

# EXTENDING THE RANGE OF ERROR ESTIMATES FOR RADIAL APPROXIMATION IN EUCLIDEAN SPACE AND ON SPHERES

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**Abstract.** We adapt Schaback's error doubling trick [R. Schaback. Improved error bounds for scattered data interpolation by radial basis functions. *Math. Comp.*, 68(225):201–216, 1999.] to give error estimates for radial interpolation of functions with smoothness lying (in some sense) between that of the usual native space and the subspace with double the smoothness. We do this for both bounded subsets of  $\mathbb{R}^d$  and spheres. As a step on the way to our ultimate goal we also show convergence of pseudoderivatives of the interpolation error.

**Key words.** multivariate interpolation, radial basis functions, error estimates, smooth functions

**AMS subject classifications.** 41A05, 41A15, 41A25, 41A30, 41A63

**1. Introduction.** In this paper we are interested in extending the range of applicability of error estimates for radial basis function interpolation in Euclidean space and on spheres. Let  $\Omega$  be a subset of  $\mathbb{R}^d$ , or the sphere. Let  $d(x, y)$  denote the distance between two points in  $\Omega$ . Let  $Y \subset \Omega$  be a finite set of points, and measure the *fill-distance* of  $Y$  in  $\Omega$  with

$$h(Y, \Omega) := \sup_{x \in \overline{\Omega}} \min_{y \in Y} d(x, y).$$

Given a univariate function  $\phi$  defined either on  $\mathbb{R}_+$  or  $[0, \pi]$ , depending on whether we are in Euclidean space or on the sphere, we form an approximation

$$S_\phi^Y(x) = \sum_{y \in Y} \alpha_y \phi(d(x, y)).$$

If the coefficients  $\alpha_y$ ,  $y \in Y$ , are determined by the interpolation conditions

$$S_\phi^Y(y) = f(y), \quad \text{for } y \in Y,$$

we refer to  $S_\phi^Y$  as the  $\phi$ -spline interpolant to  $f$  on  $Y$ .

We will be approximating functions  $f \in \mathcal{H}_\phi$ , a Hilbert space of functions which depends on the function  $\phi$ —the so-called *native space*. Later we will be more explicit about this space of functions. With this Hilbert space we have an inner product  $\langle \cdot, \cdot \rangle_\phi$ , with associated norm  $\|\cdot\|_\phi$ . We will require the following useful orthogonality and consequent *Pythagorean* property; see, e.g., [11, 13].

**PROPOSITION 1.1.** *Let  $S_\phi^Y$  be the  $\phi$ -spline interpolant to  $f$  on the point set  $Y \subset \Omega$ . Then, for all  $f \in \mathcal{H}_\phi$ ,*

1.  $\langle f - S_\phi^Y, S_\phi^Y \rangle_\phi = 0$ ;
2.  $\|f\|_\phi^2 = \|f - S_\phi^Y\|_\phi^2 + \|S_\phi^Y\|_\phi^2$ .

The usual error estimate for  $\phi$ -spline interpolants is of the form

$$|f(x) - S_\phi^Y(x)| \leq P(x, Y, \phi) \|f - S_\phi^Y\|_\phi,$$

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<sup>‡</sup>This author was partially supported by a studentship from the Engineering and Physical Sciences Research Council.

where estimation of  $P(x, Y, \phi)$ —the so-called *power function*—leads to error estimates for interpolation in terms of the fill-distance  $h(Y, \Omega)$ . For the archetypal function in  $\mathcal{H}_\phi$  we can say no more than  $\|f - S_\phi^Y\|_\phi \rightarrow 0$  as  $h(Y, \Omega) \rightarrow 0$ . However, if  $f$  has double the smoothness (in some sense to be made clear later) than the typical function, then Schaback [19] has shown how to double the convergence order of the  $\phi$ -spline interpolant.

We show how to get improved orders of convergence when the target function,  $f$ , has less smoothness than Schaback requires, but more smoothness than the typical function. We shall be doing this on the sphere (though this can be easily generalised to other two-point homogeneous spaces) in §2 and in Sobolev spaces on Euclidean space in §3. An intermediate result in both cases is to prove approximation orders for pseudoderivatives of the interpolant. We will define this notation at the appropriate place in each of the following sections.

In each case we shall be concerned with the practical scenario in which  $Y$  consists of a finite number of points. Foregoing this assumption is of theoretical interest. In particular, in Euclidean space for (perturbed) gridded data, certain improved error estimates are already known to hold for functions within the native space itself (see, e.g., [4]).

