Estimates of n–Widths of Besov Classes on Two-Point Homogeneous Manifolds

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Received 15 November 2003, revised 30 November 2003, accepted 2 December 2003
Published online 3 December 2003

Key words n–width, two point homogeneous manifold, Besov space.
MSC (2000) 41A46, 42B15

Estimates of Kolmogorov and linear n-widths of Besov classes on compact globally symmetric spaces of rank 1 (i.e. on S^d, $P^d(\mathbb{R})$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$, $P^{16}(\mathrm{Cay})$) are established. It is shown that these estimates have sharp orders in different important cases. A new characterisation of Besov spaces is also given.

1 Introduction

In the present paper we investigate the asymptotically optimal approximation of Besov classes on compact globally symmetric spaces of rank 1 (two-point homogeneous spaces) S^d, $P^d(\mathbb{R})$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$, $P^{16}(\mathrm{Cay})$. In what follows, optimal approximation will be interpreted in the sense of Kolmogorov and linear n-widths.

Estimates for Kolmogorov n-widths of Besov classes on bounded regions of Euclidean spaces can be found in [21]. The spaces of Besov type on manifolds and their equivalent characterisations have been investigated in different articles (see e.g. [22, 23, 16, 15, 5, 17]).

There are various approaches to the definition of smoothness via harmonic analysis. The basic theorem in this range of problems is the well known analog of the Littlewood-Paley theorem [12] for trigonometric series, on compact globally symmetric spaces of rank 1 by Bonami and Clerc [2]. We introduce the Besov spaces decomposing a smooth function f into a series relative to spherical harmonics and using zonal polynomials $K_n(z)$ which are natural generalizations of the de la Vallée Poussin polynomials on S^1. We prove that the Besov spaces are real interpolation of two Sobolev spaces. Our definition of Besov space is new even for the sphere S^d, $d \geq 2$.

We use sharp orders of Kolmogorov n-widths of Sobolev classes from [3] and [11], and interpolation techniques by Triebel [21] to prove asymptotic estimates for Kolmogorov and linear n-widths of Besov classes on two-point homogeneous spaces.

Suppose that A is a convex, compact, centrally symmetric subset of a Banach space X with unit ball B. The *linear* n–width of A in X is defined by

$$\delta_n(A,X) := \delta_n(A,B) := \inf_{P_n} \sup_{f\in A} \|f - P_n f\|,$$

where P_n varies over all linear operators of rank at most n that map X into itself.

The Kolmogorov n–width of A in X is defined by

$$d_n(A,X) := d_n(A,B) := \inf_{X_n} \sup_{f\in A} \inf_{g\in X_n} \|f - g\|,$$

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where X_n runs over all subspaces of X of dimension n.

For ease of notation we will write $a_n \ll b_n$ for two sequences, if $a_n \leq cb_n$ for $n \in \mathbb{N}$ and $a_n \asymp b_n$, if $c_1 b_n \leq a_n \leq c_2 b_n$ for all $n \in \mathbb{N}$ and some constants c, c_1 and c_2. Also, we shall put

$$(a)_+ := \begin{cases} a, & \text{if } a > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We shall be interested here in compact homogeneous spaces. Such manifolds of dimension d will be denoted by M^d. Each M^d can be considered as the orbit space of some compact subgroup $\mathcal{H}$ of the orthogonal group $\mathcal{G}$, that is $M^d = \mathcal{G}/\mathcal{H}$. Let $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ be the natural mapping and e be the identity of $\mathcal{G}$. The point $o = \pi(e)$ which is invariant under all motions of $\mathcal{H}$ is called the pole of M^d. On any such manifold there is an invariant Riemannian metric $d(\cdot,\cdot)$, and a measure $d\nu$ which is induced by the normalised left Haar measure on $\mathcal{G}$ and is invariant under the action of $\mathcal{G}$. The *two-point* homogeneous spaces have the following additional property. If $x, x', y, y' \in M^d$ with $d(x, y) = d(x', y')$ then there is a $g \in \mathcal{G}$ such that $x = gx'$ and $y = gy'$. Two point homogeneous spaces admit essentially only one invariant second order differential operator, the Laplace-Beltrami operator Δ. A function $Z : M^d \to \mathbb{R}$ is called zonal if $Z(h^{-1}\cdot) = Z(\cdot)$ for any $h \in \mathcal{H}$. The geometry of these spaces is in many respects similar. For example, all geodesics in a given one of these spaces are closed and have the same length $2L$. Here L is the diameter of $\mathcal{G}/\mathcal{H}$, i.e. the maximum distance between any two points. A complete classification of the two-point homogeneous spaces was given by Wang [24].

For each zonal function z on M^d, we have a univariate function $\tilde{z}$, defined on $[-1, 1]$,

$$z(x) = \tilde{z}(\cos(2\lambda d(x, o))), \quad x \in M^d,$$

where λ is either $\pi/2L$ or $\pi/4L$, depending on the homogeneous space M^d.

Let L_p be the set of all complex measurable functions φ on M^d of finite norm, given by

$$\| \varphi \|_p = \begin{cases} \left(\int_{M^d} |\varphi(x)|^p d\nu(x)\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{ess sup}\{|\varphi(x)| \mid x \in M^d\}, & \text{if } p = \infty. \end{cases}$$

Further, let $U_p = \{\varphi : \varphi \in L_p, \ \| \varphi \|_p \leq 1\}$.

Let $\tilde{z} \in L_1([-1, 1], (1-x)^\alpha (1+x)^\beta dx)$. Then, for any integrable function g we can define convolution h on M^d as the following

$$h(\cdot) = (z * g)(\cdot) = \int_{M^d} \tilde{z}(\cos(2\lambda d(\cdot, x)))g(x)d\nu(x).$$

For the convolution on M^d we have Young's inequality

$$\|z * g\|_q \leq \|z\|_p \|g\|_r, \tag{1}$$

where $1/q = 1/p + 1/r - 1$ and $1 \leq p, q, r \leq \infty$.

For each $n \in \mathbb{N}$, let H_n be the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue $-n(n + \alpha + \beta + 1)$, where α and β are numbers associated with the particular homogeneous space M^d. Let $\mathcal{T}_n = \oplus_{k=0}^n H_k$, and $\mathcal{T} = \cup_{n=0}^\infty \mathcal{T}_n$. We have $\dim \mathcal{T}_n \asymp n^d$.

The Hilbert space L_2 with usual scalar product

$$\langle f, g \rangle = \int_{M^d} f(x)g(x)d\nu(x)$$

has the decomposition

$$L_2 = \bigoplus_{n=0}^\infty H_n.$$

There is a unique real zonal eigenfunction (up to scalar multiplication) $Z_n \in H_n$ such that the orthogonal projection from L_2 onto H_n is given by the convolution operator $\varphi \mapsto Z_n * \varphi$. We have

$$\tilde{Z}_n(t) = C_n(M^d) P_n^{\alpha,\beta}(t),$$

where $P_n^{\alpha,\beta}$ are the Jacobi polynomials (see, for example Szegö [20]), and α and β can be specified for each M^d. For example, in the case of S^d, we have $\sigma = 0$ and $\rho = d-1$. Thus, $\alpha = \beta = (d-2)/2$ and the polynomials $P_n^{\alpha,\beta}$ are just multiples of the Gegenbauer polynomial $P_n^{(d-1)/2}$. The normalisation constant $C_n(M^d) \asymp n^{d/2}$ is chosen for technical reasons, and its explicit value can be found in [3].

