Estimates of $n$–Widths of Besov Classes on Two-Point Homogeneous Manifolds

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Estimates of Kolmogorov and linear $n$-widths of Besov classes on compact globally symmetric spaces of rank 1 (i.e. on $S^d$, $P^d(\mathbb{R})$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$, $P^{16}(\text{Cay})$) are established. It is shown that these estimates have sharp orders in different important cases. A new characterisation of Besov spaces is also given.

1 Introduction

In the present paper we investigate the asymptotically optimal approximation of Besov classes on compact globally symmetric spaces of rank 1 (two-point homogeneous spaces) $S^d$, $P^d(\mathbb{R})$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$, $P^{16}(\text{Cay})$. In what follows, optimal approximation will be interpreted in the sense of Kolmogorov and linear $n$-widths.

Estimates for Kolmogorov $n$-widths of Besov classes on bounded regions of Euclidean spaces can be found in [21]. The spaces of Besov type on manifolds and their equivalent characterisations have been investigated in different articles (see e.g. [22, 23, 16, 15, 5, 17]).

There are various approaches to the definition of smoothness via harmonic analysis. The basic theorem in this range of problems is the well known analog of the Littlewood-Paley theorem [12] for trigonometric series, on compact globally symmetric spaces of rank 1 by Bonami and Clerc [2]. We introduce the Besov spaces decomposing a smooth function $f$ into a series relative to spherical harmonics and using zonal polynomials $K_n(z)$ which are natural generalizations of the de la Vallée Poussin polynomials on $S^1$. We prove that the Besov spaces are real interpolation of two Sobolev spaces. Our definition of Besov space is new even for the sphere $S^d$, $d \geq 2$.

We use sharp orders of Kolmogorov $n$-widths of Sobolev classes from [3] and [11], and interpolation techniques by Triebel [21] to prove asymptotic estimates for Kolmogorov and linear $n$-widths of Besov classes on two-point homogeneous spaces.

Suppose that $A$ is a convex, compact, centrally symmetric subset of a Banach space $X$ with unit ball $B$. The linear $n$–width of $A$ in $X$ is defined by

$$\delta_n(A, X) := \delta_n(A, B) := \inf_{P_n} \sup_{f \in A} \| f - P_n f \|,$$

where $P_n$ varies over all linear operators of rank at most $n$ that map $X$ into itself.

The Kolmogorov $n$–width of $A$ in $X$ is defined by

$$d_n(A, X) := d_n(A, B) := \inf_{X_n} \inf_{g \in X_n} \| f - g \|,$$

where $X_n$ and $g$ vary over all subspaces of $A$ with dimension $n$.

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where \( X_n \) runs over all subspaces of \( X \) of dimension \( n \).

For ease of notation we will write \( a_n \leq b_n \) for two sequences, if \( a_n \leq c b_n \) for \( n \in \mathbb{N} \) and \( a_n \asymp b_n \), if \( c_1 b_n \leq a_n \leq c_2 b_n \) for all \( n \in \mathbb{N} \) and some constants \( c, c_1 \) and \( c_2 \). Also, we shall put

\[
(a)_+ := \begin{cases} a, & \text{if } a > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

We shall be interested here in compact homogeneous spaces. Such manifolds of dimension \( d \) will be denoted by \( M^d \). Each \( M^d \) can be considered as the orbit space of some compact subgroup \( \mathcal{H} \) of the orthogonal group \( \mathcal{G} \), that is \( M^d = \mathcal{G}/\mathcal{H} \). Let \( \pi : \mathcal{G} \to \mathcal{G}/\mathcal{H} \) be the natural mapping and \( \epsilon \) be the identity of \( \mathcal{G} \). The point \( o = \pi(\epsilon) \) which is invariant under all motions of \( \mathcal{H} \) is called the pole of \( M^d \). On any such manifold there is an invariant Riemannian metric \( d(\cdot, \cdot) \), and a measure \( d\nu \) which is induced by the normalised left Haar measure on \( \mathcal{G} \) and is invariant under the action of \( \mathcal{G} \). The two-point homogeneous spaces have the following additional property. If \( x, x', y, y' \in M^d \) with \( d(x, y) = d(x', y') \) then there is a \( g \in \mathcal{G} \) such that \( x = gx' \) and \( y = gy' \). Two point homogeneous spaces admit essentially only one invariant second order differential operator, the Laplace-Beltrami operator \( \Delta \). A function \( Z : M^d \to \mathbb{R} \) is called zonal if \( Z(h^{-1} \cdot) = Z(\cdot) \) for any \( h \in \mathcal{H} \). The geometry of these spaces is in many respects similar. For example, all geodesics in a given one of these spaces are closed and have the same length \( 2L \). Here \( L \) is the diameter of \( \mathcal{G}/\mathcal{H} \), i.e. the maximum distance between any two points. A complete classification of the two-point homogeneous spaces was given by Wang [24].

For each zonal function \( z \) on \( M^d \), we have a univariate function \( \tilde{z} \), defined on \([-1, 1]\),

\[
z(x) = \tilde{z}(\cos(2\lambda d(x, o))), \quad x \in M^d,
\]

where \( \lambda \) is either \( \pi/2L \) or \( \pi/4L \), depending on the homogeneous space \( M^d \).

Let \( L_p \) be the set of all complex measurable functions \( \varphi \) on \( M^d \) of finite norm, given by

\[
\| \varphi \|_p = \left\{ \begin{array}{ll}
\left( \int_{M^d} |\varphi(x)|^p d\nu(x) \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\
\text{ess sup}\{|\varphi(x)|, x \in M^d\}, & \text{if } p = \infty.
\end{array} \right.
\]

Further, let \( U_p = \{ \varphi : \varphi \in L_p, \| \varphi \|_p \leq 1 \} \).

Let \( \tilde{z} \in L_1([-1, 1]), (1-x)^\alpha(1+x)^\beta dx \). Then, for any integrable function \( g \) we can define convolution \( h \) on \( M^d \) as the following

\[
h(\cdot) = (z * g)(\cdot) = \int_{M^d} \tilde{z}(\cos(2\lambda d(\cdot, x))) g(x) d\nu(x).
\]

For the convolution on \( M^d \) we have Young’s inequality

\[
\| z * g \|_q \leq \| z \|_p \| g \|_r,
\]

where \( 1/q = 1/p + 1/r - 1 \) and \( 1 \leq p, q, r \leq \infty \).

For each \( n \in \mathbb{N} \), let \( H_n \) be the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue \(-n(n+\alpha+\beta+1)\), where \( \alpha \) and \( \beta \) are numbers associated with the particular homogeneous space \( M^d \). Let \( T_n = \bigoplus_{k=0}^n H_k \) and \( T = \bigcup_{n=0}^\infty T_n \). We have \( \dim T_n \asymp n^d \).