The goal in this paper is quite different from the desire to establish error estimates for functions possessing insufficient smoothness for admittance in the native space. In recent years, contributions in that direction has been provided by several authors, e.g., [16, 17, 12] for the sphere and [23, 3, 18] for the Euclidean case.

**2. The sphere.** Let  $\mathcal{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ . Then the geodesic distance between points  $x, y \in \mathcal{S}^d$  is  $d(x, y) = \cos^{-1} xy$ , where  $xy$  denotes the usual inner product of vectors in  $\mathbb{R}^{d+1}$ . We let  $\nu$  denote the normalised rotationally invariant measure on the sphere and define the inner product

$$\langle f, g \rangle_{L_2(\mathcal{S}^d)} := \int_{\mathcal{S}^d} f(x)g(x) d\nu(x).$$

Let  $\|\cdot\|_{L_2(\mathcal{S}^d)} := \langle \cdot, \cdot \rangle_{L_2(\mathcal{S}^d)}^{1/2}$  and let  $L_2(\mathcal{S}^d)$  denote the set of functions for which  $\|\cdot\|_{L_2(\mathcal{S}^d)} < \infty$ . Let  $P_n$  be the polynomials of degree  $n$  in  $\mathbb{R}^{d+1}$  restricted to the sphere, and let  $H_n = P_n \cap P_n^\perp$  be the space of degree  $n$  spherical harmonics. Then,  $L_2(\mathcal{S}^d)$  has the decomposition

$$L_2(\mathcal{S}^d) = \bigoplus_{n \geq 0} H_n.$$

Let  $Y_1^n, \dots, Y_{d_n}^n$  be an orthonormal basis for  $H_n$ .

Related to  $\mathcal{S}^d$  (we will see why shortly), we have the Gegenbauer polynomials  $C_n^{(\lambda)}(t)$  which are orthogonal on  $[-1, 1]$  with respect to the weight  $(1 - t^2)^{\lambda-1/2}$ . It is well known (Müller [15], for instance) that the following addition formula holds:

$$C_n^{(\lambda)}(xy) = \sum_{j=1}^{d_n} Y_j^n(x)Y_j^n(y),$$

with  $\lambda = d/2 - 1$ . The normalisation of the Gegenbauer polynomials is chosen so that there is no constant in the addition theorem. It is straightforward to see that  $C_n^{(\lambda)}$  is

the kernel of  $T_n$ , the orthogonal projector from  $L_2(\mathbb{S}^d)$  onto  $H_n$ . Thus,

$$(T_n f)(x) = \int_{\mathbb{S}^d} f(y) C_n^{(\lambda)}(xy) d\nu(y), \quad \text{for all } f \in L_2(\mathbb{S}^d).$$

The following lemma is a specialisation of a result in [11] to the sphere.

LEMMA 2.1. *For  $n \geq 0$ ,*

$$\|T_n f\|_{L_\infty(\mathbb{S}^d)} \leq \sqrt{d_n} \|T_n f\|_{L_2(\mathbb{S}^d)}, \quad \text{for all } f \in L_2(\mathbb{S}^d).$$

We will be considering interpolation using kernels of the form  $\phi(d(x, y))$  where  $\phi : [0, \pi] \rightarrow \mathbb{R}$ . We will assume that the function  $\phi$  has an expansion

$$\phi(d(x, y)) = \sum_{n \geq 0} a_n C_n^{(\lambda)}(xy),$$

where  $a_n > 0$ , for  $n = 0, 1, \dots$ , and

$$\sum_{n \geq 0} d_n a_n < \infty.$$

The first condition ensures that  $\phi$  is positive definite, and the second that it is continuous. Our analysis will take place in the native space for  $\phi$ ,  $\mathcal{H}_\phi$ , defined by

$$\mathcal{H}_\phi := \left\{ f \in L_2(\mathbb{S}^d) : \|f\|_\phi := \left( \sum_{n \geq 0} a_n^{-1} \|T_n f\|_{L_2(\mathbb{S}^d)}^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

A pseudodifferential operator  $\Lambda$  on  $\mathbb{S}^d$  is an operator which acts via multiplication by a constant on each eigenspace  $H_n$ :

$$\Lambda p_n = \lambda_n p_n, \quad p_n \in H_n, \quad n = 0, 1, \dots$$

For more information on pseudodifferential operators on spheres see, e.g., [9, 20]. We call the sequence of numbers  $\{\lambda_n\}_{n \geq 0}$  the *symbol* of  $\Lambda$ . Let  $\delta_x$  denote the point evaluation functional at  $x$ , and, when it makes sense for the functional  $\mu$ , let  $\Lambda \mu(f) = \mu(\Lambda f)$ . Let us denote by  $Y^*$  the span of the point evaluation functionals supported on  $Y$ . In Morton and Neamtu [14] the authors give error estimates for the collocation solution of pseudodifferential equations on spheres. Here we attempt, initially, to find errors in pseudoderivatives of solutions to the interpolation problem.

