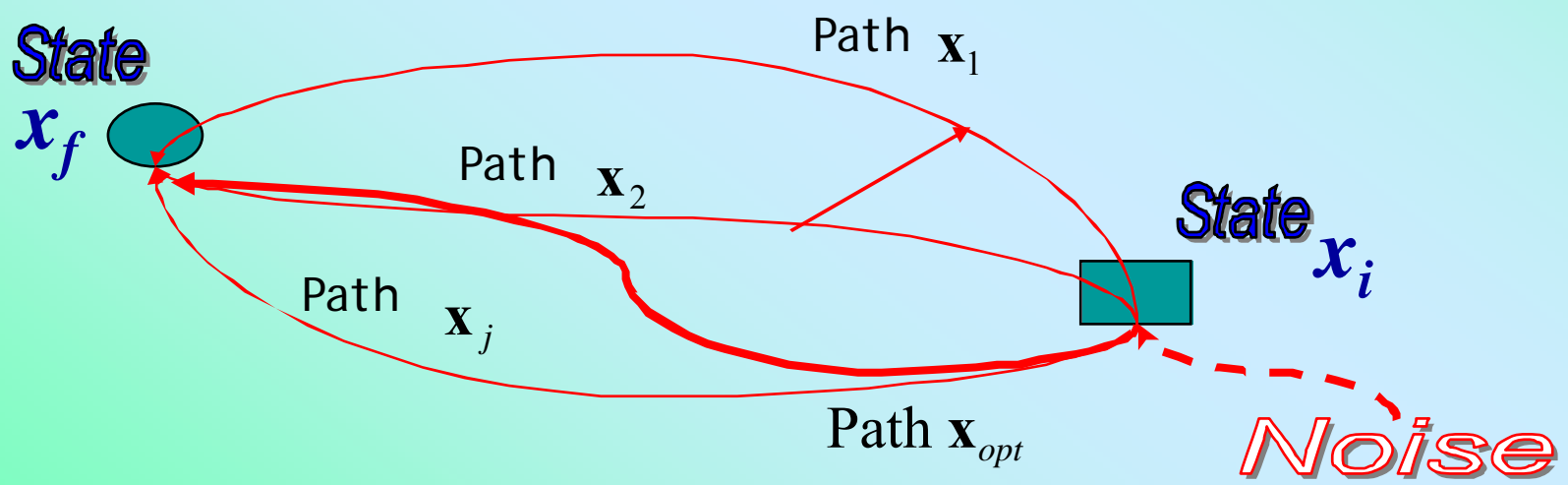
$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}, t) + \mathbf{Q}\xi(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t) \xi_\beta(s) \rangle = \mathbf{Q} \delta(t - s)$$

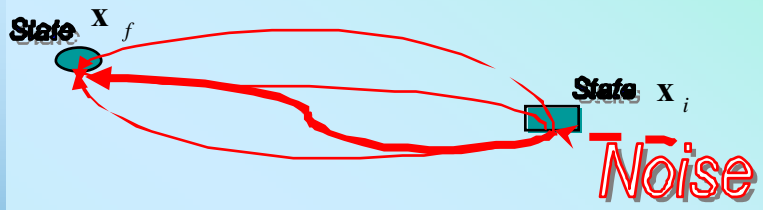
Transition probability via fluctuations paths

$$\rho(\mathbf{x}_f, t_f | \mathbf{x}_i, t_i) = \sum_j \rho[\mathbf{x}(t)_j] \approx \rho[\mathbf{x}(t)_{opt}]$$

The selection of the most probable (optimal) path



Deterministic chaos and noise: optimal path approach deterministic pattern of fluctuations



$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}, t) + \xi(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t) \xi_\beta(s) \rangle = DQ \delta(t-s)$$

The probability of fluctuational path $\rho[\mathbf{x}(t)_j]$ is related to the probability $\rho[\xi(t)_j]$ of random force to have a realization $\xi(t)_j$

For Gaussian noise:
$$\rho[\xi(t)_j] = C \exp\left(-\frac{1}{2} \int_{t_i}^{t_f} \xi(t)_j^2 dt\right) = C \exp\left(-\frac{1}{2} S\right)$$

Since the exponential form, the most probable path has a minimal $S=S_{min}$

Changing to dynamical variables:

$$\text{Action } S = S[\xi(t)] \xrightarrow[\xi(t) = \dot{\mathbf{x}} - \mathbf{K}(\mathbf{x}, t)]{\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}, t) + \xi(t)} S = [x(t)]$$

In the limit $D \rightarrow 0$,
$$\rho(\mathbf{x}_f; \mathbf{x}_i) = \rho(\mathbf{x}(t)_{opt}) = Const \times \exp\left(-\frac{S[\mathbf{x}(t)_{opt}]}{D}\right)$$

$$S_{min} = S[x_{opt}(t)] = \min \int dt (\dot{\mathbf{x}} - \mathbf{K}(\mathbf{x}, t))^2$$

Deterministic minimization problem

The initial model: the Hamiltonian for the system, the bath and coupling
In general case the dimension is infinite.

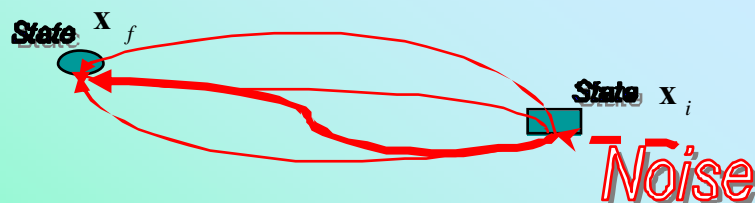
$$H = H_S + H_B + H_{SB}$$

Langevin model reduction: finite dimensional system with noise terms,
The dimension is infinite

$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}, t) + \xi(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t) \xi_\beta(s) \rangle = DQ \delta(t-s)$$

Large fluctuations reduction leads to a specific object: the optimal path as a solution of boundary value problem of the finite dimensional Hamilton system



$$S_{\min} = S[x_{opt}(t)] = \min \int dt (\dot{\mathbf{x}} - \mathbf{K}(\mathbf{x}, t))^2$$

Initial state :

$$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0, \quad t_i \rightarrow -\infty;$$

Final state :

$$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0, \quad t_f \rightarrow \infty.$$

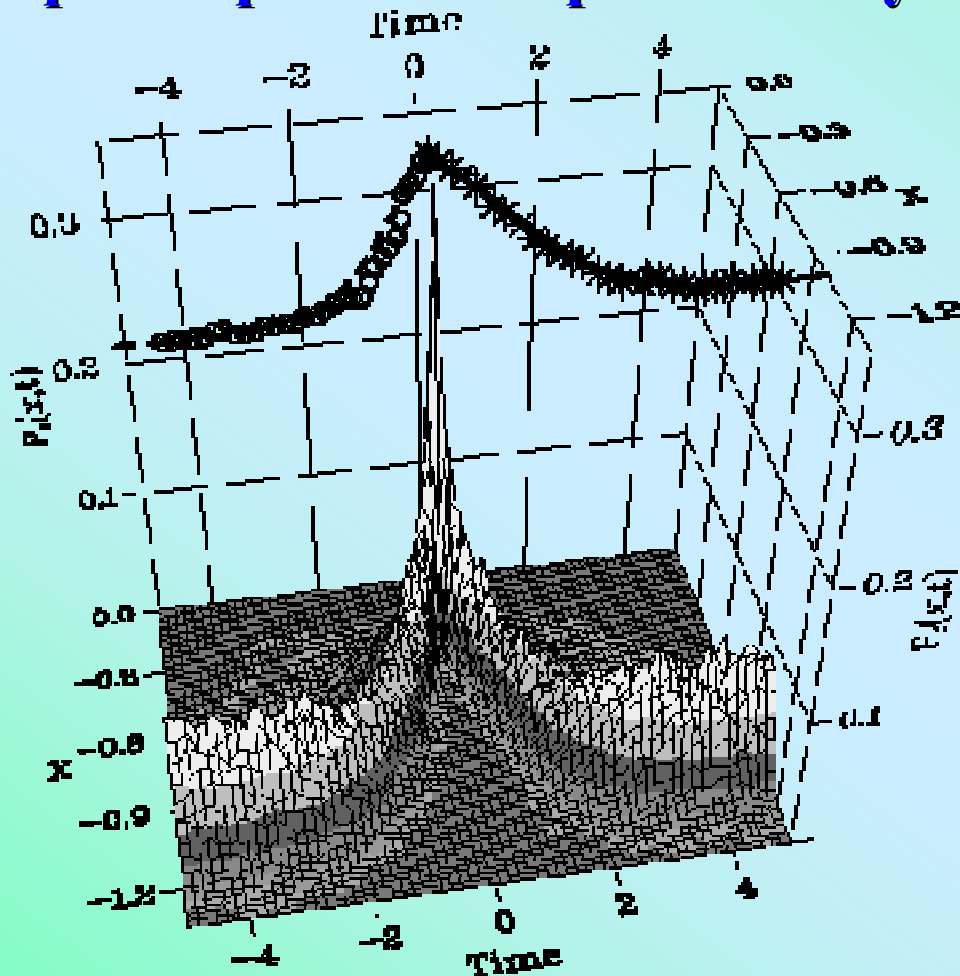
Formally the deterministic minimization problem can be formulated in the Hamiltonian form:

$$H = \frac{1}{2} \mathbf{p} \mathbf{Q} \mathbf{p} + \mathbf{p} \mathbf{K}(\mathbf{q}, t);$$

