

# Estimates of $n$ -Widths of Besov Classes on Two-Point Homogeneous Manifolds

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Estimates of Kolmogorov and linear  $n$ -widths of Besov classes on compact globally symmetric spaces of rank 1 (i.e. on  $S^d$ ,  $P^d(\mathbf{R})$ ,  $P^d(\mathbf{C})$ ,  $P^d(\mathbf{H})$ ,  $P^{16}(\text{Cay})$ ) are established. It is shown that these estimates have sharp orders in different important cases. A new characterisation of Besov spaces is also given.

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## 1 Introduction

In the present paper we investigate the asymptotically optimal approximation of Besov classes on compact globally symmetric spaces of rank 1 (two-point homogeneous spaces)  $S^d$ ,  $P^d(\mathbf{R})$ ,  $P^d(\mathbf{C})$ ,  $P^d(\mathbf{H})$ ,  $P^{16}(\text{Cay})$ . In what follows, optimal approximation will be interpreted in the sense of Kolmogorov and linear  $n$ -widths.

Estimates for Kolmogorov  $n$ -widths of Besov classes on bounded regions of Euclidean spaces can be found in [21]. The spaces of Besov type on manifolds and their equivalent characterisations have been investigated in different articles (see e.g. [22, 23, 16, 15, 5, 17]).

There are various approaches to the definition of smoothness via harmonic analysis. The basic theorem in this range of problems is the well known analog of the Littlewood-Paley theorem [12] for trigonometric series, on compact globally symmetric spaces of rank 1 by Bonami and Clerc [2]. We introduce the Besov spaces decomposing a smooth function  $f$  into a series relative to spherical harmonics and using zonal polynomials  $K_n(z)$  which are natural generalizations of the de la Vallée Poussin polynomials on  $S^1$ . We prove that the Besov spaces are real interpolation of two Sobolev spaces. Our definition of Besov space is new even for the sphere  $S^d$ ,  $d \geq 2$ .

We use sharp orders of Kolmogorov  $n$ -widths of Sobolev classes from [3] and [11], and interpolation techniques by Triebel [21] to prove asymptotic estimates for Kolmogorov and linear  $n$ -widths of Besov classes on two-point homogeneous spaces.

Suppose that  $A$  is a convex, compact, centrally symmetric subset of a Banach space  $X$  with unit ball  $B$ . The linear  $n$ -width of  $A$  in  $X$  is defined by

$$\delta_n(A, X) := \delta_n(A, B) := \inf_{P_n} \sup_{f \in A} \|f - P_n f\|,$$

where  $P_n$  varies over all linear operators of rank at most  $n$  that map  $X$  into itself.

The Kolmogorov  $n$ -width of  $A$  in  $X$  is defined by

$$d_n(A, X) := d_n(A, B) := \inf_{X_n} \sup_{f \in A} \inf_{g \in X_n} \|f - g\|,$$

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where  $X_n$  runs over all subspaces of  $X$  of dimension  $n$ .

For ease of notation we will write  $a_n \ll b_n$  for two sequences, if  $a_n \leq cb_n$  for  $n \in \mathbf{N}$  and  $a_n \asymp b_n$ , if  $c_1 b_n \leq a_n \leq c_2 b_n$  for all  $n \in \mathbf{N}$  and some constants  $c, c_1$  and  $c_2$ . Also, we shall put

$$(a)_+ := \begin{cases} a, & \text{if } a > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We shall be interested here in compact homogeneous spaces. Such manifolds of dimension  $d$  will be denoted by  $M^d$ . Each  $M^d$  can be considered as the orbit space of some compact subgroup  $\mathcal{H}$  of the orthogonal group  $\mathcal{G}$ , that is  $M^d = \mathcal{G}/\mathcal{H}$ . Let  $\pi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$  be the natural mapping and  $e$  be the identity of  $\mathcal{G}$ . The point  $o = \pi(e)$  which is invariant under all motions of  $\mathcal{H}$  is called the pole of  $M^d$ . On any such manifold there is an invariant Riemannian metric  $d(\cdot, \cdot)$ , and a measure  $d\nu$  which is induced by the normalised left Haar measure on  $\mathcal{G}$  and is invariant under the action of  $\mathcal{G}$ . The *two-point* homogeneous spaces have the following additional property. If  $x, x', y, y' \in M^d$  with  $d(x, y) = d(x', y')$  then there is a  $g \in \mathcal{G}$  such that  $x = gx'$  and  $y = gy'$ . Two point homogeneous spaces admit essentially only one invariant second order differential operator, the Laplace-Beltrami operator  $\Delta$ . A function  $Z : M^d \rightarrow \mathbf{R}$  is called zonal if  $Z(h^{-1}\cdot) = Z(\cdot)$  for any  $h \in \mathcal{H}$ . The geometry of these spaces is in many respects similar. For example, all geodesics in a given one of these spaces are closed and have the same length  $2L$ . Here  $L$  is the diameter of  $\mathcal{G}/\mathcal{H}$ , i.e. the maximum distance between any two points. A complete classification of the two-point homogeneous spaces was given by Wang [24].

For each zonal function  $z$  on  $M^d$ , we have a univariate function  $\tilde{z}$ , defined on  $[-1, 1]$ ,

$$z(x) = \tilde{z}(\cos(2\lambda d(x, o))), \quad x \in M^d,$$

where  $\lambda$  is either  $\pi/2L$  or  $\pi/4L$ , depending on the homogeneous space  $M^d$ .

Let  $L_p$  be the set of all complex measurable functions  $\varphi$  on  $M^d$  of finite norm, given by

$$\|\varphi\|_p = \begin{cases} (\int_{M^d} |\varphi(x)|^p d\nu(x))^{1/p}, & \text{if } 1 \leq p < \infty, \\ \text{ess sup}\{|\varphi(x)| \mid x \in M^d\}, & \text{if } p = \infty. \end{cases}$$

Further, let  $U_p = \{\varphi : \varphi \in L_p, \|\varphi\|_p \leq 1\}$ .

Let  $\tilde{z} \in L_1([-1, 1], (1-x)^\alpha(1+x)^\beta dx)$ . Then, for any integrable function  $g$  we can define convolution  $h$  on  $M^d$  as the following

$$h(\cdot) = (z * g)(\cdot) = \int_{M^d} \tilde{z}(\cos(2\lambda d(\cdot, x)))g(x)d\nu(x).$$

For the convolution on  $M^d$  we have Young's inequality

$$\|z * g\|_q \leq \|z\|_p \|g\|_r, \quad (1)$$

where  $1/q = 1/p + 1/r - 1$  and  $1 \leq p, q, r \leq \infty$ .

For each  $n \in \mathbf{N}$ , let  $H_n$  be the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue  $-n(n + \alpha + \beta + 1)$ , where  $\alpha$  and  $\beta$  are numbers associated with the particular homogeneous space  $M^d$ . Let  $\mathcal{T}_n = \bigoplus_{k=0}^n H_k$ , and  $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$ . We have  $\dim \mathcal{T}_n \asymp n^d$ .

The Hilbert space  $L_2$  with usual scalar product

$$\langle f, g \rangle = \int_{M^d} f(x)g(x)d\nu(x)$$

has the decomposition

$$L_2 = \bigoplus_{n=0}^{\infty} H_n.$$

There is a unique real zonal eigenfunction (up to scalar multiplication)  $Z_n \in H_n$  such that the orthogonal projection from  $L_2$  onto  $H_n$  is given by the convolution operator  $\varphi \mapsto Z_n * \varphi$ . We have

$$\tilde{Z}_n(t) = C_n(M^d)P_n^{\alpha, \beta}(t),$$

where  $P_n^{\alpha,\beta}$  are the Jacobi polynomials (see, for example Szegő [20]), and  $\alpha$  and  $\beta$  can be specified for each  $M^d$ . For example, in the case of  $S^d$ , we have  $\sigma = 0$  and  $\rho = d - 1$ . Thus,  $\alpha = \beta = (d - 2)/2$  and the polynomials  $P_n^{\alpha,\beta}$  are just multiples of the Gegenbauer polynomial  $P_n^{(d-1)/2}$ . The normalisation constant  $C_n(M^d) \asymp n^{d/2}$  is chosen for technical reasons, and its explicit value can be found in [3].

See [7, 9, 14, 18] for more information concerning the harmonic analysis on homogeneous spaces.

In the next section we will give the definition of a standard Sobolev spaces and introduce a set Besov type spaces on two-point homogeneous manifolds. In the Section 3 we demonstrate that our definition of Besov spaces generates a set of an equivalent norms. In the case of  $S^d$  our definition is equivalent to another definition of Besov spaces given in [13]. In the final section of the paper we will use interpolation results of Triebel [21] to give estimates of  $n$ -widths for our Besov spaces.

## 2 Sobolev and Besov spaces on two-point homogeneous manifolds

Let  $\Lambda = \{\lambda_k\}_{k \geq 0}$  be a sequence of complex numbers, and let  $1 \leq p, q \leq \infty$ . If for any  $\varphi \in L_p$  there is a function  $f := \Lambda \varphi \in L_q$  with formal expansion

$$f \sim \sum_{k=0}^{\infty} \lambda_k Z_k * \varphi,$$

then we shall say that  $\Lambda$  is a multiplier operator of  $(p, q)$ -type with norm  $\|\Lambda\|_{p,q} := \sup_{\varphi \in U_p} \|\Lambda \varphi\|_q$ . For  $s \in \mathbf{R}$ , let  $\Lambda^s = \{\mu_k^s\}_{k \geq 0}$ , where  $\mu_k^s = (k(k + \alpha + \beta + 1))^{s/2}$ . It was proved in [4] that, for  $s > d(1/p - 1/q)_+$ ,  $\Lambda^{-s}$  is a multiplier operator of  $(p, q)$ -type, and for each  $\varphi \in L_p$  and  $n \geq 1$ , there exists a polynomial function  $t_n(\varphi) \in \mathcal{T}_n$  such that

$$\|\Lambda^{-s} \varphi - t_n(\varphi)\|_q \ll n^{-s+d(1/p-1/q)_+} \|\varphi\|_p. \quad (2)$$

The Sobolev space  $W_p^s$ ,  $s > 0$ , is given by

$$W_p^s := \{f \in L_p : \Lambda^s f \in L_p\},$$

with norm

$$\|f\|_p^s := \|\Lambda^s f\|_p.$$

Here we have identified functions which differ by a constant, i.e., if  $f - g = \text{constant}$ , then  $f = g$  in  $W_p^s$ . The Sobolev class  $\overline{W}_p^s$ ,  $s > 0$ , is given by

$$\overline{W}_p^s := \{f \in W_p^s : \|f\|_p^s \leq 1\}.$$

It is easy to show that

$$\overline{W}_p^s = \{c + \Lambda^{-s} f : c \in \mathbf{R}, f \in U_p\}.$$

Let  $(\lambda_k^{(n)})$  be an infinite lower triangular matrix, i.e.,  $\lambda_k^{(n)} = 0$  for any  $k > n$  and  $n \in \mathbf{N}$ .

**Definition 2.1** We say that the sequence  $K = \{K_{2^n}\}_{n \in \mathbf{N}}$  of polynomial zonal functions

$$K_{2^n} = \sum_{k=0}^{2^n} \lambda_k^{(2^n)} Z_k \quad (3)$$

possesses the property  $\mathcal{K}_C$ , and we write  $K \in \mathcal{K}_C$ , if  $\lambda_k^{(2^n)} = 1$  for any  $0 \leq k \leq 2^{n-1}$  and there exists a positive constant  $C$  such that,  $\|K_{2^n}\|_1 \leq C$  for all  $n \in \mathbf{N}$ .

**Remark 2.2** In the case of  $S^d$ ,  $d \geq 2$ , the sequence of zonal polynomials  $K$  was introduced and constructed in an explicit form in [10, p. 287]. They are a natural generalization of the de la Vallée Poussin polynomial  $V_{n,2n}(t)$  on  $S^1$ ,

$$V_{n,2n}(t) = 1/2 + \sum_{k=1}^{2n} \lambda_k^{(2n)} \cos kt,$$

where  $\lambda_k^{(2n)} = 1$  for  $1 \leq k \leq n$  and  $\lambda_k^{(2n)} = (2n - k)/n$  for  $n < k \leq 2n$ . The de la Vallée Poussin polynomials  $V_{n,2n}(t)$  were used to introduce the Besov spaces on  $S^1$ .

Let  $K \in \mathcal{K}_C$ . For  $k \geq 0$  we write

$$\varphi_k = K_{2k} - K_{2k-1}, \quad k \geq 1; \quad \varphi_0 = Z_0. \quad (4)$$

**Definition 2.3** The function  $f \in L_p$  belongs to the Besov space  $B_{p,q}^s(K)$ ,  $s, p, q \in \mathbf{R}$ ,  $s > 0$ ,  $1 \leq p, q \leq \infty$ , if

$$\|f\|_{p,q}^s(K) = \left( \sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} < \infty,$$

where the functions  $\varphi_k$  are defined in (4).

It will be proved in Theorem 3.6 that the norms of  $B_{p,q}^s(K)$  and  $(W_p^{s_0}, W_p^{s_1})_{\theta,q}$  are equivalent for  $s = (1 - \theta)s_0 + \theta s_1$ ,  $s_0, s_1 > 0$ ,  $s_0 \neq s_1$ . As consequence we can conclude that the norms of  $B_{p,q}^s(K^1)$  and  $B_{p,q}^s(K^2)$  are equivalent for all  $K^1, K^2 \in \mathcal{K}_C$ , that is, the norm of  $B_{p,q}^s(K)$  does not depend on the sequence  $K \in \mathcal{K}_C$ . So that we will write  $\|\cdot\|_{p,q}^s$  instead of  $\|\cdot\|_{p,q}^s(K)$ .

As with the Sobolev spaces we will identify two functions in  $B_{p,q}^s(K)$  which differ by a constant. It is easy to see that  $B_{p,q}^s(K)$  is a normed vector space with norm  $\|\cdot\|_{p,q}^s(K)$ . We will see in Section 3 that  $B_{p,q}^s(K)$  is the interpolation space for two of the Sobolev spaces defined above, and is therefore a Banach space. Let

$$\bar{B}_{p,q}^s(K) = \{f \in L_p : \|f\|_{p,q}^s(K) \leq 1\}.$$

To give a useful sufficient conditions for the imbedding  $K \in \mathcal{K}_C$  we will need some information concerning Cesàro means. Let  $\delta \geq 0$  and  $n \in \mathbf{N}$ . We define the Cesàro kernel  $\tilde{S}_n^\delta$  by

$$\tilde{S}_n^\delta = \frac{1}{C_n^\delta} \sum_{m=0}^n C_{n-m}^\delta \tilde{Z}_m,$$

where  $C_n^\delta$  are Cesàro numbers of order  $n$  and index  $\delta$ ,

$$C_n^\delta = \frac{\Gamma(n + \delta + 1)}{\Gamma(\delta + 1)\Gamma(n + 1)} \asymp n^\delta, \quad n \rightarrow \infty \quad (5)$$

(see, e.g., [20, p. 237]). Given a sequence  $\{\lambda_k\}_{k \geq 0}$  we define the differences  $\Delta^s \lambda_k$ ,  $k, s \in \mathbf{N}$ , by  $\Delta^0 \lambda_k = \lambda_k$ ,  $\Delta^1 \lambda_k = \lambda_k - \lambda_{k+1}$ , and  $\Delta^{s+1} \lambda_k = \Delta^s \lambda_k - \Delta^s \lambda_{k+1}$ .

**Lemma 2.4** Let  $(\lambda_k^{(2^n)})$  be such that for any  $n \in \mathbf{N}$  there is such  $C' > 0$  that

$$\sum_{k=0}^{2^n - d - 1} |\Delta^{d+1} \lambda_k^{(2^n)}| k^d + \sum_{s=0}^d |\Delta^s \lambda_{2^n - s}^{(2^n)}| \vartheta_n^{(s)} < C', \quad (6)$$

where

$$\vartheta_n^{(s)} = \begin{cases} 2^{n(d-1)/2}, & 0 \leq s < (d-1)/2, \\ n 2^{n(d-1)/2}, & s = (d-1)/2, \\ 2^{ns}, & s > (d-1)/2. \end{cases}$$

Then  $K = \{K_{2^n}\}_{n \in \mathbf{N}} \in \mathcal{K}_C$ .