The Hilbert space \( L_2 \) with usual scalar product

\[
\langle f, g \rangle = \int_{M^d} f(x) g(x) d\nu(x)
\]

has the decomposition

\[
L_2 = \bigoplus_{n=0}^\infty H_n.
\]

There is a unique real zonal eigenfunction (up to scalar multiplication) \( Z_n \in H_n \) such that the orthogonal projection from \( L_2 \) onto \( H_n \) is given by the convolution operator \( \varphi \mapsto Z_n * \varphi \). We have

\[
\tilde{Z}_n(t) = C_n(M^d) P^{\alpha,\beta}_n(t),
\]
We say that the sequence $f := \Lambda$ possesses the property of a constant $C$. It is easy to show that $f$ is a multiplier operator of $(p, q)$-type with norm $\|\Lambda\|_{p, q} := \sup_{\varphi \in U_p} \|\Lambda \varphi\|_q$. For $s \in \mathbb{R}$, let $\Lambda^s = \{\mu_k^s\}_{k \geq 0}$, where $\mu_k^s = (k(k + \alpha + \beta + 1))^{s/2}$. It was proved in [4] that, for $s > d(1/p - 1/q)_+$, $\Lambda^{-s}$ is a multiplier operator of $(p, q)$-type, and for each $\varphi \in L_p$ and $n \geq 1$, there exists a polynomial function $t_n(\varphi) \in T_n$ such that
\[
\|\Lambda^{-s} \varphi - t_n(\varphi)\|_q \ll n^{-s + d(1/p - 1/q)_+} \|\varphi\|_p.
\]

The Sobolev space $W_p^s$, $s > 0$, is given by
\[
W_p^s := \{ f \in L_p : \Lambda^s f \in L_p \},
\]
with norm
\[
\|f\|_p^s := \|\Lambda^s f\|_p.
\]

Here we have identified functions which differ by a constant, i.e., if $f - g = \text{constant}$, then $f = g$ in $W_p^s$. The Sobolev class $\overline{W}_p^s$, $s > 0$, is given by
\[
\overline{W}_p^s := \{ f \in W_p^s : \|f\|_p^s \leq 1 \}.
\]

It is easy to show that
\[
\overline{W}_p^s = \{ c + \Lambda^{-s} f : c \in \mathbb{R}, f \in U_p \}.
\]

Let $(\lambda_k^{(n)})$ be an infinite lower triangular matrix, i.e., $\lambda_k^{(n)} = 0$ for any $k > n$ and $n \in \mathbb{N}$.

**Definition 2.1** We say that the sequence $K = \{K_{2^n}\}_{n \in \mathbb{N}}$ of polynomial zonal functions
\[
K_{2^n} = \sum_{k=0}^{2^n} \lambda_k^{(2^n)} Z_k
\]
possesses the property $K_C$, and we write $K \in K_C$, if $\lambda_k^{(2^n)} = 1$ for any $0 \leq k \leq 2^n - 1$ and there exists a positive constant $C$ such that, $\|K_{2^n}\|_1 \leq C$ for all $n \in \mathbb{N}$. 


Remark 2.2 In the case of \( S^d, d \geq 2 \), the sequence of zonal polynomials \( K \) was introduced and constructed in an explicit form in [10, p. 287]. They are a natural generalization of the de la Vallée Poussin polynomial \( V_{n,2n}(t) \) on \( S^1 \),

\[
V_{n,2n}(t) = 1/2 + \sum_{k=1}^{2n} \lambda_k^{(2n)} \cos kt,
\]

where \( \lambda_k^{(2n)} = 1 \) for \( 1 \leq k \leq n \) and \( \lambda_k^{(2n)} = (2n - k)/n \) for \( n < k \leq 2n \). The de la Vallée Poussin polynomials \( V_{n,2n}(t) \) were used to introduce the Besov spaces on \( S^1 \).

Let \( K \in K_C \). For \( k \geq 0 \) we write

\[
\varphi_k = K_{2^k} - K_{2^{k-1}}, \quad k \geq 1; \quad \varphi_0 = Z_0.
\]

**Definition 2.3** The function \( f \in L_p \) belongs to the Besov space \( B^s_{p,q}(K) \), \( s, p, q \in \mathbb{R} \), \( s > 0 \), \( 1 \leq p, q \leq \infty \), if

\[
\|f\|_{p,q}(K) = \left( \sum_{k=0}^{\infty} (2^{ks}) \|\varphi_k * f\|_p^q \right)^{1/q} < \infty,
\]

where the functions \( \varphi_k \) are defined in (4).

It will be proved in Theorem 3.6 that the norms of \( B^s_{p,q}(K) \) and \((W^{2s}_p, W^{s+1}_p)\) are equivalent for \( s = (1-\theta)s_0 + \theta s_1, s_0, s_1 \geq 0, s_0 \neq s_1 \). As consequence we can conclude that the norms of \( B^s_{p,q}(K^1) \) and \( B^s_{p,q}(K^2) \) are equivalent for all \( K^1, K^2 \in K_C \), that is, the norm of \( B^s_{p,q}(K) \) does not depend on the sequence \( K \in K_C \). So that we will write \( \| \cdot \|_{p,q}(K) \) instead of \( \| \cdot \|_{p,q}(K) \).

As with the Sobolev spaces we will identify two functions in \( B^s_{p,q}(K) \) which differ by a constant. It is easy to see that \( B^s_{p,q}(K) \) is a normed vector space with norm \( \| \cdot \|_{p,q}(K) \). We will see in Section 3 that \( B^s_{p,q}(K) \) is the interpolation space for two of the Sobolev spaces defined above, and is therefore a Banach space. Let

\[
B^s_{p,q}(K) = \{ f \in L_p : \|f\|_{p,q}(K) \leq 1 \}.
\]

To give a useful sufficient conditions for the imbedding \( K \in K_C \) we will need some information concerning Cesàro means. Let \( \delta \geq 0 \) and \( n \in \mathbb{N} \). We define the Cesàro kernel \( \tilde{S}^\delta_n \) by

\[
\tilde{S}^\delta_n = \frac{1}{C^\delta_n} \sum_{m=0}^{n} C^\delta_{n-m} \tilde{Z}_m,
\]

where \( C^\delta_n \) are Cesàro numbers of order \( n \) and index \( \delta \),

\[
C^\delta_n = \frac{\Gamma(n+\delta+1)}{\Gamma(\delta+1)\Gamma(n+1)} \approx n^\delta, \quad n \to \infty
\]

(see, e.g., [20, p. 237]). Given a sequence \( \{ \lambda_k \}_{k \geq 0} \) we define the differences \( \Delta^s \lambda_k, k, s \in \mathbb{N} \), by \( \Delta^0 \lambda_k = \lambda_k, \Delta^1 \lambda_k = \lambda_k - \lambda_{k+1}, \) and \( \Delta^{s+1} \lambda_k = \Delta^s \lambda_k - \Delta^s \lambda_{k+1} \).