PROPOSITION 2.2. *Let  $S_\phi^Y$  be the  $\phi$ -spline interpolant to  $f \in \mathcal{H}_\phi$  on the point set  $Y \subset \mathbb{S}^d$ . Let  $\Lambda$  be a pseudodifferential operator. Then, for each  $x \in \mathbb{S}^d$ ,*

$$|\Lambda(f - S_\phi^Y)(x)| \leq \inf_{\mu \in Y^*} \sup_{\substack{v \in \mathcal{H}_\phi \\ \|v\|_\phi = 1}} |\Lambda v(x) - \mu(v)| \|f - S_\phi^Y\|_\phi.$$

*Proof.* Since  $f(y) - S_\phi^Y(y) = 0$ ,  $y \in Y$ , we have, for any coefficients  $c_y$ ,  $y \in Y$ ,

$$\begin{aligned} |\Lambda(f - S_\phi^Y)(x)| &= \left| \Lambda(f - S_\phi^Y)(x) - \sum_{y \in Y} c_y (f(y) - S_\phi^Y(y)) \right| \\ &= \left| \Lambda(f - S_\phi^Y)(x) - \sum_{y \in Y} c_y (f(y) - S_\phi^Y(y)) \right| \frac{\|f - S_\phi^Y\|_\phi}{\|f - S_\phi^Y\|_\phi} \\ &\leq \sup_{\substack{v \in \mathcal{H}_\phi \\ \|v\|_\phi = 1}} \left| \Lambda v(x) - \sum_{y \in Y} c_y v(y) \right| \|f - S_\phi^Y\|_\phi. \end{aligned}$$

We now take the infimum over all functionals in  $Y^*$  to obtain the result.  $\square$

In what follows we will need the pseudodifferential operator  $\Lambda$  to satisfy the following assumption:

ASSUMPTION 2.3. *For all  $n \geq 0$ ,  $\lambda_n = (n(d+n-2))^s$ , for some  $s > 0$ . From Ditzian [6], if  $\Lambda$  satisfies Assumption 2.3, then for  $p \in P_n$ ,*

$$\|\Lambda p\|_{L_\infty(\mathbb{S}^d)} \leq E\lambda_n \|p\|_{L_\infty(\mathbb{S}^d)},$$

for some  $E$  independent of  $n$ .

From [10, Lemma 7] we have the following result.

LEMMA 2.4. *Let  $Y$  be a finite set of points with fill-distance  $h(Y, \mathbb{S}^d) \leq 1/(2N)$ , for some fixed  $N \in \mathbb{Z}_+$ . Then, for any linear functional  $\gamma$  on  $P_N$  with*

$$\sup_{\substack{p \in P_N \\ \|p\|_{L_\infty(\mathbb{S}^d)}=1}} |\gamma p| \leq 1,$$

there is a set of real numbers  $\{b_y\}_{y \in Y}$ , with  $\sum_{y \in Y} |b_y| \leq 2$ , such that

$$\gamma p = \sum_{y \in Y} b_y p(y), \quad \text{for all } p \in P_N.$$

Now, for a fixed  $x \in \mathbb{S}^d$ , let

$$\gamma p = \frac{\Lambda p(x)}{E\lambda_N}, \quad \text{for all } p \in P_N.$$

Then,

$$\sup_{0 \neq p \in P_N} \frac{|\gamma p|}{\|p\|_{L_\infty(\mathbb{S}^d)}} \leq 1,$$

so that, by the previous lemma, there is a set of coefficients  $\{b_y\}_{y \in Y}$ , such that

$$\gamma p = \sum_{y \in Y} b_y p(y),$$

with  $\sum_{y \in Y} |b_y| \leq 2$ . Thus, with  $c_y = E\lambda_N b_y$ , for  $y \in Y$ , we have

$$\Lambda p(x) = \sum_{y \in Y} c_y p(y), \quad \text{for all } p \in P_N, \quad (2.1)$$

where,

$$\sum_{y \in Y} |c_y| \leq 2E\lambda_N. \quad (2.2)$$

We now arrive at the first main result of this section.