See [7, 9, 14, 18] for more information concerning the harmonic analysis on homogeneous spaces.

In the next section we will give the definition of a standard Sobolev spaces and introduce a set Besov type spaces on two-point homogeneous manifolds. In the Section 3 we demonstrate that our definition of Besov spaces generates a set of an equivalent norms. In the case of S^d our definition is equivalent to another definition of Besov spaces given in [13]. In the final section of the paper we will use interpolation results of Triebel [21] to give estimates of n-widths for our Besov spaces.

2 Sobolev and Besov spaces on two-point homogeneous manifolds

Let $\Lambda = \{\lambda_k\}_{k\geq 0}$ be a sequence of complex numbers, and let $1 \leq p, q \leq \infty$. If for any $\varphi \in L_p$ there is a function $f := \Lambda\varphi \in L_q$ with formal expansion

$$f \sim \sum_{k=0}^{\infty} \lambda_k Z_k * \varphi,$$

then we shall say that Λ is a multiplier operator of (p,q)-type with norm $\|\Lambda\|_{p,q} := \sup_{\varphi \in U_p} \|\Lambda\varphi\|_q$. For $s \in \mathbb{R}$, let $\Lambda^s = \{\mu_k^s\}_{k\geq 0}$, where $\mu_k^s = (k(k+\alpha+\beta+1))^{s/2}$. It was proved in [4] that, for $s > d(1/p - 1/q)_+$, Λ^{-s} is a multiplier operator of (p,q)-type, and for each $\varphi \in L_p$ and $n \geq 1$, there exists a polynomial function $t_n(\varphi) \in \mathcal{T}_n$ such that

$$\|\Lambda^{-s}\varphi - t_n(\varphi)\|_q \ll n^{-s+d(1/p-1/q)_+} \|\varphi\|_p. \tag{2}$$

The Sobolev space W_p^s, $s > 0$, is given by

$$W_p^s := \{f \in L_p : \Lambda^s f \in L_p\},$$

with norm

$$\|f\|_p^s := \|\Lambda^s f\|_p.$$

Here we have identified functions which differ by a constant, i.e., if $f - g =$constant, then $f = g$ in W_p^s. The Sobolev class $\overline{W}_p^s$, $s > 0$, is given by

$$\overline{W}_p^s := \{f \in W_p^s : \|f\|_p^s \leq 1\}.$$

It is easy to show that

$$\overline{W}_p^s = \{c + \Lambda^{-s} f : c \in \mathbb{R},\ f \in U_p\}.$$

Let $(\lambda_k^{(n)})$ be an infinite lower triangular matrix, i.e., $\lambda_k^{(n)} = 0$ for any $k > n$ and $n \in \mathbb{N}$.

Definition 2.1 We say that the sequence $K = \{K_{2^n}\}_{n\in\mathbb{N}}$ of polynomial zonal functions

$$K_{2^n} = \sum_{k=0}^{2^n} \lambda_k^{(2^n)} Z_k \tag{3}$$

possesses the property $\mathcal{K}_C$, and we write $K \in \mathcal{K}_C$, if $\lambda_k^{(2^n)} = 1$ for any $0 \leq k \leq 2^{n-1}$ and there exists a positive constant C such that, $\|K_{2^n}\|_1 \leq C$ for all $n \in \mathbb{N}$.

Remark 2.2 In the case of S^d, $d \geq 2$, the sequence of zonal polynomials K was introduced and constructed in an explicit form in [10, p. 287]. They are a natural generalization of the de la Vallée Poussin polynomial $V_{n,2n}(t)$ on S^1,

$$V_{n,2n}(t) = 1/2 + \sum_{k=1}^{2n} \lambda_k^{(2n)} \cos kt,$$

where $\lambda_k^{(2n)} = 1$ for $1 \leq k \leq n$ and $\lambda_k^{(2n)} = (2n-k)/n$ for $n < k \leq 2n$. The de la Vallée Poussin polynomials $V_{n,2n}(t)$ were used to introduce the Besov spaces on S^1.

Let $K \in \mathcal{K}_C$. For $k \geq 0$ we write

$$\varphi_k = K_{2^k} - K_{2^{k-1}},\ k \geq 1;\ \ \varphi_0 = Z_0. \tag{4}$$

Definition 2.3 The function $f \in L_p$ belongs to the Besov space $B^s_{p,q}(K)$, $s, p, q \in \mathbb{R}$, $s > 0$, $1 \leq p, q \leq \infty$, if

$$\|f\|^s_{p,q}(K) = \left(\sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} < \infty,$$

where the functions φ_k are defined in (4).

It will be proved in Theorem 3.6 that the norms of $B^s_{p,q}(K)$ and $(W^{s_0}_p, W^{s_1}_p)_{\theta,q}$ are equivalent for $s = (1-\theta)s_0 + \theta s_1$, $s_0, s_1 > 0$, $s_0 \neq s_1$. As consequence we can conclude that the norms of $B^s_{p,q}(K^1)$ and $B^s_{p,q}(K^2)$ are equivalent for all $K^1, K^2 \in \mathcal{K}_C$, that is, the norm of $B^s_{p,q}(K)$ does not depend on the sequence$K \in \mathcal{K}_C$. So that we will write $\|\cdot\|^s_{p,q}$ instead of $\|\cdot\|^s_{p,q}(K)$.

As with the Sobolev spaces we will identify two functions in $B^s_{p,q}(K)$ which differ by a constant. It is easy to see that $B^s_{p,q}(K)$ is a normed vector space with norm $\|\cdot\|^s_{p,q}(K)$. We will see in Section 3 that $B^s_{p,q}(K)$ is the interpolation space for two of the Sobolev spaces defined above, and is therefore a Banach space. Let

$$\overline{B}^s_{p,q}(K) = \{f \in L_p : \|f\|^s_{p,q}(K) \leq 1\}.$$

To give a useful sufficient conditions for the imbedding $K \in \mathcal{K}_C$ we will need some information concerning Cesàro means. Let $\delta \geq 0$ and $n \in \mathbb{N}$. We define the Cesàro kernel $\tilde{S}^\delta_n$ by

$$\tilde{S}^\delta_n = \frac{1}{C^\delta_n} \sum_{m=0}^{n} C^\delta_{n-m} \tilde{Z}_m,$$

where C^δ_n are Cesàro numbers of order n and index δ,

$$C^\delta_n = \frac{\Gamma(n+\delta+1)}{\Gamma(\delta+1)\Gamma(n+1)} \asymp n^\delta,\ \ n \to \infty \tag{5}$$

(see, e.g., [20, p. 237]). Given a sequence $\{\lambda_k\}_{k\geq 0}$ we define the differences $\Delta^s \lambda_k$, $k, s \in \mathbb{N}$, by $\Delta^0 \lambda_k = \lambda_k$, $\Delta^1 \lambda_k = \lambda_k - \lambda_{k+1}$, and $\Delta^{s+1}\lambda_k = \Delta^s \lambda_k - \Delta^s \lambda_{k+1}$.