**Lemma 2.4** Let \( \{ \lambda_k^{(2n)} \} \) be such that for any \( n \in \mathbb{N} \) there is such \( C' \) > 0 that

\[
\sum_{k=0}^{2^n-2^{-d-1}} |\Delta^{d+1} \lambda_k^{(2n)}| k^d + \sum_{s=0}^{d} |\Delta^s \lambda^{(2n)}_{2^n-s}| q^s < C',
\]

where

\[
q^s = \begin{cases} 
2^{n(d-1)/2}, & 0 \leq s < (d-1)/2, \\
n2^{n(d-1)/2}, & s = (d-1)/2, \\
2^{ns}, & s > (d-1)/2.
\end{cases}
\]

Then \( K = \{ K_{2^n} \}_{n \in \mathbb{N}} \subset K_C \).
Proof. Applying the Abel transform \( d + 1 \) times we get
\[
K_{2^n} = \sum_{k=0}^{2^n-d-1} (\Delta^{d+1} \lambda_k^{(2^n)}) C_k^d \omega_k^d + \sum_{s=0}^{d} (\Delta^s \lambda_{2^n-s}^{(2^n)}) C_{2^n-s}^s S_{2^n-s}^s
\]
and hence
\[
\|K_{2^n}\|_1 \leq \sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{(2^n)}| C_k^d \|S_k^d\|_1 + \sum_{s=0}^{d} |\Delta^s \lambda_{2^n-s}^{(2^n)}| C_{2^n-s}^s S_{2^n-s}^s \|S_{2^n-s}^s\|_1. \tag{7}
\]

It was proved in [4, Lemma 3] that
\[
\|S_n\|_1 \ll \begin{cases} 
  n^{(d-1)/2- \delta}, & 0 \leq \delta < (d-1)/2, \\
  \log n, & \delta = (d-1)/2, \\
  1, & \delta > (d-1)/2.
\end{cases} \tag{8}
\]
Comparing (5) - (8) we get (6).

We present here two examples of sequences \( \{K_{2^n}\}_{n \in \mathbb{N}} \in \mathcal{K_C} \).

**Example 2.5** Consider the function
\[
\xi(t) = -\frac{1}{\omega} \exp \left( \frac{1}{(t-1/2)(t-1)} \right),
\]
where
\[
\omega = \int_{1/2}^1 \exp \left( \frac{1}{(t-1/2)(t-1)} \right) \, dt.
\]
Let
\[
\nu(t) = \begin{cases} 
  0, & 0 \leq t < 1/2, \\
  \xi(t), & 1/2 \leq t \leq 1, \\
  0, & t > 1.
\end{cases}
\]
and
\[
\mu(x) = 1 + \int_0^x \nu(t) dt, \quad x \geq 0.
\]
Put \( \lambda_k^{(n)} = \mu(k/n), \quad 0 \leq k \leq n \). Let \( f(x) \) be a continuous function on \( \mathbb{R} \). Then for the differences of order \( k \) we have
\[
\Delta^k f(x) = \sum_{\nu=0}^k (-1)^\nu \frac{k!}{\nu!(k-\nu)!} f(x + \nu h). \tag{9}
\]
In addition, if \( f(x) \) has \( k \) derivatives, then
\[
\Delta^k f(x) = (-1)^k \int_0^h \cdots \int_0^h f^{(k)}(x + t_1 + \cdots + t_k) \, dt_1 \cdots dt_k. \tag{10}
\]
Observe that for any integer \( k \geq 1 \) we have \( |\mu^{(k)}(x)| = 0 \) for \( 0 \leq x \leq 1/2, \quad x \geq 1, \) and
\[
|\mu^{(k)}(x)| \leq C[(x-1/2)(x-1)]^{-2(k-1)} \exp \left( \frac{1}{(x-1/2)(x-1)} \right) \tag{11}
\]
for $1/2 < x < 1$. Since $\Delta^s \lambda_k^{2^n} = \Delta^s_{2-n} \mu(k/2^n)$, it follows from (10) and (11) that $|\Delta^{d+1} \lambda_k^{2^n}| \leq C 2^{-(d+1)n}$ and therefore,

$$
\sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{2^n}| k^d \leq C.
$$

(12)

We have $\Delta^s \lambda_{2^n-s}^{(2^n)} = \mu(1) = 0$ for $s = 0$, and from (10) and (11) we have that

$$
|\Delta^s \lambda_{2^n-s}^{(2^n)}| \leq 2^{-ns} |\mu(1 - s/2^n)| \leq C_s 2^{-(s-1)n} \exp\left(\frac{2n+2}{s}\right)
$$

for $1 \leq s \leq 2^n$. Then

$$
\lim_{n \to \infty} \sum_{s=0}^{d} |\Delta^s \lambda_{2^n-s}^{(2^n)}| t_n(s) = 0.
$$

(13)

Finally, from Lemma 2.4, (12) and (13) it follows that $K \in \mathcal{K}_C$.

**Example 2.6** (see, also, [10, p. 287]) Consider the function

$$
\chi_0(t) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
0, & t > 1,
\end{cases}
$$

and for $1 \leq s \leq d$ let

$$
\chi_s(t) = 2d \int_{t}^{t+1/(2d)} \chi_{s-1}(u) du.
$$

The function $\chi_d$ is $d-1$ times continuously differentiable and positive on $[0, \infty)$. Furthermore, $\chi_d^{(d-1)}$ is Lipschitz continuous, $\chi_d(t) = 1$ for $0 \leq t \leq 1/2$, and

$$
\chi_{d}(t) = P_d(t) = \frac{(2d)^d}{d!} (1-t)^d, \quad 1 - \frac{1}{2d} \leq t \leq 1.
$$

(14)

Also $\chi_d$ is a polynomial function of degree $d$ in each interval $[t_s, t_{s-1}]$, $1 \leq s \leq d$, where $t_s = 1 - s/(2d)$. For each $n \geq 1$ consider a sequence $\{\lambda_k^{(n)}\}$ given by

$$
\lambda_k^{(n)} = \chi_d \left( \frac{k}{n} \right), \quad 0 \leq k \leq n.
$$

(15)

To apply Lemma 2.4 we need to get some bounds for the differences $|\Delta^s \lambda_k^{(2^n)}|$. We show that for $2^n \geq 2d^2$ and $0 \leq s \leq d$,

$$
|\Delta^s \lambda_{2^n-s}^{(2^n)}| \leq (2d)^{2d} 2^{-dn}
$$

(16)

and for all $n \in \mathbb{N}$, and $1 \leq k \leq 2^n$,

$$
|\Delta^{d+1} \lambda_k^{(2^n)}| \leq C(d) 2^{-dn}.
$$

(17)

Remark that for $n \geq 2d^2$ and $0 \leq s \leq d$, $(2^n-s)/2^n \geq 1-1/(2d)$, so that, setting $h = 2^{-n}$,

$$
\lambda_{2^n-s}^{(2^n)} = P_d \left( \frac{2^n-s}{2^n} \right), \quad \text{and} \quad \Delta^s \lambda_{2^n-s}^{(2^n)} = \Delta^s P_d \left( \frac{2^n-s}{2^n} \right).
$$

Using (10) we get

$$
|\Delta^s \lambda_{2^n-s}^{(2^n)}| \leq h^s \max_{t \geq (2^n-s)/n} P_d^{(s)}(t)
$$

$$
\leq 2^{-sn} \frac{(2d)^d}{(d-s)!} \left( \frac{s}{2^n} \right)^{d-s}
$$

$$
\leq (2d)^{2d} 2^{-dn}, \quad 2^n \geq 2d^2, \quad 0 \leq s \leq d,
$$

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proving (16) and consequently (13). Since $\chi_d^{(d)}$ is a piecewise continuous function on $[0, 1]$, it follows from (10) that $|\Delta_{2-n}^{d} \chi_d(k/2^n)| \leq C(d) 2^{-nd}$. Then (17) follows from the definition of differences.