THEOREM 2.5. *Let  $S_\phi^Y$  be the  $\phi$ -spline interpolant to  $f \in \mathcal{H}_\phi$ , on the point set  $Y \subset \mathbb{S}^d$ , where  $h(Y, \mathbb{S}^d) \leq 1/(2N)$ , for some fixed  $N \in \mathbb{Z}_+$ . Let  $\Lambda$  be a pseudodifferential operator with symbol  $\{\lambda_n\}_{n \geq 0}$  satisfying Assumption 2.3 and*

$$\sum_{n \geq 0} d_n \lambda_n^2 a_n < \infty.$$

Then, for  $x \in \mathbb{S}^d$ ,

$$|\Lambda(f - S_\phi^Y)(x)| \leq (1 + 2E) \left( \sum_{n \geq N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} \|f - S_\phi^Y\|_\phi.$$

*Proof.* Let us choose  $\{c_y\}_{y \in Y}$  to be the coefficients described above. Let  $v \in \mathcal{H}_\phi$  with  $\|v\|_\phi = 1$ . Then,

$$\begin{aligned} \inf_{\mu \in Y^*} |\Lambda v(x) - \mu(v)| &\leq \left| \sum_{n \geq 0} \left( \Lambda T_n v(x) - \sum_{y \in Y} c_y T_n v(y) \right) \right| \\ &= \left| \sum_{n > N} \left( \lambda_n T_n v(x) - \sum_{y \in Y} c_y T_n v(y) \right) \right|, \end{aligned}$$

by (2.1). Thus,

$$\begin{aligned} \inf_{\mu \in Y^*} |\Lambda v(x) - \mu(v)| &\leq \left| \sum_{n > N} \lambda_n T_n v(x) \right| + \left| \sum_{n > N} \sum_{y \in Y} c_y T_n v(y) \right| \\ &\leq \sum_{n > N} \left( \lambda_n + \sum_{y \in Y} |c_y| \right) \|T_n v\|_{L_\infty(\mathbb{S}^d)} \\ &\leq \sum_{n > N} \left( \lambda_n + \sum_{y \in Y} |c_y| \right) \sqrt{d_n} \|T_n v\|_{L_2(\mathbb{S}^d)}, \end{aligned}$$

using Lemma 2.1. Hence, using (2.2) and the Cauchy–Schwarz inequality,

$$\inf_{\mu \in Y^*} |\Lambda v(x) - \mu(v)| \leq \left[ \left( \sum_{n > N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} + 2E \lambda_N \left( \sum_{n > N} d_n a_n \right)^{\frac{1}{2}} \right] \|v\|_\phi,$$

and the result follows from Proposition 2.2 since  $\|v\|_\phi = 1$ , and because  $\{\lambda_n\}_{n \geq 0}$  is an increasing sequence.  $\square$

Integrating the conclusion of the previous theorem over the sphere we easily obtain  
COROLLARY 2.6. *Under the hypotheses of Theorem 2.5,*

$$\|\Lambda(f - S_\phi^Y)\|_{L_2(\mathbb{S}^d)} \leq (1 + 2E) \left( \sum_{n > N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} \|f - S_\phi^Y\|_\phi.$$

Before we give our improved error estimate we need to define a new space  $\mathcal{H}_{\Lambda\phi}$  by

$$\mathcal{H}_{\Lambda\phi} := \left\{ f \in \mathcal{H}_\phi : \|f\|_{\Lambda\phi} := \left( \sum_{n \geq 0} (\lambda_n a_n)^{-2} \|T_n f\|_{L_2(\mathbb{S}^d)}^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

THEOREM 2.7. *Let  $S_\phi^Y$  be the  $\phi$ -spline interpolant to  $f \in \mathcal{H}_\phi$  on the point set  $Y \subset \mathbb{S}^d$ , where  $h(Y, \mathbb{S}^d) \leq 1/(2N)$ , for some fixed  $N \in \mathbb{Z}_+$ . Let  $\Lambda$  be a pseudodifferential operator with symbol  $\{\lambda_n\}_{n \geq 0}$  satisfying Assumption 2.3 and*

$$\sum_{n \geq 0} d_n \lambda_n^2 a_n < \infty.$$

Then, for  $f \in \mathcal{H}_{\Lambda\phi}$  and for all  $x \in \mathbb{S}^d$ ,

$$|f(x) - S_\phi^Y(x)| \leq (1 + 2E) \left( \sum_{n>N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} \left( \sum_{n>N} d_n a_n \right)^{\frac{1}{2}} \|f\|_{\Lambda\phi}.$$