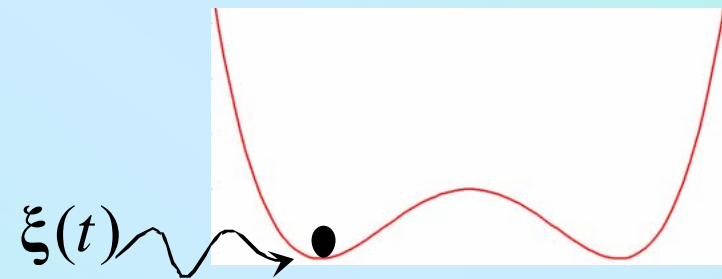
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

Optimal path approach deterministic pattern of fluctuations

Optimal paths are experimentally observable (Dykman'92)



The prehistory probability of transition between states of bistable oscillator (electronic experiment)



Optimal paths are essentially deterministic trajectories

Lorenz system

$$\sigma = 10, \quad b = 8/3, \quad r = 24.08$$

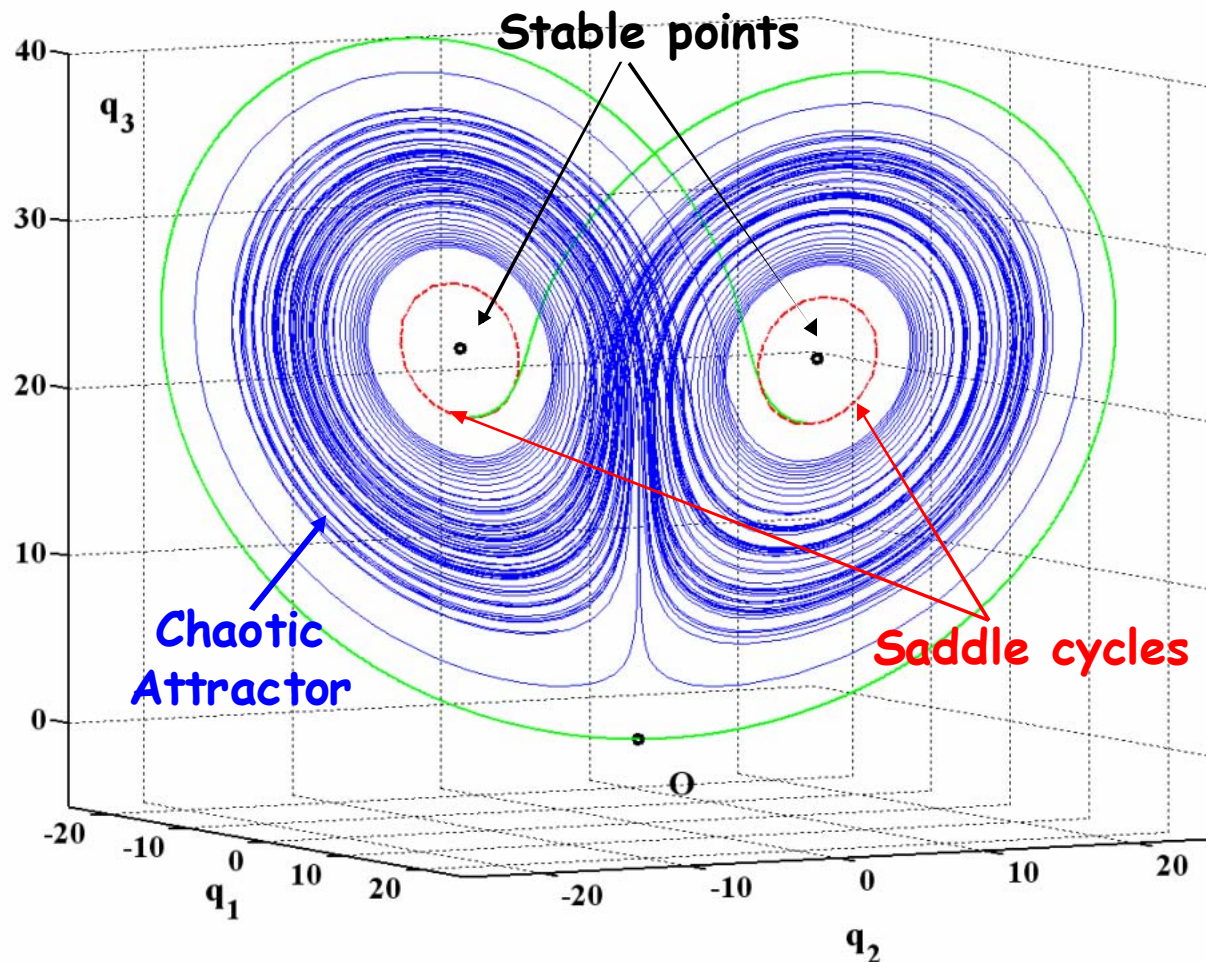
$$\dot{q}_1 = \sigma(q_2 - q_1)$$

$$\dot{q}_2 = r q_1 - q_2 - q_1 q_3$$

$$\dot{q}_3 = q_1 q_2 - b q_3 + \sqrt{2D} \xi(t)$$

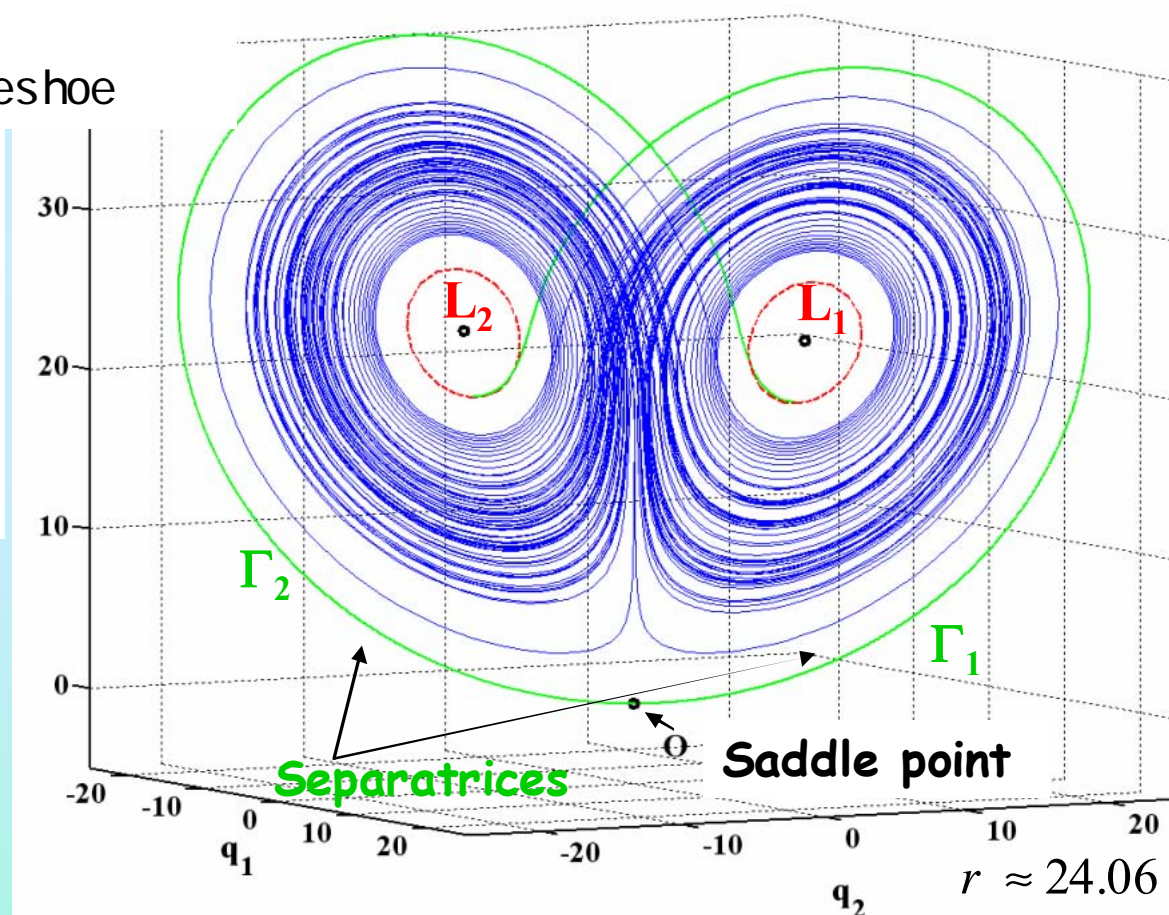
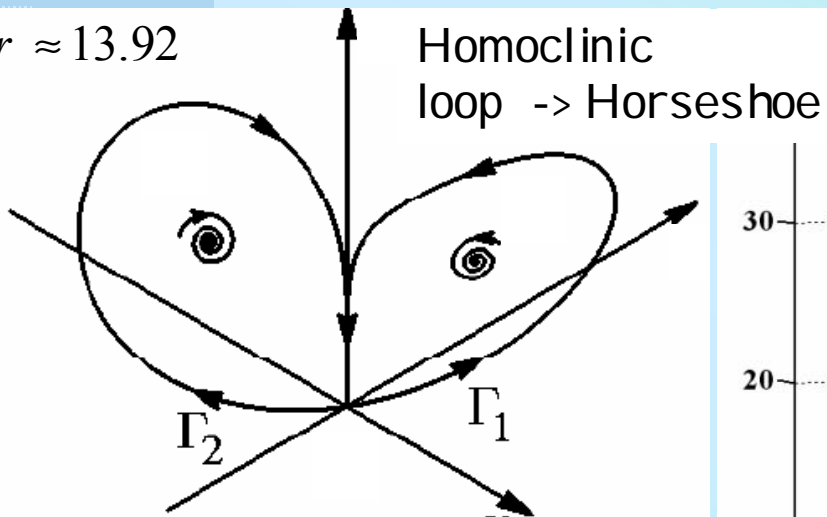
Consider noise-induced escape from the chaotic attractor to the stable point in the limit $D \rightarrow 0$