Lemma 2.4 *Let $(\lambda_k^{(2^n)})$ be such that for any $n \in \mathbb{N}$ there is such $C' > 0$ that*

$$\sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{(2^n)}| k^d + \sum_{s=0}^{d} |\Delta^s \lambda^{(2^n)}_{2^n - s}| \vartheta^{(s)}_n < C', \tag{6}$$

where

$$\vartheta^{(s)}_n = \begin{cases} 2^{n(d-1)/2}, & 0 \leq s < (d-1)/2, \\ n 2^{n(d-1)/2}, & s = (d-1)/2, \\ 2^{ns}, & s > (d-1)/2. \end{cases}$$

Then $K = \{K_{2^n}\}_{n \in \mathbb{N}} \in \mathcal{K}_C$.

P r o o f. Applying the Abel transform $d+1$ times we get

$$K_{2^n} = \sum_{k=0}^{2^n-d-1} (\Delta^{d+1}\lambda_k^{(2^n)}) C_k^d S_k^d + \sum_{s=0}^{d} (\Delta^s \lambda_{2^n-s}^{(2^n)}) C_{2^n-s}^s S_{2^n-s}^s$$

and hence

$$\|K_{2^n}\|_1 \leq \sum_{k=0}^{2^n-d-1} |\Delta^{d+1}\lambda_k^{(2^n)}| C_k^d \|S_k^d\|_1 + \sum_{s=0}^{d} |\Delta^s \lambda_{2^n-s}^{(2^n)}| C_{2^n-s}^s \|S_{2^n-s}^s\|_1. \tag{7}$$

It was proved in [4, Lemma 3] that

$$\|S_n^\delta\|_1 \ll \begin{cases} n^{(d-1)/2-\delta}, & 0 \leq \delta < (d-1)/2, \\ \log n, & \delta = (d-1)/2, \\ 1, & \delta > (d-1)/2. \end{cases} \tag{8}$$

Comparing (5) - (8) we get (6). □

We present here two examples of sequences $\{K_{2^n}\}_{n\in\mathbb{N}} \in \mathcal{K}_C$.

Example 2.5 Consider the function

$$\xi(t) = -\frac{1}{\omega} \exp\left(\frac{1}{(t-1/2)(t-1)}\right),$$

where

$$\omega = \int_{1/2}^{1} \exp\left(\frac{1}{(t-1/2)(t-1)}\right) dt.$$

Let

$$\nu(t) = \begin{cases} 0, & 0 \leq t < 1/2, \\ \xi(t), & 1/2 \leq t \leq 1, \\ 0, & t > 1 \end{cases}$$

and

$$\mu(x) = 1 + \int_0^x \nu(t) dt, \quad x \geq 0.$$

Put $\lambda_k^{(n)} = \mu(k/n)$, $0 \leq k \leq n$. Let $f(x)$ be a continuous function on $\mathbb{R}$. Then for the differences of order k we have

$$\Delta_h^k f(x) = \sum_{\nu=0}^{k} (-1)^\nu \frac{k!}{\nu!(k-\nu)!} f(x+\nu h). \tag{9}$$

In addition, if $f(x)$ has k derivatives, then

$$\Delta_h^k f(x) = (-1)^k \int_0^h \cdots \int_0^h f^{(k)}(x+t_1+\cdots+t_k) dt_1 \cdots dt_k \tag{10}$$

Observe that for any integer $k \geq 1$ we have $|\mu^{(k)}(x)| = 0$ for $0 \leq x \leq 1/2$, $x \geq 1$, and

$$|\mu^{(k)}(x)| \leq C[(x-1/2)(x-1)]^{-2(k-1)} \exp\left(\frac{1}{(x-1/2)(x-1)}\right) \tag{11}$$

for $1/2 < x < 1$. Since $\Delta^s \lambda_k^{2^n} = \Delta_{2^{-n}}^s \mu(k/2^n)$, it follows from (10) and (11) that $|\Delta^{d+1}\lambda_k^{(2^n)}| \le C2^{-(d+1)n}$ and therefore,

$$\sum_{k=0}^{2^n-d-1} |\Delta^{d+1}\lambda_k^{(2^n)}| k^d \le C. \tag{12}$$

We have $\Delta^s \lambda_{2^n-s}^{(2^n)} = \mu(1) = 0$ for $s = 0$, and from (10) and (11) we have that

$$|\Delta^s \lambda_{2^n-s}^{(2^n)}| \le 2^{-ns} \left|\mu\left(1 - \frac{s}{2^n}\right)\right| \le \frac{C}{s} 2^{-(s-1)n} \exp\left(-\frac{2^{n+2}}{s}\right)$$

for $1 \le s \le 2^n$. Then

$$\lim_{n\to\infty} \sum_{s=0}^{d} |\Delta^s \lambda_{2^n-s}^{(2^n)}| \vartheta_n^{(s)} = 0. \tag{13}$$

Finally, from Lemma 2.4, (12) and (13) it follows that $K \in \mathcal{K}_C$.

Example 2.6 (see, also, [10, p. 287]) Consider the function

$$\chi_0(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1, \end{cases}$$

and for $1 \le s \le d$ let

$$\chi_s(t) = 2d \int_t^{t+1/(2d)} \chi_{s-1}(u)du.$$

The function χ_d is $d-1$ times continuously differentiable and positive on $[0,\infty)$. Furthermore, $\chi_d^{(d-1)}$ is Lipschitz continuous, $\chi_d(t) = 1$ for $0 \le t \le 1/2$, and

$$\chi_d(t) = P_d(t) = \frac{(2d)^d}{d!}(1-t)^d, \;\; 1 - \frac{1}{2d} \le t \le 1. \tag{14}$$

Also χ_d is a polynomial function of degree d in each interval $[t_s, t_{s-1}]$, $1 \le s \le d$, where $t_s = 1 - s/(2d)$. For each $n \ge 1$ consider a sequence $\{\lambda_k^{(n)}\}_{k=0}^n$ given by

$$\lambda_k^{(n)} = \chi_d\left(\frac{k}{n}\right), \; 0 \le k \le n. \tag{15}$$

To apply Lemma 2.4 we need to get some bounds for the differences $|\Delta^s \lambda_k^{(2^n)}|$. We show that for $2^n \ge 2d^2$ and $0 \le s \le d$,

$$|\Delta^s \lambda_{2^n-s}^{(2^n)}| \le (2d)^{2d} 2^{-dn} \tag{16}$$

and for all $n \in \mathbb{N}$, and $1 \le k \le 2^n$,

$$|\Delta^{d+1} \lambda_k^{(2^n)}| \le C(d) 2^{-dn}. \tag{17}$$

Remark that for $n \ge 2d^2$ and $0 \le s \le d$, $(2^n - s)/2^n \ge 1 - 1/(2d)$, so that, setting $h = 2^{-n}$,

$$\lambda_{2^n-s}^{(2^n)} = P_d\left(\frac{2^n-s}{2^n}\right), \quad \text{and} \quad \Delta^s \lambda_{2^n-s}^{(2^n)} = \Delta_h^s P_d\left(\frac{2^n-s}{2^n}\right).$$

Using (10) we get

$$\begin{aligned} |\Delta^s \lambda_{2^n-s}^{(2^n)}| &\le h^s \max_{t \ge (2^n-s)/n} P_d^{(s)}(t) \\ &\le 2^{-sn} \frac{(2d)^d}{(d-s)!} \left(\frac{s}{2^n}\right)^{d-s} \\ &\le (2d)^{2d} 2^{-dn}, \;\; 2^n \ge 2d^2, \; 0 \le s \le d, \end{aligned}$$

proving (16) and consequently (13). Since $\chi_d^{(d)}$ is a piecewise continuous function on $[0,1]$, it follows from (10) that $|\Delta^d_{2^{-n}}\chi_d(k/2^n)| \le C(d)2^{-nd}$. Then (17) follows from the definition of differences.