*Proof.* Firstly, using Proposition 1.1 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|f - S_\phi^Y\|_\phi^2 &= \langle f - S_\phi^Y, f \rangle_\phi \\ &= \sum_{n \geq 0} a_n^{-1} \langle T_n(f - S_\phi^Y), T_n f \rangle_{L_2(\mathbb{S}^d)} \\ &\leq \left( \sum_{n \geq 0} \lambda_n^2 \|T_n(f - S_\phi^Y)\|_{L_2(\mathbb{S}^d)}^2 \right)^{\frac{1}{2}} \left( \sum_{n \geq 0} (\lambda_n a_n)^{-2} \|T_n f\|_{L_2(\mathbb{S}^d)}^2 \right)^{\frac{1}{2}} \\ &= \|\Lambda(f - S_\phi^Y)\|_{L_2(\mathbb{S}^d)} \|f\|_{\Lambda\phi} \\ &\leq (1 + 2E) \left( \sum_{n>N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} \|f - S_\phi^Y\|_\phi \|f\|_{\Lambda\phi}, \end{aligned}$$

using Corollary 2.6. Cancelling a factor of  $\|f - S_\phi^Y\|_\phi$  from both sides yields

$$\|f - S_\phi^Y\|_\phi \leq (1 + 2E) \left( \sum_{n>N} d_n \lambda_n^2 a_n \right)^{\frac{1}{2}} \|f\|_{\Lambda\phi}.$$

We can now employ the standard error estimate taken from Jetter, Stöckler and Ward [10] (our Theorem 2.5 with  $\lambda_n = 1$  for all  $n$ ) to give the required result.  $\square$

**3. The Euclidean case.** Our attention now turns to  $\phi$ -spline interpolants of the form

$$S_\phi^Y(x) = \sum_{y \in Y} \alpha_y \phi(|x - y|),$$

where  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ . We will conduct our analysis for positive definite basis functions  $\phi \in L_1(\mathbb{R}^d)$  whose Fourier transform satisfy, for some  $s > 0$ ,

$$C_1(1 + |x|)^{-2s} \leq \widehat{\phi}(x) \leq C_2(1 + |x|)^{-2s}, \quad (3.1)$$

for some positive constants  $C_1$  and  $C_2$ , for example, the Sobolev splines [7] or piecewise polynomial compactly supported radial functions of minimal degree [21]. The exposition contained in this section can be readily adapted to include the polyharmonic splines [8] as well. In that case, the  $\phi$ -spline interpolant must be augmented by a polynomial  $p$  with the extra degrees of freedom taken up by the side conditions

$$\sum_{y \in Y} \alpha_y q(y) = 0,$$

where  $q$  is polynomial of the same degree (or less) as  $p$ .

For a domain  $\Omega \subset \mathbb{R}^d$  let  $L_2(\Omega)$  denote the usual space of square-integrable functions on  $\Omega$  with inner product  $\langle \cdot, \cdot \rangle_{L_2(\Omega)}$  and norm  $\|\cdot\|_{L_2(\Omega)}$ . For  $k \in \mathbb{Z}_+$ , the integer-order Sobolev space is defined by

$$\mathcal{H}_k := \left\{ f \in L_2(\mathbb{R}^d) : D^\alpha f \in L_2(\mathbb{R}^d) \text{ for all } |\alpha| \leq k \right\},$$

with  $D^\alpha$  understood in the distributional sense, which carries the inner product

$$\langle f, g \rangle_k := \langle f, g \rangle_{L_2(\mathbb{R}^d)} + (f, g)_k,$$

where  $(f, g)_k$  denotes the Sobolev semi-inner product

$$(f, g)_k := \sum_{|\alpha|=k} c_\alpha^{(k)} \int_{\mathbb{R}^d} (D^\alpha f)(x) (\overline{D^\alpha g})(x) dx,$$

with associated semi-norm  $|\cdot|_k := (\cdot, \cdot)_k^{1/2}$ . The coefficients  $c_\alpha^{(k)}$  have been chosen so that

$$\sum_{|\alpha|=k} c_\alpha^{(k)} x^{2\alpha} = |x|^{2k}.$$

We can write the semi-norm, using the Fourier transform, in the alternative form

$$|f|_k^2 = \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{2k} dx,$$

which facilitates the definition of fractional-order Sobolev space,  $\mathcal{H}_s$ , for  $s > 0$ , which has the semi-norm

$$|f|_s^2 := \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{2s} dx. \quad (3.2)$$

The space  $\mathcal{H}_s$  is complete with respect to

$$\|f\|_s := \begin{cases} \left( \|f\|_{L_2(\mathbb{R}^d)}^2 + |f|_s^2 \right)^{\frac{1}{2}} & \text{if } s \in \mathbb{Z}_+, \\ \left( \|f\|_{[s]}^2 + |f|_s^2 \right)^{\frac{1}{2}} & \text{otherwise,} \end{cases}$$

and, whenever we have  $s > d/2$ ,  $\mathcal{H}_s$  is continuously embedded in the continuous functions. The native space for  $\phi$  satisfying (3.1) is equivalent to  $\mathcal{H}_s$ .

