MA2081 – Vector Calculus and Fluid Dynamics

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Course Information

Module Webpage

http://www.math.le.ac.uk/PEOPLE/eg64/TEACHING/MA2081/MA2081.html

Contact Details

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Module Information

Prerequisites

▶ MA1001 Multivariate Calculus
▶ MA1002 Vectors and ODEs
▶ MA1051 Newtonian Dynamics

Assessment

▶ 2 hour examination: 60%
▶ 10 weekly problem sheets 20%
▶ 10 computer practicals: 20%
Problem classes / Computer Practicals

**Weekly problem classes**

Manolis Georgoulis
Venue: BEN LT5
Time: Mondays 1:30-2:30pm (starting in 2 weeks)

**Weekly computer practicals**

Manolis Georgoulis
Venue: CW305
Time: Tuesdays 2:30pm (starting TOMORROW)
Module Description

▶ Vector differential and integral Calculus
  ▶ computing arc lengths, surface areas and volumes
  ▶ computing tangents and integrals over useful domains
  ▶ the implicit function theorem

▶ Applications to continuum mechanics and fluid dynamics
  ▶ calculate physical quantities, such as velocity, acceleration, energy, work done by a force, etc.
  ▶ modelling of physical phenomena, such as fluid flow, waves, heat transfer, etc.

▶ Computer simulations
  ▶ simple numerical computations using (mainly) MATLAB
  ▶ nice pictures!
Reading List

Vector Calculus:

► Calculus, Edwards and Penney, 6th edition

Accessible and colourful.

► Advanced Engineering Mathematics, Kreyszig, 8th edition

Less worked examples but covers the material thoroughly; applications oriented. (It includes more topics than Calculus.)

► Vector Calculus, Marsden and Tromba

Rigorous and enjoyable but slightly demanding.

Fluid Dynamics:

► Elementary Fluid Dynamics, Acheson

We shall only cover Chapters 1 and 3.
Let’s get started then!
... with a bit of Geometry.
Vectors in Space

Figure: Components of a vector in space
Vector Algebra

Let \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \), \( \mathbf{c} = (c_1, c_2, c_3) \) vectors, and \( \lambda \) a scalar:

\[
\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = \mathbf{b} + \mathbf{a}
\]

\[
(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})
\]

\[
\lambda \mathbf{a} = (\lambda a_1, \lambda a_2, \lambda a_3)
\]

\[
\mathbf{a} + \mathbf{0} = \mathbf{a}, \quad \text{for} \quad \mathbf{0} = (0, 0, 0)
\]

\[
-\mathbf{a} = (-1)\mathbf{a}
\]

Length of a vector:

\[
|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]
Unit Vector Notation

Figure: The representation $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$
Scalar Product (a.k.a. dot product, inner product ...)

Let \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \) vectors, and \( \gamma \) the angle between them

Scalar Product:

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma \\
= a_1 b_1 + a_2 b_2 + a_3 b_3
\]

Properties:

\[
\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \\
\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 \\
(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}
\]

If \( \mathbf{a} \) perpendicular to \( \mathbf{b} \) then \( \mathbf{a} \cdot \mathbf{b} = 0 \)
Scalar Product – Application

Work done by a force:

Consider a constant force $\vec{F}$ acting on a body that results to a displacement $\vec{d}$ of the body. What is the work $W$ done by $\vec{F}$?

$$W = \vec{F} \cdot \vec{d}$$
Vector Product (a.k.a. cross product, wedge product ...)

Let \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \) vectors, and \( \gamma \) the angle between them.

Vector Product:

\[
\mathbf{a} \times \mathbf{b} = \mathbf{v}
\]

with

\[
\mathbf{v} = (a_2 b_3 - a_3 b_2, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2)
\]

and length

\[
|\mathbf{v}| = |\mathbf{a}| |\mathbf{b}| \sin \gamma
\]
Area of parallelogram:

Consider \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \) vectors. Then the area of the parallelogram having \( \mathbf{a} \) and \( \mathbf{b} \) as edges is given by

\[
\text{Area} = |\mathbf{a} \times \mathbf{b}|
\]

**BIG IDEA:** Area can be described using vector product!
Moment of a force:

Let $F$ a force acting on a body. Body at distance $r$ from a point $O$. The moment vector $\mathbf{m}$ of the force $F$ about the point $O$ is

$$\mathbf{m} = r \times F$$

Moment describes the rotation about the point $O$ caused by $F$.

BIG IDEA: Rotation can be quantified using vector product!
Vector Product – Properties

The formula

\[ \mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - b_3a_1, a_1b_2 - b_1a_2) \]

\[ = (a_2b_3 - a_3b_2)i + (a_3b_1 - b_3a_1)j + (a_1b_2 - b_1a_2)k \]

seems hard to remember...

Let us see if we can relate it to something we know already!

**Keyword:** Determinants!

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]
Vector Product – Properties

Hence

\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \]

Other properties:

\[ \mathbf{a} \times \mathbf{a} = 0 \]

\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \] \quad antisymmetry

\[ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \]

...it is not the same as

\[ \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b} \]

because of antisymmetry!
Find the moment \( \vec{m} \) of the force \( \vec{F} = (3, 2, 4) \) about the point \( O = (0, 0, 0) \) of a body located at \( \vec{r} = (1, 2, 4) \) from \( O \).

Answer:

\[
(3, 2, 4) \times (1, 2, 4) = ... = (0, 8, -4)
\]
Plane perpendicular to a vector

**Problem:** How to describe a plane that goes through a point \((x_0, y_0, z_0)\) in space and it perpendicular to a vector \(\vec{a} = (a_1, a_2, a_3)\)?

**Solution:**

\[
a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0
\]
Polar Coordinates

Sometimes it is hard to describe curves or surfaces in Cartesian coordinates. For example, a circle with centre (0, 0) and radius 1 cannot be represented as a function $y = f(x)$...

Let us think of an alternative way, that might be useful in this case...

![Figure: Polar coordinates](image-url)
Polar Coordinates

Hence

\[ x = r \cos \theta, \quad y = r \sin \theta \]

So we have the change of variables

\[(x, y) \rightarrow (r, \theta) \quad \text{with} \quad 0 \leq \theta < 2\pi\]

Example: Find \((x, y) = (1, 1)\) in polar coordinates.

Solution: We have \((r, \theta) = (\sqrt{2}, \frac{\pi}{4})\).

In general:

\[ r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \]