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SYMPLECTIC INTEGRATION OF HAMILTONIAN SYSTEMS WITH ADDITIVE NOISE*

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Abstract. Hamiltonian systems with additive noise possess the property of preserving symplectic structure. Numerical methods with the same property are constructed for such systems. Special attention is paid to systems with separable Hamiltonians and to second-order differential equations with additive noise. Some numerical tests are presented.

Key words. Hamiltonian systems with additive noise, symplectic integration, mean-square methods for stochastic differential equations

AMS subject classifications. 60H10, 65C30, 65P10

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1. Introduction.

The problem of preserving integral invariants in numerical integration of deterministic differential equations is of great current interest (see, e.g., [13, 3, 5, 12, 15, 16] and references therein). The phase flows of some classes of stochastic systems possess the property of phase-volume preservation or possess the stronger property of preserving symplectic structure (symplecticness) [2, 8]. For instance, Hamiltonian equations with additive noise make up a rather wide and important class of systems having these properties. In the present paper, we construct special numerical methods which preserve symplectic structure in such stochastic systems.

Consider the Cauchy problem for systems of stochastic differential equations (SDEs) in the sense of Stratonovich,

$$dX = a(t, X)dt + \sum_{r=1}^{m} b_r(t, X) \circ dw_r(t), \qquad X(t_0) = x, \tag{1.1}$$

where X, $a(t, x^1, \ldots, x^d)$, $b_r(t, x^1, \ldots, x^d)$ are d-dimensional column-vectors with the components X^i, a^i, b^i_r, $i = 1, \ldots, d$, and where $w_r(t)$, $r = 1, \ldots, m$, are independent standard Wiener processes.

We denote by $X(t; t_0, x) = X(t; t_0, x^1, \ldots, x^d)$, $t_0 \le t \le t_0 + T$, the solution of problem (1.1). A more detailed notation is $X(t; t_0, x; \omega)$, where ω is an elementary event. It is known that $X(t; t_0, x; \omega)$ is a phase flow for almost every ω. See its properties in, e.g., [2, 6, 4, 8].

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Let us write a system of SDEs of even dimension $d = 2n$ in the form

$$dP = f(t,P,Q)dt + \sum_{r=1}^{m} \sigma_r(t,P,Q) \circ dw_r(t), \qquad P(t_0) = p, \tag{1.2}$$

$$dQ = g(t,P,Q)dt + \sum_{r=1}^{m} \gamma_r(t,P,Q) \circ dw_r(t), \qquad Q(t_0) = q.$$

Here P, Q, f, g, σ_r, γ_r are n-dimensional column-vectors.

Consider the differential 2-form

$$\omega^2 = dp \wedge dq = dp^1 \wedge dq^1 + \cdots + dp^n \wedge dq^n. \tag{1.3}$$

We are interested in systems (1.2) such that the transformation $(p,q) \mapsto (P,Q)$ preserves symplectic structure [1] as follows:

$$dP \wedge dQ = dp \wedge dq, \tag{1.4}$$

i.e., when the sum of the oriented areas of projections of a two-dimensional surface onto the coordinate planes $(p^1,q^1),\dots,(p^n,q^n)$ is an integral invariant. As a consequence, all external powers of the 2-form are invariant for such systems as well. The case of the nth external power gives preservation of phase volume. To avoid confusion, we note that the differentials in (1.2) and (1.4) have different meanings. In (1.2), P, Q are treated as functions of time, and p, q are fixed parameters, while differentiation in (1.4) is made with respect to the initial data p, q.

Phase flows of deterministic Hamiltonian systems (i.e., when $\sigma_r = 0$, $\gamma_r = 0$, $r = 1,\dots,m$, and when there is a function $H(t,p,q)$ such that $f^i = -\partial H/\partial q^i$, $g^i = \partial H/\partial p^i$, $i = 1,\dots,n$) are known to preserve symplectic structure. It turns out (see [2] and section 2 of the present paper) that if there are functions $H(t,p,q)$, $H_r(t,p,q)$, $r = 1,\dots,m$, such that

$$f^i = -\partial H/\partial q^i, \qquad g^i = \partial H/\partial p^i, \tag{1.5}$$
$$\sigma_r^i = -\partial H_r/\partial q^i, \quad \gamma_r^i = \partial H_r/\partial p^i, \quad i = 1,\dots,n, \quad r = 1,\dots,m,$$

then the phase flow of (1.2) preserves symplectic structure. Obviously, the phase flow of a Hamiltonian system with additive noise preserves symplectic structure.

A one-step mean-square approximation $\bar{X}(t+h;t,x)$, $t_0 \le t < t+h \le t_0+T$ of (1.1) is constructed, depending on t, x, h, and $\{w_1(\vartheta)-w_1(t),\dots,w_m(\vartheta)-w_m(t);\ t \le \vartheta \le t+h\}$. Using the one-step approximation, we recurrently obtain the approximation X_k, $k = 0,\dots,N$, $t_{k+1} - t_k = h_{k+1}$, $t_N = t_0 + T$ as follows:

$$X_0 = X(t_0),\ X_{k+1} = \bar{X}(t_{k+1};t_k,X_k).$$

For simplicity, we take $t_{k+1} - t_k = h = T/N$. Note that X_0 may be random. A mean-square method for (1.2) based on the one-step approximation $\bar{P} = \bar{P}(t+h;t,p,q)$, $\bar{Q} = \bar{Q}(t+h;t,p,q)$ preserves symplectic structure (said to be symplectic or Hamiltonian) if

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq\,. \tag{1.6}$$

In section 3, we construct some symplectic methods of mean-square order 1 and order 3/2 for general Hamiltonian systems with additive noise. We propose more effective methods for systems with Hamiltonians of a special form. We consider the case of separable Hamiltonians $H(t,p,q) = V(p) + U(t,q)$ in section 4 and the case of Hamiltonians $H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q)$ with M a constant, symmetric, invertible matrix in section 5. In addition, symplectic methods for Hamiltonian systems with small additive noise can be found in the preprint [10]. These methods are constructed using the results from [11]. Let us emphasize that all the derived methods are efficient with respect to simulation of the used random variables. Section 6 is devoted to numerical experiments which demonstrate the superiority of the proposed symplectic methods over long periods of time in comparison with nonsymplectic methods.

Symplectic methods for Hamiltonian systems with multiplicative noise are in progress. Hamiltonian weak methods will be considered in later publications.

2. Preservation of symplectic structure. Consider system (1.2). Our aim is to indicate a class of stochastic systems, which preserve symplectic structure, i.e., satisfy condition (1.4).

Using the formula of change of variables in differential forms, we obtain

$$\begin{aligned} &dP \wedge dQ = dP^1 \wedge dQ^1 + \cdots + dP^n \wedge dQ^n \\ &= \sum_{k=1}^{n} \sum_{l=k+1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial p^k}\frac{\partial Q^i}{\partial p^l} - \frac{\partial P^i}{\partial p^l}\frac{\partial Q^i}{\partial p^k} \right) dp^k \wedge dp^l \\ &+ \sum_{k=1}^{n} \sum_{l=k+1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial q^k}\frac{\partial Q^i}{\partial q^l} - \frac{\partial P^i}{\partial q^l}\frac{\partial Q^i}{\partial q^k} \right) dq^k \wedge dq^l \\ &+ \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial p^k}\frac{\partial Q^i}{\partial q^l} - \frac{\partial P^i}{\partial q^l}\frac{\partial Q^i}{\partial p^k} \right) dp^k \wedge dq^l. \end{aligned}$$

Hence, the phase flow of (1.2) preserves symplectic structure if and only if

$$\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(p^k, p^l)} = 0, \qquad k \neq l, \tag{2.1}$$

$$\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(q^k, q^l)} = 0, \qquad k \neq l, \tag{2.2}$$

and

$$\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(p^k, q^l)} = \delta_{kl}, \qquad k, l = 1, \dots, n. \tag{2.3}$$

Introduce the notation

$$P_p^{ik} = \frac{\partial P^i}{\partial p^k}, \;\; P_q^{ik} = \frac{\partial P^i}{\partial q^k}, \;\; Q_p^{ik} = \frac{\partial Q^i}{\partial p^k}, \;\; Q_q^{ik} = \frac{\partial Q^i}{\partial q^k}.$$

For a fixed k, we obtain that P_p^{ik}, Q_p^{ik}, $i = 1, \dots, n$, obey the following system of SDEs:

$$dP_p^{ik} = \sum_{\alpha=1}^{n} \left(\frac{\partial f^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha} Q_p^{\alpha k} \right) dt + \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \sigma_r^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha} Q_p^{\alpha k} \right) \circ dw_r, \tag{2.4}$$

$$P_p^{ik}(t_0) = \delta_{ik},$$

$$dQ_p^{ik} = \sum_{\alpha=1}^{n}\left(\frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha k}\right)dt + \sum_{r=1}^{m}\sum_{\alpha=1}^{n}\left(\frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha k}\right)\circ dw_r,$$
$$Q_p^{ik}(t_0) = 0.$$

Analogously, for a fixed k, P_q^{ik}, Q_q^{ik}, $i = 1, \dots, n$, satisfy the system

$$(2.5)\quad dP_q^{ik} = \sum_{\alpha=1}^{n}\left(\frac{\partial f^i}{\partial p^\alpha}P_q^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha}Q_q^{\alpha k}\right)dt + \sum_{r=1}^{m}\sum_{\alpha=1}^{n}\left(\frac{\partial \sigma_r^i}{\partial p^\alpha}P_q^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_q^{\alpha k}\right)\circ dw_r,$$
$$P_q^{ik}(t_0) = 0,$$

$$dQ_q^{ik} = \sum_{\alpha=1}^{n}\left(\frac{\partial g^i}{\partial p^\alpha}P_q^{\alpha k} + \frac{\partial g^i}{\partial q^\alpha}Q_q^{\alpha k}\right)dt + \sum_{r=1}^{m}\sum_{\alpha=1}^{n}\left(\frac{\partial \gamma_r^i}{\partial p^\alpha}P_q^{\alpha k} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_q^{\alpha k}\right)\circ dw_r,$$
$$Q_q^{ik}(t_0) = \delta_{ik}.$$

The coefficients in (2.4) and (2.5) are calculated at (t, P, Q) with $P = P(t) = [P^1(t; t_0, p, q), \dots, P^n(t; t_0, p, q)]^\top$, where $Q = Q(t) = [Q^1(t; t_0, p, q), \dots, Q^n(t; t_0, p, q)]^\top$ is a solution to (1.2).

Consider condition (2.1). Clearly,

$$\frac{D(P^i(t_0), Q^i(t_0))}{D(p^k, p^l)} = \frac{D(p^i, q^i)}{D(p^k, p^l)} = 0.$$

Therefore, (2.1) is fulfilled if and only if

$$(2.6)\qquad \sum_{i=1}^{n} d\frac{D(P^i(t), Q^i(t))}{D(p^k, p^l)} = 0.$$

Due to (2.4), we get

$$\begin{aligned} d\frac{\partial P^i}{\partial p^k}\frac{\partial Q^i}{\partial p^l} &= dP_p^{ik}(t)Q_p^{il}(t) \\ &= \sum_{\alpha=1}^{n}\left[\left(\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha k}\right)Q_p^{il} + \left(\frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha l} + \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha l}\right)P_p^{ik}\right]dt \\ &+ \sum_{r=1}^{m}\sum_{\alpha=1}^{n}\left[\left(\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha k}\right)Q_p^{il} + \left(\frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha l} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha l}\right)P_p^{ik}\right]\circ dw_r\ . \end{aligned}$$

Then (2.6) holds if and only if the following equalities take place:

$$\begin{aligned} (2.7)\quad \sum_{i=1}^{n}\sum_{\alpha=1}^{n}\Bigg(&\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha k}Q_p^{il} + \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha k}Q_p^{il} + \frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha l}P_p^{ik} + \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha l}P_p^{ik} \\ &-\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha l}Q_p^{ik} - \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha l}Q_p^{ik} - \frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha k}P_p^{il} - \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha k}P_p^{il}\Bigg) = 0, \end{aligned}$$

$$\begin{aligned} (2.8)\quad \sum_{i=1}^{n}\sum_{\alpha=1}^{n}\Bigg(&\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha k}Q_p^{il} + \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha k}Q_p^{il} + \frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha l}P_p^{ik} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha l}P_p^{ik} \\ &-\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha l}Q_p^{ik} - \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha l}Q_p^{ik} - \frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha k}P_p^{il} - \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha k}P_p^{il}\Bigg) = 0,\ r = 1, \dots, m. \end{aligned}$$