Now, let $I_k^n = [k/2^n, (k+d+1)/2^n]$, $t_s = 1 - s/(2d)$, $n \in \mathbb{N}$, $1 \le k \le 2^n$, $1 \le s \le d+1$, and let

$$E_{n,d} = \{k : \{t_1, t_2, \ldots, t_{d+1}\} \cap I_k^n \neq \emptyset,\ 1 \le k \le 2^n\}\,.$$

If k is not in $E_{n,d}$, than $\chi_d^{(d+1)}(t) = 0$ for $t \in I_k^n$ and hence $\Delta^{d+1}\lambda_k^{(2^n)} = \Delta^{d+1}_{2^{-n}}\chi_d(k/2^n) = 0$. Therefore

$$\left\{k : \Delta^{d+1}\lambda_k^{(2^n)} \neq 0,\ 0 \le k \le 2^n\right\} \subset E_{n,d}.$$

Since $\mathrm{Card}\,(E_{n,d}) \le (d+1)(d+2)$ for all $n \in \mathbb{N}$, using (17) we get

$$\sum_{k=0}^{2^n-d-1} |\Delta^{d+1}\lambda_k^{(2^n)}|k^d \le C(d)\sum_{k\in E_{n,d}}\left(\frac{k}{2^n}\right)^d \le C(d)(d+1)(d+2).$$

Finally from Lemma 2.4 it follows that $K = \{K_{2^n}\}_{n\in\mathbb{N}} \in \mathcal{K}_C$.

3 Besov spaces as interpolation spaces

Two complex Banach spaces A_0 and A_1 are called an interpolation pair $\bar{A} = (A_0, A_1)$ if there exists a Hausdorff topological vector space in which A_0 and A_1 are continuously embedded. Then, the following spaces and quantities are well-defined:

$$\begin{aligned}
\Delta(\bar{A}) &= A_0 \cap A_1;\\
\|a\|_{\Delta(\bar{A})} &= \max(\|a\|_{A_0}, \|a\|_{A_1}), \quad a \in \Delta(\bar{A});\\
\Sigma(\bar{A}) &= A_0 + A_1 = \{a_0 + a_1 : a_0 \in A_0, a_1 \in A_1\};\\
\|a\|_{\Sigma(\bar{A})} &= \inf_{a=a_0+a_1\in\Sigma(\bar{A})}(\|a_0\|_{A_0} + \|a_1\|_{A_1});\\
K(t,a) = K(t,a;\bar{A}) &= \inf_{a=a_0+a_1\in\Sigma(\bar{A})}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad 0 < t < \infty;\\
J(t,a) = J(t,a;\bar{A}) &= \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad 0 < t < \infty,\ a \in \Delta(\bar{A}).
\end{aligned}$$

For a given $t > 0$, $\|\cdot\|_{\Sigma(\bar{A})}$ and $K(t,\cdot)$ are equivalent norms on $\Sigma(\bar{A})$, and $\|\cdot\|_{\Delta(\bar{A})}$ and $J(t,\cdot)$ are equivalent norms on $\Delta(\bar{A})$.

Let $0 < \theta < 1$, $1 \le q \le \infty$, and let $\Phi_{\theta,q}$ be the functional defined by

$$\Phi_{\theta,q}(\varphi(t)) = \begin{cases} \left(\int_0^\infty (t^{-\theta}\varphi(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty,\\ \operatorname{ess\,sup}_{t>0} t^{-\theta}\varphi(t), & q = \infty,\end{cases}$$

where φ is a non-negative measurable function.

Given $a \in \Sigma(\bar{A})$ we define

$$\|a\|_{\theta,q;K} = \Phi_{\theta,q}(K(t,a)). \tag{18}$$

The set

$$\mathcal{K}_{\theta,q}(\bar{A}) = \{a \in \Sigma(\bar{A}) : \|a\|_{\theta,q;K} < \infty\}$$

is a Banach space with the norm $\|\cdot\|_{\theta,q;K}$, and is called the interpolation space of the pair $\bar{A}$ by the K-method. If $1 \le q < \infty$ then $\Delta(\bar{A})$ is dense in $\mathcal{K}_{\theta,q}(\bar{A})$.

We define the interpolation space of the pair $\bar{A}$ by the J-method as the set $\mathcal{J}_{\theta,q}(\bar{A})$, of all $a \in \Sigma(\bar{A})$ which can be represented by

$$a = \int_0^\infty u(t)\frac{dt}{t}, \tag{19}$$

where u is a measurable function taking values in $\Delta(\bar{A})$, convergence in the integral is in $\Sigma(\bar{A})$, and

$$\Phi_{\theta,q}(J(t,u(t))) < \infty. \tag{20}$$

The set $\mathcal{J}_{\theta,q}(\bar{A})$ is a Banach space with the norm

$$\|a\|_{\theta,q;J} = \inf_{u} \Phi_{\theta,q}(J(t,u(t))), \tag{21}$$

where the infimum is taken over all u such that (19) and (20) hold. If $1 \leq q < \infty$ then $\Delta(\bar{A})$ is dense in $\mathcal{J}_{\theta,q}(\bar{A})$.

Both the J- and K-method are called real interpolation methods.

A direct consequence of the definition is that, for any $a \in \Sigma(\bar{A})$, the function $t \mapsto K(t,a)$ is positive, increasing, concave, and for $t, s > 0$,

$$K(t,a) \leq \max(1, t/s) K(s,a). \tag{22}$$

Similarly, for any $a \in \Delta(\bar{A})$, the function $t \mapsto J(t,a)$ is positive, increasing, convex, and for $t, s > 0$,

$$J(t,a) \leq \max(1, t/s) J(s,a). \tag{23}$$

Theorem 3.1 *(*[1, p. 44]*) Let $\bar{A} = (A_0, A_1)$ be an interpolation pair, and let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then, $\mathcal{J}_{\theta,q}(\bar{A}) = \mathcal{K}_{\theta,q}(\bar{A})$ with equivalence of norms.*

From this point to the end of the paper we will not distinguish $\mathcal{J}_{\theta,q}(\bar{A})$ and $\mathcal{K}_{\theta,q}(\bar{A})$, and will denote them both by $(A_0, A_1)_{\theta,q}$. Both norms $\|\cdot\|_{\theta,q;J}$ and $\|\cdot\|_{\theta,q;K}$ will be denoted by $\|\cdot\|_{(A_0,A_1)_{\theta,q}}$.