Proof. Applying the Abel transform  $d + 1$  times we get

$$K_{2^n} = \sum_{k=0}^{2^n-d-1} (\Delta^{d+1} \lambda_k^{(2^n)}) C_k^d S_k^d + \sum_{s=0}^d (\Delta^s \lambda_{2^n-s}^{(2^n)}) C_{2^n-s}^s S_{2^n-s}^s$$

and hence

$$\|K_{2^n}\|_1 \leq \sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{(2^n)}| C_k^d \|S_k^d\|_1 + \sum_{s=0}^d |\Delta^s \lambda_{2^n-s}^{(2^n)}| C_{2^n-s}^s \|S_{2^n-s}^s\|_1. \quad (7)$$

It was proved in [4, Lemma 3] that

$$\|S_n^\delta\|_1 \ll \begin{cases} n^{(d-1)/2-\delta}, & 0 \leq \delta < (d-1)/2, \\ \log n, & \delta = (d-1)/2, \\ 1, & \delta > (d-1)/2. \end{cases} \quad (8)$$

Comparing (5) - (8) we get (6).  $\square$

We present here two examples of sequences  $\{K_{2^n}\}_{n \in \mathbf{N}} \in \mathcal{K}_C$ .

**Example 2.5** Consider the function

$$\xi(t) = -\frac{1}{\omega} \exp\left(\frac{1}{(t-1/2)(t-1)}\right),$$

where

$$\omega = \int_{1/2}^1 \exp\left(\frac{1}{(t-1/2)(t-1)}\right) dt.$$

Let

$$\nu(t) = \begin{cases} 0, & 0 \leq t < 1/2, \\ \xi(t), & 1/2 \leq t \leq 1, \\ 0, & t > 1 \end{cases}$$

and

$$\mu(x) = 1 + \int_0^x \nu(t) dt, \quad x \geq 0.$$

Put  $\lambda_k^{(n)} = \mu(k/n)$ ,  $0 \leq k \leq n$ . Let  $f(x)$  be a continuous function on  $\mathbf{R}$ . Then for the differences of order  $k$  we have

$$\Delta_h^k f(x) = \sum_{\nu=0}^k (-1)^\nu \frac{k!}{\nu!(k-\nu)!} f(x + \nu h). \quad (9)$$

In addition, if  $f(x)$  has  $k$  derivatives, then

$$\Delta_h^k f(x) = (-1)^k \int_0^h \cdots \int_0^h f^{(k)}(x + t_1 + \cdots + t_k) dt_1 \cdots dt_k \quad (10)$$

Observe that for any integer  $k \geq 1$  we have  $|\mu^{(k)}(x)| = 0$  for  $0 \leq x \leq 1/2$ ,  $x \geq 1$ , and

$$|\mu^{(k)}(x)| \leq C[(x-1/2)(x-1)]^{-2(k-1)} \exp\left(\frac{1}{(x-1/2)(x-1)}\right) \quad (11)$$

for  $1/2 < x < 1$ . Since  $\Delta^s \lambda_k^{2^n} = \Delta_{2^{-n}}^s \mu(k/2^n)$ , it follows from (10) and (11) that  $|\Delta^{d+1} \lambda_k^{(2^n)}| \leq C 2^{-(d+1)n}$  and therefore,

$$\sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{(2^n)}| k^d \leq C. \quad (12)$$

We have  $\Delta^s \lambda_{2^n-s}^{(2^n)} = \mu(1) = 0$  for  $s = 0$ , and from (10) and (11) we have that

$$|\Delta^s \lambda_{2^n-s}^{(2^n)}| \leq 2^{-ns} \left| \mu \left( 1 - \frac{s}{2^n} \right) \right| \leq \frac{C}{s} 2^{-(s-1)n} \exp \left( -\frac{2^{n+2}}{s} \right)$$

for  $1 \leq s \leq 2^n$ . Then

$$\lim_{n \rightarrow \infty} \sum_{s=0}^d |\Delta^s \lambda_{2^n-s}^{(2^n)}| \vartheta_n^{(s)} = 0. \quad (13)$$

Finally, from Lemma 2.4, (12) and (13) it follows that  $K \in \mathcal{K}_C$ .

**Example 2.6** (see, also, [10, p. 287]) Consider the function

$$\chi_0(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1, \end{cases}$$

and for  $1 \leq s \leq d$  let

$$\chi_s(t) = 2d \int_t^{t+1/(2d)} \chi_{s-1}(u) du.$$

The function  $\chi_d$  is  $d-1$  times continuously differentiable and positive on  $[0, \infty)$ . Furthermore,  $\chi_d^{(d-1)}$  is Lipschitz continuous,  $\chi_d(t) = 1$  for  $0 \leq t \leq 1/2$ , and

$$\chi_d(t) = P_d(t) = \frac{(2d)^d}{d!} (1-t)^d, \quad 1 - \frac{1}{2d} \leq t \leq 1. \quad (14)$$

Also  $\chi_d$  is a polynomial function of degree  $d$  in each interval  $[t_s, t_{s-1}]$ ,  $1 \leq s \leq d$ , where  $t_s = 1 - s/(2d)$ . For each  $n \geq 1$  consider a sequence  $\{\lambda_k^{(n)}\}_{k=0}^n$  given by

$$\lambda_k^{(n)} = \chi_d \left( \frac{k}{n} \right), \quad 0 \leq k \leq n. \quad (15)$$

To apply Lemma 2.4 we need to get some bounds for the differences  $|\Delta^s \lambda_k^{(2^n)}|$ . We show that for  $2^n \geq 2d^2$  and  $0 \leq s \leq d$ ,

$$|\Delta^s \lambda_{2^n-s}^{(2^n)}| \leq (2d)^{2d} 2^{-dn} \quad (16)$$

and for all  $n \in \mathbf{N}$ , and  $1 \leq k \leq 2^n$ ,

$$|\Delta^{d+1} \lambda_k^{(2^n)}| \leq C(d) 2^{-dn}. \quad (17)$$

Remark that for  $n \geq 2d^2$  and  $0 \leq s \leq d$ ,  $(2^n - s)/2^n \geq 1 - 1/(2d)$ , so that, setting  $h = 2^{-n}$ ,

$$\lambda_{2^n-s}^{(2^n)} = P_d \left( \frac{2^n - s}{2^n} \right), \quad \text{and} \quad \Delta^s \lambda_{2^n-s}^{(2^n)} = \Delta_h^s P_d \left( \frac{2^n - s}{2^n} \right).$$

Using (10) we get

$$\begin{aligned} |\Delta^s \lambda_{2^n-s}^{(2^n)}| &\leq h^s \max_{t \geq (2^n-s)/n} P_d^{(s)}(t) \\ &\leq 2^{-sn} \frac{(2d)^d}{(d-s)!} \left( \frac{s}{2^n} \right)^{d-s} \\ &\leq (2d)^{2d} 2^{-dn}, \quad 2^n \geq 2d^2, \quad 0 \leq s \leq d, \end{aligned}$$

proving (16) and consequently (13). Since  $\chi_d^{(d)}$  is a piecewise continuous function on  $[0, 1]$ , it follows from (10) that  $|\Delta_{2^{-n}}^d \chi_d(k/2^n)| \leq C(d)2^{-nd}$ . Then (17) follows from the definition of differences.

Now, let  $I_k^n = [k/2^n, (k+d+1)/2^n]$ ,  $t_s = 1 - s/(2d)$ ,  $n \in \mathbf{N}$ ,  $1 \leq k \leq 2^n$ ,  $1 \leq s \leq d+1$ , and let

$$E_{n,d} = \{k : \{t_1, t_2, \dots, t_{d+1}\} \cap I_k^n \neq \emptyset, 1 \leq k \leq 2^n\}.$$

If  $k$  is not in  $E_{n,d}$ , then  $\chi_d^{(d+1)}(t) = 0$  for  $t \in I_k^n$  and hence  $\Delta^{d+1} \lambda_k^{(2^n)} = \Delta_{2^{-n}}^{d+1} \chi_d(k/2^n) = 0$ . Therefore

$$\left\{k : \Delta^{d+1} \lambda_k^{(2^n)} \neq 0, 0 \leq k \leq 2^n\right\} \subset E_{n,d}.$$

Since  $\text{Card}(E_{n,d}) \leq (d+1)(d+2)$  for all  $n \in \mathbf{N}$ , using (17) we get

$$\sum_{k=0}^{2^n-d-1} |\Delta^{d+1} \lambda_k^{(2^n)}| k^d \leq C(d) \sum_{k \in E_{n,d}} \left(\frac{k}{2^n}\right)^d \leq C(d)(d+1)(d+2).$$

Finally from Lemma 2.4 it follows that  $K = \{K_{2^n}\}_{n \in \mathbf{N}} \in \mathcal{K}_C$ .

### 3 Besov spaces as interpolation spaces

Two complex Banach spaces  $A_0$  and  $A_1$  are called an interpolation pair  $\bar{A} = (A_0, A_1)$  if there exists a Hausdorff topological vector space in which  $A_0$  and  $A_1$  are continuously embedded. Then, the following spaces and quantities are well-defined:

$$\begin{aligned} \Delta(\bar{A}) &= A_0 \cap A_1; \\ \|a\|_{\Delta(\bar{A})} &= \max(\|a\|_{A_0}, \|a\|_{A_1}), \quad a \in \Delta(\bar{A}); \\ \Sigma(\bar{A}) &= A_0 + A_1 = \{a_0 + a_1 : a_0 \in A_0, a_1 \in A_1\}; \\ \|a\|_{\Sigma(\bar{A})} &= \inf_{a=a_0+a_1 \in \Sigma(\bar{A})} (\|a_0\|_{A_0} + \|a_1\|_{A_1}); \\ K(t, a) = K(t, a; \bar{A}) &= \inf_{a=a_0+a_1 \in \Sigma(\bar{A})} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad 0 < t < \infty; \\ J(t, a) = J(t, a; \bar{A}) &= \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad 0 < t < \infty, \quad a \in \Delta(\bar{A}). \end{aligned}$$

For a given  $t > 0$ ,  $\|\cdot\|_{\Sigma(\bar{A})}$  and  $K(t, \cdot)$  are equivalent norms on  $\Sigma(\bar{A})$ , and  $\|\cdot\|_{\Delta(\bar{A})}$  and  $J(t, \cdot)$  are equivalent norms on  $\Delta(\bar{A})$ .

Let  $0 < \theta < 1$ ,  $1 \leq q \leq \infty$ , and let  $\Phi_{\theta,q}$  be the functional defined by

$$\Phi_{\theta,q}(\varphi(t)) = \begin{cases} \left(\int_0^\infty (t^{-\theta} \varphi(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \leq q < \infty, \\ \text{ess sup}_{t>0} t^{-\theta} \varphi(t), & q = \infty, \end{cases}$$

where  $\varphi$  is a non-negative measurable function.

Given  $a \in \Sigma(\bar{A})$  we define

$$\|a\|_{\theta,q;K} = \Phi_{\theta,q}(K(t, a)). \tag{18}$$

The set

$$\mathcal{K}_{\theta,q}(\bar{A}) = \{a \in \Sigma(\bar{A}) : \|a\|_{\theta,q;K} < \infty\}$$

is a Banach space with the norm  $\|\cdot\|_{\theta,q;K}$ , and is called the interpolation space of the pair  $\bar{A}$  by the  $K$ -method. If  $1 \leq q < \infty$  then  $\Delta(\bar{A})$  is dense in  $\mathcal{K}_{\theta,q}(\bar{A})$ .

We define the interpolation space of the pair  $\bar{A}$  by the  $J$ -method as the set  $\mathcal{J}_{\theta,q}(\bar{A})$ , of all  $a \in \Sigma(\bar{A})$  which can be represented by

$$a = \int_0^\infty u(t) \frac{dt}{t}, \tag{19}$$

where  $u$  is a measurable function taking values in  $\Delta(\bar{A})$ , convergence in the integral is in  $\Sigma(\bar{A})$ , and

$$\Phi_{\theta,q}(J(t, u(t))) < \infty. \quad (20)$$

The set  $\mathcal{J}_{\theta,q}(\bar{A})$  is a Banach space with the norm

$$\|a\|_{\theta,q;J} = \inf_u \Phi_{\theta,q}(J(t, u(t))), \quad (21)$$

where the infimum is taken over all  $u$  such that (19) and (20) hold. If  $1 \leq q < \infty$  then  $\Delta(\bar{A})$  is dense in  $\mathcal{J}_{\theta,q}(\bar{A})$ .

Both the  $J$ - and  $K$ -method are called real interpolation methods.

A direct consequence of the definition is that, for any  $a \in \Sigma(\bar{A})$ , the function  $t \mapsto K(t, a)$  is positive, increasing, concave, and for  $t, s > 0$ ,

$$K(t, a) \leq \max(1, t/s)K(s, a). \quad (22)$$

Similarly, for any  $a \in \Delta(\bar{A})$ , the function  $t \mapsto J(t, a)$  is positive, increasing, convex, and for  $t, s > 0$ ,

$$J(t, a) \leq \max(1, t/s)J(s, a). \quad (23)$$

**Theorem 3.1** ([1, p. 44]) *Let  $\bar{A} = (A_0, A_1)$  be an interpolation pair, and let  $0 < \theta < 1$  and  $1 \leq q \leq \infty$ . Then,  $\mathcal{J}_{\theta,q}(\bar{A}) = \mathcal{K}_{\theta,q}(\bar{A})$  with equivalence of norms.*

From this point to the end of the paper we will not distinguish  $\mathcal{J}_{\theta,q}(\bar{A})$  and  $\mathcal{K}_{\theta,q}(\bar{A})$ , and will denote them both by  $(A_0, A_1)_{\theta,q}$ . Both norms  $\|\cdot\|_{\theta,q;J}$  and  $\|\cdot\|_{\theta,q;K}$  will be denoted by  $\|\cdot\|_{(A_0, A_1)_{\theta,q}}$ .

Given a linear operator  $T \in L(A, B)$ , where  $A$  and  $B$  are Banach spaces, we denote

$$\|T\|_{A,B} = \sup_{a \in A, \|a\| \leq 1} \|Ta\|_B.$$

Suppose that  $T$  is a linear operator from  $A_0 + A_1$  into  $B$ , and let  $T_i$  be the restriction of  $T$  to  $A_i$ ,  $i = 0, 1$ . If  $T_i \in L(A_i, B)$ ,  $i = 0, 1$ , then

$$\|T\|_{(A_0, A_1)_{\theta,q}, B} \leq (\|T_0\|_{A_0, B})^{1-\theta} (\|T_1\|_{A_1, B})^\theta. \quad (24)$$

Let  $0 < \theta < 1$ ,  $1 < r < \infty$ , and  $1 \leq q \leq \infty$ . We will denote by  $\lambda^{\theta,r,q}$ , the space of all sequences  $(\alpha_k)_{k \in \mathbb{Z}}$ , such that

$$\|(\alpha_k)\|_{\lambda^{\theta,r,q}} = \begin{cases} \left( \sum_{k \in \mathbb{Z}} (r^{-k\theta} |\alpha_k|)^q \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{k \in \mathbb{Z}} r^{-k\theta} |\alpha_k|, & q = \infty, \end{cases}$$

is finite.

The next two lemmas can be found for  $r = 2$  in [1].

**Lemma 3.2** *Let  $\bar{A} = (A_0, A_1)$  be an interpolation pair and let  $a \in \Sigma(\bar{A})$ . Then,  $a \in (A_0, A_1)_{\theta,q}$  if and only if  $(K(r^k, a))_{k \in \mathbb{Z}} \in \lambda^{\theta,r,q}$ . Moreover, we have*

$$r^{-\theta} (\log r)^{1/q} \| (K(r^k, a)) \|_{\lambda^{\theta,r,q}} \leq \|a\|_{(A_0, A_1)_{\theta,q}} \leq r (\log r)^{1/q} \| (K(r^k, a)) \|_{\lambda^{\theta,r,q}}. \quad (25)$$

**Lemma 3.3** *Let  $\bar{A} = (A_0, A_1)$  be an interpolation pair and let  $a \in \Sigma(\bar{A})$ . Then  $a \in (A_0, A_1)_{\theta,q}$  if and only if there exists  $u_k \in \Delta(\bar{A})$ ,  $k \in \mathbb{Z}$ , with*

$$a = \sum_{k \in \mathbb{Z}} u_k \quad (26)$$

(convergence inside  $\Sigma(\bar{A})$ ), and such that  $(J(r^k, u_k)) \in \lambda^{\theta,r,q}$ . Moreover,

$$\begin{aligned} (\log r)^{-1+1/q} r^{-\theta} \inf_{u_k} \| (J(r^k, u_k)) \|_{\lambda^{\theta,r,q}} &\leq \|a\|_{(A_0, A_1)_{\theta,q}} \\ &\leq r (\log r)^{-1+1/q} \inf_{u_k} \| (J(r^k, u_k)) \|_{\lambda^{\theta,r,q}}. \end{aligned} \quad (27)$$

Now suppose  $0 < s_1 < s_2$  and  $1 \leq p \leq \infty$ . It follows from (2) that  $\Lambda^{s_1-s_2}$  is bounded on  $L_p$  and hence, for  $f \in W_p^{s_2}$ ,

$$\|f\|_p^{s_1} = \|\Lambda^{s_1-s_2} \Lambda^{s_2} f\|_p \leq C_p \|f\|_p^{s_2} < \infty.$$

Therefore  $W_p^{s_2} \subset W_p^{s_1}$  for  $0 < s_1 < s_2$  and  $1 \leq p \leq \infty$ . The space  $\mathcal{T} = \cup_{n=1}^{\infty} \mathcal{T}_n$  is dense in  $L_p$ ,  $1 \leq p < \infty$ , and hence is also dense in  $W_p^s$ ,  $s > 0$ ,  $1 \leq p < \infty$ . Since  $B_{p,q}^s$  is the interpolation of two Sobolev spaces (see Theorem 3.6) and  $\mathcal{T}$  is dense in these Sobolev spaces, it follows that  $\mathcal{T}$  is also dense in  $B_{p,q}^s$ ,  $s > 0$ ,  $1 \leq p < \infty$ , and  $1 \leq q \leq \infty$ .

The next result is Bernstein's inequality and is proved in [6].

**Theorem 3.4** For all  $s, p \in \mathbf{R}$ ,  $s > 0$ ,  $1 \leq p \leq \infty$ ,

$$\left\| \sum_{k=1}^m \mu_k^s Z_k * f \right\|_p \leq C_s m^s \|f\|_p, \quad f \in \mathcal{T}_m, \quad m \geq 1. \quad (28)$$

**Corollary 3.5** Let  $K \in \mathcal{K}_C$ ,  $s, p \in \mathbf{R}$ ,  $s > 0$ , and  $1 \leq p \leq \infty$ . Then

$$\|\Lambda^s(\varphi_k * f)\|_p \leq C_s 2^{ks} \|\varphi_k * f\|_p, \quad f \in L_p, \quad k \geq 0 \quad (29)$$

and

$$\|\varphi_k * f\|_p \leq C_s 2^{-ks} \|\Lambda^s f\|_p, \quad f \in W_p^s, \quad k \geq 0. \quad (30)$$

*Proof.* Let us fix  $f \in L_p$ ,  $1 \leq p \leq \infty$ , and  $k \geq 0$ . We have that  $\varphi_k * f \in \mathcal{T}_{2^k}$ . Hence, by (28),

$$\|\Lambda^s(\varphi_k * f)\|_p \leq C_s 2^{ks} \|\varphi_k * f\|_p.$$

Now if  $f \in W_p^s$ , let  $t_k$  be a polynomial of degree  $2^{k-1}$  satisfying (2) for  $\varphi = \Lambda^s f$ . Then

$$\|f - t_k\|_p \ll 2^{-ks} \|\Lambda^s f\|_p,$$

and  $K_{2^k} * t_k = t_k$ . Therefore, by (1), (2) and Definition 2.1 we get

$$\begin{aligned} \|f - K_{2^k} * f\|_p &\leq \|f - t_k\|_p + \|K_{2^k} * (f - t_k)\|_p \\ &\leq (1 + \|K_{2^k}\|_1) \|f - t_k\|_p \\ &\leq C 2^{-ks} \|\Lambda^s f\|_p \end{aligned}$$

and hence by (4)

$$\begin{aligned} \|\varphi_k * f\|_p &\leq \|f - K_{2^k} * f\|_p + \|f - K_{2^{k-1}} * f\|_p \\ &\leq C 2^{-ks} \|\Lambda^s f\|_p. \end{aligned}$$

□

**Theorem 3.6** Let  $K \in \mathcal{K}_C$ . If  $0 < s_0 < s_1$  then

$$B_{p,q}^{s_1} \subset B_{p,q}^{s_0}, \quad 1 \leq p, q \leq \infty. \quad (31)$$

If  $1 \leq q_0 < q_1 \leq \infty$  then

$$B_{p,q_0}^s \subset B_{p,q_1}^s, \quad s > 0, \quad 1 \leq p \leq \infty. \quad (32)$$

Moreover,

$$B_{p,1}^s \subset W_p^s \subset B_{p,\infty}^s, \quad s > 0, \quad 1 \leq p \leq \infty. \quad (33)$$

If  $s_0, s_1 > 0$  and  $s_0 \neq s_1$ , then

$$(W_p^{s_0}, W_p^{s_1})_{\theta,q} = B_{p,q}^s, \quad 1 \leq p, q \leq \infty, \quad 0 < \theta < 1, \quad (34)$$

with equivalence of norms, where  $s = (1 - \theta)s_0 + \theta s_1$ .

Proof. The inclusions (31) and (32) follow directly from the definition of Besov space. Now, from (29), we have

$$\begin{aligned} \|f\|_p^s &\leq \sum_{k=0}^{\infty} \|\Lambda^s(\varphi_k * f)\|_p \\ &\leq C_s \sum_{k=0}^{\infty} 2^{ks} \|\varphi_k * f\|_p \\ &= C_s \|f\|_{p,1}^s. \end{aligned}$$

From (30),

$$\begin{aligned} \|f\|_{p,\infty}^s &= \sup_{k \in \mathbf{N}} 2^{ks} \|\varphi_k * f\|_p \\ &\leq C_s \sup_{k \in \mathbf{N}} 2^{ks} 2^{-ks} \|\Lambda^s f\|_p \\ &= C_s \|f\|_p^s. \end{aligned}$$

The inclusions in (33) follow from the previous two inequalities.

We now prove (34). Consider  $0 < s_0 < s_1$ . Let  $f \in (W_p^{s_0}, W_p^{s_1})_{\theta,q}$ , with  $f = f_0 + f_1$ ,  $f_i \in W_p^{s_i}$ ,  $i = 0, 1$ . From (30),

$$\begin{aligned} \|\varphi_k * f\|_p &\leq \|\varphi_k * f_0\|_p + \|\varphi_k * f_1\|_p \\ &\leq C_0 2^{-ks_0} \|\Lambda^{s_0} f_0\|_p + C_1 2^{-ks_1} \|\Lambda^{s_1} f_1\|_p \\ &\leq C 2^{-ks_0} (\|f_0\|_p^{s_0} + 2^{k(s_0-s_1)} \|f_1\|_p^{s_1}), \end{aligned}$$

and hence

$$\|\varphi_k * f\|_p \leq C 2^{-ks_0} K(2^{k(s_0-s_1)}, f).$$

Thus, putting  $r = 2^{s_1-s_0}$  we get

$$2^{ks} \|\varphi_k * f\|_p \leq C 2^{k\theta(s_1-s_0)} K(2^{k(s_0-s_1)}, f) = C r^{k\theta} K(r^{-k}, f),$$

so that

$$\left( \sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} \leq C \left( \sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^q \right)^{1/q}.$$

Applying Lemma 3.2 we find

$$\begin{aligned} \|f\|_{p,q}^s &= \left( \sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} \\ &\leq C \left( \sum_{k=0}^{\infty} (r^{k\theta} K(r^{-k}, f))^q \right)^{1/q} \\ &\leq C \|f\|_{(W_p^{s_0}, W_p^{s_1})_{\theta,q}}. \end{aligned}$$