Now, let $I_k^n = [k/2^n, (k + d + 1)/2^n]$, $t_s = 1 - s/(2d)$, $n \in \mathbb{N}$, $1 \leq k \leq 2^n$, $1 \leq s \leq d + 1$, and let

$$E_{n,d} = \{ k : \{ t_1, t_2, \ldots, t_{d+1} \} \cap I_k^n \neq \emptyset, 1 \leq k \leq 2^n \}.$$ 

If $k$ is not in $E_{n,d}$, than $\chi_d^{(d+1)}(t) = 0$ for $t \in I_k^n$ and hence $\Delta_{d+1}^{d+1} \chi_d^{(2^n)} = \Delta_{2-n}^{d+1} \chi_d^{(k/2^n)} = 0$. Therefore

$$\{ k : \Delta_{d+1}^{d+1} \chi_d^{(2^n)} \neq 0, 0 \leq k \leq 2^n \} \subset E_{n,d}.$$ 

Since $\text{Card}(E_{n,d}) \leq (d+1)(d+2)$ for all $n \in \mathbb{N}$, using (17) we get

$$\sum_{k=0}^{2^n-d-1} |\Delta_{d+1}^{d+1} \chi_d^{(2^n)}| k^d \leq C(d) \sum_{k \in E_{n,d}} \left( \frac{k}{2^n} \right)^d \leq C(d)(d+1)(d+2).$$

Finally from Lemma 2.4 it follows that $K = \{ K_{2^n} \}_{n \in \mathbb{N}} \in \mathcal{K}_C$.

### 3 Besov spaces as interpolation spaces

Two complex Banach spaces $A_0$ and $A_1$ are called an interpolation pair $\bar{A} = (A_0, A_1)$ if there exists a Hausdorff topological vector space in which $A_0$ and $A_1$ are continuously embedded. Then, the following spaces and quantities are well-defined:

$$\Delta(\bar{A}) = A_0 \cap A_1;$$

$$\|a\|_{\Delta(\bar{A})} = \max(\|a\|_{A_0}, \|a\|_{A_1}), \quad a \in \Delta(\bar{A});$$

$$\Sigma(\bar{A}) = A_0 + A_1 = \{ a_0 + a_1 : a_0 \in A_0, a_1 \in A_1 \};$$

$$\|a\|_{\Sigma(\bar{A})} = \inf_{a = a_0 + a_1 \in \Sigma(\bar{A})} (\|a_0\|_{A_0} + \|a_1\|_{A_1});$$

$$K(t, a) = K(t, a; \bar{A}) = \inf_{a = a_0 + a_1 \in \Sigma(\bar{A})} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad 0 < t < \infty;$$

$$J(t, a) = J(t, a; \bar{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad 0 < t < \infty, \quad a \in \Delta(\bar{A}).$$

For a given $t > 0$, $\| \cdot \|_{\Sigma(\bar{A})}$ and $K(t, \cdot)$ are equivalent norms on $\Sigma(\bar{A})$, and $\| \cdot \|_{\Delta(\bar{A})}$ and $J(t, \cdot)$ are equivalent norms on $\Delta(\bar{A})$.

Let $0 < \theta < 1, 1 \leq q \leq \infty$, and let $\Phi_{\theta, q}$ be the functional defined by

$$\Phi_{\theta, q}(\varphi(t)) = \left\{ \begin{array}{ll}
\left( \int_0^\infty (t^{-\theta} \varphi(t))^q \frac{dt}{t} \right)^{1/q}, & 1 \leq q < \infty, \\
\text{ess sup}_{t > 0} t^{-\theta} \varphi(t), & q = \infty,
\end{array} \right.$$ 

where $\varphi$ is a non-negative measurable function.

Given $a \in \Sigma(\bar{A})$ we define

$$\|a\|_{\theta, q; K} = \Phi_{\theta, q}(K(t, a)). \quad (18)$$

The set

$$\mathcal{K}_{\theta, q}(\bar{A}) = \{ a \in \Sigma(\bar{A}) : \|a\|_{\theta, q; K} < \infty \}$$

is a Banach space with the norm $\| \cdot \|_{\theta, q; K}$, and is called the interpolation space of the pair $\bar{A}$ by the $K$-method. If $1 \leq q < \infty$ then $\Delta(\bar{A})$ is dense in $\mathcal{K}_{\theta, q}(\bar{A})$.

We define the interpolation space of the pair $\bar{A}$ by the $J$-method as the set $\mathcal{J}_{\theta, q}(\bar{A})$, of all $a \in \Sigma(\bar{A})$ which can be represented by

$$a = \int_0^\infty u(t) \frac{dt}{t}, \quad (19)$$
where $u$ is a measurable function taking values in $\Delta(\bar{A})$, convergence in the integral is in $\Sigma(\bar{A})$, and

$$\Phi_{\theta,q}(J(t, u(t))) < \infty.$$  

(20)

The set $\mathcal{J}_{\theta,q}(\bar{A})$ is a Banach space with the norm

$$\|a\|_{\theta,q,J} = \inf_u \Phi_{\theta,q}(J(t, u(t))),$$

(21)

where the infimum is taken over all $u$ such that (19) and (20) hold. If $1 \leq q < \infty$ then $\Delta(\bar{A})$ is dense in $\mathcal{J}_{\theta,q}(\bar{A})$.

Both the $J$- and $K$-method are called real interpolation methods.

A direct consequence of the definition is that, for any $a \in \Sigma(\bar{A})$, the function $t \mapsto K(t, a)$ is positive, increasing, concave, and for $t, s > 0$,

$$K(t, a) \leq \max(1, t/s)K(s, a).$$  

Similarly, for any $a \in \Delta(\bar{A})$, the function $t \mapsto J(t, a)$ is positive, increasing, convex, and for $t, s > 0$,

$$J(t, a) \leq \max(1, t/s)J(s, a).$$

(23)

**Theorem 3.1** ([1, p. 44]) Let $\bar{A} = (A_0, A_1)$ be an interpolation pair, and let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then, $\mathcal{J}_{\theta,q}(\bar{A}) = K_{\theta,q}(\bar{A})$ with equivalence of norms.

From this point to the end of the paper we will not distinguish $\mathcal{J}_{\theta,q}(\bar{A})$ and $K_{\theta,q}(\bar{A})$, and will denote them both by $(A_0, A_1)_{\theta,q}$. Both norms $\|\cdot\|_{\theta,q,J}$ and $\|\cdot\|_{\theta,q,K}$ will be denoted by $\|\cdot\|_{(A_0, A_1)_{\theta,q}}$.

Given a linear operator $T \in L(A, B)$, where $A$ and $B$ are Banach spaces, we denote

$$\|T\|_{A,B} = \sup_{a \in A, \|a\| \leq 1} \|Ta\|_B.$$

Suppose that $T$ is a linear operator from $A_0 + A_1$ into $B$, and let $T_i$ be the restriction of $T$ to $A_i$, $i = 0, 1$. If $T_i \in L(A_i, B), i = 0, 1$, then

$$\|T\|_{(A_0, A_1)_{\theta,q}} \leq (\|T_0\|_{A_0, B}^{1-\theta} (\|T_1\|_{A_1, B})^{\theta}.$$

(24)

Let $0 < \theta < 1$, $1 < r < \infty$, and $1 \leq q \leq \infty$. We will denote by $\lambda^{\theta, r, q}$, the space of all sequences $(\alpha_k)_{k \in \mathbb{Z}}$, such that

$$\|\alpha\|_{\lambda^{\theta, r, q}} = \left\{ \left( \sum_{k \in \mathbb{Z}} (r^{-k\theta} |\alpha_k|)^q \right)^{1/q}, \quad 1 \leq q < \infty, \right.$$  

$$\sup_{k \in \mathbb{Z}} r^{-k\theta} |\alpha_k|, \quad q = \infty,$$

is finite.