Given a linear operator $T \in L(A,B)$, where A and B are Banach spaces, we denote

$$\|T\|_{A,B} = \sup_{a \in A,\, \|a\| \leq 1} \|Ta\|_B.$$

Suppose that T is a linear operator from $A_0 + A_1$ into B, and let T_i be the restriction of T to A_i, $i = 0, 1$. If $T_i \in L(A_i, B)$, $i = 0, 1$, then

$$\|T\|_{(A_0,A_1)_{\theta,q},B} \leq \left(\|T_0\|_{A_0,B}\right)^{1-\theta} \left(\|T_1\|_{A_1,B}\right)^{\theta}. \tag{24}$$

Let $0 < \theta < 1$, $1 < r < \infty$, and $1 \leq q \leq \infty$. We will denote by $\lambda^{\theta,r,q}$, the space of all sequences $(\alpha_k)_{k \in \mathbb{Z}}$, such that

$$\|(\alpha_k)\|_{\lambda^{\theta,r,q}} = \begin{cases} \left(\sum_{k \in \mathbb{Z}} (r^{-k\theta} |\alpha_k|)^q \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{k \in \mathbb{Z}} r^{-k\theta} |\alpha_k|, & q = \infty, \end{cases}$$

is finite.

The next two lemmas can be found for $r = 2$ in [1].

Lemma 3.2 *Let $\bar{A} = (A_0, A_1)$ be an interpolation pair and let $a \in \Sigma(\bar{A})$. Then, $a \in (A_0, A_1)_{\theta,q}$ if and only if $(K(r^k, a))_{k \in \mathbb{Z}} \in \lambda^{\theta,r,q}$. Moreover, we have*

$$r^{-\theta} (\log r)^{1/q} \|(K(r^k, a))\|_{\lambda^{\theta,r,q}} \leq \|a\|_{(A_0,A_1)_{\theta,q}} \leq r (\log r)^{1/q} \|(K(r^k, a))\|_{\lambda^{\theta,r,q}}. \tag{25}$$

Lemma 3.3 *Let $\bar{A} = (A_0, A_1)$ be an interpolation pair and let $a \in \Sigma(\bar{A})$. Then $a \in (A_0, A_1)_{\theta,q}$ if and only if there exists $u_k \in \Delta(\bar{A})$, $k \in \mathbb{Z}$, with*

$$a = \sum_{k \in \mathbb{Z}} u_k \tag{26}$$

(convergence inside $\Sigma(\bar{A})$), and such that $(J(r^k, u_k)) \in \lambda^{\theta,r,q}$. Moreover,

$$\begin{aligned} (\log r)^{-1+1/q} r^{-\theta} \inf_{u_k} \|(J(r^k, u_k))\|_{\lambda^{\theta,r,q}} \quad &\leq \quad \|a\|_{(A_0,A_1)_{\theta,q}} \\ &\leq r (\log r)^{-1+1/q} \inf_{u_k} \|(J(r^k, u_k))\|_{\lambda^{\theta,r,q}}. \end{aligned} \tag{27}$$

Now suppose $0 < s_1 < s_2$ and $1 \le p \le \infty$. It follows from (2) that $\Lambda^{s_1 - s_2}$ is bounded on L_p and hence, for $f \in W_p^{s_2}$,

$$\|f\|_p^{s_1} = \|\Lambda^{s_1-s_2}\Lambda^{s_2} f\|_p \le C_p \|f\|_p^{s_2} < \infty.$$

Therefore $W_p^{s_2} \subset W_p^{s_1}$ for $0 < s_1 < s_2$ and $1 \le p \le \infty$. The space $\mathcal{T} = \cup_{n=1}^{\infty} \mathcal{T}_n$ is dense in L_p, $1 \le p < \infty$, and hence is also dense in W_p^s, $s > 0$, $1 \le p < \infty$. Since $B_{p,q}^s$ is the interpolation of two Sobolev spaces (see Theorem 3.6) and $\mathcal{T}$ is dense in these Sobolev spaces, it follows that $\mathcal{T}$ is also dense in $B_{p,q}^s$, $s > 0$, $1 \le p < \infty$, and $1 \le q \le \infty$.

The next result is Bernstein's inequality and is proved in [6].

Theorem 3.4 *For all* $s, p \in \mathbb{R}$*,* $s > 0$*,* $1 \le p \le \infty$*,*

$$\left\| \sum_{k=1}^{m} \mu_k^s Z_k * f \right\|_p \le C_s m^s \|f\|_p, \quad f \in \mathcal{T}_m,\ m \ge 1. \tag{28}$$

Corollary 3.5 *Let* $K \in \mathcal{K}_C$*,* $s, p \in \mathbb{R}$*,* $s > 0$*, and* $1 \le p \le \infty$*. Then*

$$\|\Lambda^s(\varphi_k * f)\|_p \le C_s 2^{ks} \|\varphi_k * f\|_p, \quad f \in L_p,\ k \ge 0 \tag{29}$$

and

$$\|\varphi_k * f\|_p \le C_s 2^{-ks} \|\Lambda^s f\|_p, \quad f \in W_p^s,\ k \ge 0. \tag{30}$$

P r o o f. Let us fix $f \in L_p$, $1 \le p \le \infty$, and $k \ge 0$. We have that $\varphi_k * f \in \mathcal{T}_{2^k}$. Hence, by (28),

$$\|\Lambda^s(\varphi_k * f)\|_p \le C_s 2^{ks} \|\varphi_k * f\|_p.$$

Now if $f \in W_p^s$, let t_k be a polynomial of degree 2^{k-1} satisfying (2) for $\varphi = \Lambda^s f$. Then

$$\|f - t_k\|_p \ll 2^{-ks} \|\Lambda^s f\|_p,$$

and $K_{2^k} * t_k = t_k$. Therefore, by (1), (2) and Definition 2.1 we get

$$\begin{aligned} \|f - K_{2^k} * f\|_p &\le \|f - t_k\|_p + \|K_{2^k} * (f - t_k)\|_p \\ &\le (1 + \|K_{2^k}\|_1)\|f - t_k\|_p \\ &\le C 2^{-ks} \|\Lambda^s f\|_p \end{aligned}$$

and hence by (4)

$$\begin{aligned} \|\varphi_k * f\|_p &\le \|f - K_{2^k} * f\|_p + \|f - K_{2^{k-1}} * f\|_p \\ &\le C 2^{-ks} \|\Lambda^s f\|_p. \end{aligned}$$

□

Theorem 3.6 *Let* $K \in \mathcal{K}_C$*. If* $0 < s_0 < s_1$ *then*

$$B_{p,q}^{s_1} \subset B_{p,q}^{s_0}, \quad 1 \le p, q \le \infty. \tag{31}$$

If $1 \le q_0 < q_1 \le \infty$ *then*

$$B_{p,q_0}^s \subset B_{p,q_1}^s, \quad s > 0,\ 1 \le p \le \infty. \tag{32}$$

Moreover,

$$B_{p,1}^s \subset W_p^s \subset B_{p,\infty}^s, \quad s > 0,\ 1 \le p \le \infty. \tag{33}$$

If $s_0, s_1 > 0$ *and* $s_0 \ne s_1$*, then*

$$(W_p^{s_0}, W_p^{s_1})_{\theta,q} = B_{p,q}^s, \quad 1 \le p, q \le \infty,\ 0 < \theta < 1, \tag{34}$$

with equivalence of norms, where $s = (1-\theta)s_0 + \theta s_1$*.*

P r o o f. The inclusions (31) and (32) follow directly from the definition of Besov space. Now, from (29), we have

$$\begin{aligned} \|f\|_p^s &\le \sum_{k=0}^{\infty} \|\Lambda^s(\varphi_k * f)\|_p \\ &\le C_s \sum_{k=0}^{\infty} 2^{ks} \|\varphi_k * f\|_p \\ &= C_s \|f\|_{p,1}^s. \end{aligned}$$

From (30),

$$\begin{aligned} \|f\|_{p,\infty}^s &= \sup_{k\in\mathbb{N}} 2^{ks}\|\varphi_k * f\|_p \\ &\le C_s \sup_{k\in\mathbb{N}} 2^{ks} 2^{-ks} \|\Lambda^s f\|_p \\ &= C_s \|f\|_p^s. \end{aligned}$$

The inclusions in (33) follow from the previous two inequalities.