The task is to determine the most probable (optimal) escape path



Lorenz Attractor

$r \approx 13.92$

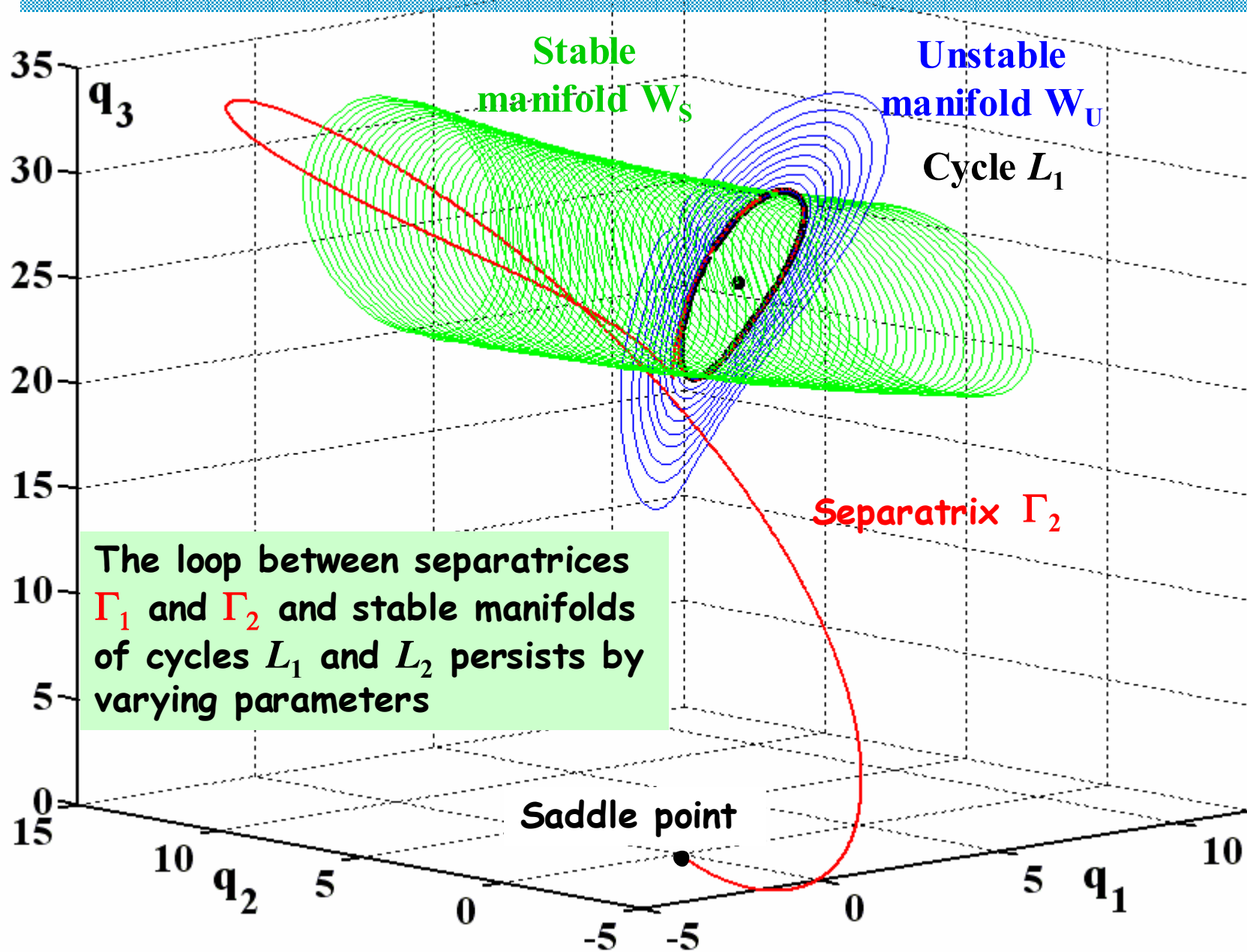


The saddle point and its separatrices belong to chaotic attractor and form "bad set" or non-hyperbolic part of the attractor

Loops between separatrices Γ_1 and Γ_2 and stable manifolds of cycles L_1 and L_2 generate The Lorenz attractor - quasi-hyperbolic attractor



Chaos: Quasi-hyperbolic Attractor





The prehistory approach

$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}, t) + \xi(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t) \xi_\beta(s) \rangle = DQ \delta(t-s)$$

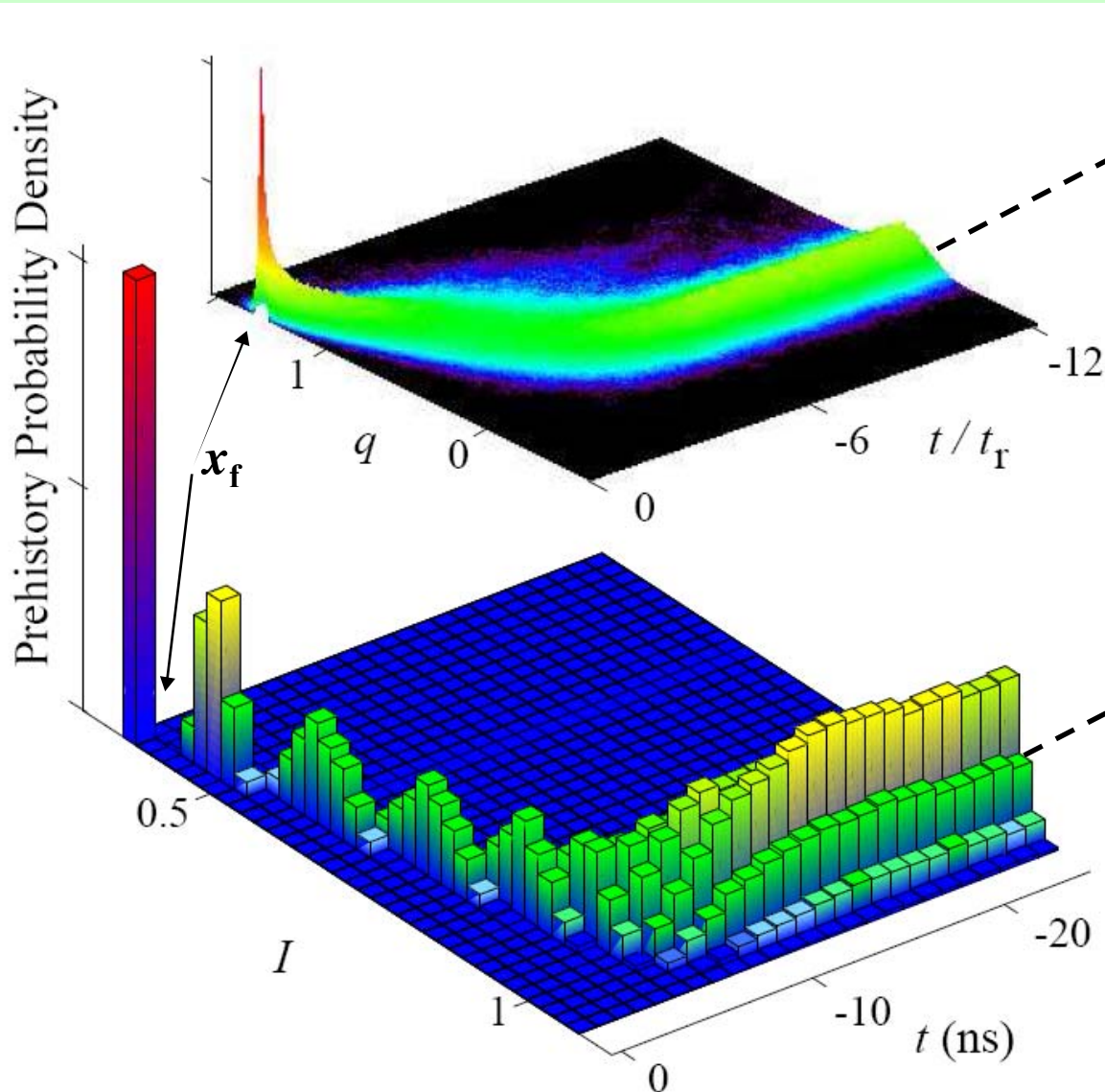
1. Select the regime $D \rightarrow 0$
i.e. rare large fluctuations

$$t_{relax} \ll t_{activ}$$

2. Record all trajectories $\mathbf{x}_j(t)$ arrived to the final state and build the prehistory probability density $p_h(\mathbf{x}, t)$