It is not difficult to check that if the functions $f^i(t,p,q)$, $g^i(t,p,q)$ are such that

$$\frac{\partial f^i}{\partial p^\alpha} + \frac{\partial g^\alpha}{\partial q^i} = 0, \quad \frac{\partial f^i}{\partial q^\alpha} = \frac{\partial f^\alpha}{\partial q^i}, \quad \frac{\partial g^i}{\partial p^\alpha} = \frac{\partial g^\alpha}{\partial p^i}, \quad i, \alpha = 1, \dots, n, \tag{2.9}$$

then (2.7) holds, and if the functions $\sigma_r^i(t,p,q)$, $\gamma_r^i(t,p,q)$, $r = 1, \dots, m$, are such that

$$\frac{\partial \sigma_r^i}{\partial p^\alpha} + \frac{\partial \gamma_r^\alpha}{\partial q^i} = 0, \quad \frac{\partial \sigma_r^i}{\partial q^\alpha} = \frac{\partial \sigma_r^\alpha}{\partial q^i}, \quad \frac{\partial \gamma_r^i}{\partial p^\alpha} = \frac{\partial \gamma_r^\alpha}{\partial p^i}, \quad i, \alpha = 1, \dots, n, \tag{2.10}$$

then (2.8) holds. Thus, if relations (2.9)–(2.10) take place, then condition (2.1) is fulfilled.

Condition (2.2) also holds when (2.9)–(2.10) are true. This can be proved analogously by using (2.5) instead of (2.4).

Now consider condition (2.3). Clearly,

$$\sum_{i=1}^n \frac{D(P^i(t_0), Q^i(t_0))}{D(p^k, q^l)} = \sum_{i=1}^n \frac{D(p^i, q^i)}{D(p^k, q^l)} = \delta_{kl}.$$

Then condition (2.3) is fulfilled if and only if

$$\sum_{i=1}^n d\frac{D(P^i(t), Q^i(t))}{D(p^k, q^l)} = 0.$$

Using the same arguments again, we prove that the relations (2.9)–(2.10) ensure this condition as well.

Finally, noting that relations (1.5) imply (2.9)–(2.10), we obtain the following proposition (cf. [2]).

THEOREM 2.1. *The phase flow of the system of SDEs*

$$dP^i = -\frac{\partial H}{\partial q^i}(t,P,Q)dt - \sum_{r=1}^m \frac{\partial H_r}{\partial q^i}(t,P,Q) \circ dw_r(t),$$
$$dQ^i = \frac{\partial H}{\partial p^i}(t,P,Q)dt + \sum_{r=1}^m \frac{\partial H_r}{\partial p^i}(t,P,Q) \circ dw_r(t), \qquad i = 1, \dots, n,$$

with Hamiltonians $H(t,p,q)$, $H_r(t,p,q)$, $r = 1, \dots, m$, *preserves symplectic structure.*

COROLLARY 2.2. *The phase flow of a Hamiltonian system with additive noise preserves symplectic structure.*

3. Symplectic mean-square methods for general Hamiltonian systems with additive noise. In this section we consider the general Hamiltonian system with additive noise

$$\begin{aligned} dP &= f(t,P,Q)dt + \sum_{r=1}^m \sigma_r(t)dw(t), \qquad P(t_0) = p, \\ dQ &= g(t,P,Q)dt + \sum_{r=1}^m \gamma_r(t)dw(t), \qquad Q(t_0) = q, \end{aligned} \tag{3.1}$$

$$f^i = -\partial H/\partial q^i, \quad g^i = \partial H/\partial p^i, \qquad i = 1, \dots, n, \tag{3.2}$$

where P, Q, f, g, σ_r, γ_r are n-dimensional column-vectors, $w_r(t)$, $r = 1, \ldots, m$, are independent standard Wiener processes, and $H(t, p, q)$ is a Hamiltonian.

In this paper we suppose that H is sufficiently smooth and that first derivatives of f and g are bounded. Moreover, we also require boundedness of some higher order derivatives of f and g. At the same time, we emphasize that these conditions are not necessary and the methods obtained are more widely applicable.

3.1. First-order methods. Consider the two-parameter family of implicit methods

$$\begin{aligned}\mathcal{P} &= P_k + hf(t_k + \beta h, \alpha\mathcal{P} + (1-\alpha)P_k, (1-\alpha)\mathcal{Q} + \alpha Q_k),\\ \mathcal{Q} &= Q_k + hg(t_k + \beta h, \alpha\mathcal{P} + (1-\alpha)P_k, (1-\alpha)\mathcal{Q} + \alpha Q_k),\end{aligned} \tag{3.3}$$

$$P_{k+1} = \mathcal{P} + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r, \quad Q_{k+1} = \mathcal{Q} + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r, \qquad k = 0, \ldots, N-1, \tag{3.4}$$

where $\Delta_k w_r(h) := w_r(t_k + h) - w_r(t_k)$ and the parameters $\alpha, \beta \in [0, 1]$.

When $\sigma_r = 0$, $\gamma_r = 0$, $r = 1, \ldots, m$, this family coincides with the known family of symplectic methods for deterministic Hamiltonian systems (see [16]).

The following lemma guarantees the unique solvability of (3.3) with respect to $\mathcal{P}$, $\mathcal{Q}$ for any P_k, Q_k, and sufficiently small h.

LEMMA 3.1. *Let $F(x; c, s)$ be a continuous d-dimensional vector-function depending on $x \in R^d$, $c \in R^d$, and $s \in S$, where S is a set from an R^l. Suppose F has the first partial derivatives $\partial F^i/\partial x^j$, $i, j = 1, \ldots, d$, which are uniformly bounded in $R^d \times R^d \times S$. Then there is an $h_0 > 0$ such that the equation*

$$x = c + hF(x; c, s) + \nu \tag{3.5}$$

is uniquely solvable with respect to x for $0 < h \le h_0$ and any $c \in R^d$, $\nu \in R^d$, $s \in S$. The solution of (3.5) *can be found by the method of simple iteration with an arbitrary initial approximation.*

The proof of this lemma is not difficult and is therefore omitted. The next lemma is true for system (3.1) with arbitrary f and g (i.e., f and g may not obey condition (3.2)).

LEMMA 3.2. *The mean-square order of the methods* (3.3)–(3.4) *for system* (3.1) *is equal to* 1.

The proof is based on comparison of the one-step approximation of methods (3.3)–(3.4) with the one-step approximation of the Euler method (see details in [10]).

As remarked in our introduction, the method based on a one-step approximation $\tilde{P} = \tilde{P}(t+h; t, p, q)$, $\tilde{Q} = \tilde{Q}(t+h; t, p, q)$ preserves symplectic structure if its one-step approximation satisfies condition (1.6). The one-step approximation $\tilde{P}$, $\tilde{Q}$ of methods (3.3)–(3.4) is such that $d\tilde{P} = d\mathcal{P}$, $d\tilde{Q} = d\mathcal{Q}$. Hence $d\tilde{P} \wedge d\tilde{Q} = d\mathcal{P} \wedge d\mathcal{Q}$. The relations for $\mathcal{P}$, $\mathcal{Q}$ coincide with those for the one-step approximation corresponding to the deterministic symplectic method [16]. Therefore, methods (3.3)–(3.4) are symplectic as well. From here and Lemma 3.2, we get the theorem.

