Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 199 (2007) 159 – 171
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Perturbed kernel approximation on homogeneous manifolds

J. Levesley[a], X. Sun[b,*,1]

[a]Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK
[b]Department of Mathematics, Missouri State University, Springfield, MO 65804, USA

Received 30 November 2004; received in revised form 11 April 2005

Dedicated to the memory of Will Light

Abstract

Current methods for interpolation and approximation within a native space rely heavily on the strict positive-definiteness of the underlying kernels. If the domains of approximation are the unit spheres in euclidean spaces, then zonal kernels (kernels that are invariant under the orthogonal group action) are strongly favored. In the implementation of these methods to handle real world problems, however, some or all of the symmetries and positive-definiteness may be lost in digitalization due to small random errors that occur unpredictably during various stages of the execution. Perturbation analysis is therefore needed to address the stability problem encountered. In this paper we study two kinds of perturbations of positive-definite kernels: small random perturbations and perturbations by Dunkl's intertwining operators [C. Dunkl, Y. Xu, Orthogonal polynomials of several variables, Encyclopedia of Mathematics and Its Applications, vol. 81, Cambridge University Press, Cambridge, 2001]. We show that with some reasonable assumptions, a small random perturbation of a strictly positive-definite kernel can still provide vehicles for interpolation and enjoy the same error estimates. We examine the actions of the Dunkl intertwining operators on zonal (strictly) positive-definite kernels on spheres. We show that the resulted kernels are (strictly) positive-definite on spheres of lower dimensions.

Keywords: Native space; Strict positive-definiteness; Perturbation; Dunkl's intertwining operator; Zonal kernels; Spherical harmonics

1. Overview

There has been a significant amount of recent interest in approximation on spheres and more exotic manifolds. Exploration of the existence and convergence properties of interpolants generated by zonal kernels on the sphere by Wahba [26], Freeden [5], Xu and Cheney [27] provided much of the impetus for this work. The papers of Dyn et al. [4] and Schaback [23] provided a framework for the analysis of positive-definite kernel interpolation on general manifolds, and in a series of papers Levesley and Ragozin [10,11,20] exploited the group structure underlying homogeneous spaces, and studied the similar approximation problems on homogeneous manifolds. In all the work to date the authors have assumed that the kernels involved are strictly positive-definite, or conditionally strictly positive-definite. The exact meaning of the strict-positive-definiteness will be given in the next section. On the unit sphere S^{d-1} of the

* Corresponding author.
E-mail address: XSun@MissouriState.edu (X. Sun).
[1] Partially supported by the Engineering and Physical Science Research Council (EPSRC) of Great Britain, and an SMSU Faculty Summer Research Fellowship.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.04.073

d-dimensional space $\mathbb{R}^d$, zonal kernels are strongly favored because of their simple structure. (A zonal kernel on S^{d-1} is of the form $\phi(xy)$, where ϕ is a univariate function, and xy is the usual inner product of x and y.) In the implementation of these methods to handle real world problems, however, some or all of the symmetries and positive-definiteness may be lost in digitalization due to small random errors that occur unpredictably during various stages of the execution. Many problems occur in domains that are not exact spheres, but rather some kind of perturbation of the spheres, like the surface of the earth. These problems have thus motivated us to consider approximating with a kernel that is not positive-definite. The existing theory in the area does not have available answers to these questions. Perturbation analysis is therefore needed to address these issues. We consider here first the situation where the kernel employed for interpolation arises from a compact perturbation of an integral operator. We show that under some reasonable assumptions, a small random perturbation of a strictly positive-definite kernel can still provide vehicles for interpolation and enjoy the same orders of error estimates.

In some other situations, it is preferable to use kernels that are only invariant under a certain finite reflection group as opposed to zonal kernels that are invariant under the entire orthogonal group. For example, in the simulation of the movement of snow flakes, using kernels having only hexagonal symmetry yields much better effect. To obtain strictly positive-definite kernels that have only finite reflection group symmetry, we apply the Dunkl [3] intertwining operator to positive-definite zonal kernels on spheres. We refer to this scheme as "perturbation by the intertwining operator". We demonstrate that the new kernels (after the action of the intertwining operator) have the desired symmetry, and we show that they are strictly positive-definite on spheres of lower dimensions. Interestingly, the move leads to a new proof of Dunkl's conjecture that the intertwining operator is a positive operator. The first proof of Dunkl's conjecture was given by Rösler [22]. In the present paper, however, we choose not to elaborate further on the positivity issue, instead we devote our attention to perturbation theory.

The paper is organized as follows. In Section 2, we introduce notations and necessary background material for the analysis in subsequent sections. In Section 3, we study the effect of small random perturbations on strict positive-definite kernels using integral operator theory. In Section 4, we study perturbation by Dunkl's intertwining operators. Sections 3 and 4 can be read independently of each other.

2. Introduction

Let M be a d-dimensional homogeneous space of a compact Lie group G. Then (see [15]) we may assume that $G \subset \mathrm{O}(d+r)$, the orthogonal group on $\mathbb{R}^{d+r}$ for some integer $r \geqslant 0$. Thus $M = \{gp : g \in G\}$ where $p \in M$ is a non-zero vector. If $H = \{g \in G : gp = p\}$, then $M \cong G/H$. For technical reasons we will restrict ourselves to considering the *reflexive spaces*. That is, for any given pair $x, y \in M$ there exists $g \in G$ such that $gx = y$ and $gy = x$.

Let $d(x, y)$ be the geodesic distance between $x, y, \in M$ induced by the embedding of M in $\mathbb{R}^{d+r}$ (see e.g. [19] for details). On the spheres this corresponds to the usual geodesic distance. A real valued function $\mathscr{K}(x, y)$ defined on $M \times M$ is called a positive-definite kernel on M if for every nonempty finite subset $Y \subset M$, and arbitrary real numbers c_y, $y \in Y$, we have

$$\sum_{x \in Y} \sum_{y \in Y} c_x c_y \mathscr{K}(x, y) \geqslant 0.$$

Shall the above inequality become strict whenever the points y are distinct, and not all the c_y are zero, then the kernel $\mathscr{K}$ is called strictly positive-definite. A kernel $\mathscr{K}$ is called G-invariant if $\mathscr{K}(gx, gy) = \mathscr{K}(x, y)$ for all $x, y \in M$ and $g \in G$. For example, if $M := S^{d-1}$, and $G := \mathrm{O}(d)$, then all the G-invariant kernels have the form $\phi(xy)$, where $\phi : [-1, 1] \to \mathbb{R}$, and xy denotes the usual inner product of x and y. A kernel of the form $\phi(xy)$ is often called a zonal kernel on the sphere in the literature. If a zonal kernel $\phi(xy)$ is positive-definite, then the univariate function ϕ is called a positive-definite function on S^{d-1}. Schoenberg [24] proved the following remarkable result characterizing all the positive-definite functions on S^{d-1}.