Suppose conversely that  $f \in B_{p,q}^s$ . From (29),

$$\begin{aligned} J(2^{k(s_0-s_1)}, \varphi_k * f) &= \max(\|\Lambda^{s_0}(\varphi_k * f)\|_p, 2^{k(s_0-s_1)} \|\Lambda^{s_1}(\varphi_k * f)\|_p) \\ &\leq \max(C_{s_0} 2^{ks_0} \|\varphi_k * f\|_p, C_{s_1} 2^{k(s_0-s_1)} 2^{ks_1} \|\varphi_k * f\|_p) \\ &= C_s 2^{ks_0} \|\varphi_k * f\|_p, \end{aligned}$$

and hence

$$2^{k(s-s_0)} J(2^{k(s_0-s_1)}, \varphi_k * f) \leq C_s 2^{ks} \|\varphi_k * f\|_p.$$

Thus, again putting  $r = 2^{s_1 - s_0}$ , we get

$$r^{\theta k} J(r^{-k}, \varphi_k * f) \leq C_s 2^{ks} \|\varphi_k * f\|_p, \quad k \geq 0. \quad (35)$$

Consider the sequence  $(u_k)_{k \in \mathbb{Z}}$  defined by

$$u_k = \begin{cases} 0, & k \geq 1, \\ \varphi_{-k} * f, & k \leq 0. \end{cases}$$

Since  $s_0 < s_1$  we have that  $W_p^{s_1} \subset W_p^{s_0}$ , and hence  $W_p^{s_0} + W_p^{s_1} = W_p^{s_0}$ . From (29) and Hölder's inequality

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \|u_k\|_p^{s_0} &= \sum_{k=0}^{\infty} \|\Lambda^{s_0}(\varphi_k * f)\|_p \\ &\leq C_{s_0} \sum_{k=0}^{\infty} 2^{ks_0} \|\varphi_k * f\|_p \\ &= C_{s_0} \sum_{k=0}^{\infty} 2^{k\theta(s_0 - s_1)} 2^{ks} \|\varphi_k * f\|_p \\ &\leq C_{s_0} \left( \sum_{k=0}^{\infty} r^{-k\theta q'} \right)^{1/q'} \|f\|_{p,q}^s < \infty, \end{aligned}$$

where  $1/q + 1/q' = 1$ . Therefore, we may conclude that the series  $\sum_{k \in \mathbb{Z}} u_k$  converges to  $f \in W_p^{s_0}$ . Now, by (35)

$$\begin{aligned} \left( \sum_{k \in \mathbb{Z}} (r^{-k\theta} J(r^k, u_k))^q \right)^{1/q} &= \left( \sum_{k=0}^{\infty} (r^{k\theta} J(r^{-k}, \varphi_k * f))^q \right)^{1/q} \\ &\leq C_s \left( \sum_{k=0}^{\infty} (2^{ks} \|\varphi_k * f\|_p)^q \right)^{1/q} \\ &= C_s \|f\|_{p,q}^s. \end{aligned}$$

Thus, by Lemma 3.2, we can conclude that  $f \in (W_p^{s_0}, W_p^{s_1})_{\theta, q}$  and that

$$\|f\|_{(W_p^{s_0}, W_p^{s_1})_{\theta, q}} \leq C_s \|f\|_{p,q}^s.$$

□

**Remark 3.7** Let  $s > d(1/p - 1/q)_+$ ,  $1 \leq p, q \leq \infty$ , and  $n \in \mathbf{N}$ . Consider the operator  $I_n$ , defined on  $L_p$ , by  $I_n f = f - t_n(\Lambda^s f)$ , where  $t_n(\varphi) = G_n * \varphi$ , where  $G_n = \sum_{k=1}^n (\Delta^{N+1} \lambda_k^{-s}) C_k^N S_k^N$ ,  $N = (d+1)/2$  if  $d$  is odd and  $N = (d+2)/2$  if  $d$  is even. From (2) we have that

$$\|I_n\|_{W_p^s, L_q} \leq C_s n^{-s+d(1/p-1/q)_+}. \quad (36)$$

Now, consider  $s, p, r, q \in \mathbf{R}$  with  $1 \leq p, q, r \leq \infty$ , and  $s > d(1/p - 1/q)_+$ . Let  $s_0, s_1 \in \mathbf{R}$  such that  $s_1 > s > s_0 > d(1/p - 1/q)_+$ , and let  $\theta \in (0, 1)$ , such that  $s = (1 - \theta)s_0 + \theta s_1$ . From (34) we have  $B_{p,r}^s = (W_p^{s_0}, W_p^{s_1})_{\theta, r}$ . Then, from (24) and (36),

$$\begin{aligned} \|I_n\|_{B_{p,r}^s, L_q} &\leq \left( \|I_n\|_{W_p^{s_0}, L_q} \right)^{1-\theta} \left( \|I_n\|_{W_p^{s_1}, L_q} \right)^{\theta} \\ &\leq C_s n^{-s+d(1/p-1/q)_+}. \end{aligned} \quad (37)$$

**Remark 3.8** Consider the multiplier sequence  $\dot{\Lambda}^s = \{\dot{\lambda}_k^s\}_{k \geq 0}$ ,  $\dot{\lambda}_k^s = (1 + k(k + \alpha + \beta + 1))^{s/2}$ ,  $s > 0$ . The Sobolev spaces can also be defined using  $\dot{\Lambda}^s$  rather than  $\Lambda^s$ . We define

$$\dot{W}_p^s := \left\{ f \in L_p : \dot{\Lambda}^s f \in L_p \right\},$$

with norm

$$\|f\|_p^s := \|\dot{\Lambda}^s f\|_p.$$

From [4, Theorem 2],  $\dot{\Lambda}^{-s}$  is bounded from  $L_p$  to  $L_q$  for  $s > d(1/p - 1/q)_+$ . The inequality (2) also holds for  $\dot{\Lambda}^{-s}$ ,  $s > 0$ . Since Bernstein's inequality (Theorem 3.4) holds for the sequence  $\dot{\lambda}_k^s$  (see [6]), the same is true for Corollary 3.5.

Now, for  $s, p, q \in \mathbf{R}$ ,  $s > 0$ ,  $1 \leq p, q \leq \infty$ , and  $f \in L_p$ , we define

$$\|f\|_{p,q}^s := \|f\|_p + \|f\|_{p,q}^s,$$

and

$$\dot{B}_{p,q}^s := \{f \in L_p : \|f\|_{p,q}^s < \infty\}.$$

In this case we have that  $\|\cdot\|_p^s$  and  $\|\cdot\|_{p,q}^s$  are norms in  $\dot{W}_p^s$  and  $\dot{B}_{p,q}^s$  respectively, when we consider these spaces as subspaces of  $L_p$ .

Theorem 3.6 holds for the spaces  $\dot{W}_p^s$  and  $\dot{B}_{p,q}^s$ , with minor changes in the proof. In the proof we consider a sequence  $(u_k)_{k \in \mathbf{Z}}$  as in the proof of Theorem 3.6, only with the minor change  $u_1 = Z_0 * f$ . We need the inequalities

$$C_1 r^{-1} J(r, f) \leq \|f\|_p \leq C_2 K(1, f), \quad 1 \leq p \leq \infty, \quad f \in L_p.$$

**Remark 3.9** Besov spaces on the unit sphere  $S^d$  in  $\mathbf{R}^{d+1}$  were studied in [13], where the authors give a list of equivalent norms for Besov spaces. We will show that the Besov spaces considered in [13] and the spaces  $\dot{B}_{p,q}^s$  have equivalent norms.

The Sobolev spaces considered in [13] are the spaces  $\dot{W}_p^s$  given in Remark 3.8. Let us denote by  $B_{p,q}^{*s}$  the Besov spaces in [13],  $s > 0$ ,  $1 \leq p, q \leq \infty$ . The following two equivalent norms for  $B_{p,q}^{*s}$ , among others, are given in [13]:

$$\begin{aligned} (1) \|f\|_{p,q}^s &= \|f\|_p + \left( \sum_{m=1}^{\infty} m^{sq-1} E_m(f)_p^q \right)^{1/q}, \\ (2) \|f\|_{p,q}^s &= \|f\|_p + \left( \sum_{m=1}^{\infty} m^{sq-1} K_r(\pi/m, f)_p^q \right)^{1/q}, \quad s < 2r, \quad r = 1, 2, \dots, \end{aligned}$$

where  $E_m(f)_p$  is the best approximation of a function  $f \in L_p$  by spherical polynomials of degree  $\leq m$ , and  $K_r(\pi/m, f)_p$  is given in terms of the  $K$ -functional by

$$K_r(\pi/m, f)_p := K(\pi^{2r} m^{-2r}, f; L_p, \dot{W}_p^{2r}).$$

It is not difficult to show (see Lemma 3.2) that the norm  $(2) \|f\|_{p,q}^s$  is the norm of the interpolation space  $(L_p, \dot{W}_p^{2r})_{s/2r, q}$ , that is,  $B_{p,q}^{*s} = (L_p, \dot{W}_p^{2r})_{s/2r, q}$ . It was also proved in [13] that

$$B_{p,1}^{*s} \subset \dot{W}_p^s \subset B_{p,\infty}^{*s}, \quad s > 0, \quad 1 \leq p \leq \infty.$$

By the definition of the class  $\mathcal{H}(\theta, L_p, \dot{W}_p^{2r})$  given in Section 4, we can conclude that

$$\dot{W}_p^s \in \mathcal{H}(s/2r, L_p, \dot{W}_p^{2r}), \quad 1 \leq p \leq \infty, \quad s < 2r.$$

Consider now  $s, s_0, s_1 \in \mathbf{R}$  with  $0 < s_0 < s < s_1$ , and  $s = (1 - \eta)s_0 + \eta s_1$ ,  $\eta \in (0, 1)$ . Then

$$\dot{W}_p^{s_i} \in \mathcal{H}(s_i/2r, L_p, \dot{W}_p^{2r}), \quad i = 0, 1,$$

and from (34)

$$\dot{B}_{p,q}^s = (\dot{W}_p^{s_0}, \dot{W}_p^{s_1})_{\eta,q}.$$

Applying the Reiteration Theorem (see [1, p. 50]) we can conclude that

$$\dot{B}_{p,q}^s = (\dot{W}_p^{s_0}, \dot{W}_p^{s_1})_{\eta,q} = (L_p, \dot{W}_p^{2r})_{s/2r,q} = \dot{B}_{p,q}^*.$$

#### 4 Estimates of $n$ -widths of Besov classes

For  $s > d(1/p - 1/q)_+$  the operator  $\Lambda^{-s}$  is bounded from  $L_p$  to  $L_q$  and therefore,  $W_p^s = \Lambda^{-s}(L_p) \subset L_q$ . Now let  $s_0, s_1 > d(1/p - 1/q)_+$  with  $s_0 < s < s_1$ , and let  $0 < \theta < 1$  such that  $s = (1 - \theta)s_0 + \theta s_1$ . Then, by (34),  $B_{p,r}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,r} \subset W_p^{s_0} \subset L_q$ . Thus  $B_{p,r}^s \subset L_q$ , for  $s > d(1/p - 1/q)_+$ , and  $1 \leq p, q, r \leq \infty$ .

Let  $\bar{A} = (A_0, A_1)$  be an interpolation pair of Banach spaces,  $U_{A_i}$  be the unit ball of  $A_i$ ,  $i = 0, 1$ , and let  $B$  be another Banach space such that  $A_0, A_1 \subset B$ . In Triebel [21] sufficient conditions were given on Kolmogorov  $n$ -widths  $d_n(U_{A_i}, B)$ ,  $i = 0, 1$ , for obtaining estimates for the Kolmogorov  $n$ -widths  $d_n(U_{(A_0, A_1)_{\theta,r}}, B)$ . To get our results we will use sharp orders for the Kolmogorov  $n$ -widths  $d_n(W_p^s, L_q)$  found in [3, 11], the property (34), which says that a Besov space is an interpolation space of Sobolev spaces, and Triebel's theorem mentioned above.

We will denote the unit ball of a Banach space  $A$  by  $U_A$ . If  $\bar{A} = (A_0, A_1)$  is an interpolation pair and  $(A_0, A_1)_{\theta,1} \subset A \subset (A_0, A_1)_{\theta,\infty}$ , we write  $A \in \mathcal{H}(\theta, A_0, A_1)$ .

**Theorem 4.1** (see [21]) *Let  $\bar{A} = (A_0, A_1)$  be an interpolation pair of Banach spaces and let  $B$  be a Banach space such that  $A_0, A_1 \subset B$ . Suppose that there exist  $\alpha_i \geq 0$  and  $C_i > 0$ ,  $i = 0, 1$ , such that*

$$d_n(U_{A_i}, B) \leq C_i n^{-\alpha_i}, \quad n \in \mathbf{N}, \quad i = 0, 1. \quad (38)$$

Suppose also that there exists  $\tilde{\theta} \in (0, 1)$  and  $\tilde{A} \in \mathcal{H}(\tilde{\theta}, A_0, A_1)$  such that

$$d_n(U_{\tilde{A}}, B) \geq C n^{-\tilde{\alpha}}, \quad n \in \mathbf{N}, \quad (39)$$

where  $0 < \tilde{\alpha} = (1 - \tilde{\theta})\alpha_0 + \tilde{\theta}\alpha_1$ , and  $C > 0$ . If  $\theta \in [0, 1]$ ,  $A \in \mathcal{H}(\theta, A_0, A_1)$ , and  $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$ , then

$$d_n(U_A, B) \asymp n^{-\alpha}, \quad n \in \mathbf{N}. \quad (40)$$

**Theorem 4.2** (see [3, 4, 11]) *Let  $p, q, s \in \mathbf{R}$ .*

: (i) *If  $1 \leq p = q \leq \infty$  or  $2 \leq q \leq p < \infty$ , and  $s > 0$ , then*

$$d_n(\overline{W}_p^s, L_q) \asymp n^{-s/d}. \quad (41)$$

: (ii) *If  $1 \leq p \leq q \leq 2$  and  $s > d(1/p - 1/q)$ , then*

$$d_n(\overline{W}_p^s, L_q) \asymp n^{-s/d+1/p-1/q}. \quad (42)$$

: (iii) *If  $2 \leq p \leq q < \infty$  and  $s > d/2$ , then*

$$d_n(\overline{W}_p^s, L_q) \asymp n^{-s/d}. \quad (43)$$

: (iv) *If  $1 \leq p \leq 2 \leq q < \infty$  and  $s > d/p$ , then*

$$d_n(\overline{W}_p^s, L_q) \asymp n^{-s/d+1/p-1/2}. \quad (44)$$

: (v) If  $1 < q \leq p \leq 2$  and  $s > 0$ , then

$$d_n(\overline{W}_p^s, L_q) \asymp n^{-s/d}. \quad (45)$$

**Theorem 4.3** Let  $p, q, s, r \in \mathbb{R}$ .

: (i) If  $1 \leq p = q \leq \infty$  or  $2 \leq q \leq p < \infty$ ,  $s > 0$ , and  $1 \leq r \leq \infty$ , then

$$d_n(\overline{B}_{p,r}^s, L_q) \asymp \delta_n(\overline{B}_{p,r}^s, L_q) \asymp n^{-s/d}. \quad (46)$$

: (ii) If  $1 \leq p \leq q \leq 2$ ,  $s > d(1/p - 1/q)$ , and  $1 \leq r \leq \infty$ , then

$$d_n(\overline{B}_{p,r}^s, L_q) \asymp \delta_n(\overline{B}_{p,r}^s, L_q) \asymp n^{-s/d+1/p-1/q}. \quad (47)$$

: (iii) If  $2 \leq p \leq q < \infty$ ,  $s > d/2$ , and  $1 \leq r \leq \infty$ , then

$$d_n(\overline{B}_{p,r}^s, L_q) \asymp n^{-s/d}. \quad (48)$$

: (iv) If  $1 \leq p \leq 2 \leq q < \infty$ ,  $s > d/p$ , and  $1 \leq r \leq \infty$ , then

$$d_n(\overline{B}_{p,r}^s, L_q) \asymp n^{-s/d+1/p-1/2}. \quad (49)$$

: (v) If  $1 < q \leq p \leq 2$ ,  $s > 0$ , and  $1 \leq r \leq \infty$ , then

$$d_n(\overline{B}_{p,r}^s, L_q) \asymp \delta_n(\overline{B}_{p,r}^s, L_q) \asymp n^{-s/d}. \quad (50)$$