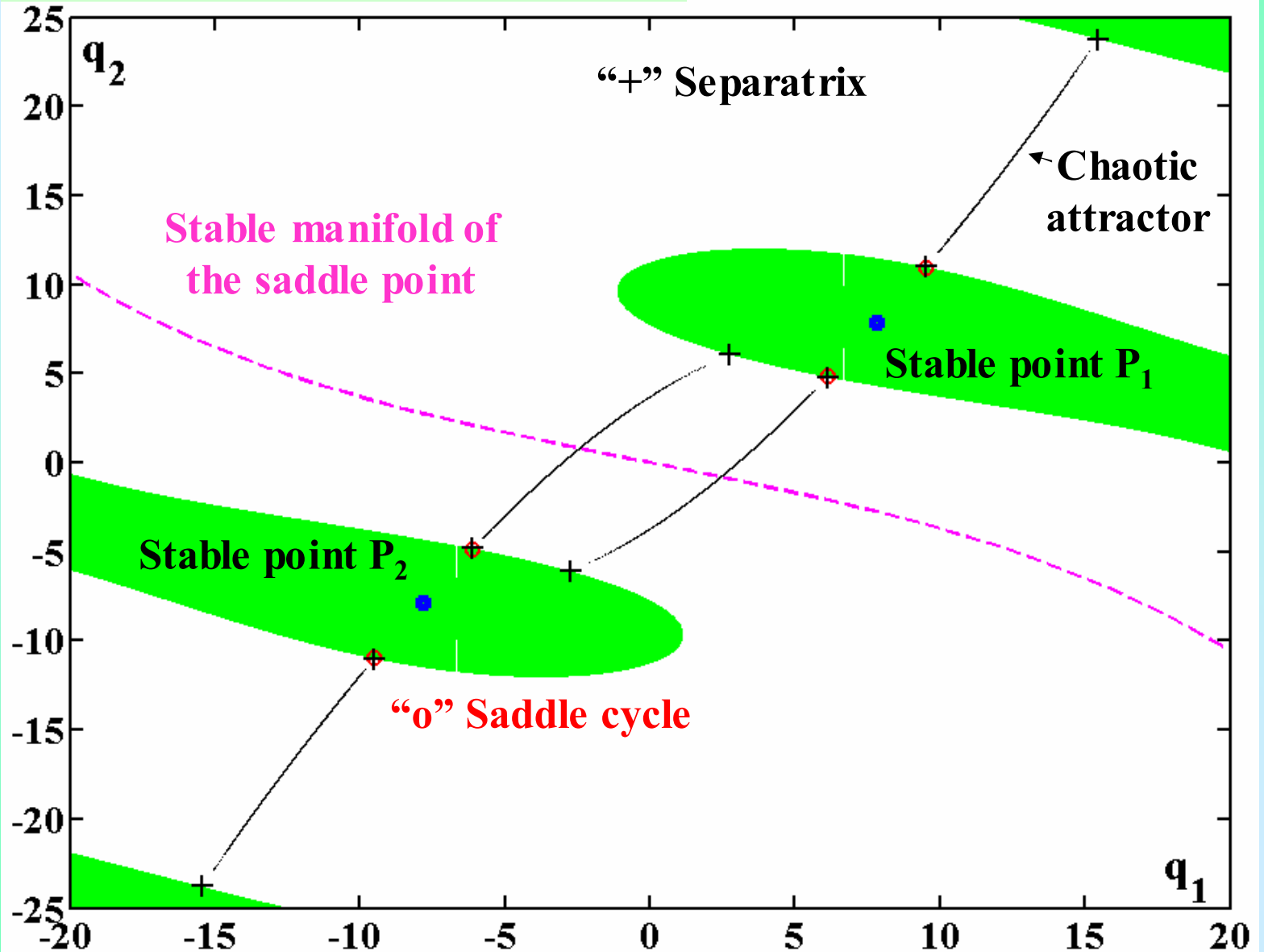
The maximum of the density corresponds to the **most probable (optimal) path**

3. Simultaneously noise realizations $\xi(t)$ are collected and give us the **optimal fluctuational force**



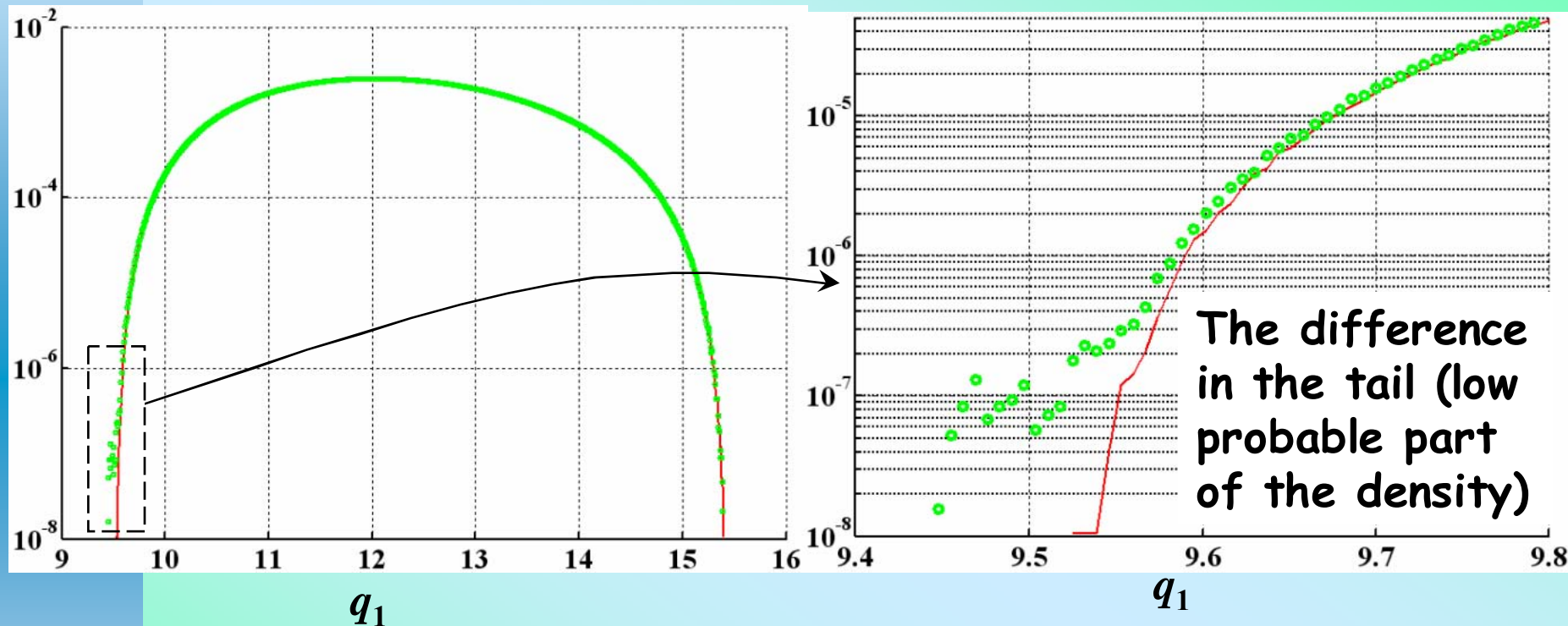
Chaos: Quasi-hyperbolic Attractor

The Poincaré section $q_3 = r - 1$



Chaos and Noise: Quasi-hyperbolic Attractor

Probability density $p(q_1)$ for Poincaré section $q_3 = r - 1$



The difference in the tail (low probable part of the density)

— $D=0$ In the absence of noise

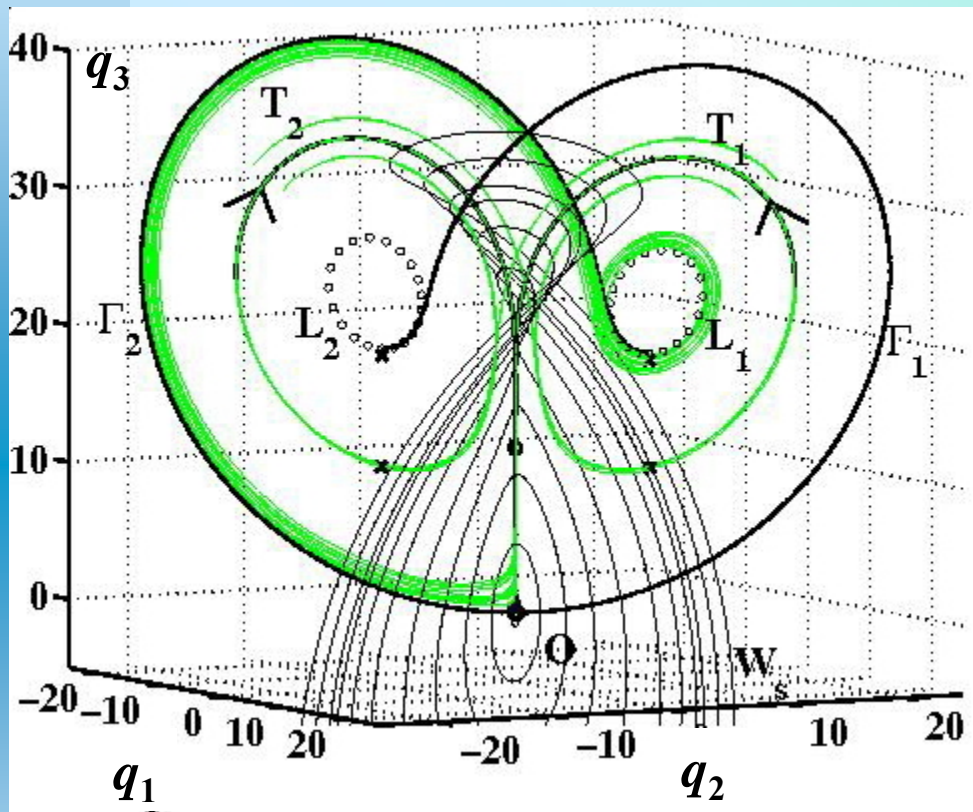
●●●●● $D=0.001$ In the presence of noise

Noise does not change significantly the probability density

Are noise-induced tail important?