Theorem 1. *In order that ϕ be positive-definite on S^{d-1}, it is necessary and sufficient that ϕ have the Gegenbauer polynomial* (*see* [16,25]) *expansion*

$$\phi(t) = \sum_{n=0}^{\infty} a_n C_n^{(\lambda)}(t), \quad t \in [-1, 1]$$

where $\lambda=(d-3)/2$, $a_n \geqslant 0$ *for all* $n=0,1,2,\ldots,$ *and* $\sum_{n=0}^{\infty} a_n C_n^{(\lambda)}(1)<\infty$. *Here the Gegenbauer polynomials* $C_n^{(\lambda)}$ *are normalized so that*

$$\int_{-1}^{1}(C_n^{(\lambda)}(t))^2(1-t^2)^{(d-3)/2}\,\mathrm{d}t=1.$$

Xu and Cheney [27] showed that the kernel $\phi(xy)$ is strictly positive-definite on S^{d-1} if all the coefficients in the Gegenbauer polynomial expansion of ϕ are positive. More information on strictly positive-definite functions on spheres can be found in [2,21] and the references therein. Using strictly positive-definite zonal kernels, various approximation schemes have been established, and in some cases, optimal error estimates are obtained; see, for example, [4,6,8,12,14,17,18,23]. Omission of any other important works in the field is a matter of ignorance and negligence of the authors of the present paper, and it is, however, by no means intentional. On a general d-dimensional homogeneous space M of a compact Lie group G, the expansion of a G-invariant kernel is more delicate.

Let μ be a G-invariant measure on M (which may be taken as an appropriately normalized "surface" measure). Then, for $f, g \in L_2$, we define the following inner product with respect to μ:

$$[f,g]=\int_M fg\,\mathrm{d}\mu.$$

Let P_n be the space of polynomials in $d+r$ variables of degree n restricted on M. Then the *harmonic polynomials* of degree n on M are $H_n := P_n \cap P_{n-1}^{\perp}$. We can (uniquely) decompose H_n into irreducible G-invariant subspaces $H_{n,k}$, $k=1,\ldots,v_n$. Let $Y_{n,k}^1,\ldots,Y_{n,k}^{d_{n,k}}$ be any orthonormal basis for $H_{n,k}$, and set

$$Q_{n,k}(x,y):=\sum_{j=1}^{d_{n,k}} Y_{n,k}^j(x)Y_{n,k}^j(y).$$

Then $Q_{n,k}$ is the unique G-invariant kernel for the orthogonal projection $T_{n,k}$ onto $H_{n,k}$:

$$T_{n,k}f(x)=\int_M Q_{n,k}(x,y)f(y)\,\mathrm{d}\mu(y).$$

Any G-invariant kernel $\mathscr{K}$ has an associated integral operator

$$T_{\mathscr{K}}f(x)=\int_M \mathscr{K}(x,y)f(y)\,\mathrm{d}\mu(y).$$

On a reflexive space, all such operators commute, which allows us to show that any G-invariant kernel $\mathscr{K}$ has the spectral decomposition

$$\mathscr{K}(x,y)\sim\sum_{n=0}^{\infty}\sum_{k=1}^{v_n} a_{n,k}(\mathscr{K})Q_{n,k}(x,y),$$

where

$$a_{n,k}(\mathscr{K})=\frac{1}{d_{n,k}}\int_M \mathscr{K}(x,y)Q_{n,k}(x,y)\,\mathrm{d}\mu(y).$$

Note that the above integral does not depend on x. Throughout the paper, we assume that $a_{n,k}(\mathscr{K})>0$ for all n, k, and

$$\sum_{n=0}^{\infty}\sum_{k=1}^{v_n} d_{n,k}a_{n,k}(\mathscr{K})<\infty. \tag{2.1}$$

The above conditions insure that $\mathscr{K}$ is continuous on $M\times M$, and is strictly positive-definite. Let the kernel ψ : $M\times M\to\mathbb{R}$, the square root of $\mathscr{K}$, be defined by

$$\psi(x,y)=\sum_{n=0}^{\infty}\sum_{k=1}^{v_n}(a_{n,k}(\mathscr{K}))^{1/2}Q_{n,k}(x,y). \tag{2.2}$$

For every fixed y, let ψ_y denote the function $x \mapsto \psi(x, y)$. Then for any $x, y \in M$, we have

$$T_\psi \psi_y(x) = \int_M \psi(x, z)\psi(z, y)\, \mathrm{d}\mu(z) = \mathscr{K}(x, y). \tag{2.3}$$

Also, for $f \in L_2$,

$$T_{n,k} T_\psi f = T_\psi T_{n,k} f = (a_{n,k}(\mathscr{K}))^{1/2} T_{n,k} f. \tag{2.4}$$

We can develop a Fourier series expansion for any function $f \in L_1$:

$$f \sim \sum_{n=0}^{\infty} \sum_{k=1}^{v_n} T_{n,k} f.$$

Since $a_{n,k}(\mathscr{K}) > 0$, we can define the following bilinear form:

$$\langle f, g\rangle_{\mathscr{K}} = \sum_{n=0}^{\infty} \sum_{k=1}^{v_n} (a_{n,k}(\mathscr{K}))^{-1} [T_{n,k} f, T_{n,k} g].$$

We use $\|f\|_{\mathscr{K}}$ to denote $\langle f, f\rangle_{\mathscr{K}}^{1/2}$. The Hilbert space of functions $\mathscr{N}_{\mathscr{K}}$ defined by

$$\mathscr{N}_{\mathscr{K}} = \{f \in L_2 : \|f\|_{\mathscr{K}} < \infty\}$$

is called the native space of $\mathscr{K}$. Let Y be a finite subset of M. For $g \in \mathscr{N}_{\mathscr{K}}$, the $\mathscr{K}$-spline interpolant $s_{\mathscr{K}}^Y[g]$ of g on Y is defined by

$$s_{\mathscr{K}}^Y[g](x) = \sum_{y \in Y} \alpha_y \mathscr{K}(x, y),$$

where the coefficients α_y, $y \in Y$, are determined by the interpolation conditions

$$s_{\mathscr{K}}^Y[g](y) = g(y), \quad y \in Y. \tag{2.5}$$

Note that $s_{\mathscr{K}}^Y[g]$ is uniquely defined because of the strict positive-definiteness of $\mathscr{K}$. In the literature, the error estimate for $|g(x) - s_{\mathscr{K}}^Y[g](x)|$ is often gauged by the so-called "filling-distance", $\tau(Y)$, of Y in M, defined by

$$\tau(Y) = \max_{x \in M} \min_{y \in Y} d(y, x).$$

The following error estimate is established in [10] using the "norming set" approach developed by Jetter et al. [8], and a duality argument developed by Morton and Neamtu [14].