We now prove (34). Consider $0 < s_0 < s_1$. Let $f \in (W_p^{s_0}, W_p^{s_1})_{\theta,q}$, with $f = f_0 + f_1$, $f_i \in W_p^{s_i}$, $i = 0, 1$. From (30),

$$\begin{aligned} \|\varphi_k * f\|_p &\le \|\varphi_k * f_0\|_p + \|\varphi_k * f_1\|_p \\ &\le C_0 2^{-ks_0}\|\Lambda^{s_0} f_0\|_p + C_1 2^{-ks_1}\|\Lambda^{s_1} f_1\|_p \\ &\le C 2^{-ks_0}(\|f_0\|_p^{s_0} + 2^{k(s_0-s_1)}\|f_1\|_p^{s_1}), \end{aligned}$$

and hence

$$\|\varphi_k * f\|_p \le C 2^{-ks_0} K(2^{k(s_0-s_1)}, f).$$

Thus, putting $r = 2^{s_1-s_0}$ we get

$$2^{ks}\|\varphi_k * f\|_p \le C 2^{k\theta(s_1-s_0)} K(2^{k(s_0-s_1)}, f) = C r^{k\theta} K(r^{-k}, f),$$

so that

$$\left(\sum_{k=0}^{\infty} (2^{ks}\|\varphi_k * f\|_p)^q\right)^{1/q} \le C \left(\sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^q\right)^{1/q}.$$

Applying Lemma 3.2 we find

$$\begin{aligned} \|f\|_{p,q}^s &= \left(\sum_{k=0}^{\infty} (2^{ks}\|\varphi_k * f\|_p)^q\right)^{1/q} \\ &\le C \left(\sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^q\right)^{1/q} \\ &\le C \|f\|_{(W_p^{s_0}, W_p^{s_1})_{\theta,q}}. \end{aligned}$$

Suppose conversely that $f \in B_{p,q}^s$. From (29),

$$\begin{aligned} J(2^{k(s_0-s_1)}, \varphi_k * f) &= \max(\|\Lambda^{s_0}(\varphi_k * f)\|_p, 2^{k(s_0-s_1)}\|\Lambda^{s_1}(\varphi_k * f)\|_p) \\ &\le \max(C_{s_0} 2^{ks_0}\|\varphi_k * f\|_p, C_{s_1} 2^{k(s_0-s_1)} 2^{ks_1}\|\varphi_k * f\|_p) \\ &= C_s 2^{ks_0}\|\varphi_k * f\|_p, \end{aligned}$$

and hence

$$2^{k(s-s_0)} J(2^{k(s_0-s_1)}, \varphi_k * f) \le C_s 2^{ks}\|\varphi_k * f\|_p.$$

Thus, again putting $r = 2^{s_1 - s_0}$, we get

$$r^{\theta k} J(r^{-k}, \varphi_k * f) \leq C_s 2^{ks} \|\varphi_k * f\|_p, \quad k \geq 0. \tag{35}$$

Consider the sequence $(u_k)_{k\in\mathbb{Z}}$ defined by

$$u_k = \begin{cases} 0, & k \geq 1, \\ \varphi_{-k} * f, & k \leq 0. \end{cases}$$

Since $s_0 < s_1$ we have that $W_p^{s_1} \subset W_p^{s_0}$, and hence $W_p^{s_0} + W_p^{s_1} = W_p^{s_0}$. From (29) and Hölder's inequality

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|u_k\|_p^{s_0} &= \sum_{k=0}^{\infty} \|\Lambda^{s_0}(\varphi_k * f)\|_p \\ &\leq C_{s_0} \sum_{k=0}^{\infty} 2^{ks_0} \|\varphi_k * f\|_p \\ &= C_{s_0} \sum_{k=0}^{\infty} 2^{k\theta(s_0 - s_1)} 2^{ks} \|\varphi_k * f\|_p \\ &\leq C_{s_0} \left(\sum_{k=0}^{\infty} r^{-k\theta q'} \right)^{1/q'} \|f\|_{p,q}^s < \infty, \end{aligned}$$

where $1/q + 1/q' = 1$. Therefore, we may conclude that the series $\sum_{k\in\mathbb{Z}} u_k$ converges to $f \in W_p^{s_0}$. Now, by (35)

$$\begin{aligned} \left(\sum_{k\in\mathbb{Z}} (r^{-k\theta} J(r^k, u_k))^q \right)^{1/q} &= \left(\sum_{k=0}^{\infty} (r^{k\theta} J(r^{-k}, \varphi_k * f))^q \right)^{1/q} \\ &\leq C_s \left(\sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} \\ &= C_s \|f\|_{p,q}^s. \end{aligned}$$

Thus, by Lemma 3.2, we can conclude that $f \in (W_p^{s_0}, W_p^{s_1})_{\theta,q}$ and that

$$\|f\|_{(W_p^{s_0}, W_p^{s_1})_{\theta,q}} \leq C_s \|f\|_{p,q}^s.$$

□

Remark 3.7 Let $s > d(1/p - 1/q)_+$, $1 \leq p, q \leq \infty$, and $n \in \mathbb{N}$. Consider the operator I_n, defined on L_p, by $I_n f = f - t_n(\Lambda^s f)$, where $t_n(\varphi) = G_n * \varphi$, where $G_n = \sum_{k=1}^{n} (\Delta^{N+1} \lambda_k^{-s}) C_k^N S_k^N$, $N = (d+1)/2$ if d is odd and $N = (d+2)/2$ if d is even. From (2) we have that

$$\|I_n\|_{W_p^s, L_q} \leq C_s n^{-s + d(1/p - 1/q)_+}. \tag{36}$$

Now, consider $s, p, r, q \in \mathbb{R}$ with $1 \leq p, q, r \leq \infty$, and $s > d(1/p - 1/q)_+$. Let $s_0, s_1 \in \mathbb{R}$ such that $s_1 > s > s_0 > d(1/p - 1/q)_+$, and let $\theta \in (0, 1)$, such that $s = (1-\theta)s_0 + \theta s_1$. From (34) we have $B_{p,r}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,r}$. Then, from (24) and (36),

$$\begin{aligned} \|I_n\|_{B_{p,r}^s, L_q} &\leq \left(\|I_n\|_{W_p^{s_0}, L_q} \right)^{1-\theta} \left(\|I_n\|_{W_p^{s_1}, L_q} \right)^{\theta} \\ &\leq C_s n^{-s + d(1/p - 1/q)_+}. \end{aligned} \tag{37}$$

Remark 3.8 Consider the multiplier sequence $\dot{\Lambda}^s = \{\dot{\lambda}_k^s\}_{k\geq 0}$, $\dot{\lambda}_k^s = (1+k(k+\alpha+\beta+1))^{s/2}$, $s>0$. The Sobolev spaces can also be defined using $\dot{\Lambda}^s$ rather than Λ^s. We define

$$\dot{W}_p^s := \left\{ f \in L_p : \dot{\Lambda}^s f \in L_p \right\},$$

with norm

$$\|f\dot{\|}_p^s := \|\dot{\Lambda}^s f\|_p.$$

From [4, Theorem 2], $\dot{\Lambda}^{-s}$ is bounded from L_p to L_q for $s > d(1/p-1/q)_+$. The inequality (2) also holds for $\dot{\Lambda}^{-s}$, $s>0$. Since Bernstein's inequality (Theorem 3.4) holds for the sequence $\dot{\lambda}_k^s$ (see [6]), the same is true for Corollary 3.5.