Chaos and Noise: Quasi-hyperbolic Attractor

Escape from quasi-hyperbolic attractor



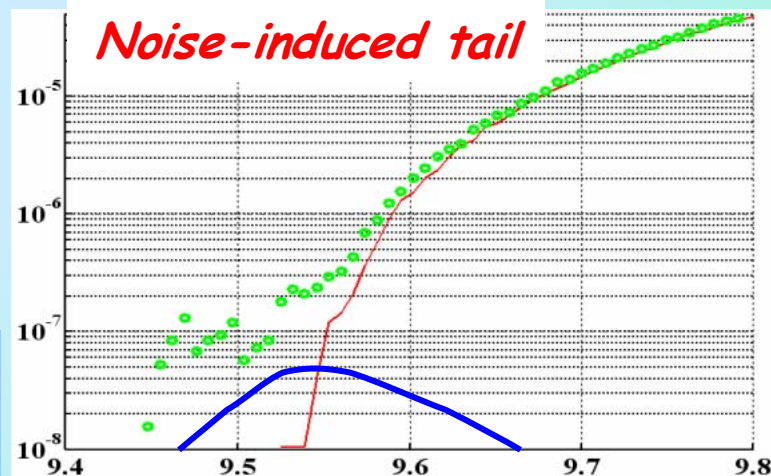
— Escape trajectories

W_s is the stable manifold and Γ_1 and Γ_2 are separatrices of the saddle point O

L_1 and L_2 are saddle cycles

T_1 and T_2 are trajectories which are tangent to W_s

The escape process is connected with the non-hyperbolic structure of attractor: stable and unstable manifolds of the saddle point

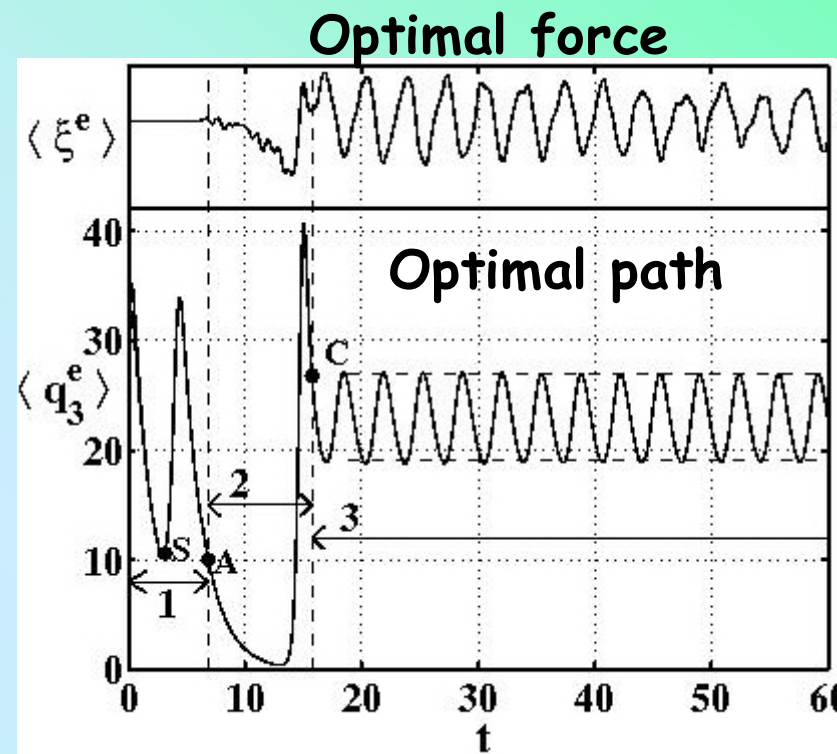
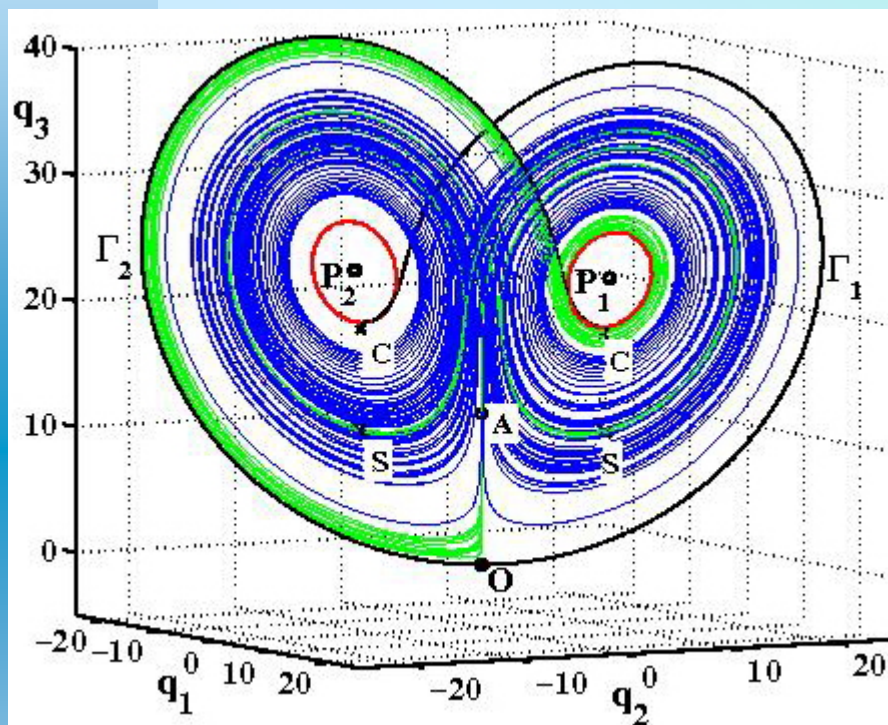


— The distribution of escape trajectories (exit distribution)



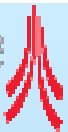
Chaos and Noise: Quasi-hyperbolic Attractor

The optimal path and fluctuational force from analysis of fluctuations prehistory



Three parts:

- 1) Deterministic part, the force equals to zero; The point A is the initial state x_i
- 2) Noise-assisted motion along stable and unstable manifolds of the saddle point
- 3) Slow diffusion to overcome the deterministic drift of unstable manifold of the saddle cycle and cross the cycle



Chaos and Noise: Non-hyperbolic Attractor

Non-autonomous nonlinear oscillator

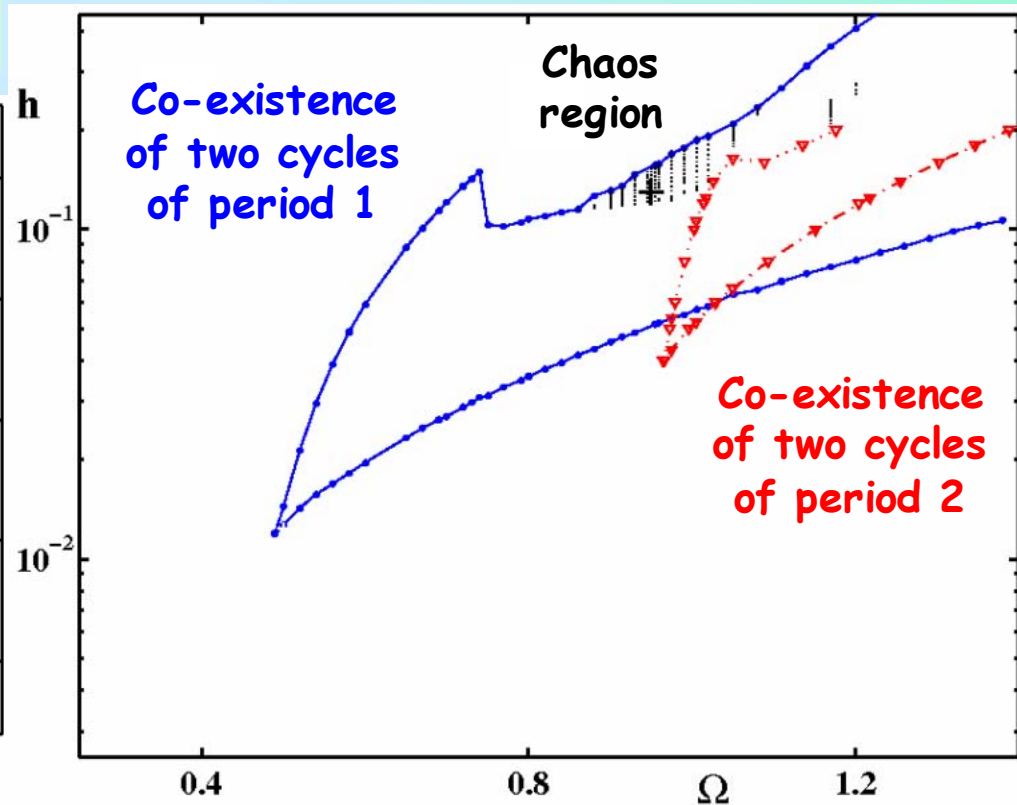
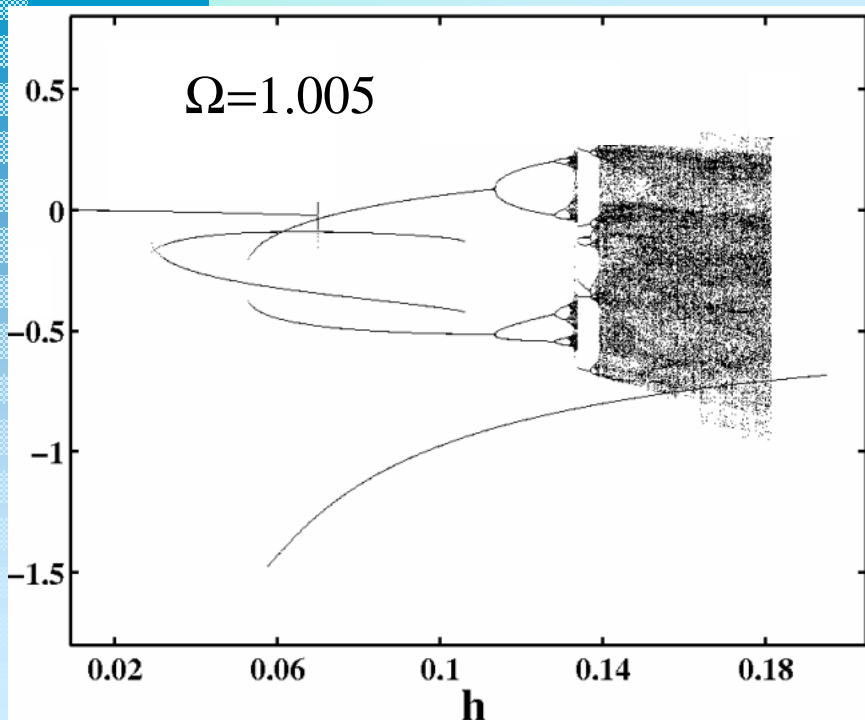
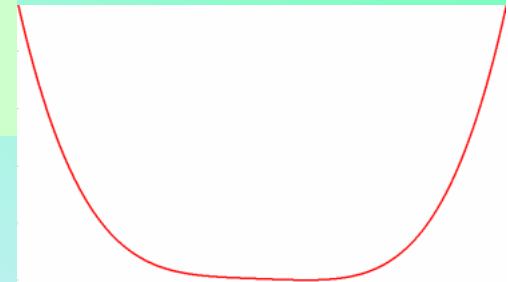
$$\ddot{q} + \Gamma \dot{q} + \frac{\partial U(q, t)}{\partial q} = \sqrt{2D} \xi(t)$$