Proposition 2. *Let $s_{\mathscr{K}}^Y[g]$ be the $\mathscr{K}$-spline interpolant to $g \in \mathscr{N}_{\mathscr{K}}$ on the finite point set $Y \subset M$. Then there is a constant $B > 0$ such that for all $\tau(Y) < B/N$,*

$$|f(x) - s_{\mathscr{K}}^Y[g](x)| \leqslant 3 \left(\sum_{n>N} \sum_{k=1}^{v_n} a_{n,k} d_{n,k} \right)^{1/2} \|f\|_{\mathscr{K}}, \quad x \in M. \tag{2.6}$$

To obtain an error estimate in terms of $\tau(Y)$ from (2.6), one first estimates the sum $(\sum_{n>N} \sum_{k=1}^{v_n} a_{n,k} d_{n,k})^{1/2}$, and then uses the relation $\tau(Y) = \mathscr{O}(N^{-1})$.

In this paper, we are interested in doing interpolation on a finite set $Y \subset M$ using a kernel ρ that is a perturbation of a G-invariant, strictly positive-definite kernel $\mathscr{K}$. The results will be stated and proved in the next two sections.

3. Small random perturbations

Let ψ be as defined in Eq. (2.2). Consider the integral operator $T_\psi : L_2 \to \mathcal{N}_{\mathcal{K}}$ given by

$$T_\psi f(x) = \int_M \psi(x, z) f(z)\, \mathrm{d}\mu(z), \quad x \in M.$$

Lemma 3. *T_ψ is an isometry from L_2 to $\mathcal{N}_{\mathcal{K}}$.*

Proof. Let $f \in L_2$. Then, using (2.4), we have

$$\begin{aligned}\|T_\psi f\|_{\mathcal{K}} &= \left\{\sum_{n=0}^{\infty}\sum_{k=1}^{v_n}(a_{n,k}(\mathcal{K}))^{-1}\|T_{n,k}T_\psi f\|_2^2\right\}^{1/2}\\ &= \left\{\sum_{n=0}^{\infty}\sum_{k=1}^{v_n}(a_{n,k}(\mathcal{K}))^{-1}\|(a_{n,k}(\mathcal{K}))^{1/2}T_{n,k} f\|_2^2\right\}^{1/2}\\ &= \left\{\sum_{n=0}^{\infty}\sum_{k=1}^{v_n}\|T_{n,k} f\|_2^2\right\}^{1/2}\\ &= \|f\|_2. \qquad \square\end{aligned}$$

Now, for $g \in \mathcal{N}_{\mathcal{K}}$, consider the integral equation

$$T_\psi f(x) = \int_M \psi(x, z) f(z)\, \mathrm{d}\mu(z) = g(x). \tag{3.1}$$

Given a finite set of points $Y \subset M$ we approximate $f \in L_2$ by

$$f_Y(z) := \sum_{y\in Y} \alpha_y \psi(z, y), \tag{3.2}$$

where we determine the coefficients α_y, $y \in Y$, by the collocation conditions

$$T_\psi f(x) = g(x), \quad x \in Y.$$

Substituting (3.2) into the last equation, and using (2.3), we get

$$\sum_{y\in Y} \alpha_y T_\psi \psi(x, y) = \sum_{y\in Y} \alpha_y \mathcal{K}(x, y) = g(x), \quad x \in Y.$$

This is a square linear system of order $|Y|$. We know this interpolation problem is uniquely solvable because $\mathcal{K}$ is strictly positive-definite.

Suppose that a "small" random perturbation occurs in the integral equation (3.1). We intend to apply some standard integral equation techniques to show that the solution of the perturbed interpolation problem converges at the same rate as the standard positive-definite kernel interpolation problem.

Definition 4. Let X and X' be normed vector spaces. An operator $A : X \to X'$ is compact if, for every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, the sequence $\{Ax_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in X'.

We require the following standard results concerning linear operators. These can be found in, for example, [9]. We denote the identity operator by I.

Proposition 5. *Let X and X' be normed vector spaces.*

(a) *Let $B : X \to X'$ be a compact linear operator. If $I + B$ is injective then $I + B$ has a bounded inverse.*
(b) *The product of a compact operator and a bounded operator is also compact.*

(c) *Let $B : X \to X'$ be a bounded linear operator with bounded inverse. Suppose that the sequence of linear operators B_n, $n \in \mathbb{N}$, are such that* $\lim_{n\to\infty} \|B_n - B\|_{X\to X'} = 0$. *Then, there exists an $N \in \mathbb{N}$ such that, for all $n > N$,*

$$\|B_n^{-1}\|_{X'\to X} \leqslant C\|B^{-1}\|_{X'\to X},$$

for some positive constant C, independent of n.

(d) *Let X' be a Banach space, and $B_n : X \to X'$, $n \in \mathbb{N}$ a sequence of compact linear operators. If* $\lim_{n\to\infty} \|B_n - B\|_{X\to X'} = 0$, *then B is also compact.*

(e) *If $B : X \to X'$ has finite dimensional range, then B is compact.*

In order to prove the existence of a unique solution to the perturbed problem we will need to introduce pseudodifferential operators and associated function spaces.