The next two lemmas can be found for $r = 2$ in [1].

**Lemma 3.2** Let $\bar{A} = (A_0, A_1)$ be an interpolation pair and let $a \in \Sigma(\bar{A})$. Then, $a \in (A_0, A_1)_{\theta,q}$ if and only if $(K(r^k, a))_{k \in \mathbb{Z}} \in \lambda^{\theta, r, q}$. Moreover, we have

$$r^{-\theta}(\log r)^{1/q}\|K(r^k, a)\|_{\lambda^{\theta, r, q}} \leq \|a\|_{(A_0, A_1)_{\theta,q}} \leq r(\log r)^{1/q}\|K(r^k, a)\|_{\lambda^{\theta, r, q}}.$$  

(25)

**Lemma 3.3** Let $\bar{A} = (A_0, A_1)$ be an interpolation pair and let $a \in \Sigma(\bar{A})$. Then $a \in (A_0, A_1)_{\theta,q}$ if and only if there exists $u_k \in \Delta(\bar{A}), k \in \mathbb{Z}$, with

$$a = \sum_{k \in \mathbb{Z}} u_k$$

(convergence inside $\Sigma(\bar{A})$), and such that $(J(r^k, u_k)) \in \lambda^{\theta, r, q}$. Moreover,

$$r^{-\theta}(\log r)^{-1+1/q}\inf_{u_k} \|J(r^k, u_k)\|_{\lambda^{\theta, r, q}} \leq \|a\|_{(A_0, A_1)_{\theta,q}} \leq r(\log r)^{-1+1/q}\inf_{u_k} \|J(r^k, u_k)\|_{\lambda^{\theta, r, q}}.$$  

(27)
Now suppose \( 0 < s_1 < s_2 \) and \( 1 \leq p \leq \infty \). It follows from (2) that \( \Lambda^{s_1-s_2} \) is bounded on \( L_p \) and hence, for \( f \in W_p^{s_2} \),
\[
\| f \|_{p, s_1} = \| \Lambda^{s_1-s_2} f \|_p \leq C_p \| f \|_{p, s_2} < \infty.
\]
Therefore \( W_p^{s_2} \subset W_p^{s_1} \) for \( 0 < s_1 < s_2 \) and \( 1 \leq p \leq \infty \). The space \( T = \bigcup_{n=1}^{\infty} T_n \) is dense in \( L_p \), \( 1 \leq p < \infty \), and hence is also dense in \( W_p^s \), \( s > 0 \), \( 1 \leq p < \infty \). Since \( B_{p,q}^s \) is the interpolation of two Sobolev spaces (see Theorem 3.6) and \( T \) is dense in these Sobolev spaces, it follows that \( T \) is also dense in \( B_{p,q}^s \), \( s > 0 \), \( 1 \leq p < \infty \), and \( 1 \leq q \leq \infty \).

The next result is Bernstein’s inequality and is proved in [6].

**Theorem 3.4** For all \( s, p \in \mathcal{R}, s > 0, 1 \leq p \leq \infty \),
\[
\left\| \sum_{k=1}^{m} \mu_k Z_k * f \right\|_p \leq C_s m^s \| f \|_p, \quad f \in T_m, \ m \geq 1. \tag{28}
\]

**Corollary 3.5** Let \( K \in \mathcal{K}_C \), \( s, p \in \mathcal{R}, s > 0, \) and \( 1 \leq p \leq \infty \). Then
\[
\| \Lambda^s (\varphi_k * f) \|_p \leq C_s 2^{ks} \| \varphi_k * f \|_p, \quad f \in L_p, \ k \geq 0 \tag{29}
\]
and
\[
\| \varphi_k * f \|_p \leq C_s 2^{-ks} \| \Lambda^s f \|_p, \quad f \in W_p^s, \ k \geq 0. \tag{30}
\]

**Proof.** Let us fix \( f \in L_p \), \( 1 \leq p \leq \infty \), and \( k \geq 0 \). We have that \( \varphi_k * f \in T_{2^k} \). Hence, by (28),
\[
\| \Lambda^s (\varphi_k * f) \|_p \leq C_s 2^{ks} \| \varphi_k * f \|_p.
\]
Now if \( f \in W_p^s \), let \( t_k \) be a polynomial of degree \( 2^{k-1} \) satisfying (2) for \( \varphi = \Lambda^s f \). Then
\[
\| f - t_k \|_p \leq 2^{-ks} \| \Lambda^s f \|_p,
\]
and \( K_{2^k} * t_k = t_k \). Therefore, by (1), (2) and Definition 2.1 we get
\[
\| f - K_{2^k} * f \|_p \leq \| f - t_k \|_p + \| K_{2^k} * (f - t_k) \|_p \\
\leq (1 + \| K_{2^k} \|_1) \| f - t_k \|_p \\
\leq C 2^{-ks} \| \Lambda^s f \|_p
\]
and hence by (4)
\[
\| \varphi_k * f \|_p \leq \| f - K_{2^k} * f \|_p + \| f - K_{2^{s-1}} * f \|_p \\
\leq C 2^{-ks} \| \Lambda^s f \|_p.
\]

**Theorem 3.6** Let \( K \in \mathcal{K}_C \). If \( 0 < s_0 < s_1 \) then
\[
B_{p,q}^{s_1} \subset B_{p,q}^{s_0}, \quad 1 \leq p, q \leq \infty. \tag{31}
\]
If \( 1 \leq q_0 < q_1 \leq \infty \) then
\[
B_{p,q_0}^{s_1} \subset B_{p,q_1}^{s}, \quad s > 0, 1 \leq p \leq \infty. \tag{32}
\]
Moreover,
\[
B_{p,1}^{s} \subset W_{p}^{s} \subset B_{p,\infty}^{s}, \quad s > 0, 1 \leq p \leq \infty. \tag{33}
\]
If \( s_0, s_1 > 0 \) and \( s_0 \neq s_1 \), then
\[
(W_{p,q}^{s_1}, W_{p}^{s_1})_{\theta,q} = B_{p,q}^{s_{0}}, \quad 1 \leq p, q \leq \infty, \ 0 < \theta < 1, \tag{34}
\]
with equivalence of norms, where \( s = (1 - \theta)s_0 + \theta s_1 \).
The inclusions (31) and (32) follow directly from the definition of Besov space. Now, from (29), we have
\[
\| f \|_{p}^{s} \leq \sum_{k=0}^{\infty} \| \Lambda^{s} (\varphi_{k} * f) \|_{p}
\]
\[
\leq C_{s} \sum_{k=0}^{\infty} 2^{ks} \| \varphi_{k} * f \|_{p}
\]
\[
= C_{s} \| f \|_{p,1}^{s}.
\]
From (30),
\[
\| f \|_{p,\infty}^{s} = \sup_{k \in \mathbb{N}} 2^{ks} \| \varphi_{k} * f \|_{p}
\]
\[
\leq C_{s} \sup_{k \in \mathbb{N}} 2^{ks} 2^{-ks} \| \Lambda^{s} f \|_{p}
\]
\[
= C_{s} \| f \|_{p}^{s}.
\]
The inclusions in (33) follow from the previous two inequalities.