THEOREM 3.3. *Methods* (3.3)–(3.4) *for system* (3.1)–(3.2) *preserve symplectic structure and have the first mean-square order of convergence.*

Now consider another generalization of the same family of deterministic symplectic methods to system (3.1),

$$
\begin{aligned}
P_{k+1} &= P_k + hf(t_k+\beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k) \\
&\quad + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r, \\
Q_{k+1} &= Q_k + hg(t_k+\beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k) \\
&\quad + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r, \qquad k = 0,\dots,N-1,
\end{aligned} \tag{3.6}
$$

with the parameters $\alpha, \beta \in [0,1]$.

For sufficiently small h, equations (3.6) are uniquely solvable with respect to P_{k+1}, Q_{k+1} according to Lemma 3.1.

THEOREM 3.4. *Method* (3.6) *for system* (3.1)–(3.2) *preserves symplectic structure and has the first mean-square order of convergence.*

Proof. Comparing the one-step approximation of method (3.6) with the one-step approximation of the Euler method, one can establish that the mean-square order of method (3.6) is equal to 1.

Now we check symplecticness of the method. Let $\tilde{P}$, $\tilde{Q}$ be the one-step approximation corresponding to method (3.6). Introduce

$$
\begin{aligned}
\hat{p} = p + \alpha \sum_{r=1}^{m} \sigma_r(t)\Delta w_r, \quad \hat{q} = q + (1-\alpha)\sum_{r=1}^{m} \gamma_r(t)\Delta w_r, \\
\hat{P} = \tilde{P} - (1-\alpha)\sum_{r=1}^{m} \sigma_r(t)\Delta w_r, \quad \hat{Q} = \tilde{Q} - \alpha\sum_{r=1}^{m} \gamma_r(t)\Delta w_r \,.
\end{aligned}
$$

We have

$$
\begin{aligned}
\hat{P} &= \hat{p} + hf(t+\beta h, \alpha\hat{P} + (1-\alpha)\hat{p}, (1-\alpha)\hat{Q} + \alpha\hat{q}), \\
\hat{Q} &= \hat{q} + hg(t+\beta h, \alpha\hat{P} + (1-\alpha)\hat{p}, (1-\alpha)\hat{Q} + \alpha\hat{q}).
\end{aligned}
$$

The relations for $\hat{P}$, $\hat{Q}$ coincide with the one-step approximation corresponding to the symplectic deterministic method. Therefore, $d\hat{P} \wedge d\hat{Q} = d\hat{p} \wedge d\hat{q}$. Further, it is obvious that $d\hat{P} \wedge d\hat{Q} = d\tilde{P} \wedge d\tilde{Q}$ and $d\hat{p} \wedge d\hat{q} = dp \wedge dq$. Consequently, $d\tilde{P} \wedge d\tilde{Q} = dp \wedge dq$; i.e., method (3.6) is symplectic. ∎

3.2. Methods of order 3/2. For $i = 1,\dots,s$, consider the relations

$$
\mathcal{P}_i = p + h\sum_{j=1}^{s} \alpha_{ij} f(t+c_j h, \mathcal{P}_j, \mathcal{Q}_j) + \varphi_i, \;\; \mathcal{Q}_i = q + h\sum_{j=1}^{s} \alpha_{ij} g(t+c_j h, \mathcal{P}_j, \mathcal{Q}_j) + \psi_i, \tag{3.7}
$$

$$
\bar{P} = p + h\sum_{i=1}^{s} \beta_i f(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i) + \eta, \qquad \bar{Q} = q + h\sum_{i=1}^{s} \beta_i g(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i) + \zeta, \tag{3.8}
$$

where φ_i, ψ_i, η, ζ do not depend on p and q, the parameters α_{ij} and β_i satisfy the conditions

$$\beta_i \alpha_{ij} + \beta_j \alpha_{ji} - \beta_i \beta_j = 0, \qquad i, j = 1, \dots, s, \tag{3.9}$$

and c_i are arbitrary parameters.

Equations (3.7) are uniquely solvable with respect to $\mathcal{P}_i$, $\mathcal{Q}_i$, $i = 1, \dots, s$, for any p, q, φ_i, ψ_i, η, ζ, and sufficiently small h according to Lemma 3.1.

If $\varphi_i = \psi_i = \eta = \zeta = 0$, the relations (3.7)–(3.8) coincide with a general form of s-stage Runge–Kutta (RK) methods for deterministic differential equations. It is known (see, e.g., Theorem 6.1 in [13]) that the symplectic condition $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$ holds for $\bar{P}$, $\bar{Q}$ from (3.7)–(3.8) with (3.9) and $\varphi_i = \psi_i = \eta = \zeta = 0$. Let us check the case of arbitrary φ_i, ψ_i, η, ζ.

LEMMA 3.5. *Relations* (3.7)–(3.8) *with condition* (3.9) *preserve symplectic structure, i.e.,* $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$.

Proof. We generalize the proof of Theorem 6.1 in [13]. Introduce the temporary notation $f_i = f(t + c_i h, \mathcal{P}_i, \mathcal{Q}_i)$, $g_i = g(t + c_i h, \mathcal{P}_i, \mathcal{Q}_i)$. Differentiate (3.7) and form the exterior products

(3.10)
$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq + h \sum_{i=1}^{s} \beta_i \, df_i \wedge dq + h \sum_{j=1}^{s} \beta_j \, dp \wedge dg_j + h^2 \sum_{i,j=1}^{s} \beta_i \beta_j \, df_i \wedge dg_j,$$

$$df_i \wedge d\mathcal{Q}_i = df_i \wedge dq + h \sum_{j=1}^{s} \alpha_{ij} \, df_i \wedge dg_j, \tag{3.11}$$

$$d\mathcal{P}_j \wedge dg_j = dp \wedge dg_j + h \sum_{i=1}^{s} \alpha_{ji} \, df_i \wedge dg_j. \tag{3.12}$$