Definition 6. A pseudodifferential operator Λ is one which acts via scalar multiplication on each of the G-invariant spaces $H_{n,k}$. It is specified by its eigenvalues $\lambda_{n,k}$, defined by

$$\Lambda p_{n,k} = \lambda_{n,k} p_{n,k}, \quad p_{n,k} \in H_{n,k}, \quad k = 1, \ldots, v_n, \ n = 0, 1, \ldots.$$

The set $\{\lambda_{n,k}, k = 1, \ldots, v_n, n = 0, 1, \ldots\}$ is called the symbol of Λ. Assume $\lambda_{n,k} \neq 0$ for all n, k. The function space $\mathcal{N}_{\Lambda,\mathscr{K}}$ associated with Λ, $\mathscr{K}$ is defined as

$$\mathcal{N}_{\Lambda,\mathscr{K}} := \left\{ f : \|f\|_{\Lambda,\mathscr{K}} = \left\{ \sum_{n=0}^{\infty} \sum_{k=1}^{v_n} (\lambda_{n,k} a_{n,k}(\mathscr{K}))^{-2} \|T_{n,k} f\|_2^2 \right\}^{1/2} < \infty \right\}.$$

For example, the operator T_ψ is a pseudodifferential operator with symbol

$$\{(a_{n,k}(\mathscr{K}))^{1/2}, \ k = 1, \ldots, v_n, \ n = 0, 1, \ldots\}.$$

Let $c(x, y)$ be a (not necessarily G-invariant) kernel and define the operator $\mathscr{C}$ by

$$\mathscr{C} f(x) = \int_M c(x, z) f(z) \, \mathrm{d}\mu(z).$$

Lemma 7. *Assume that* $\limsup_{n\to\infty} \max_{1\leqslant k\leqslant v_n} a_{n,k}(\mathscr{K})\lambda_{n,k}^2 = 0$, *and that the operator $\mathscr{C} : L_2 \to \mathcal{N}_{\Lambda,\mathscr{K}}$ is bounded. Then the operator $\mathscr{C} : L_2 \to \mathcal{N}_{\mathscr{K}}$ is compact.*

Proof. We begin by showing that $\mathcal{N}_{\Lambda,\mathscr{K}}$ is compactly embedded in $\mathcal{N}_{\mathscr{K}}$. We denote the embedding operator by I. Define the operator $S_N : \mathcal{N}_{\Lambda,\mathscr{K}} \to \mathcal{N}_{\mathscr{K}}$ by

$$S_N f = \sum_{n=0}^{N} \sum_{k=1}^{v_n} T_{n,k} f.$$

Then, by Proposition 5(e) S_N is compact because it is finite-dimensional. Also, for $f \in \mathcal{N}_{\Lambda,\mathscr{K}}$

$$\begin{aligned} \|(I - S_N) f\|_{\mathscr{K}} &= \left\{ \sum_{n=N+1}^{\infty} \sum_{k=1}^{v_n} (a_{n,k}(\mathscr{K}))^{-1} \|T_{n,k} f\|_2^2 \right\}^{1/2} \\ &\leqslant \sup_{n>N} \max_{1\leqslant k\leqslant v_n} a_{n,k}(\mathscr{K})\lambda_{n,k}^2 \left\{ \sum_{n=N+1}^{\infty} \sum_{k=1}^{v_n} (\lambda_{n,k} a_{n,k}(\mathscr{K}))^{-2} \|T_{n,k} f\|_2^2 \right\}^{1/2} \\ &\leqslant \sup_{n>N} \max_{1\leqslant k\leqslant v_n} a_{n,k}(\mathscr{K})\lambda_{n,k}^2 \|f\|_{\Lambda,\mathscr{K}}. \end{aligned}$$

Thus, by using the assumptions, we see that the embedding operator is the norm limit of a sequence of compact operators, and therefore by Proposition 5(d) it is itself compact. We complete the proof by observing that $\mathscr{C} : L_2 \to \mathscr{N}_{\mathscr{K}}$ may be viewed as the product of $\mathscr{C} : L_2 \to \mathscr{N}_{\Lambda,\mathscr{K}}$ with $I : \mathscr{N}_{\Lambda,\mathscr{K}} \to \mathscr{N}_{\mathscr{K}}$. Proposition 5(b) tells us that this product of a compact operator and a bounded operator is also compact. $\square$

Consider the perturbed integral equation

$$(T_\psi + \mathscr{C})f = g \in \mathscr{N}_{\mathscr{K}}. \tag{3.3}$$

Since T_ψ has a bounded inverse we can pre-multiply the above equation by T_ψ^{-1} to obtain

$$(I + T_\psi^{-1}\mathscr{C})f = T_\psi^{-1}g \in L_2. \tag{3.4}$$

Now, since T_ψ^{-1} is bounded and $\mathscr{C}$ is compact, by Proposition 5(b), $T_\psi^{-1}\mathscr{C}$ is also compact. Then, so long as $I + T_\psi^{-1}\mathscr{C}$ (equivalently $T_\psi + \mathscr{C}$) is injective then $I + T_\psi^{-1}\mathscr{C}$ has a bounded inverse and (3.4) has a unique solution in L_2 by Proposition 5(a).

We approximate f in Eq. (3.4) by writing

$$f(x) \approx f_Y(x) = \sum_{y\in Y} \alpha_y \psi(x, y).$$

Let us denote by V_Y the linear space spanned by the Y-translates of $\mathscr{K}$, i.e.,

$$V_Y = \text{span}\{\mathscr{K}(\cdot, y) : y \in Y\}.$$

Define the linear projection operator $P_Y : \mathscr{N}_{\mathscr{K}} \to V_Y : \gamma \mapsto P_Y\gamma$ for each $\gamma \in \mathscr{N}_{\mathscr{K}}$, where $P_Y\gamma$ is uniquely determined by the interpolation condition,

$$P_Y\gamma(y) = \gamma(y) \quad y \in Y.$$

Then, we can write the collocation solution of (3.4) in the form

$$P_Y(T_\psi + \mathscr{C})f_Y = P_Y g.$$

This may be rewritten in the form

$$Z_Y(I + T_\psi^{-1}\mathscr{C})f_Y = Z_Y\overline{g}, \tag{3.5}$$

where $Z_Y = T_\psi^{-1}P_Y T_\psi$, and $\overline{g} = T_\psi^{-1}g$. The last equation can be viewed as a projection method solution to the second kind integral equation (3.4), with projection operator Z_Y.

If we pre-multiply (3.4) by Z_Y, then we obtain

$$Z_Y(I + T_\psi^{-1}\mathscr{C})f = Z_Y\overline{g}.$$

Now, from (3.5), and using the fact that $Z_Y f_Y = f_Y$, we see that

$$\begin{aligned} f - Z_Y f &= f - (Z_Y\overline{g} - Z_Y T_\psi^{-1}\mathscr{C}f) \\ &= f + Z_Y T_\psi^{-1}\mathscr{C}f - Z_Y f_Y - Z_Y T_\psi^{-1}\mathscr{C}f_Y \\ &= f - f_Y + Z_Y T_\psi^{-1}\mathscr{C}(f - f_Y) \\ &= (I + Z_Y T_\psi^{-1}\mathscr{C})(f - f_Y). \end{aligned} \tag{3.6}$$

To proceed to state and prove our major result of this section, we will need the following technical assumption that is satisfied by a wide class of pseudodifferential operators on the spheres; see [12,14].

Assumption 8. Suppose $g \in \mathcal{N}_{\Lambda,\mathcal{K}}$. Then,

$$\|g - s_{\mathcal{K}}^{Y}[g]\|_{\mathcal{K}} \leqslant B(Y)\|g\|_{\Lambda,\mathcal{K}},$$

where $B(Y) \to 0$ as $\tau(Y) \to 0$.

Proposition 9. *Under Assumption* 8, *we have the following result. For sufficiently small* $\tau(Y)$, $I + Z_Y T_\psi^{-1}\mathscr{C} : L_2 \to L_2$ *has an inverse that is bounded independently of* Y.

Proof. Assumption 8 implies that $\|I - P_Y\|_{\mathcal{N}_{\Lambda,\mathcal{K}} \to \mathcal{N}_{\mathcal{K}}} \to 0$ as $\tau(Y) \to 0$. Also,

$$\begin{aligned}
&\|(I + T_\psi^{-1}\mathscr{C}) - (I + Z_Y T_\psi^{-1}\mathscr{C})\|_2 \\
&\quad = \|T_\psi^{-1}\mathscr{C} - Z_Y T_\psi^{-1}\mathscr{C}\|_2 \\
&\quad = \|T_\psi^{-1}(I - P_Y)\mathscr{C}\|_2 \\
&\quad \leqslant \|T_\psi^{-1}\|_{\mathcal{N}_{\mathcal{K}} \to L_2}\|I - P_Y\|_{\mathcal{N}_{\Lambda,\mathcal{K}} \to \mathcal{N}_{\mathcal{K}}}\|\mathscr{C}\|_{L_2 \to \mathcal{N}_{\Lambda,\mathcal{K}}} \\
&\quad \leqslant C\|I - P_Y\|_{\mathcal{N}_{\Lambda,\mathcal{K}} \to \mathcal{N}_{\mathcal{K}}}
\end{aligned}$$

by Lemma 3. The result follows from Assumption 8 and Proposition 5(c). □

We have shown that under some reasonably administered conditions the perturbed interpolation problem has a unique solution. Next, we show that the convergence rate of the perturbed interpolation is the same as that of the interpolation problem using the positive-definite kernel $\mathcal{K}$. To do this we need to use the following result, communicated by Light and Brownlee [13].

Lemma 10. *Let* $q \in \mathcal{N}_{\mathcal{K}}$ *satisfy* $q(y) = 0$, $y \in Y$. *Then, for any* $x \in M$,

$$|q(x)| \leqslant C\left(\sum_{n>N}\sum_{k=1}^{v_n} a_{n,k}d_{n,k}\right)^{1/2}\|q\|_{\mathcal{K}}.$$

Proof. Let $s_{\mathcal{K}}^{Y}[q]$ be the $\mathcal{K}$-spline interpolant to q. Then, since $q = 0$ on Y, $s_{\mathcal{K}}^{Y}[q] = 0$. Substituting this into Proposition 2 gives the required result. □

Theorem 11. *Let* $\rho(x, y) = \mathcal{K}(x, y) + c(x, y)$, *where* $\mathcal{K}$ *is a strictly positive-definite kernel satisfying the conditions in* (2.1) *and* $c(x, y)$ *is the kernel of a bounded integral operator* $\mathscr{C} : L_2 \to \mathcal{N}_{\Lambda,\mathcal{K}}$. *Assume that* $T_\psi + \mathscr{C}$ *is injective. Then, for sufficiently small* $\tau(Y)$, *the* ρ-*spline interpolant* $S_\rho^Y[g]$ *to* $g \in \mathcal{N}_{\mathcal{K}}$ *exists and is unique. Furthermore, there is a constant* $B > 0$ *such that whenever* $\tau(Y) < B/N$, *we have*

$$|g(x) - s_\rho^Y(g)(x)| \leqslant C\left(\sum_{n>N}\sum_{k=1}^{v_n} a_{n,k}d_{n,k}\right)^{1/2}\|g\|_{\mathcal{K}}, \quad x \in M,$$

for some C *independent of* x, g, *and* Y.

Proof. Applying Proposition 9 to (Eq. (3.6)) we see that, for sufficiently small $\tau(Y)$,

$$\begin{aligned}
\|f - f_Y\|_2 &\leqslant C\|f - Z_Y f\|_2 \\
&\leqslant \|T_\psi^{-1}(T_\psi f - P_Y T_\psi f)\|_2 \\
&\leqslant \|T_\psi f - P_Y T_\psi f\|_{\mathcal{K}} \\
&\leqslant \|T_\psi f\|_{\mathcal{K}} \\
&= \|f\|_2.
\end{aligned}$$

Let us write $\mathscr{D} = T_\psi + \mathscr{C}$. Now, $s_\rho^Y[g] = \mathscr{D} f_Y$, so that

$$\begin{aligned}\|g - s_\rho^Y[g]\|_{\mathscr{K}} &= \|\mathscr{D}(f - f_Y)\|_{\mathscr{K}} \\ &\leqslant C\|f - f_Y\|_2 \\ &\leqslant C\|f\|_2 \\ &= C\|\mathscr{D}^{-1}g\|_2 \leqslant C\|g\|_{\mathscr{K}}.\end{aligned}$$

We now apply Lemma 10 to obtain the result. □

Remark 12. It is straightforward to slightly modify the proof of Theorem 11 to cover the case when

$$\rho(x, y) = \eta(y)\mathscr{K}(x, y) + c(x, y),$$

where the kernels $\mathscr{K}$ and c are as defined in Theorem 11, and η is a strictly positive, smooth function.

4. Perturbations by intertwining operators

In this section, we restrict our attention to the unit sphere S^{d-1} in the d-dimensional euclidean space $\mathbb{R}^d$. We begin with a brief introduction to the theory of h-spherical harmonics associated to reflection groups developed by Dunkl, Xu and others. Readers interested in more details are referred to the recently published book by Dunkl and Xu [3].

For a nonzero vector $v \in \mathbb{R}^d$, define the reflection σ_v by

$$\sigma_v x := x - 2(\langle x, v\rangle/|v|^2)v, \quad x \in \mathbb{R}^d.$$

It is easy to see that $\sigma_v v = -v$, and that $\sigma_v x = x$ if and only if $\langle x, v\rangle = 0$. Any nonempty finite set of reflections generates a subgroup of $\mathrm{O}(d)$, the orthogonal group on $\mathbb{R}^d$. Under certain fully characterized conditions, the group generated is finite, and is called a finite reflection group or a Coxeter group; see [7]. Suppose $\mathscr{G}$ is a Coxeter group with the set of reflections $\{\sigma_1, \ldots, \sigma_m\}$. Choose a set of vectors $\{v_1, \ldots, v_m\} \subset \mathbb{R}^d$, such that $\sigma_i = \sigma_{v_i}$ for $i = 1, 2, \ldots, m$, and $|v_i| = |v_j|$ whenever σ_i is conjugate to σ_j in $\mathscr{G}$. The set $\mathscr{R} := \{v_i : 1 \leqslant i \leqslant m\}$ is called a root system for $\mathscr{G}$. We consider weighted measures on S^{d-1} of the form $h_\alpha^2\,\mathrm{d}\omega$, where $\mathrm{d}\omega$ denotes the usual rotational invariant measure on S^{d-1}, and

$$h_\alpha(x) := \prod_{i=1}^m |\langle x, v_i\rangle|^{\alpha_i}, \quad \alpha_i \geqslant 0,$$

with $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in $\mathscr{G}$. It is obvious to see that the function h_α is $\mathscr{G}$-invariant, i.e.,

$$h_\alpha(\sigma x) = h_\alpha(x) \quad \text{for each } \sigma \in \mathscr{G} \text{ and all } x \in S^{d-1}.$$

Definition 13 (*Dunkl operators*). For $i = 1, \ldots, d$, the first order differential-difference operator T_i is defined by

$$T_i f(x) := \partial_i f(x) + \sum_{j=1}^m \alpha_j \frac{f(x) - f(\sigma_j x)}{\langle x, v_j\rangle}\langle v_j, e_i\rangle,$$

where ∂_i, e_i, $1 \leqslant i \leqslant d$, are the partial derivative operators and the standard unit vectors of $\mathbb{R}^d$, respectively. The h-Laplacian, which plays the role similar to that of the classical Laplacian, is defined by

$$\Delta_h := \sum_{i=1}^d T_i^2.$$

Let Π_n denote the space of all the homogeneous polynomials of degree n, and let $\Pi := \bigoplus_{i=0}^\infty \Pi_n$. Let $\mathscr{H}_n^h := \Pi_n \cap (\ker \Delta_h)$. Polynomials in $\mathscr{H}_n^h$ are called (homogeneous) h-harmonic polynomials of degree n, and their restrictions to the spheres are called (homogeneous) h-spherical harmonics of degree n. Dunkl, Xu and others have developed a rich theory for the h-spherical harmonics analogous to the classical spherical harmonics. Some highlights of the development are summarized as follows.

1. The dimension of $\mathscr{H}_n^h$ is the same as that of the classical homogeneous spherical harmonics of degree n:
$$v(n,d) := \dim \Pi_n - \dim \Pi_{n-2} = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$
2. There exists a unique decomposition of Π_n in the form $\Pi_n = \bigoplus_{j=0}^{\lfloor n/2 \rfloor} |x|^2 \mathscr{H}_{n-2j}^h$. That is, for each $p \in \Pi_n$, and each $j = 0, 1, \ldots, \lfloor n/2 \rfloor$, there is a unique $p_{n-2j} \in \mathscr{H}_{n-2j}^h$, such that
$$p(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} |x|^2 p_{n-2j}(x). \tag{4.1}$$
3. If $p \in \Pi_n$, then
$$\int_{S^{d-1}} pqh_\alpha^2 \, d\omega = 0 \quad \text{for all } q \in \bigoplus_{j=1}^{n-1} P_j$$
if and only if $\Delta_h p = 0$.

Definition 14. The intertwining operator V is a linear operator from Π to Π uniquely defined by the following three conditions:
$$V\Pi_n \subset \Pi_n, \quad V1 = 1, \quad T_i V = V\partial_i, \quad 1 \leqslant i \leqslant d.$$

Rösler [22] proved that the intertwining operator is a positive operator. The intertwining operator plays a crucial role in the analysis of the h-spherical harmonics. However, closed forms for the intertwining operator V are known only for several special reflection groups. For example, if $\mathscr{G} = Z_2 \times \cdots \times Z_2$, then Xu showed that the intertwining operator is an integral operator. We remind readers that all the above stated results (including their proofs) can be found in [3]. It is well-understood that V is a bounded operator under the uniform topology on compact subsets of $\mathbb{R}^d$, and therefore it is a bounded operator on $C(S^{d-1})$. Let $S_{n,k}^h$, $k = 1, \ldots, v(n,d)$, be an orthonormal basis of $\mathscr{H}_n^h$ with respect to the inner product
$$\langle p, q \rangle_h = H_\alpha \int_{S^{d-1}} pqh_\alpha^2 \, d\omega, \quad p, q \in \mathscr{H}_n^h,$$
where H_α is the normalization constant defined by $H_\alpha^{-1} = \int_{S^{d-1}} h_\alpha^2 \, d\omega$. Let
$$P_n^h(x,y) := \sum_{k=1}^{v(n,d)} S_{n,k}^h(x) S_{n,k}^h(y).$$
Then $P_n^h(x,y)$ is the reproducing kernel of $\mathscr{H}_n^h$. Xu [3] has proved the following result:
$$P_n^h(x,y) = \frac{n + |\alpha|_1 + (d-3)/2}{|\alpha|_1 + (d-3)/2} [VC_n^{(|\alpha|_1+(d-3)/2)}(\cdot y)](x), \tag{4.2}$$
where $|\alpha|_1 := \sum_{i=1}^m |\alpha_i|$, with the Gegenbauer polynomials normalized so that
$$\int_{-1}^1 (C_n^{(\lambda)}(t))^2 (1-t^2)^{(d-3)/2} \, dt = 1.$$

Let ϕ be a positive-definite function on S^{d-1}. We are interested in the kernel $(V\phi(\cdot y))(x)$. For the convenience of writing, we denote the kernel by $\mathscr{K}_\phi^h(x,y)$. It is natural to think that the kernel $\mathscr{K}_\phi^h(x,y)$ is a perturbation of the zonal kernel $\phi(xy)$ by the action of the intertwining operator V. It will become clear in the proof of Theorem 15 that the kernel $\mathscr{K}^h_\phi(x, y)$ is $\mathscr{G}$-invariant.