Now, for $s,p,q \in \mathbb{R}$, $s>0$, $1 \leq p,q \leq \infty$, and $f \in L_p$, we define

$$\|f\dot{\|}_{p,q}^s := \|f\|_p + \|f\|_{p,q}^s,$$

and

$$\dot{B}_{p,q}^s := \{f \in L_p : \|f\dot{\|}_{p,q}^s < \infty\}.$$

In this case we have that $\|\cdot\dot{\|}_p^s$ and $\|\cdot\dot{\|}_{p,q}^s$ are norms in $\dot{W}_p^s$ and $\dot{B}_{p,q}^s$ respectively, when we consider these spaces as subspaces of L_p.

Theorem 3.6 holds for the spaces $\dot{W}_p^s$ and $\dot{B}_{p,q}^s$, with minor changes in the proof. In the proof we consider a sequence $(u_k)_{k\in\mathbb{Z}}$ as in the proof of Theorem 3.6, only with the minor change $u_1 = Z_0 * f$. We need the inequalities

$$C_1 r^{-1} J(r,f) \leq \|f\|_p \leq C_2 K(1,f), \quad 1 \leq p \leq \infty,\ f \in L_p.$$

Remark 3.9 Besov spaces on the unit sphere S^d in $\mathbb{R}^{d+1}$ were studied in [13], where the authors give a list of equivalent norms for Besov spaces. We will show that the Besov spaces considered in [13] and the spaces $\dot{B}_{p,q}^s$ have equivalent norms.

The Sobolev spaces considered in [13] are the spaces $\dot{W}_p^s$ given in Remark 3.8. Let us denote by $\overset{*}{B}{}_{p,q}^s$ the Besov spaces in [13], $s>0$, $1 \leq p,q \leq \infty$. The following two equivalent norms for $\overset{*}{B}{}_{p,q}^s$, among others, are given in [13]:

$$\begin{aligned} {}^{(1)}\|f\|_{p,q}^s &= \|f\|_p + \left(\sum_{m=1}^{\infty} m^{sq-1} E_m(f)_p^q \right)^{1/q}, \\ {}^{(2)}\|f\|_{p,q}^s &= \|f\|_p + \left(\sum_{m=1}^{\infty} m^{sq-1} K_r(\pi/m,f)_p^q \right)^{1/q}, \quad s<2r,\ r=1,2,\dots, \end{aligned}$$

where $E_m(f)_p$ is the best approximation of a function $f \in L_p$ by spherical polynomials of degree $\leq m$, and $K_r(\pi/m,f)_p$ is given in terms of the K-functional by

$$K_r(\pi/m,f)_p := K(\pi^{2r}m^{-2r}, f; L_p, \dot{W}_p^{2r}).$$

It is not difficult to show (see Lemma 3.2) that the norm ${}^{(2)}\|f\|_{p,q}^s$ is the norm of the interpolation space $(L_p, \dot{W}_p^{2r})_{s/2r,q}$, that is, $\overset{*}{B}{}_{p,q}^s = (L_p, \dot{W}_p^{2r})_{s/2r,q}$. It was also proved in [13] that

$$\overset{*}{B}{}_{p,1}^s \subset \dot{W}_p^s \subset \overset{*}{B}{}_{p,\infty}^s, \quad s>0,\ 1 \leq p \leq \infty.$$

By the definition of the class $\mathcal{H}(\theta, L_p, \dot{W}_p^{2r})$ given in Section 4, we can conclude that

$$\dot{W}_p^s \in \mathcal{H}(s/2r, L_p, \dot{W}_p^{2r}), \quad 1 \leq p \leq \infty,\ s<2r.$$

Consider now $s, s_0, s_1 \in \mathbb{R}$ with $0 < s_0 < s < s_1$, and $s = (1-\eta)s_0 + \eta s_1$, $\eta \in (0,1)$. Then

$$\dot{W}_p^{s_i} \in \mathcal{H}(s_i/2r, L_p, \dot{W}_p^{2r}), \quad i = 0, 1,$$

and from (34)

$$\dot{B}_{p,q}^s = (\dot{W}_p^{s_0}, \dot{W}_p^{s_1})_{\eta,q}.$$

Applying the Reiteration Theorem (see [1, p. 50]) we can conclude that

$$\dot{B}_{p,q}^s = (\dot{W}_p^{s_0}, \dot{W}_p^{s_1})_{\eta,q} = (L_p, \dot{W}_p^{2r})_{s/2r,q} = \overset{*}{B}{}_{p,q}^s .$$

4 Estimates of n–widths of Besov classes

For $s > d(1/p - 1/q)_+$ the operator Λ^{-s} is bounded from L_p to L_q and therefore, $W_p^s = \Lambda^{-s}(L_p) \subset L_q$. Now let $s_0, s_1 > d(1/p - 1/q)_+$ with $s_0 < s < s_1$, and let $0 < \theta < 1$ such that $s = (1-\theta)s_0 + \theta s_1$. Then, by (34), $B_{p,r}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,r} \subset W_p^{s_0} \subset L_q$. Thus $B_{p,r}^s \subset L_q$, for $s > d(1/p - 1/q)_+$, and $1 \le p, q, r \le \infty$.

Let $\bar{A} = (A_0, A_1)$ be an interpolation pair of Banach spaces, U_{A_i} be the unit ball of A_i, $i = 0, 1$, and let B be another Banach space such that $A_0, A_1 \subset B$. In Triebel [21] sufficient conditions were given on Kolmogorov n-widths $d_n(U_{A_i}, B)$, $i = 0, 1$, for obtaining estimates for the Kolmogorov n-widths $d_n(U_{(A_0,A_1)_{\theta,r}}, B)$. To get our results we will use sharp orders for the Kolmogorov n-widths $d_n(W_p^s, L_q)$ found in [3, 11], the property (34), which says that a Besov space is an interpolation space of Sobolev spaces, and Triebel's theorem mentioned above.

We will denote the unit ball of a Banach space A by U_A. If $\bar{A} = (A_0, A_1)$ is an interpolation pair and $(A_0, A_1)_{\theta,1} \subset A \subset (A_0, A_1)_{\theta,\infty}$, we write $A \in \mathcal{H}(\theta, A_0, A_1)$.