$$U(q, t) = \frac{\omega_0^2}{2} q^2 + \frac{\beta}{3} q^3 + \frac{\gamma}{4} q^4 + q h \sin \Omega t$$

$$\Gamma = 0.05 \quad \omega_0 = 0.597 \quad \beta = \gamma = 1$$

The motion is underdamped

The potential $U(q)$
is monostable



Chaos and Noise: Non-hyperbolic Attractor

The co-existence of chaotic attractor and the limit cycle

$h = 0.13 \quad \Omega = 0.95$

Initial state: ?

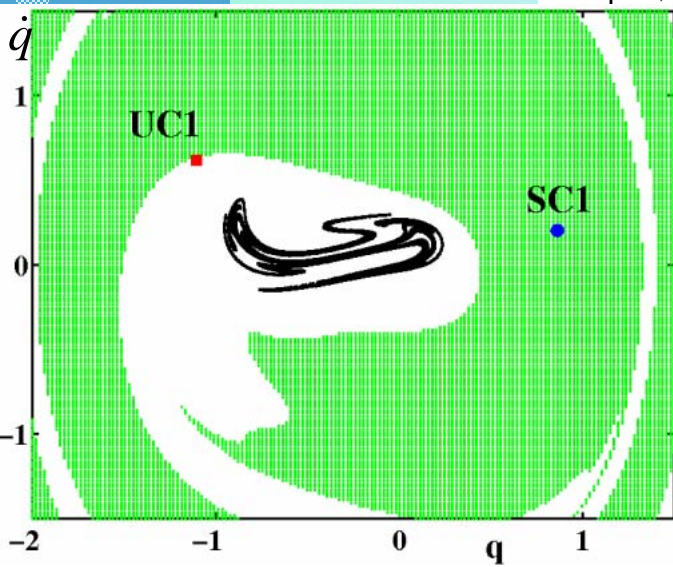
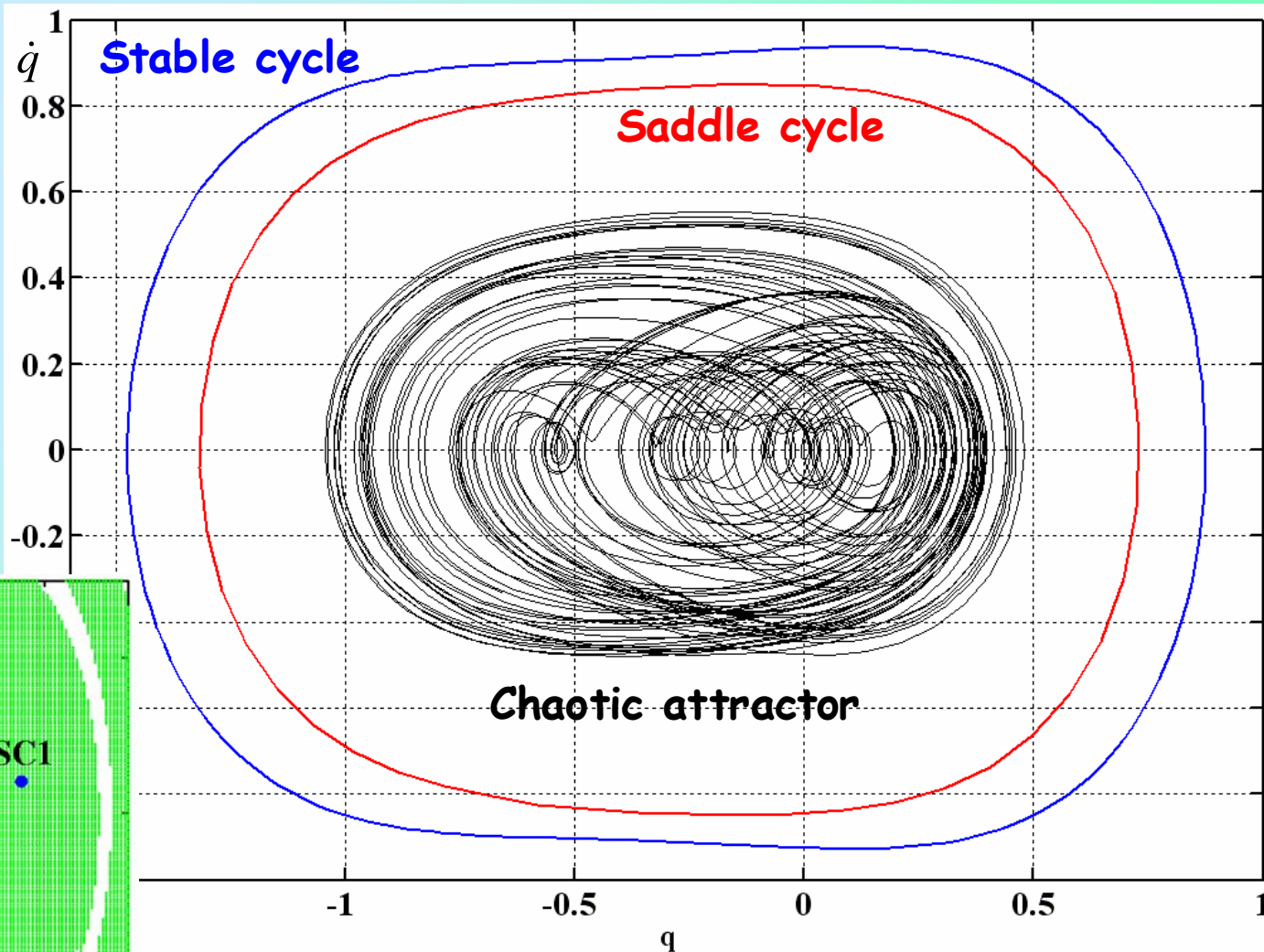
$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0,$

$t_i \rightarrow -\infty;$

Final state: The stable
cycle

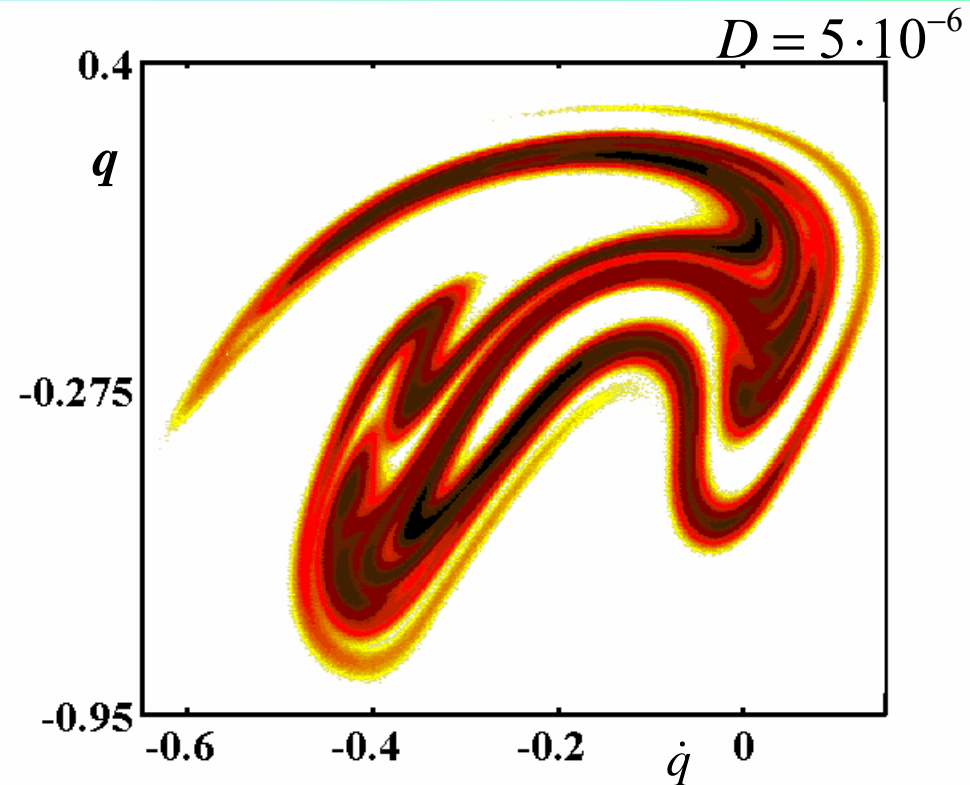
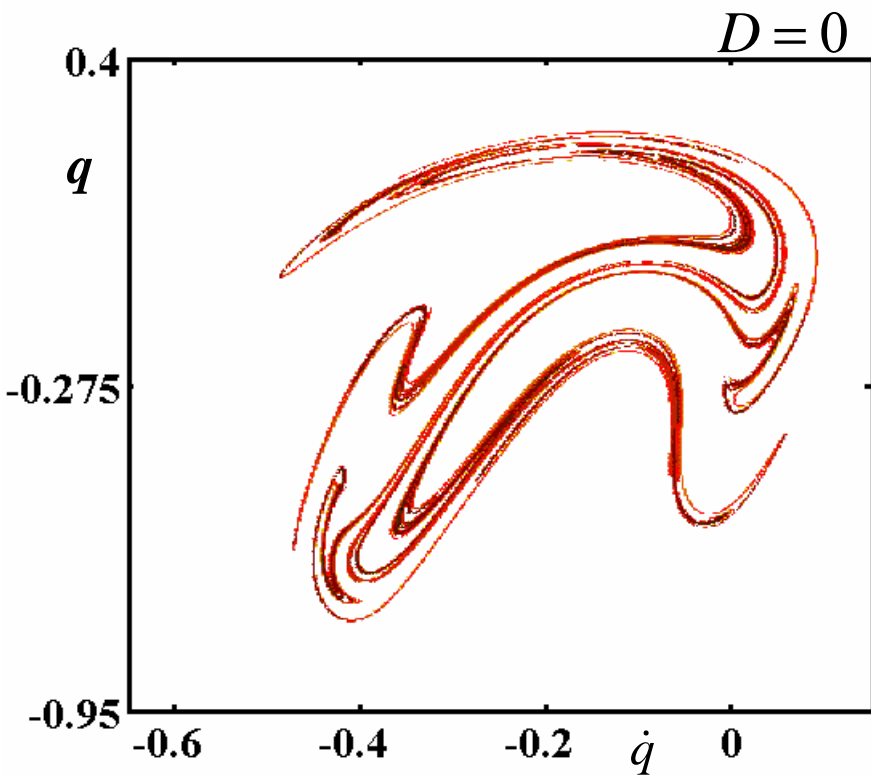
$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0,$