We now wish to make local definitions of our function spaces, which we shall denote by  $\mathcal{H}_s(\Omega)$ . For  $s \in \mathbb{Z}_+$  the definition should be clear. In what follows we also need the local fractional-order Sobolev spaces:

$$\mathcal{H}_s(\Omega) := \left\{ f \in \mathcal{H}_{[s]}(\Omega) : \|f\|_{s,\Omega} := \left( \|f\|_{[s],\Omega}^2 + |f|_{s,\Omega}^2 \right)^{\frac{1}{2}} < \infty \right\},$$

where  $|f|_{s,\Omega}$  is the local fractional-order Sobolev semi-norm obtained by rewriting (3.2) in an equivalent direct form, i.e., not defined through the Fourier transform of  $f$  (see, e.g., Adams [1, p. 214]). For our analysis we find it more useful to exploit an equivalent wavelet representation for the local Sobolev norm [5].

To introduce this equivalent norm we stipulate that the bounded domain,  $\Omega$ , admits a local multiresolution of closed subspaces  $\{V_n(\Omega)\}_{n \geq 0}$  of  $L_2(\Omega)$ :

$$V_0(\Omega) \subset V_1(\Omega) \subset \cdots \subset L_2(\Omega), \quad \overline{\bigcup_{n \geq 0} V_n(\Omega)} = L_2(\Omega).$$

Cohen et al. [5] give sufficient conditions on  $\Omega$  to admit such a local multiresolution. In particular, for  $d = 2$ , those domains whose boundaries have certain piecewise Lipschitz smoothness are admissible. The following is an incidence of [5, Theorem 4.2].

**THEOREM 3.1.** *Suppose  $\Omega \subset \mathbb{R}^d$  is a bounded domain that admits a local multi-resolution  $\{V_n(\Omega)\}_{n \geq 0}$  for  $L_2(\Omega)$ . For  $n \geq 0$ , let  $Q_n^\Omega$  denote the orthogonal projection from  $L_2(\Omega)$  onto  $W_n(\Omega) = V_n(\Omega) \ominus V_{n-1}(\Omega)$  with the convention that  $V_{-1}(\Omega) = \{0\}$ . For each  $s \geq 0$ , let  $\Lambda_s$  be the pseudodifferential operator on  $\Omega$  defined via*

$$\Lambda_s := \sum_{n \geq 0} 2^{ns} Q_n^\Omega.$$

*Then, there exists positive constants  $C_1$  and  $C_2$  such that, for all  $f \in \mathcal{H}_s(\Omega)$ ,*

$$C_1 \|\Lambda_s f\|_{L_2(\Omega)} \leq \|f\|_{s,\Omega} \leq C_2 \|\Lambda_s f\|_{L_2(\Omega)}.$$

Now, let us return to the task at hand. For  $\phi$  satisfying (3.1), we will denote the  $\phi$ -spline interpolant on the point set  $Y$  by  $S_\phi^Y$ . The standard error estimate in this context is

$$|f(x) - S_\phi^Y(x)| \leq Ch^{s-d/2} \|f - S_\phi^Y\|_s, \quad (3.3)$$

see [22]. If  $f$  is smoother (and satisfies some boundary conditions) we can get a better rate of convergence.

We will exploit the fact that  $\mathcal{H}_\mu(\Omega)$ , for  $0 < \mu < s$ , is an interpolation space lying between  $L_2(\Omega)$  and  $\mathcal{H}_s(\Omega)$  (see Bergh and Löfström [2, p. 131]). We can then use the standard interpolation theorem concerning the norms of operators bounded on the extreme spaces to infer a bound on the norm for the interpolation space. For further information on interpolation spaces the reader can consult, e.g., Bergh and Löfström [2]. We use the following interpolation theorem.