Now using (3.11)–(3.12), find the expressions for $df_i \wedge dq$ and $dp \wedge dg_j$ and substitute them in (3.10) as follows:

$$\begin{aligned} d\bar{P} \wedge d\bar{Q} = dp \wedge dq &+ h \sum_{i=1}^{s} \beta_i \left(df_i \wedge d\mathcal{Q}_i + d\mathcal{P}_i \wedge dg_i \right) \\ &+ h^2 \sum_{i,j=1}^{s} \left(\beta_i \beta_j - \beta_i \alpha_{ij} - \beta_j \alpha_{ji} \right) df_i \wedge dg_j. \end{aligned} \tag{3.13}$$

The last term in the right-hand side vanishes owing to (3.9). Consider the second term in the right-hand side of (3.13). We have

$$\begin{aligned} df_i \wedge d\mathcal{Q}_i + d\mathcal{P}_i \wedge dg_i &= \sum_{k=1}^{n} \left(df_i^k \wedge d\mathcal{Q}_i^k + d\mathcal{P}_i^k \wedge dg_i^k \right) \\ = \sum_{k,l=1}^{n} &\left(\frac{\partial f_i^k}{\partial p^l} \, d\mathcal{P}_i^l \wedge d\mathcal{Q}_i^k + \frac{\partial f_i^k}{\partial q^l} \, d\mathcal{Q}_i^l \wedge d\mathcal{Q}_i^k + \frac{\partial g_i^k}{\partial p^l} \, d\mathcal{P}_i^k \wedge d\mathcal{P}_i^l + \frac{\partial g_i^k}{\partial q^l} \, d\mathcal{P}_i^k \wedge d\mathcal{Q}_i^l \right). \end{aligned}$$

Taking into account that the exterior product is skew-symmetric and f and g satisfy condition (1.5), it is not difficult to see that this expression vanishes. Returning to (3.13), we obtain $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$. ∎

The next lemma is used in Theorem 3.7 for the Hamiltonian system (3.1)–(3.2). However, this lemma is of interest for arbitrary systems with additive noise as well (see Remark 3.2 below). So, consider the system with additive noise

$$dX = a(t,X)dt + \sum_{r=1}^{m} b_r(t)dw_r(t), \qquad X(t_0) = X_0, \tag{3.14}$$

and introduce the following parametric family of one-step approximations for (3.14):

(3.15)

$$\mathsf{X}_1 = x + \frac{\alpha}{2}ha\left(t + \frac{\alpha}{2}h, \mathsf{X}_1\right) + \sum_{r=1}^{m} b_r(t)\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right),$$

$$\mathsf{X}_2 = x + \alpha ha\left(t + \frac{\alpha}{2}h, \mathsf{X}_1\right) + \frac{1-\alpha}{2}ha\left(t + \frac{1+\alpha}{2}h, \mathsf{X}_2\right) + \sum_{r=1}^{m} b_r(t)\left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right),$$

$$\bar{X} = x + h\left[\alpha a\left(t + \frac{\alpha}{2}h, \mathsf{X}_1\right) + (1-\alpha)a\left(t + \frac{1+\alpha}{2}h, \mathsf{X}_2\right)\right] + \sum_{r=1}^{m} b_r(t)\Delta w_r + \sum_{r=1}^{m} b_r'(t) I_{0r},$$

where

$$\Delta w_r := w_r(t+h) - w_r(t), \qquad I_{0r} := \int_t^{t+h} (\vartheta - t)\, dw_r(\vartheta), \tag{3.16}$$

$$J_{r0} := \frac{1}{h}\int_t^{t+h} \left(w_r(\vartheta) - w_r(t)\right) d\vartheta,$$

and the parameters α, λ_1, λ_2, μ_1, μ_2 are such that

$$\alpha\lambda_1 + (1-\alpha)\lambda_2 = 1, \qquad \alpha\mu_1 + (1-\alpha)\mu_2 = 0, \tag{3.17}$$

$$\alpha\left(\frac{\lambda_1^2}{3} + \lambda_1\mu_1 + \mu_1^2\right) + (1-\alpha)\left(\frac{\lambda_2^2}{3} + \lambda_2\mu_2 + \mu_2^2\right) = \frac{1}{2}. \tag{3.18}$$

For example, the following set of parameters satisfies (3.17)–(3.18):

$$\alpha = \frac{1}{2}, \quad \lambda_1 = \lambda_2 = 1, \quad \mu_1 = -\mu_2 = \frac{1}{\sqrt{6}}. \tag{3.19}$$

Note that random variables Δw_r and J_{r0} are of the same mean-square order $O(h^{1/2})$.

LEMMA 3.6. *The method for the system with additive noise* (3.14) *based on the one-step approximation* (3.15) *with conditions* (3.17)–(3.18) *is of mean-square order* 3/2.

Proof. Due to properties of the Wiener process and Itô integrals, we get

(3.20)

$$E\Delta w_i = 0, \quad E\Delta w_i \Delta w_j = \delta_{ij}h, \quad E\Delta w_i \Delta w_j \Delta w_k = 0, \quad E\left(\Delta w_i\right)^4 = 3h^2,$$

$$EJ_{i0} = 0, \quad EJ_{i0}J_{j0} = \delta_{ij}\frac{h}{3}, \quad EJ_{i0}J_{j0}J_{k0} = 0, \quad E\left(J_{i0}\right)^4 = \frac{h^2}{3},$$

$$E\Delta w_i J_{j0} = \delta_{ij}\frac{h}{2}, \quad E\Delta w_i \Delta w_j J_{k0} = 0, \quad E\Delta w_i J_{j0} J_{k0} = 0.$$

Let $\Delta\mathsf{X}_i := \mathsf{X}_i - x$, $i = 1, 2$. We have

$$|E\Delta\mathsf{X}_i| = O(h),\ E\left(\Delta\mathsf{X}_i\right)^{2l} = O(h^l),\ l = 1,2,3,4,\ i = 1,2\,,\ \left|E\left(\Delta\mathsf{X}_i\right)^3\right| = O(h^2). \tag{3.21}$$

Expand (3.15) as follows:

$$\Delta\mathsf{X}_1 = \frac{\alpha}{2} ha(t,x) + \sum_{r=1}^{m} b_r(t)\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right) + \rho_1, \tag{3.22}$$

$$\Delta\mathsf{X}_2 = \frac{1+\alpha}{2} ha(t,x) + \sum_{r=1}^{m} b_r(t)\left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right) + \rho_2, \tag{3.23}$$

$$\begin{aligned} \bar{X} = x &+ \sum_{r=1}^{m} b_r(t)\Delta w_r + \sum_{r=1}^{m} b_r'(t) I_{0r} + ha(t,x) \\ &+ h\sum_{i=1}^{d} \frac{\partial a}{\partial x^i}(t,x)\left(\alpha\Delta\mathsf{X}_1^i + (1-\alpha)\Delta\mathsf{X}_2^i\right) + \frac{h^2}{2}\frac{\partial a}{\partial t}(t,x) \\ &+ \frac{h}{2}\sum_{i,j=1}^{d} \frac{\partial^2 a}{\partial x^i \partial x^j}(t,x)\left(\alpha\Delta\mathsf{X}_1^i\Delta\mathsf{X}_1^j + (1-\alpha)\Delta\mathsf{X}_2^i\Delta\mathsf{X}_2^j\right) + \bar{\rho}. \end{aligned} \tag{3.24}$$