$t_f \rightarrow \infty.$



Chaos and Noise: Non-hyperbolic Attractor

Probability density $p(q, \dot{q})$ for Poincaré section $\Omega t = 0$



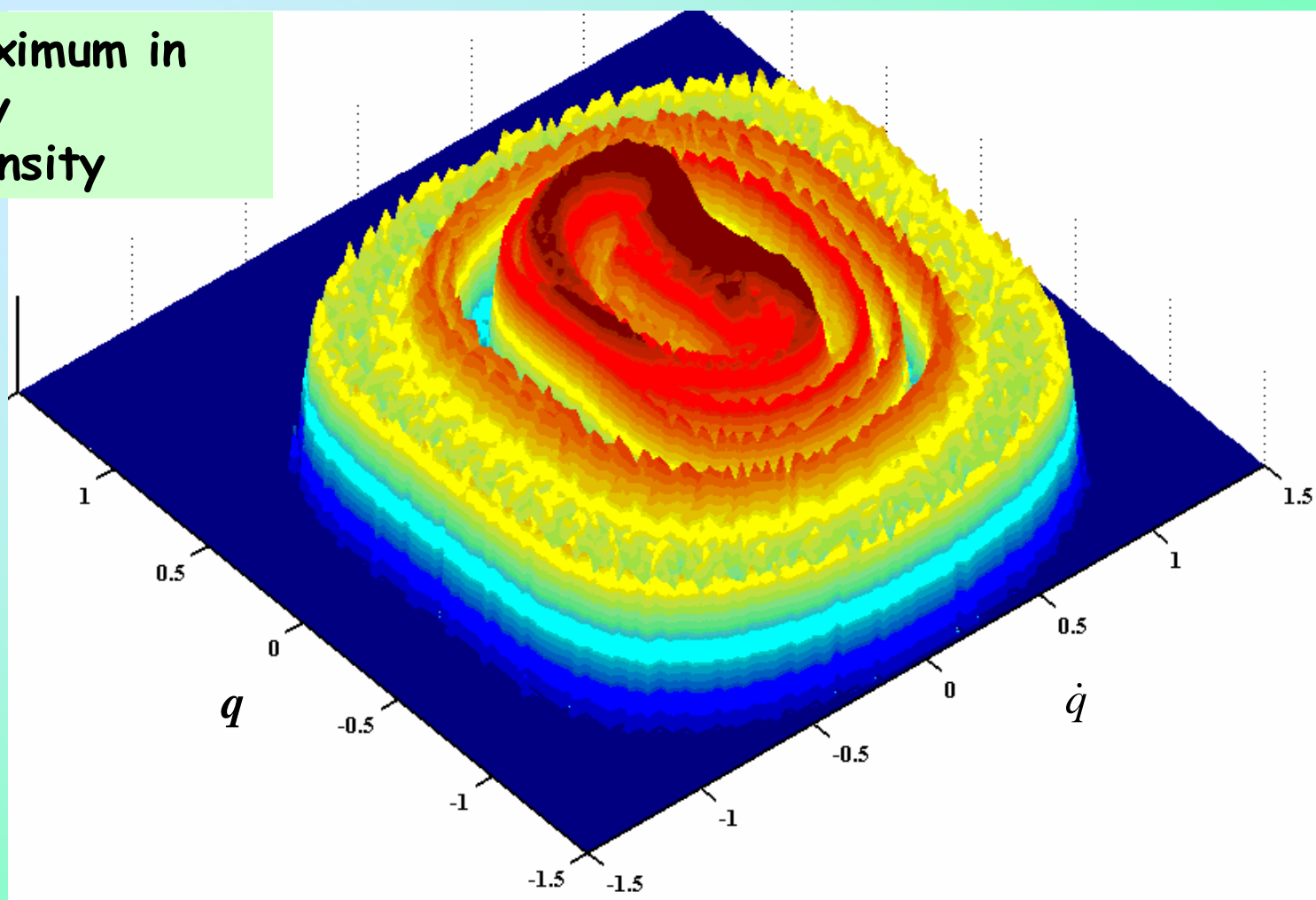
A weak noise significantly changes the probability density

Chaos and Noise: Non-hyperbolic Attractor

The prehistory probability density $p_h(q, \dot{q}, t)$

$$D = 5 \cdot 10^{-4}$$

There is a maximum in the prehistory probability density

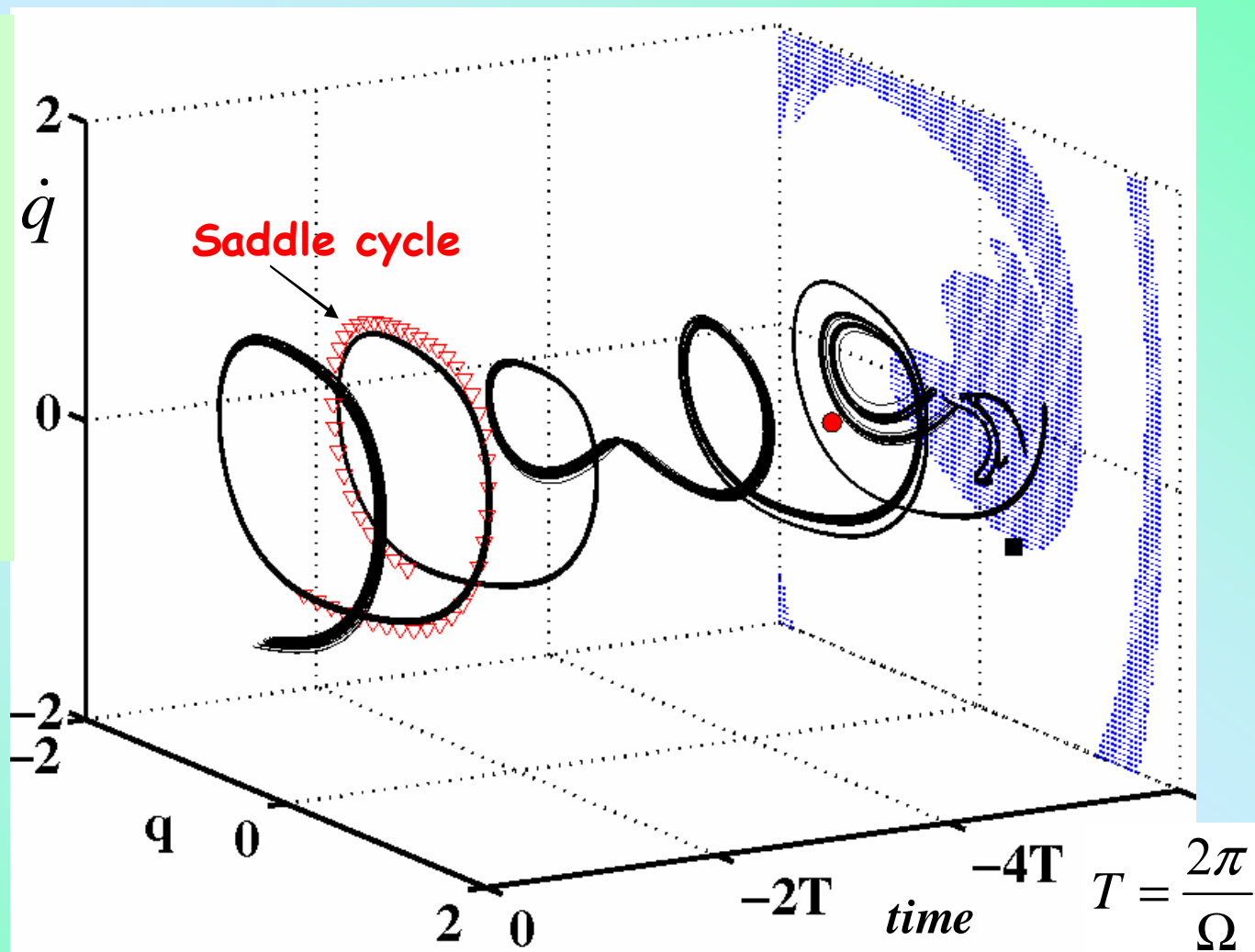


Chaos and Noise: Non-hyperbolic Attractor

Escape trajectories follow a narrow tube

Q: Do any sets form the escape path?