**PROPOSITION 3.2.** *Let  $0 < \mu < s$ . Further suppose that  $T : \mathcal{H}_s(\Omega) \rightarrow L_2(\Omega)$ , and  $T : \mathcal{H}_s(\Omega) \rightarrow \mathcal{H}_s(\Omega)$  is a bounded operator. Then,*

$$\|T\|_{\mathcal{H}_s(\Omega) \rightarrow \mathcal{H}_\mu(\Omega)} \leq \|T\|_{\mathcal{H}_s(\Omega) \rightarrow L_2(\Omega)}^{1-\mu/s} \|T\|_{\mathcal{H}_s(\Omega) \rightarrow \mathcal{H}_s(\Omega)}^{\mu/s}.$$

Since we can write the  $\mathcal{H}_s$ -norm in entirely direct form, we are at liberty to utilise Duchon's localisation technique [8] to enhance the standard error estimate (3.3). Therefore, if  $\Omega$  is bounded and satisfies an interior cone condition then, for  $f \in \mathcal{H}_s(\Omega)$ ,  $s > d/2$ , and sufficiently small  $h = h(Y, \Omega)$ ,

$$\|f - S_\phi^Y\|_{L_2(\Omega)} \leq Ch^s \|f - S_\phi^Y\|_{s,\Omega}.$$

Writing  $Tf = f - S_\phi^Y$ , we see, using the last proposition, that, for  $0 < \mu < s$ ,

$$\begin{aligned} \|f - S_\phi^Y\|_{\mu,\Omega} &\leq (Ch^s \|f - S_\phi^Y\|_{s,\Omega})^{1-\mu/s} \|f - S_\phi^Y\|_{s,\Omega}^{\mu/s} \\ &= Ch^{s-\mu} \|f - S_\phi^Y\|_{s,\Omega}. \end{aligned} \quad (3.4)$$

We can now prove our main result of this section, which is a generalisation of that of Schaback [19].

**THEOREM 3.3.** *Suppose  $\Omega \subset \mathbb{R}^d$  is bounded, satisfies an interior cone condition and admits a local multiresolution. Let  $s > d/2$  and let  $S_\phi^Y$  be the  $\phi$ -spline interpolant to  $f \in \mathcal{H}_s$  on the point set  $Y \subset \Omega$ . Suppose further that  $f \in \mathcal{H}_\nu$ , for  $s < \nu \leq 2s$ , and*

that  $f$  is compactly supported in  $\Omega$ . Then there exists  $C > 0$ , independent of  $f$  and  $h = h(Y, \Omega)$ , such that for all  $x \in \Omega$  and sufficiently small  $h$ ,

$$|f(x) - S_\phi^Y(x)| \leq Ch^{\nu-d/2} \|f\|_{\nu, \Omega}.$$

*Proof.* From Proposition 1.1 we know that

$$\langle f - S_\phi^Y, S_\phi^Y \rangle_s = 0,$$

so that

$$\|f - S_\phi^Y\|_s^2 = \langle f - S_\phi^Y, f \rangle_s \leq C \langle f - S_\phi^Y, f \rangle_{s, \Omega},$$

where we have used the compact support of  $f$  in  $\Omega$ . Now, the equivalent norm from Theorem 3.1 gives us

$$\begin{aligned} \|f - S_\phi^Y\|_s^2 &\leq C \langle \Lambda_s(f - S_\phi^Y), \Lambda_s f \rangle_{L_2(\Omega)} \\ &= C \sum_{n \geq 0} 4^{ns} \langle Q_n^\Omega(f - S_\phi^Y), Q_n^\Omega f \rangle_{L_2(\Omega)}, \end{aligned}$$

and successive applications of the continuous and discrete Cauchy–Schwarz inequality yields

$$\begin{aligned} \|f - S_\phi^Y\|_s^2 &\leq C \sum_{n \geq 0} \|2^{n(2s-\nu)} Q_n^\Omega(f - S_\phi^Y)\|_{L_2(\Omega)} \|2^{n\nu} Q_n^\Omega f\|_{L_2(\Omega)} \\ &\leq C \left( \sum_{n \geq 0} \|2^{n(2s-\nu)} Q_n^\Omega(f - S_\phi^Y)\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \left( \sum_{n \geq 0} \|2^{n\nu} Q_n^\Omega f\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}} \\ &= C \|\Lambda_{2s-\nu}(f - S_\phi^Y)\|_{L_2(\Omega)} \|\Lambda_\nu f\|_{L_2(\Omega)}. \end{aligned}$$

Thus, using the norm equivalence from Theorem 3.1 again together with (3.4), we have

$$\begin{aligned} \|f - S_\phi^Y\|_s^2 &\leq C \|f - S_\phi^Y\|_{2s-\nu, \Omega} \|f\|_{\nu, \Omega} \\ &\leq Ch^{\nu-s} \|f - S_\phi^Y\|_{s, \Omega} \|f\|_{\nu, \Omega} \\ &\leq Ch^{\nu-s} \|f - S_\phi^Y\|_s \|f\|_{\nu, \Omega}, \end{aligned}$$

and cancelling one power of  $\|f - S_\phi^Y\|_s$  gives

$$\|f - S_\phi^Y\|_s \leq Ch^{\nu-s} \|f\|_{\nu, \Omega}.$$

The result follows by substitution into the standard error estimate (3.3).  $\square$

**Acknowledgements.** We are grateful to Professor Mikhail Shubin for useful input into questions of pseudodifferential operators and alternative Sobolev space semi-norms.