Using (3.20)–(3.21), one can obtain

$$|E\rho_i| = O(h^2), \quad \left|E\rho_i^l\Delta\mathsf{X}_i^k\right| = O(h^2), \quad E\rho_i^2 = O(h^3), \tag{3.25}$$

$$|E\bar{\rho}| = O(h^3), \quad E\bar{\rho}^2 = O(h^5). \tag{3.26}$$

Substituting (3.22)–(3.23) in (3.24) and using (3.17), we get

$$\begin{aligned} \bar{X} = x &+ \sum_{r=1}^{m} b_r\Delta w_r + \sum_{r=1}^{m} b_r' I_{0r} + ha + \frac{h^2}{2}\frac{\partial a}{\partial t} + \frac{h^2}{2}\sum_{i=1}^{d}\frac{\partial a}{\partial x^i}a^i \\ &+ h\sum_{r=1}^{m}\sum_{i=1}^{d} b_r^i \frac{\partial a}{\partial x^i} J_{r0} + \frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i\partial x^j} b_r^i b_r^j + R; \end{aligned} \tag{3.27}$$

$$\begin{aligned} R = \frac{h}{2}\sum_{r,l=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i\partial x^j} b_r^i b_l^j \cdot &\left[\alpha\left(\lambda_1 J_{r0} + \mu_1\Delta w_r\right)\left(\lambda_1 J_{l0} + \mu_1 \Delta w_l\right)\right. \\ &+ \left.(1-\alpha)\left(\lambda_2 J_{r0} + \mu_2\Delta w_r\right)\left(\lambda_2 J_{l0} + \mu_2\Delta w_l\right)\right] - \frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i\partial x^j} b_r^i b_r^j + \rho, \end{aligned}$$

where the coefficients and their derivatives are calculated at (t, x) and where ρ satisfies the same relations as $\bar{\rho}$ (see (3.26)).

Relations (3.20) and (3.18) imply

$$
\begin{aligned}
(3.28)\qquad & E[\alpha\,(\lambda_1 J_{r0} + \mu_1 \Delta w_r)\,(\lambda_1 J_{l0} + \mu_1 \Delta w_l) \\
& \quad + (1-\alpha)\,(\lambda_2 J_{r0} + \mu_2 \Delta w_r)\,(\lambda_2 J_{l0} + \mu_2 \Delta w_l)] = \frac{h}{2}\delta_{rl}.
\end{aligned}
$$

Using relations (3.20), (3.25)–(3.26), and (3.28), it is not difficult to get that

$$
(3.29)\qquad |ER| = O(h^3), \qquad \left(ER^2\right)^{1/2} = O(h^2).
$$

Now, comparing (3.27) with the one-step approximation of the standard method of mean-square order 3/2 for systems with additive noise [9, p. 37] (see also [7]), we obtain that method (3.15) is of mean-square order 3/2. □

REMARK 3.1. *It is very unusual that the direct expansion of* (3.15) *does not contain the habitual term* $\frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i \partial x^j} b_r^i b_r^j$. *A similar term in the expansion contains some combinations of* Δw_r *and* J_{r0} *instead of* h. *This is necessary for a method conserving symplectic structure. At the same time, this new reception allows us to construct new RK methods for general* (*not only Hamiltonian*) *stochastic systems with additive noise* (*see the next remark*).

REMARK 3.2. *In a way similar to how method* (3.15) *was obtained, it is not difficult to construct new explicit RK methods of mean-square order* 3/2 *for an arbitrary system of differential equations with additive noise* (3.14). *For instance, we obtain the following explicit RK method of order* 3/2 *for* (3.14):

$$
\begin{aligned}
X_{k+1} = X_k &+ \sum_{r=1}^{m} b_r(t_k)\Delta_k w_r + \frac{h}{2}\left[a\left(t_k, X_k + \sum_{r=1}^{m} b_r(t_k)\cdot\left((J_{r0})_k + \frac{1}{\sqrt{6}}\Delta_k w_r\right)\right)\right. \\
&\left. + \, a\left(t_k + h, X_k + h a(t_k, X_k) + \sum_{r=1}^{m} b_r(t_k)\cdot\left((J_{r0})_k - \frac{1}{\sqrt{6}}\Delta_k w_r\right)\right)\right] \\
&+ \sum_{r=1}^{m} b_r'(t_k)(I_{0r})_k, \qquad k = 0, \dots, N-1.
\end{aligned}
$$

Note that if we apply this method, as well as any other explicit RK method, to (3.1)–(3.2), *it will not preserve symplectic structure.*

Now consider the parametric family of methods for the Hamiltonian system with additive noise (3.1),

$$
\begin{aligned}
(3.30)\quad \mathcal{P}_1 &= P_k + \frac{\alpha}{2} h f\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + \sum_{r=1}^{m} \sigma_r(t_k)\left(\lambda_1 \left(J_{r0}\right)_k + \mu_1 \Delta_k w_r\right), \\
\mathcal{Q}_1 &= Q_k + \frac{\alpha}{2} h g\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + \sum_{r=1}^{m} \gamma_r(t_k)\left(\lambda_1 \left(J_{r0}\right)_k + \mu_1 \Delta_k w_r\right);
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_2 = P_k &+ \alpha h f\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + \frac{1-\alpha}{2} h f\left(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2\right) \\
&+ \sum_{r=1}^{m} \sigma_r(t_k)\left(\lambda_2 \left(J_{r0}\right)_k + \mu_2 \Delta_k w_r\right);
\end{aligned}
$$

$$\mathcal{Q}_2 = Q_k + \alpha h g\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + \frac{1-\alpha}{2}hg\left(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2\right)$$
$$+ \sum_{r=1}^{m} \gamma_r(t_k)\left(\lambda_2 \left(J_{r0}\right)_k + \mu_2 \Delta_k w_r\right);$$

$$P_{k+1} = P_k + h\left[\alpha f\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + (1-\alpha) f\left(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2\right)\right]$$
$$+ \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k)\left(I_{0r}\right)_k;$$

$$Q_{k+1} = Q_k + h\left[\alpha g\left(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1\right) + (1-\alpha) g\left(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2\right)\right]$$
$$+ \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \gamma_r'(t_k)\left(I_{0r}\right)_k,$$

where the parameters α, λ_1, λ_2, μ_1, μ_2 satisfy (3.17)–(3.18).