Theorem 15. *Let ϕ be a positive-definite function on $S^{d'-1}(d' > d)$. Then for a properly chosen α satisfying $d' = |\alpha|_1 + (d-3)/2$ (the choice of α is not unique) the kernel $\mathscr{K}^h_\phi(x, y)$ is positive-definite on S^{d-1}. Furthermore, if all the coefficients in the Gegenbauer polynomials expansion of ϕ*

$$\phi(t) = \sum_{n=0}^{\infty} a_n C_n^{(d'-3)/2}(t) \tag{4.3}$$

are positive, then the kernel $\mathscr{K}^h_\phi(x, y)$ is strictly positive-definite on S^{d-1}.

Proof. Letting $t = xy$, $x, y \in S^{d'-1}$, in Eq. (4.3) we get

$$\phi(xy) = \sum_{n=0}^{\infty} a_n C_n^{(d'-3)/2}(xy). \tag{4.4}$$

The series in Eq. (4.4) converges uniformly for $(x, y) \in S^{d'-1} \times S^{d'-1}$. Since the intertwining operator V is bounded on $C(S^{d'-1})$, we can apply V on both sides of Eq. (4.4). The uniform convergence enables us to apply V term by term to the right-hand side of Eq. (4.4) to obtain that

$$\mathscr{K}^h_\phi(x, y) = \sum_{n=0}^{\infty} a_n V C_n^{(d'-3)/2}(xy).$$

Choose a proper α so that $d' = |\alpha|_1 + (d-3)/2$ (the choice of α is not unique). We then restrict $x, y \in S^{d-1}$, and use Eq. (4.2) to write

$$\mathscr{K}^h_\phi(x, y) = \sum_{n=0}^{\infty} a_n \left(\frac{n + |\alpha|_1 + (d-3)/2}{|\alpha|_1 + (d-3)/2} \right)^{-1} P_n^h(x, y). \tag{4.5}$$

Being the reproducing kernel of $\mathscr{H}_n^h$, $P_n^h(x, y)$ is $\mathscr{G}$-invariant and positive-definite on S^{d-1}. It then follows from Eq. (4.5) that $\mathscr{K}^h_\phi(x, y)$ is $\mathscr{G}$-invariant and positive-definite on S^{d-1}. The first part of the theorem is now proved. To prove the second part, let Y be a finite subset of S^{d-1}. Assume that $c_x (x \in Y)$ are $|Y|$ real numbers such that

$$\sum_{x \in Y} \sum_{y \in Y} c_x c_y \mathscr{K}^h_\phi(x, y) = 0.$$

This leads to

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n \left(\frac{n + |\alpha|_1 + (d-3)/2}{|\alpha|_1 + (d-3)/2} \right)^{-1} \sum_{x \in Y} \sum_{y \in Y} c_x c_y P_n^h(x, y) \\
&\quad = \sum_{n=0}^{\infty} a_n \left(\frac{n + |\alpha|_1 + (d-3)/2}{|\alpha|_1 + (d-3)/2} \right)^{-1} \sum_{k=1}^{v(n,d)} \left(\sum_{x \in Y} c_x S_{n,k}^h(x) \right)^2 \\
&\quad = 0.
\end{aligned} \tag{4.6}$$

Therefore, we have

$$\sum_{x \in Y} c_x S_{n,k}^h(x) = 0 \quad \text{for all } n = 0, 1, \ldots, \text{ and all } k = 1, 2, \ldots, v(n, d).$$

Using the decomposition result Eq. (4.1), we can conclude that for every polynomial P,

$$\sum_{x\in Y} c_x p(x) = 0,$$

implying all the c_x ($x \in Y$) are zero. This shows that the kernel $\mathscr{K}^h_\phi$ is strictly positive-definite. □

The inconvenience stemming from the discrepancy of d and d' and the choice of α in the equation $d' = |\alpha|_1 + (d-3)/2$ can be circumvented by using zonal kernels of the form $\phi(xy)$ that are positive-definite on S^{d-1} for all $d = 2, 3, \ldots$. Such kernels have also been characterized by Schoenberg [24] whose result asserts that in order that a zonal kernel of the form $\phi(xy)$ be positive-definite on S^{d-1} for all $d = 2, 3, \ldots$ it is necessary and sufficient that the univariate function ϕ have the series expansion

$$\phi(t) = \sum_{n=0}^{\infty} a_n t^n, \quad t \in [-1, 1], \tag{4.7}$$

where $a_n \geqslant 0$, and $\sum_{n=0}^{\infty} a_n < \infty$. Schoenberg called such univariate function ϕ positive-definite on the unit sphere of the Hilbert space l_2. The following theorem can be proved in a similar way as in that of Theorem 15.

Theorem 16. *Let ϕ be a positive-definite function on the unit sphere of l_2. Then the kernel $\mathscr{K}^h_\phi(x, y)$ is positive-definite on the unit sphere of l_2. If we assume furthermore that all the coefficients in the expansion of ϕ (see Eq. (4.7)) are positive, then the kernel $\mathscr{K}^h_\phi(x, y)$ is strictly positive-definite on S^{d-1} for all $d = 1, 2, \ldots$.*