We now prove (34). Consider \( 0 < s_{0} < s_{1} \). Let \( f \in (W^{s_{0}}, W^{s_{1}})_{p,q} \), with \( f = f_{0} + f_{1}, f_{i} \in W^{s_{i}} \), \( i = 0, 1 \).

From (30),
\[
\| \varphi_{k} * f \|_{p} \leq \| \varphi_{k} * f_{0} \|_{p} + \| \varphi_{k} * f_{1} \|_{p}
\]
\[
\leq C_{0} 2^{-ks_{0}} \| \Lambda^{s_{0}} f_{0} \|_{p} + C_{1} 2^{-ks_{1}} \| \Lambda^{s_{1}} f_{1} \|_{p}
\]
\[
\leq C_{2} 2^{-ks_{0}} (\| f_{0} \|_{p} + 2^{k(s_{0} - s_{1})} \| f_{1} \|_{p}^{s_{1}}),
\]
and hence
\[
\| \varphi_{k} * f \|_{p} \leq C_{2} 2^{-ks_{0}} K(2^{k(s_{0} - s_{1})}, f).
\]
Thus, putting \( r = 2^{s_{1} - s_{0}} \) we get
\[
2^{ks} \| \varphi_{k} * f \|_{p} \leq C_{2}^{k(\theta(s_{1} - s_{0}))} K(2^{k(s_{0} - s_{1})}, f) = C_{R}^{k\theta} K(r^{-k}, f),
\]
so that
\[
(\sum_{k=0}^{\infty} (2^{ks} \| \varphi_{k} * f \|_{p})^{q})^{1/q} \leq C \left( \sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^{q} \right)^{1/q}.
\]
Applying Lemma 3.2 we find
\[
\| f \|_{p,q}^{s} = \left( \sum_{k=0}^{\infty} (2^{ks} \| \varphi_{k} * f \|_{p})^{q} \right)^{1/q}
\]
\[
\leq C \left( \sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^{q} \right)^{1/q}
\]
\[
\leq C \| f \|_{(W^{s_{0}}, W^{s_{1}})}^{s}_{p,q}.
\]
Suppose conversely that \( f \in B_{p,q}^{s} \). From (29),
\[
J(2^{k(s_{0} - s_{1})}, \varphi_{k} * f) = \max (\| \Lambda^{s_{0}} (\varphi_{k} * f) \|_{p}, 2^{k(s_{0} - s_{1})} \| \Lambda^{s_{1}} (\varphi_{k} * f) \|_{p})
\]
\[
\leq \max (C_{s_{0}} 2^{ks_{0}} \| \varphi_{k} * f \|_{p}, C_{s_{1}} 2^{k(s_{0} - s_{1})} 2^{ks_{1}} \| \varphi_{k} * f \|_{p})
\]
\[
= C_{s} 2^{ks_{0}} \| \varphi_{k} * f \|_{p},
\]
and hence
\[
2^{k(s_{0} - s)} J(2^{k(s_{0} - s_{1})}, \varphi_{k} * f) \leq C_{s} 2^{ks} \| \varphi_{k} * f \|_{p}.
\]
Thus, again putting \( r = 2^{s_1 - s_0} \), we get
\[
\rho_k J(r^{-k}, \varphi_k * f) \leq C_s 2^{ks} \| \varphi_k * f \|_p, \quad k \geq 0. \tag{35}
\]

Consider the sequence \( (u_k)_{k \in \mathbb{Z}} \) defined by
\[
u_k = \begin{cases} 0, & k \geq 1, \\
\varphi_{-k} * f, & k \leq 0.
\end{cases}
\]

Since \( s_0 < s_1 \) we have that \( W_p^{s_0} \subset W_p^{s_0} \), and hence \( W_p^{s_0} + W_p^{s_1} = W_p^{s_0} \). From (29) and Hölder’s inequality
\[
\sum_{k \in \mathbb{Z}} \| u_k \|_{W_p^{s_0}} = \sum_{k=0}^{\infty} \| \Lambda^{s_0} (\varphi_k * f) \|_p
\leq C_{s_0} \sum_{k=0}^{\infty} 2^{ks_0} \| \varphi_k * f \|_p
\leq C_{s_0} \sum_{k=0}^{\infty} 2^{k\theta(s_0-s_1)} 2^{ks} \| \varphi_k * f \|_p
\leq C_{s_0} \left( \sum_{k=0}^{\infty} r^{-k\theta} \right)^{1/q'} \| f \|_{p,q}^s < \infty,
\]

where \( 1/q + 1/q' = 1 \). Therefore, we may conclude that the series \( \sum_{k \in \mathbb{Z}} u_k \) converges to \( f \in W_p^{s_0} \). Now, by (35)
\[
\left( \sum_{k \in \mathbb{Z}} (r^{-k\theta} J(r^{-k}, u_k))^q \right)^{1/q} = \left( \sum_{k=0}^{\infty} (r^{k\theta} J(r^{-k}, \varphi_k * f))^q \right)^{1/q}
\leq C_s \left( \sum_{k=0}^{\infty} (2^{ks} \| \varphi_k * f \|_p)^q \right)^{1/q}
= C_s \| f \|_{p,q}^s.
\]

Thus, by Lemma 3.2, we can conclude that \( f \in (W_p^{s_0}, W_p^{s_1})_{\theta,q} \) and that
\[
\| f \|_{(W_p^{s_0}, W_p^{s_1})_{\theta,q}} \leq C_s \| f \|_{p,q}^s.
\]

**Remark 3.7** Let \( s > d(1/p - 1/q)_+ \), \( 1 \leq p, q \leq \infty \), and \( n \in \mathbb{N} \). Consider the operator \( I_n \), defined on \( L_p \), by \( I_n f = f - t_n (\Lambda^n f) \), where \( t_n (\varphi) = G_n * \varphi \), where \( G_n = \sum_{k=1}^{n} (\Delta^{N+1} \lambda_k^{-s}) C_k^N S_k^N \), \( N = (d + 1)/2 \) if \( d \) is odd and \( N = (d + 2)/2 \) if \( d \) is even. From (2) we have that
\[
\| I_n \|_{W_p^{s_0}, L_q} \leq C_s n^{-s+d(1/p-1/q)_+}. \tag{36}
\]

Now, consider \( s, p, r, q \in \mathbb{R} \) with \( 1 \leq p, q, r \leq \infty \), and \( s > d(1/p - 1/q)_+ \). Let \( s_0, s_1 \in \mathbb{R} \) such that \( s_1 > s > s_0 > d(1/p - 1/q)_+ \), and let \( \theta \in (0, 1) \), such that \( s = (1-\theta)s_0 + \theta s_1 \). From (34) we have \( B_{p, r} = (W_p^{s_0}, W_p^{s_1})_{\theta, r} \). Then, from (24) and (36),
\[
\| I_n \|_{B_{p, r}, L_q} \leq \left( \| I_n \|_{W_p^{s_0}, L_q} \right)^{1-\theta} \left( \| I_n \|_{W_p^{s_1}, L_q} \right)^{\theta}
\leq C_s n^{-s+d(1/p-1/q)_+}. \tag{37}
\]
Consider the multiplier sequence $\hat{\Lambda}^s = \{\hat{\Lambda}_k^s\}_{k \geq 0}$, $\hat{\Lambda}_k^s = (1 + k(k + \alpha + \beta + 1))^{s/2}$, $s > 0$. The Sobolev spaces can also be defined using $\hat{\Lambda}^s$ rather than $\Lambda^s$. We define

$$W_p^s := \{f \in L_p : \hat{\Lambda}^s f \in L_p\},$$

with norm

$$\|f\|_p^s := \|\hat{\Lambda}^s f\|_p.$$

From [4, Theorem 2], $\hat{\Lambda}^{-s}$ is bounded from $L_p$ to $L_q$ for $s > d(1/p - 1/q)_+$. The inequality (2) also holds for $\hat{\Lambda}^{-s}$, $s > 0$. Since Bernstein’s inequality (Theorem 3.4) holds for the sequence $\hat{\Lambda}_k^s$ (see [6]), the same is true for Corollary 3.5.