**Proof.** Consider  $1 \leq p, r \leq \infty$ ,  $s, s_0, s_1 > 0$ , and  $\theta \in (0, 1)$ , with  $s = (1 - \theta)s_0 + \theta s_1$ . From (34) we have  $B_{p,1}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,1}$ ,  $B_{p,\infty}^s = (W_p^{s_0}, W_p^{s_1})_{\theta,\infty}$ , and hence, from (33),

$$(W_p^{s_0}, W_p^{s_1})_{\theta,1} \subset W_p^s \subset (W_p^{s_0}, W_p^{s_1})_{\theta,\infty}.$$

Therefore

$$W_p^s \in \mathcal{H}(\theta, W_p^{s_0}, W_p^{s_1}). \quad (51)$$

Now from (32) we have  $B_{p,1}^s \subset B_{p,r}^s \subset B_{p,\infty}^s$ , and hence, from (34),

$$(W_p^{s_0}, W_p^{s_1})_{\theta,1} \subset B_{p,r}^s \subset (W_p^{s_0}, W_p^{s_1})_{\theta,\infty}.$$

Thus,

$$B_{p,r}^s \in \mathcal{H}(\theta, W_p^{s_0}, W_p^{s_1}). \quad (52)$$

Let  $s, p, q, r$  be as in (i) and choose  $s_0, s_1 \in \mathbb{R}$  such that  $0 < s_0 < s < s_1$ . Let  $\theta \in (0, 1)$  with  $s = (1 - \theta)s_0 + \theta s_1$ . From (41) we have

$$d_n(\overline{W}_p^{s_i}, L_q) \leq Cn^{-s_i/d}, \quad n \in \mathbb{N}, \quad i = 0, 1. \quad (53)$$

From (51) we have that  $W_p^s \in \mathcal{H}(\theta, W_p^{s_0}, W_p^{s_1})$ , and, from (41),

$$d_n(\overline{W}_p^s, L_q) \geq Cn^{-s/d}, \quad n \in \mathbb{N}. \quad (54)$$

Then (46) for Kolmogorov  $n$ -widths follows from Theorem 4.1, (53), and (54).

Now let  $s, p, q, r$  be as in (ii) and choose  $s_0$  and  $s_1$  such that  $s_1 > s > s_0 > d(1/p - 1/q)$ . Let  $\theta \in (0, 1)$  such that  $s = (1 - \theta)s_0 + \theta s_1$ . From (42) we have

$$d_n(\overline{W}_p^{s_i}, L_q) \leq Cn^{-s_i/d+1/p-1/q}, \quad n \in \mathbb{N}, \quad i = 0, 1. \quad (55)$$

From (51) we have that  $W_p^s \in \mathcal{H}(\theta, W_p^{s_0}, W_p^{s_1})$ , and, from (42),

$$d_n(\overline{W}_p^s, L_q) \geq Cn^{-s/d+1/p-1/q}, \quad n \in \mathbf{N}. \quad (56)$$

Then (47) for Kolmogorov  $n$ -widths follows from Theorem 4.1, (55), and (56).

The proofs of (48), (49), and (50) for Kolmogorov  $n$ -widths follow in the same way as those of (46) and (47) above.

Let  $s > d(1/p - 1/q)_+$ , and  $n \in \mathbf{N}$ . From (37) we have that

$$\delta_n^d(\overline{B}_{p,r}^s, L_q) \leq \|I_n\|_{B_{p,r}^s, L_q} \leq C_s n^{-s+d(1/p-1/q)_+},$$

and hence

$$\delta_n(\overline{B}_{p,r}^s, L_q) \leq C_s n^{-s/d+(1/p-1/q)_+}, \quad n \in \mathbf{N}. \quad (57)$$

The upper bounds for linear  $n$ -widths in (46), (47) and (50) follow from (57). The lower bounds follow from the inequality  $d_n(A, X) \leq \delta_n(A, X)$ .  $\square$

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## References

- [1] J. Bergh and J. Löfstrom, *Interpolation Spaces* (Springer-Verlag, 1976).
- [2] A. Bonami and J.-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183**, 223–263 (1973).
- [3] B. Bordin, A. K. Kushpel, J. Levesley and S. Tozoni,  $n$ -Widths of multiplier operators on two-point homogeneous spaces, In *Approximation Theory IX*, Vol. 1, Theoretical Aspects, C. Chui and L. L. Schumaker (eds.), Vanderbilt University Press, Nashville TN (1998), pp. 23–30.
- [4] B. Bordin, A. K. Kushpel, J. Levesley and S. Tozoni, Estimates of  $n$ -widths of Sobolev's classes on compact globally symmetric spaces of rank one, *J. Funct. Anal.* **202**, 307–326 (2003).
- [5] Z. Ciesielski and T. Figiel, Spline approximation and Besov spaces on compact manifolds, *Studia Math.* **75**, 13–36 (1982).
- [6] Z. Ditzian, Fractional derivatives and best approximation, *Acta Math. Hungar.* **81**, 323–348 (1998).
- [7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York, 1953).
- [8] R. Gangolli, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, *Ann. Inst. H. Poincaré* **3**, 121–226 (1967).
- [9] T. Koornwinder, The addition formula for Jacobi polynomials and spherical harmonics, *SIAM J. Appl. Math.* **25**, 236–246 (1973).
- [10] A. K. Kushpel, Inequalities for discrete norms with scattered centers of polynomials on  $S^d$ , In *47th Seminário Brasileiro de Análise*, São José dos Campos, SP, Brazil (1998), pp. 283–292.
- [11] A. K. Kushpel and S. A. Tozoni, Sharp orders of  $n$ -widths of Sobolev's classes on compact globally symmetric spaces of rank 1, In *54th Seminário Brasileiro de Análise*, São José do Rio Preto, SP, Brazil (2001), pp. 293–303. Relatório de Pesquisa 55/02 (2002), in [www.ime.unicamp.br/rel\\_pesq/relatorio.html](http://www.ime.unicamp.br/rel_pesq/relatorio.html).
- [12] J. E. Littlewood and R. E. A. C. Paley, Theorems on Fourier series and power series, *Proc. London Math. Soc.* **42**, 52–89 (1937).
- [13] P. I. Lizorkin and Kh. P. Rustamov, On some equivalent norms of the Nikol'skii-Besov space on the sphere, *Dokl. Akad. Nauk. SSSR* **322**, 228–232 (1992); *Soviet Math. Dokl.* **45**, 58–62 (1992).
- [14] F. D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington D. C., 1962).
- [15] S. S. Platonov, An approach to the theory of spaces of Nikol'skii-Besov type on homogeneous manifolds, *Fundam. Prikl. Mat.* **6**, 207–223 (2000).
- [16] H.-J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces* (John Wiley and Sons, 1987).
- [17] L. Skrzypczak, Some equivalent norms in Sobolev-Besov spaces on symmetric Riemannian manifolds, *J. London Math. Soc.* **53**, 569–581 (1996).
- [18] S. L. Sobolev, *Introduction to the Theory of Cubature Formulae* (Nauka, Moscow, 1974).
- [19] R. S. Strichartz, Multipliers for spherical harmonic expansions, *Trans. Amer. Math. Soc.* **167**, 115–124 (1972).
- [20] G. Szegö, *Orthogonal Polynomials* (AMS, New York, 1939).

- [21] H. Triebel, Interpolation properties of  $\epsilon$ -entropy and diameters, Geometric characteristics of imbedding for function spaces of Sobolev-Besov type, *Math. USSR Sbornik* **27**, 23–37 (1975).
- [22] H. Triebel, Characterisation of function spaces on a complete Riemannian manifold with bounded geometry, *Math. Nachr.* **130**, 321–346 (1987).
- [23] H. Triebel, *Theory of function spaces, II. Monographs in Mathematics*, 84 (Birkhäuser Verlag, 1992).
- [24] H. C. Wang, Two-point homogeneous spaces, *Ann. of Math.* **55**, 177–191 (1952).