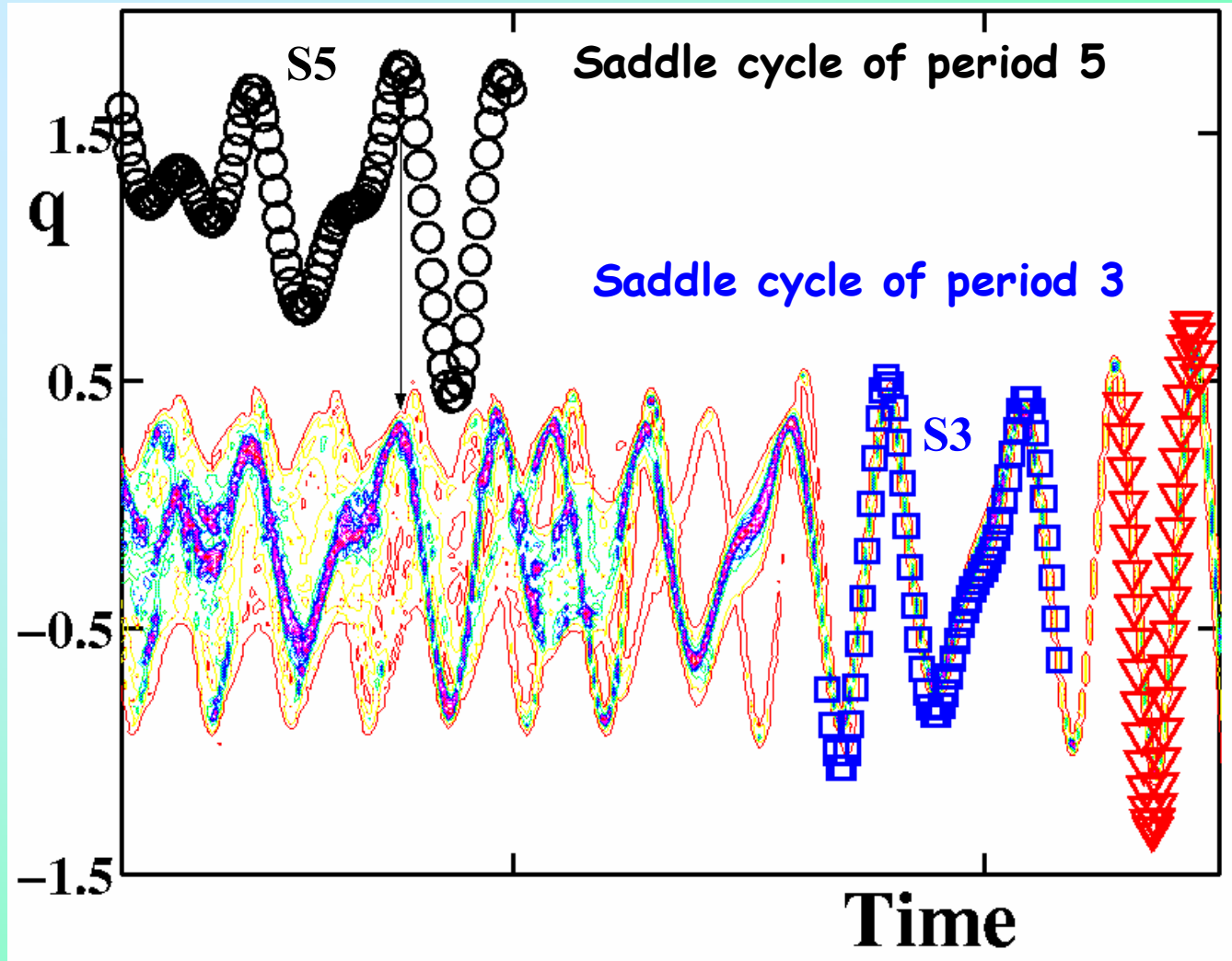
To answer we take initial conditions along the path and try to localize any sets



Chaos and Noise: Non-hyperbolic Attractor

The prehistory probability density $p_h(q, t)$

Saddle cycles form the escape path





Chaos and Noise: Non-hyperbolic Attractor

The escape is a sequence of jumps between saddle cycles.

Escape trajectory is a heteroclinic trajectories connected saddle cycles of Hamilton system.

$$H = \frac{1}{2} \mathbf{p} \mathbf{Q} \mathbf{p} + \mathbf{p} \mathbf{K}(\mathbf{q}, t);$$

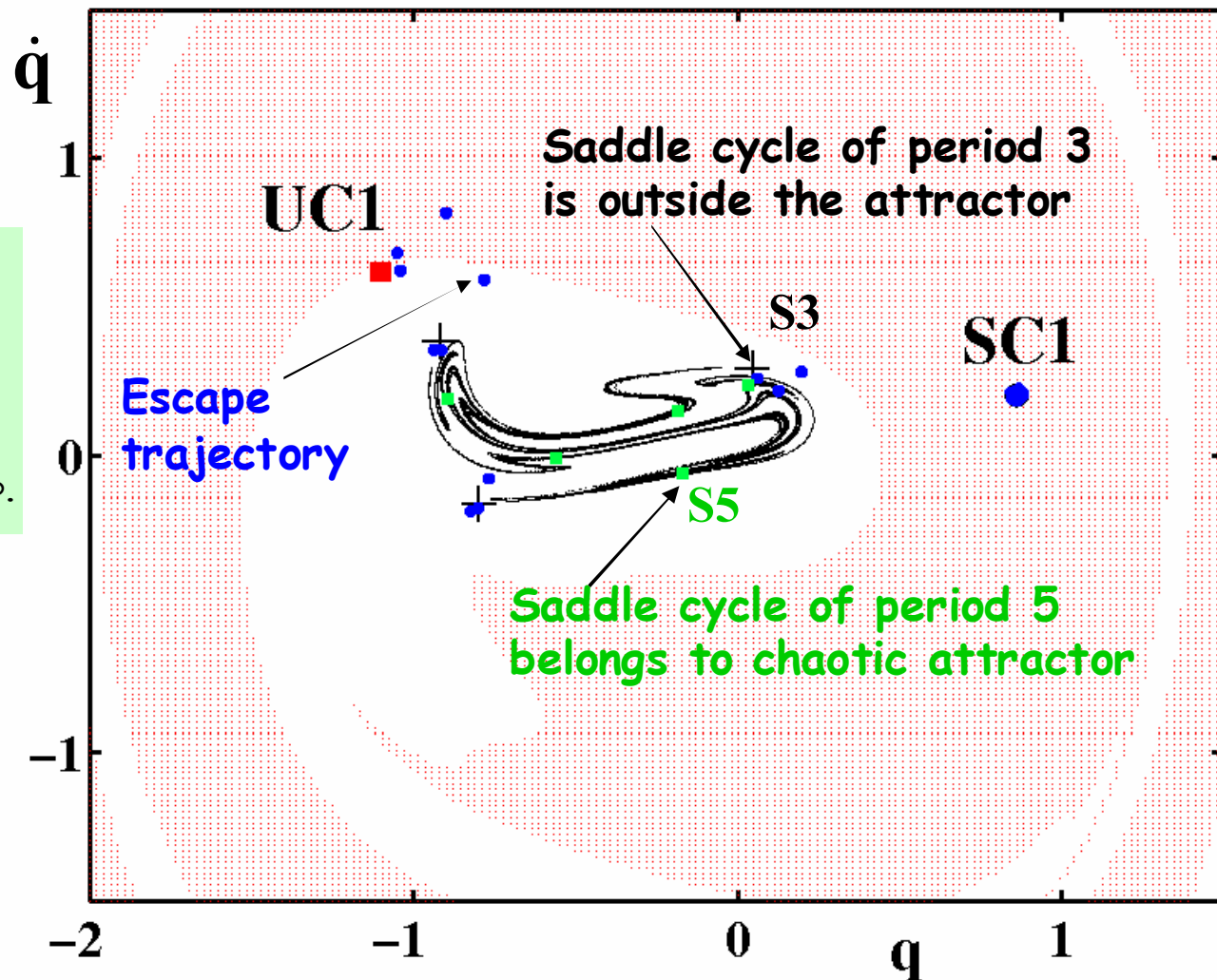
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

Initial state: **cycle S5**

$$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0, \quad t_i \rightarrow -\infty;$$

Final state: **cycle UC1**

$$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0, \quad t_f \rightarrow \infty.$$





Large Fluctuations

Summary

For a quasi-hyperbolic attractor, its non-hyperbolic part plays an essential role in the escape process.

For a non-hyperbolic attractor, saddle cycles embedded in the attractor and basin of attraction are important. Escape from a non-hyperbolic attractor occurs in a sequence of jumps between saddle cycles.

For both types of chaotic attractor we can select specific sets which are connected with Large Fluctuations and the most probable paths. Thus the description of large fluctuations is reduced to specification of a particular trajectory (the optimal path).