Now, for $s, p, q \in \mathbb{R}$, $s > 0$, $1 \leq p, q \leq \infty$, and $f \in L_p$, we define

$$\|f\|_{p,q}^s := \|f\|_p + \|f\|_{p,q}^s,$$

and

$$\check{B}_{p,q}^s := \{f \in L_p : \|f\|_{p,q}^s < \infty\}.$$

In this case we have that $\| \cdot \|_p$ and $\| \cdot \|_{p,q}$ are norms in $W_p^s$ and $\check{B}_{p,q}^s$ respectively, when we consider these spaces as subspaces of $L_p$.

Theorem 3.6 holds for the spaces $W_p^s$ and $\check{B}_{p,q}^s$, with minor changes in the proof. In the proof we consider a sequence $(u_k)_{k \in \mathbb{Z}}$ as in the proof of Theorem 3.6, only with the minor change $u_1 = Z_0 \ast f$. We need the inequalities

$$C_1 r^{-1} J(r, f) \leq \|f\|_p \leq C_2 K(1, f), \quad 1 \leq p \leq \infty, \quad f \in L_p.$$

Remark 3.9 Besov spaces on the unit sphere $S^d$ in $\mathbb{R}^{d+1}$ were studied in [13], where the authors give a list of equivalent norms for Besov spaces. We will show that the Besov spaces considered in [13] and the spaces $\check{B}_{p,q}^s$ have equivalent norms.

The Sobolev spaces considered in [13] are the spaces $\check{W}_p^s$ given in Remark 3.8. Let us denote by $\check{B}_{p,q}^s$ the Besov spaces in [13], $s > 0$, $1 \leq p, q \leq \infty$. The following two equivalent norms for $\check{B}_{p,q}^s$, among others, are given in [13]:

$$\begin{align*}
(1) \|f\|_{p,q}^s &= \|f\|_p + \left(\sum_{m=1}^{\infty} m^{sq-1} E_m(f)_p^q\right)^{1/q}, \\
(2) \|f\|_{p,q}^s &= \|f\|_p + \left(\sum_{m=1}^{\infty} m^{sq-1} K_r(\pi/m, f)_p^q\right)^{1/q}, \quad s < 2r, \quad r = 1, 2, \ldots,
\end{align*}$$

where $E_m(f)_p$ is the best approximation of a function $f \in L_p$ by spherical polynomials of degree $\leq m$, and $K_r(\pi/m, f)_p$ is given in terms of the $K$-functional by

$$K_r(\pi/m, f)_p := K(\pi^{2r} m^{-2r}, f; L_p, \check{W}_p^{2r}).$$

It is not difficult to show (see Lemma 3.2) that the norm (2) $\|f\|_{p,q}^s$ is the norm of the interpolation space $(L_p, \check{W}_p^{2r})_{s/2r,q}$, that is, $\check{B}_{p,q}^s = (L_p, \check{W}_p^{2r})_{s/2r,q}$. It was also proved in [13] that

$$\check{B}_{p,1}^s \subset \check{W}_p^s \subset \check{B}_{p,\infty}^s, \quad s > 0, \quad 1 \leq p \leq \infty.$$

By the definition of the class $\mathcal{H}(\theta, L_p, \check{W}_p^{2r})$ given in Section 4, we can conclude that

$$\check{W}_p^s \subset \mathcal{H}(s/2r, L_p, \check{W}_p^{2r}), \quad 1 \leq p \leq \infty, \quad s < 2r.$$
Consider now \( s, s_0, s_1 \in \mathbb{R} \) with \( 0 < s_0 < s < s_1 \), and \( s = (1 - \eta)s_0 + \eta s_1, \eta \in (0,1) \). Then

\[
\dot{W}^s_p \in \mathcal{H}(s_i/2r, L_p, \dot{W}^{2r}_p), \quad i = 0, 1,
\]

and from (34)

\[
\dot{B}^s_{p,q} = (W^{s_0}_p, W^{s_1}_p)_{\eta,q}.
\]

Applying the Reiteration Theorem (see [1, p. 50]) we can conclude that

\[
\dot{B}^s_{p,q} = (W^{s_0}_p, W^{s_1}_p)_{\eta,q} = (L_p, W^{2r}_p)_{s/2r,q} = B^s_{p,q}.
\]

### 4 Estimates of \( n \)-widths of Besov classes

For \( s > d(1/p - 1/q)_+ \) the operator \( \Lambda^{-s} \) is bounded from \( L_p \) to \( L_q \) and therefore, \( W^s_p = \Lambda^{-s}(L_p) \subset L_q \). Now let \( s_0, s_1 > d(1/p - 1/q)_+ \) with \( s_0 < s < s_1 \), and let \( 0 < \theta < 1 \) such that \( s = (1 - \theta)s_0 + \theta s_1 \). Then, by (34), \( B^s_{p,r} = (W^{s_0}_p, W^{s_1}_p)_{\theta,r} \subset W^s_p \subset L_p \). Thus \( B^s_{p,r} \subset L_q \), for \( s > d(1/p - 1/q)_+ \), and \( 1 \leq p, q, r \leq \infty \).

Let \( \bar{A} = (A_0, A_1) \) be an interpolation pair of Banach spaces, \( U_A \), be the unit ball of \( A_i, i = 0, 1 \), and let \( B \) be another Banach space such that \( A_0, A_1 \subset B \). In Triebel [21] sufficient conditions were given on Kolmogorov \( n \)-widths \( d_n(U_A, B) \), \( i = 0, 1 \), for obtaining estimates for the Kolmogorov \( n \)-widths \( d_n(U_{(A_0, A_1)_{\theta,i}}, B) \). To get our results we will use sharp orders for the Kolmogorov \( n \)-widths \( d_n(W^s_p, L_q) \) found in [3, 11], the property (34), which says that a Besov space is an interpolation space of Sobolev spaces, and Triebel’s theorem mentioned above.

We will denote the unit ball of a Banach space \( A \) by \( U_A \). If \( \bar{A} = (A_0, A_1) \) is an interpolation pair and \( (A_0, A_1)_{\theta,1} \subset A \subset (A_0, A_1)_{\theta,\infty} \), we write \( A \in \mathcal{H}(\theta, A_0, A_1) \).