Under $\sigma_r \equiv 0$, $\gamma_r \equiv 0$, $r = 1, \dots, m$, method (3.30) is reduced to the well-known second-order symplectic RK method for deterministic Hamiltonian systems (see, e.g., [13, p. 101]). Let us note that, using this deterministic method with $\alpha = 0$ (the midpoint rule), another implicit 3/2-order method for Hamiltonian systems with noise was proposed in [17]—however, without preserving symplectic structure.

The one-step approximation corresponding to method (3.30) is of the form (3.15). Therefore, due to Lemma 3.6, method (3.30) is of mean-square order 3/2. Moreover, this one-step approximation is of the form (3.7) with $s = 2$ and

$$\varphi_1 = \sum_{r=1}^{m} \sigma_r\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right), \qquad \varphi_2 = \sum_{r=1}^{m} \sigma_r\left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right),$$
$$\psi_1 = \sum_{r=1}^{m} \gamma_r\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right), \qquad \psi_2 = \sum_{r=1}^{m} \gamma_r\left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right),$$
$$\eta = \sum_{r=1}^{m} \sigma_r \Delta w_r + \sum_{r=1}^{m} \sigma_r' I_{0r}, \qquad \zeta = \sum_{r=1}^{m} \gamma_r \Delta w_r + \sum_{r=1}^{m} \gamma_r' I_{0r},$$

$\alpha_{11} = \dfrac{\alpha}{2}$, $\alpha_{12} = 0$, $\alpha_{21} = \alpha$, $\alpha_{22} = \dfrac{1-\alpha}{2}$, $\beta_1 = \alpha$, $\beta_2 = 1 - \alpha$, $c_1 = \dfrac{\alpha}{2}$, $c_2 = \dfrac{1+\alpha}{2}$.

This set of parameters α_{ij}, β_i, $i, j = 1, 2$, satisfies conditions (3.9). Then due to Lemma 3.5, the method (3.30) is symplectic. Thus we obtain the following theorem.

THEOREM 3.7. *Under conditions* (3.17)–(3.18) *on the parameters, method* (3.30) *for system* (3.1)–(3.2) *preserves symplectic structure and has mean-square order* 3/2.

REMARK 3.3. *Formula* (3.30) *contains the random variables* $\Delta_k w_r(h)$, $(J_{r0})_k$, $(I_{0r})_k$, *the joint distribution of which is Gaussian. They can be simulated at each step by* $2m$ *independent* $N(0,1)$*-distributed random variables* ξ_{rk} *and* η_{rk}, $r = 0, \dots, m$ *as follows:*

$$\Delta_k w_r(h) = \sqrt{h}\xi_{rk}, \qquad (J_{r0})_k = \sqrt{h}\left(\xi_{rk}/2 + \eta_{rk}/\sqrt{12}\right), \tag{3.31}$$
$$(I_{0r})_k = h^{3/2}\left(\xi_{rk}/2 - \eta_{rk}/\sqrt{12}\right).$$

So, method (3.30) *can be rewritten in the constructive form.*

4. Symplectic mean-square methods in the case of a separable Hamiltonian. In this section we consider the Hamiltonian system with additive noise (3.1), a Hamiltonian of which has the special structure

$$H(t,p,q) = V(p) + U(t,q). \tag{4.1}$$

Such Hamiltonians are called separable. We note that it is not difficult to consider a slightly more general separable Hamiltonian $H(t,p,q) = V(t,p) + U(t,q)$, but we restrict ourselves here to (4.1). In the case of separable Hamiltonian (4.1), system (3.1) takes the partitioned form

$$\begin{aligned} dP &= f(t,Q)dt + \sum_{r=1}^{m} \sigma_r(t)dw_r(t), \qquad P(t_0) = p, \\ dQ &= g(P)dt + \sum_{r=1}^{m} \gamma_r(t)dw_r(t), \qquad Q(t_0) = q, \end{aligned} \tag{4.2}$$

where $f^i = -\partial U/\partial q^i$, $g^i = \partial V/\partial p^i$, $i = 1, \dots, n$.

Obviously, the implicit symplectic methods from the previous section can be applied to the partitioned system (4.2), and these methods take a more simple form in this case (we do not write them here). We recall that there are no explicit symplectic RK methods for the general system (3.1)–(3.2). However, for the partitioned system (4.2) it is possible to construct explicit symplectic methods just as in the deterministic case [16, 12, 13].

4.1. Explicit first-order methods. On the basis of the known family of deterministic partitioned RK (PRK) methods [16, 12, 13], we construct the family of explicit partitioned methods for stochastic system (4.2) as follows:

$$\begin{aligned} \mathcal{Q} &= Q_k + \alpha h g(P_k), \qquad \mathcal{P} = P_k + h f(t_k + \alpha h, \mathcal{Q}), \\ Q_{k+1} &= \mathcal{Q} + (1-\alpha) h g(\mathcal{P}) + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r, \\ P_{k+1} &= \mathcal{P} + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r, \qquad k = 0, \dots, N-1. \end{aligned} \tag{4.3}$$

Since the expressions for dP_{k+1}, dQ_{k+1} coincide with those corresponding to the deterministic symplectic method, then method (4.3) is symplectic. Further, it is not difficult to show that method (4.3) has the first mean-square order of accuracy. As a result, we obtain the following theorem.

THEOREM 4.1. *The explicit partitioned method* (4.3) *for system* (4.2) *preserves symplectic structure and has the first mean-square order of convergence.*

REMARK 4.1. *In the special cases of* $\alpha = 0$ *and* $\alpha = 1$, *method* (4.3) *takes a more simple form. In these cases it requires evaluation of each of the coefficients* f, g *only once per step.*

REMARK 4.2. *It is possible to propose other symplectic first-order methods for* (4.2) *on the basis of the same deterministic PRK methods as above. For instance, the* *method*

$$\mathcal{Q} = Q_k + \alpha h g(P_k) + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r, \tag{4.4}$$

$$\mathcal{P} = P_k + hf(t_k + \alpha h, \mathcal{Q}) + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r,$$

$$Q_{k+1} = \mathcal{Q} + (1-\alpha)hg(\mathcal{P}),\ P_{k+1} = \mathcal{P}, \qquad k = 0, \dots, N-1,$$

is of first mean-square order and symplectic.