Theorem 4.1 *(see* [21]*) Let $\bar{A} = (A_0, A_1)$ be an interpolation pair of Banach spaces and let B be a Banach space such that $A_0, A_1 \subset B$. Suppose that there exist $\alpha_i \ge 0$ and $C_i > 0$, $i = 0, 1$, such that*

$$d_n(U_{A_i}, B) \le C_i n^{-\alpha_i}, \quad n \in \mathbb{N},\ i = 0, 1. \tag{38}$$

Suppose also that there exists $\tilde{\theta} \in (0,1)$ and $\tilde{A} \in \mathcal{H}(\tilde{\theta}, A_0, A_1)$ such that

$$d_n(U_{\tilde{A}}, B) \ge C n^{-\tilde{\alpha}}, \quad n \in \mathbb{N}, \tag{39}$$

where $0 < \tilde{\alpha} = (1-\tilde{\theta})\alpha_0 + \tilde{\theta}\alpha_1$, and $C > 0$. If $\theta \in [0,1]$, $A \in \mathcal{H}(\theta, A_0, A_1)$, and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, then

$$d_n(U_A, B) \asymp n^{-\alpha}, \quad n \in \mathbb{N}. \tag{40}$$

Theorem 4.2 *(see* [3, 4, 11]*) Let $p, q, s \in \mathbb{R}$.*

: *(i) If $1 \le p = q \le \infty$ or $2 \le q \le p < \infty$, and $s > 0$, then*

$$d_n(\overline{W}{}_p^s, L_q) \asymp n^{-s/d}. \tag{41}$$

: *(ii) If $1 \le p \le q \le 2$ and $s > d(1/p - 1/q)$, then*

$$d_n(\overline{W}{}_p^s, L_q) \asymp n^{-s/d + 1/p - 1/q}. \tag{42}$$

: *(iii) If $2 \le p \le q < \infty$ and $s > d/2$, then*

$$d_n(\overline{W}{}_p^s, L_q) \asymp n^{-s/d}. \tag{43}$$

: *(iv) If $1 \le p \le 2 \le q < \infty$ and $s > d/p$, then*

$$d_n(\overline{W}{}_p^s, L_q) \asymp n^{-s/d + 1/p - 1/2}. \tag{44}$$

: *(v) If* $1 < q \le p \le 2$ *and* $s > 0$*, then*

$$d_n(\overline{W}^s_p, L_q) \asymp n^{-s/d}. \tag{45}$$

Theorem 4.3 *Let* $p, q, s, r \in \mathbb{R}$*.*

: *(i) If* $1 \le p = q \le \infty$ *or* $2 \le q \le p < \infty$*,* $s > 0$*, and* $1 \le r \le \infty$*, then*

$$d_n(\overline{B}^s_{p,r}, L_q) \asymp \delta_n(\overline{B}^s_{p,r}, L_q) \asymp n^{-s/d}. \tag{46}$$

: *(ii) If* $1 \le p \le q \le 2$*,* $s > d(1/p - 1/q)$*, and* $1 \le r \le \infty$*, then*

$$d_n(\overline{B}^s_{p,r}, L_q) \asymp \delta_n(\overline{B}^s_{p,r}, L_q) \asymp n^{-s/d+1/p-1/q}. \tag{47}$$

: *(iii) If* $2 \le p \le q < \infty$*,* $s > d/2$*, and* $1 \le r \le \infty$*, then*

$$d_n(\overline{B}^s_{p,r}, L_q) \asymp n^{-s/d}. \tag{48}$$

: *(iv) If* $1 \le p \le 2 \le q < \infty$*,* $s > d/p$*, and* $1 \le r \le \infty$*, then*

$$d_n(\overline{B}^s_{p,r}, L_q) \asymp n^{-s/d+1/p-1/2}. \tag{49}$$

: *(v) If* $1 < q \le p \le 2$*,* $s > 0$*, and* $1 \le r \le \infty$*, then*

$$d_n(\overline{B}^s_{p,r}, L_q) \asymp \delta_n(\overline{B}^s_{p,r}, L_q) \asymp n^{-s/d}. \tag{50}$$

Proof. Consider $1 \le p, r \le \infty$, $s, s_0, s_1 > 0$, and $\theta \in (0,1)$, with $s = (1-\theta)s_0 + \theta s_1$. From (34) we have $B^s_{p,1} = (W^{s_0}_p, W^{s_1}_p)_{\theta,1}$, $B^s_{p,\infty} = (W^{s_0}_p, W^{s_1}_p)_{\theta,\infty}$, and hence, from (33),

$$(W^{s_0}_p, W^{s_1}_p)_{\theta,1} \subset W^s_p \subset (W^{s_0}_p, W^{s_1}_p)_{\theta,\infty}.$$

Therefore

$$W^s_p \in \mathcal{H}(\theta, W^{s_0}_p, W^{s_1}_p). \tag{51}$$

Now from (32) we have $B^s_{p,1} \subset B^s_{p,r} \subset B^s_{p,\infty}$, and hence, from (34),

$$(W^{s_0}_p, W^{s_1}_p)_{\theta,1} \subset B^s_{p,r} \subset (W^{s_0}_p, W^{s_1}_p)_{\theta,\infty}.$$

Thus,

$$B^s_{p,r} \in \mathcal{H}(\theta, W^{s_0}_p, W^{s_1}_p). \tag{52}$$

Let s, p, q, r be as in (i) and choose $s_0, s_1 \in \mathbb{R}$ such that $0 < s_0 < s < s_1$. Let $\theta \in (0,1)$ with $s = (1-\theta)s_0 + \theta s_1$. From (41) we have

$$d_n(\overline{W}^{s_i}_p, L_q) \le C n^{-s_i/d}, \quad n \in \mathbb{N},\ i = 0, 1. \tag{53}$$

From (51) we have that $W^s_p \in \mathcal{H}(\theta, W^{s_0}_p, W^{s_1}_p)$, and, from (41),

$$d_n(\overline{W}^s_p, L_q) \ge C n^{-s/d}, \quad n \in \mathbb{N}. \tag{54}$$

Then (46) for Kolmogorov n-widths follows from Theorem 4.1, (53), and (54).

Now let s, p, q, r be as in (ii) and choose s_0 and s_1 such that $s_1 > s > s_0 > d(1/p - 1/q)$. Let $\theta \in (0,1)$ such that $s = (1-\theta)s_0 + \theta s_1$. From (42) we have

$$d_n(\overline{W}^{s_i}_p, L_q) \le C n^{-s_i/d+1/p-1/q}, \quad n \in \mathbb{N},\ i = 0, 1. \tag{55}$$

From (51) we have that $W_p^s \in \mathcal{H}(\theta, W_p^{s_0}, W_p^{s_1})$, and, from (42),

$$d_n(\overline{W}_p^s, L_q) \geq C n^{-s/d+1/p-1/q}, \quad n \in \mathbb{N}. \tag{56}$$

Then (47) for Kolmogorov n-widths follows from Theorem 4.1, (55), and (56).

The proofs of (48), (49), and (50) for Kolmogorov n-widths follow in the same way as those of (46) and (47) above.

Let $s > d(1/p - 1/q)_+$, and $n \in \mathbb{N}$. From (37) we have that

$$\delta_{n^d}(\bar{B}_{p,r}^s, L_q) \leq \|I_n\|_{B_{p,r}^s, L_q} \leq C_s n^{-s+d(1/p-1/q)_+},$$

and hence

$$\delta_n(\bar{B}_{p,r}^s, L_q) \leq C_s n^{-s/d+(1/p-1/q)_+}, \quad n \in \mathbb{N}. \tag{57}$$

The upper bounds for linear n-widths in (46), (47) and (50) follow from (57). The lower bounds follow from the inequality $d_n(A, X) \leq \delta_n(A, X)$. □

Acknowledgements The first and the third authors were supported in part by FAPESP/Brazil, with the Grants 03/10393-8 and 01/05876-4 respectively.

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