**Theorem 4.1** (see [21]) Let \( \bar{A} = (A_0, A_1) \) be an interpolation pair of Banach spaces and let \( B \) be a Banach space such that \( A_0, A_1 \subset B \). Suppose that there exist \( \alpha_i \geq 0 \) and \( C_i > 0 \), \( i = 0, 1 \), such that

\[
d_n(U_{A_i}, B) \leq C_i n^{-\alpha_i}, \quad n \in \mathbb{N}, \quad i = 0, 1.
\]

Suppose also that there exists \( \tilde{\theta} \in (0,1) \) and \( \tilde{A} \in \mathcal{H}(\tilde{\theta}, A_0, A_1) \) such that

\[
d_n(U_{\tilde{A}}, B) \geq C n^{-\tilde{\alpha}}, \quad n \in \mathbb{N},
\]

where \( 0 < \tilde{\alpha} = (1 - \tilde{\theta})\alpha_0 + \tilde{\theta} \alpha_1 \), and \( C > 0 \). If \( \theta \in [0,1] \), \( A \in \mathcal{H}(\theta, A_0, A_1) \), and \( \alpha = (1 - \theta)\alpha_0 + \theta \alpha_1 \), then

\[
d_n(U_A, B) \asymp n^{-\alpha}, \quad n \in \mathbb{N}.
\]

**Theorem 4.2** (see [3, 4, 11]) Let \( p, q, s \in \mathbb{R} \).

\( : (i) \) If \( 1 \leq p = q \leq \infty \) or \( 2 \leq q \leq p < \infty \), and \( s > 0 \), then

\[
d_n(W^s_p, L_q) \asymp n^{-s/d}.
\]

\( : (ii) \) If \( 1 \leq p \leq q \leq 2 \) and \( s > d(1/p - 1/q) \), then

\[
d_n(W^s_p, L_q) \asymp n^{-s/d+1/p-1/q}.
\]

\( : (iii) \) If \( 2 \leq p \leq q < \infty \) and \( s > d/2 \), then

\[
d_n(W^s_p, L_q) \asymp n^{-s/d}.
\]

\( : (iv) \) If \( 1 \leq p \leq 2 \leq q < \infty \) and \( s > d/p \), then

\[
d_n(W^s_p, L_q) \asymp n^{-s/d+1/p-1/2}.
\]
If \(1 < q \leq p \leq 2\) and \(s > 0\), then
\[
d_n(W_p^s, L_q) \asymp n^{-s/d}.
\] (45)

**Theorem 4.3** Let \(p, q, s, r \in \mathbb{R}\).

1. If \(1 \leq p = q \leq \infty\) or \(2 \leq q \leq p < \infty\), \(s > 0\), and \(1 \leq r \leq \infty\), then
\[
d_n(B_{p,r}^s, L_q) \asymp \delta_n(B_{p,r}^s, L_q) \asymp n^{-s/d}.
\] (46)

2. If \(1 \leq p \leq q \leq 2, s > d(1/p - 1/q)\), and \(1 \leq r \leq \infty\), then
\[
d_n(B_{p,r}^s, L_q) \asymp \delta_n(B_{p,r}^s, L_q) \asymp n^{-s/d+1/p-1/q}.
\] (47)

3. If \(2 \leq p \leq q < \infty, s > d/2, \) and \(1 \leq r \leq \infty\), then
\[
d_n(B_{p,r}^s, L_q) \asymp n^{-s/d}.
\] (48)

4. If \(1 \leq p \leq 2 < q < \infty, s > d/p, \) and \(1 \leq r \leq \infty\), then
\[
d_n(B_{p,r}^s, L_q) \asymp n^{-s/d+1/p-1/2}.
\] (49)

5. If \(1 < q \leq p \leq 2, s > 0\), and \(1 \leq r \leq \infty\), then
\[
d_n(B_{p,r}^s, L_q) \asymp \delta_n(B_{p,r}^s, L_q) \asymp n^{-s/d}.
\] (50)

**Proof.** Consider \(1 \leq p, r \leq \infty, s, s_0, s_1 > 0, \) and \(\theta \in (0,1)\), with \(s = (1-\theta)s_0 + \theta s_1\). From (34) we have
\[
B_{p,1}^s = (W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,1}, \quad B_{p,\infty}^s = (W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,\infty},
\]
and hence, from (33),
\[
(W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,1} \subset B_{p,1}^s \subset (W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,\infty}.
\]
Therefore
\[
W_p^s \in \mathcal{H}(\theta, W_{p,0}^{s_0}, W_{p,1}^{s_1}).
\] (51)

Now from (32) we have \(B_{p,1}^s \subset B_{p,r}^s \subset B_{p,\infty}^s\), and hence, from (34),
\[
(W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,1} \subset B_{p,r}^s \subset (W_{p,0}^{s_0}, W_{p,1}^{s_1})_{\theta,\infty}.
\]
Thus,
\[
B_{p,r}^s \in \mathcal{H}(\theta, W_{p,0}^{s_0}, W_{p,1}^{s_1}).
\] (52)

Let \(s, p, q, r\) be as in (i) and choose \(s_0, s_1 \in \mathbb{R}\) such that \(0 < s_0 < s < s_1\). Let \(\theta \in (0,1)\) with \(s = (1-\theta)s_0 + \theta s_1\). From (41) we have
\[
d_n(W_{p,0}^{s_0}, L_q) \leq Cn^{-s_0/d}, \quad n \in \mathbb{N}, \ i = 0, 1.
\] (53)

From (51) we have that \(W_p^s \in \mathcal{H}(\theta, W_{p,0}^{s_0}, W_{p,1}^{s_1})\), and, from (41),
\[
d_n(W_{p,1}^{s_1}, L_q) \geq Cn^{-s_1/d}, \quad n \in \mathbb{N}.
\] (54)

Then (46) for Kolmogorov \(n\)-widths follows from Theorem 4.1, (53), and (54).

Now let \(s, p, q, r\) be as in (ii) and choose \(s_0\) and \(s_1\) such that \(s_1 > s > s_0 > d(1/p - 1/q)\). Let \(\theta \in (0,1)\) such that \(s = (1-\theta)s_0 + \theta s_1\). From (42) we have
\[
d_n(W_{p,0}^{s_0}, L_q) \leq Cn^{-s_0/d+1/p-1/q}, \quad n \in \mathbb{N}, \ i = 0, 1.
\] (55)
From (51) we have that \( W^s_p \in \mathcal{H}(\theta, W^a_p, W^{a+}_p) \), and, from (42),
\[
d_n(W^s_p, L_q) \geq C n^{-s/d + 1/p - 1/q}, \quad n \in \mathbb{N}.
\]
(56)

Then (47) for Kolmogorov \( n \)-widths follows from Theorem 4.1, (55), and (56).

The proofs of (48), (49), and (50) for Kolmogorov \( n \)-widths follow in the same way as those of (46) and (47) above.

Let \( s > d(1/p - 1/q) \), and \( n \in \mathbb{N} \). From (37) we have that
\[
\delta_n(\hat{B}^a_{p,r}, L_q) \leq \|I_n\|_{B^{a+}_{p,r}, L_q} \leq C s n^{-s/d + (1/p - 1/q)} + \frac{1}{n},
\]
and hence
\[
\delta_n(\hat{B}^a_{p,r}, L_q) \leq C s n^{-s/d + (1/p - 1/q)}, \quad n \in \mathbb{N}.
\]
(57)

The upper bounds for linear \( n \)-widths in (46), (47) and (50) follow from (57). The lower bounds follow from the inequality \( d_n(A, X) \leq \delta_n(A, X) \).

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References