Let $\mathscr{K}^h_\phi(x, y)$ be a strictly positive-definite kernel given by

$$\mathscr{K}^h_\phi(x, y) = \sum_{n=0}^{\infty} b_n P^h_n(x, y)$$

where all b_n are positive, and the series converges uniformly on $S^{d-1} \times S^{d-1}$. We know from the proof of Theorem 15 that $\mathscr{K}^h_\phi(x, y)$ is a $\mathscr{G}$-invariant strictly positive kernel. We can use a well-known procedure (see [1]) to build a reproducing kernel Hilbert space $\mathscr{N}^h_\phi$ of continuous functions on S^{d-1} with $\mathscr{K}^h_\phi(x, y)$ being the reproducing kernel. The Hilbert space thus built $\mathscr{N}^h_\phi$ is precisely

$$\left\{ f = \sum_{n=0}^{\infty} \sum_{k=0}^{v(n,d)} \hat{f}^h_{n,k} S^h_{n,k} : \sum_{n=0}^{\infty} b_n^{-1} \sum_{k=0}^{v(n,d)} (\hat{f}^h_{n,k})^2 < \infty \right\},$$

with inner product

$$\langle f, g\rangle_{\mathscr{N}^h_\phi} = \sum_{n=0}^{\infty} b_n^{-1} \sum_{k=0}^{v(n,d)} \hat{f}^h_{n,k}\, \hat{g}^h_{n,k}.$$

Here $\hat{f}^h_{n,k}$, the h-Fourier coefficients, are defined by

$$\hat{f}^h_{n,k} = \int_{S^{d-1}} f\, S^h_{n,k}\, h^2_\alpha \,\mathrm{d}\omega.$$

The space $\mathscr{N}^h_\phi$ is often called the native space associated with the kernel $\mathscr{K}^h_\phi$ in the approximation theory community. Let Y be a finite subset of S^{d-1}, and let $f \in \mathscr{N}^h_\phi$. Then Theorem 15 guarantees that there is a unique function $s^Y_h[f]$ in $\operatorname{span}\{\mathscr{K}^h_\phi(\cdot, x) : x \in Y\}$ that interpolates f on Y. Also, $s^Y_h[f]$ is the orthogonal projection of f onto the $|Y|$-dimensional subspace: span$\{\mathscr{K}^h_\phi(\cdot, x) : x \in Y\}$. Using the norming set approach introduced by Jetter et al. [8] and improved by Morton and Neamtu [14], we can prove the following error estimate:

Proposition 17. *Let N be an arbitrary natural number, and let Y be a finite subset of S^{d-1} such that $\delta(Y) \leqslant \frac{1}{2}N$. Suppose that $\mathscr{K}^h_\phi$ is strictly positive-definite as given in Eq.* (4.7). *Then for each $f \in \mathscr{N}^h_\phi$ and all $x \in S^{d-1}$, we have*

$$|f(x) - s_h^Y[f](x)| \leqslant 3 \sum_{n>N} b_n \|f - s_h^Y[f]\|_{\mathscr{N}^h_\phi}.$$

We omit the proof due to the similarity to those in the above-mentioned references.

Acknowledgements

Both authors of the present paper thank the referees for pointing out some oversights and inaccuracies in the initial version of the paper. X. Sun also thanks EPSRC (UK) and the University of Leicester for their support and hospitality.

References

[1] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950) 337–404.
[2] D. Chen, V.A. Menegatto, X. Sun, A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Soc. 131 (2003) 2733–2740.
[3] C. Dunkl, Y. Xu, Orthogonal polynomials of several variables, Encyclopedia of Mathematics and Its Applications, vol. 81, Cambridge University Press, Cambridge, 2001.
[4] N. Dyn, J.F. Narcowich, J.D. Ward, Variational principles and Sobolev-type estimates for generalized interpolation on a Riemannian manifold, Constr. Approx. 15 (1999) 175–208.
[5] W. Freeden, Spherical spline interpolation—basic theory and computational aspects, J. Comput. Appl. Math. 11 (1984) 367–375.
[6] M. von Golitschek, W. Light, Interpolation by polynomials and radial basis functions on spheres, Constr. Approx. 17 (2001) 1–18.
[7] L.C. Grove, C.T. Benson, Finite Reflection Groups, second ed., Springer, New York, 1984.
[8] K. Jetter, J. Stöckler, J.D. Ward, Error estimates for scattered data interpolation on spheres, Math. Comp. 68 (1999) 733–747.
[9] R. Kress, Linear Integral Equations, Applied Mathematical Sciences, vol. 82, Springer, Berlin, 1999.
[10] J. Levesley, C. Odell, D.L. Ragozin, Variational interpolation on homogeneous spaces: the norming set approach, preprint.
[11] J. Levesley, D.L. Ragozin, Radial basis interpolation on homogeneous manifolds: convergence rates, preprint.
[12] J. Levesley, X. Sun, Approximation in rough native space using shifts of smooth kernels on spheres, J. Approx. Theory 133 (2005) 269–283.
[13] W.A. Light, R. Brownlee, private communication, Leicester University, New York, 2002.
[14] T.M. Morton, M. Neamtu, Error bounds for solving pseudodifferential equations on spheres by collocation with zonal kernels, J. Approx. Theory 114 (2002) 242–268.
[15] G.D. Mostow, Equivariant embeddings in Euclidean space, Ann. of Math. 65 (1957) 432–446.
[16] C. Müller, Spherical harmonics, Lecture Notes in Mathematics, vol. 17, Springer, Berlin, Heidelberg, 1966.
[17] F.J. Narcowich, R. Schaback, J.D. Ward, Approximation in Sobolev spaces by kernel expansions, J. Approx. Theory 114 (2002) 70–83.
[18] F.J. Narcowich, J.D. Ward, Scattered data interpolation on sphere: error estimates and locally supported basis functions, SIAM J. Math. Anal. 33 (2002) 1393–1410.
[19] D.L. Ragozin, Polynomial approximation on compact manifolds and homogeneous spaces, Trans. Amer. Math. Soc. 150 (1970) 41–53.
[20] D.L. Ragozin, J. Levesley, The density of translates of zonal kernels on compact homogeneous spaces, J. Approx. Theory 103 (2000) 252–268.
[21] A. Ron, X. Sun, Strictly positive definite functions on spheres, Math. Comp. 65 (1996) 1513–1530.
[22] M. Rösler, Positivity of Dunkl's intertwining operator, Duke Math. J. 98 (1999) 445–463.
[23] R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, Math. Comp. 68 (1999) 201–216.
[24] I.J. Schoenberg, Positive definite functions on spheres, Duke Math. J. 9 (1942) 96–108.
[25] H. Szegö, Orthogonal Polynomials, fourth ed., American Mathematical Society Colloquium Publications, vol. 23, Providence, RI, 1975.
[26] G. Wahba, Spline interpolation and smoothing on the sphere, SIAM J. Sci. Statist. Comput. 2 (1981) 5–16.
[27] Y. Xu, E.W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc. 116 (1992) 977–981.