#### REFERENCES

- [1] R. A. ADAMS, *Sobolev spaces*, Academic Press, New York, 1975. Pure and Applied Mathematics, Vol. 65.

- [2] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [3] R. A. BROWNLEE AND W. LIGHT, *Approximation orders for interpolation by surface splines to rough functions*, IMA J. Numer. Anal., 24 (2004), pp. 179–192.
- [4] M. D. BUHMANN, *Radial basis functions: theory and implementations*, vol. 12 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 2003.
- [5] A. COHEN, W. DAHMEN, AND R. DEVORE, *Multiscale decompositions on bounded domains*, Trans. Amer. Math. Soc., 352 (2000), pp. 3651–3685.
- [6] Z. DITZIAN, *Fractional derivatives and best approximation*, Acta Math. Hungar., 81 (1998), pp. 323–348.
- [7] J. G. DIX AND R. D. OGDEN, *An interpolation scheme with radial basis in Sobolev spaces  $H^s(\mathbf{R}^n)$* , Rocky Mountain J. Math., 24 (1994), pp. 1319–1337.
- [8] J. DUCHON, *Sur l'erreur d'interpolation des fonctions de plusieurs variables par les  $D^m$ -splines*, RAIRO Anal. Numér., 12 (1978), pp. 325–334, vi.
- [9] W. FREEDEN, T. GERVEN, AND M. SCHREINER, *Constructive approximation on the sphere*, Numerical Mathematics and Scientific Computation with Applications to Geomathematics, Oxford Univ. Press, New York, NY, 1998.
- [10] K. JETTER, J. STÖCKLER, AND J. D. WARD, *Error estimates for scattered data interpolation on spheres*, Math. Comp., 68 (1999), pp. 733–747.
- [11] J. LEVESLEY, C. ODELL, AND D. L. RAGOZIN, *Scattered data on homogeneous manifolds: the norming set approach*, in Curve and surface fitting (Saint-Malo, 2002), J.-L. Merrien A. Cohen and L. L. Schumaker, eds., Mod. Methods Math., Nashboro Press, Brentwood, TN, 2003, pp. 269–278.
- [12] J. LEVESLEY AND X. SUN, *Approximation in rough native spaces by shifts of smooth kernels on spheres*, J. Approx. Theory, 133 (2005), pp. 269–283.
- [13] W. A. LIGHT AND H. WAYNE, *On power functions and error estimates for radial basis function interpolation*, J. Approx. Theory, 92 (1998), pp. 245–266.
- [14] T. M. MORTON AND M. NEAMTU, *Error bounds for solving pseudodifferential equations on spheres by collocation with zonal kernels*, J. Approx. Theory, 114 (2002), pp. 242–268.
- [15] C. MÜLLER, *Spherical harmonics*, vol. 17 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1966.
- [16] F. J. NARCOWICH, R. SCHABACK, AND J. D. WARD, *Approximations in Sobolev spaces by kernel expansions*, J. Approx. Theory, 114 (2002), pp. 70–83.
- [17] F. J. NARCOWICH AND J. D. WARD, *Scattered data interpolation on spheres: error estimates and locally supported basis functions*, SIAM J. Math. Anal., 33 (2002), pp. 1393–1410 (electronic).
- [18] ———, *Scattered-data interpolation on  $\mathbb{R}^n$ : error estimates for radial basis and band-limited functions*, SIAM J. Math. Anal., 36 (2004), pp. 284–300 (electronic).
- [19] R. SCHABACK, *Improved error bounds for scattered data interpolation by radial basis functions*, Math. Comp., 68 (1999), pp. 201–216.
- [20] S. L. SVENSSON, *Pseudodifferential operators - a new approach to the boundary problems of physical geodesy*, Manuscripta Geod., 8 (1983), pp. 1–40.
- [21] H. WENDLAND, *Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree*, Adv. Comput. Math., 4 (1995), pp. 389–396.
- [22] Z. M. WU AND R. SCHABACK, *Local error estimates for radial basis function interpolation of scattered data*, IMA J. Numer. Anal., 13 (1993), pp. 13–27.
- [23] J. YOON, *Interpolation by radial basis functions on Sobolev space*, J. Approx. Theory, 112 (2001), pp. 1–15.