4.2. Explicit methods of order 3/2. In this section, using specificity of system (4.2), we construct a 3/2-order symplectic *explicit* RK method (other symplectic methods for (4.2) are given in [10]).

Introduce the relations (cf. (3.7)–(3.8))

$$\mathcal{P}_i = p + h\sum_{j=1}^{s} \alpha_{ij} f(t + c_j h, \mathcal{Q}_j) + \varphi_i, \tag{4.5}$$

$$\mathcal{Q}_i = q + h\sum_{j=1}^{s} \hat{\alpha}_{ij} g(\mathcal{P}_j) + \psi_i, \qquad i = 1, \dots, s,$$

$$\bar{P} = p + h\sum_{i=1}^{s} \beta_i f(t + c_i h, \mathcal{Q}_i) + \eta, \qquad \bar{Q} = q + h\sum_{i=1}^{s} \hat{\beta}_i g(\mathcal{P}_i) + \zeta, \tag{4.6}$$

where φ_i, ψ_i, η, ζ do not depend on p and q; the parameters α_{ij}, $\hat{\alpha}_{ij}$, β_i, and $\hat{\beta}_i$ satisfy the conditions

$$\beta_i\hat{\alpha}_{ij} + \hat{\beta}_j\alpha_{ji} - \beta_i\hat{\beta}_j = 0, \qquad i, j = 1, \dots, s; \tag{4.7}$$

and c_i are arbitrary parameters.

If $\varphi_i = \psi_i = \eta = \zeta = 0$, the relations (4.5)–(4.6) coincide with a general form of s-stage PRK methods for deterministic differential equations (see, e.g., [13, p. 34]). It is known [16, 13] that the symplectic condition holds for $\bar{P}$, $\bar{Q}$ from (4.5)–(4.6) with (4.7) in the case of $\varphi_i = \psi_i = \eta = \zeta = 0$. By a generalization of the proof of Theorem 6.2 in [13] (see also Lemma 3.5 of this paper), it is not difficult to prove the following lemma.

LEMMA 4.2. *Relations* (4.5)–(4.6) *with condition* (4.7) *preserve symplectic structure, i.e.,* $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$.

Introduce the parametric family of 2-stage explicit PRK methods for system (4.2) as follows:

$$\mathcal{Q}_1 = Q_k + \sum_{r=1}^{m} \gamma_r(t_k)\left(\hat{\lambda}_1 (J_{r0})_k + \hat{\mu}_1 \Delta_k w_r\right), \tag{4.8}$$

$$\mathcal{P}_1 = P_k + h\beta_1 f(t_k + c_1 h, \mathcal{Q}_1) + \sum_{r=1}^{m} \sigma_r(t_k)\left(\lambda_1 (J_{r0})_k + \mu_1 \Delta_k w_r\right),$$

$$\mathcal{Q}_2 = Q_k + h\hat{\beta}_1 g(\mathcal{P}_1) + \sum_{r=1}^{m} \gamma_r(t_k)\left(\hat{\lambda}_2 (J_{r0})_k + \hat{\mu}_2 \Delta_k w_r\right),$$

$$\mathcal{P}_2 = P_k + h\sum_{i=1}^{2} \beta_i f(t_k + c_i h, \mathcal{Q}_i) + \sum_{r=1}^{m} \sigma_r(t_k)\left(\lambda_2 (J_{r0})_k + \mu_2 \Delta_k w_r\right),$$

$$P_{k+1} = P_k + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k)(I_{0r})_k + h\sum_{i=1}^{2} \beta_i f(t_k + c_i h, \mathcal{Q}_i), \tag{4.9}$$

$$Q_{k+1} = Q_k + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \gamma_r'(t_k)(I_{0r})_k + h\sum_{i=1}^{2} \hat{\beta}_i g(\mathcal{P}_i),$$

where the parameters β_i, $\hat{\beta}_i$, c_i, λ_i, $\hat{\lambda}_i$, μ_i, $\hat{\mu}_i$, $i = 1, 2$, satisfy the conditions

$$\beta_1 + \beta_2 = 1, \quad \hat{\beta}_1 + \hat{\beta}_2 = 1, \quad \beta_2\hat{\beta}_1 = 1/2, \quad c_1 = 0, \quad c_2 = \hat{\beta}_1, \tag{4.10}$$

$$\beta_1\hat{\mu}_1 + \beta_2\hat{\mu}_2 = 0, \qquad \hat{\beta}_1\mu_1 + \hat{\beta}_2\mu_2 = 0, \tag{4.11}$$

$$\beta_1\hat{\lambda}_1 + \beta_2\hat{\lambda}_2 = 1, \qquad \hat{\beta}_1\lambda_1 + \hat{\beta}_2\lambda_2 = 1,$$

$$\beta_1\left(\frac{\hat{\lambda}_1^2}{3} + \hat{\lambda}_1\hat{\mu}_1 + \hat{\mu}_1^2\right) + \beta_2\left(\frac{\hat{\lambda}_2^2}{3} + \hat{\lambda}_2\hat{\mu}_2 + \hat{\mu}_2^2\right) = \frac{1}{2},$$

$$\hat{\beta}_1\left(\frac{\lambda_1^2}{3} + \lambda_1\mu_1 + \mu_1^2\right) + \hat{\beta}_2\left(\frac{\lambda_2^2}{3} + \lambda_2\mu_2 + \mu_2^2\right) = \frac{1}{2},$$

and Δw_r, I_{0r}, J_{r0} are defined in (3.16).

For example, the following set of parameters satisfies (4.10)–(4.11):

$$\beta_1 = \frac{1}{4}, \quad \beta_2 = \frac{3}{4}, \quad \hat{\beta}_1 = \frac{2}{3}, \quad \hat{\beta}_2 = \frac{1}{3}, \tag{4.12}$$

$$\lambda_1 = \lambda_2 = \hat{\lambda}_1 = \hat{\lambda}_2 = 1, \quad \mu_1 = \frac{1}{2\sqrt{3}}, \quad \mu_2 = -\frac{1}{\sqrt{3}}, \quad \hat{\mu}_1 = \frac{1}{\sqrt{2}}, \quad \hat{\mu}_2 = -\frac{1}{3\sqrt{2}}.$$

Note that in the deterministic case the family of methods (4.8)–(4.9) with conditions (4.10) on the parameters coincides with the family of 2-stage second-order deterministic PRK methods [13].

It is not difficult to see that method (4.8)–(4.9) has the form of (4.5)–(4.6) and that its parameters satisfy conditions (4.7). Then, Lemma 4.2 implies that this method preserves symplectic structure. Using ideas of the proof of Lemma 3.6, we establish that method (4.8)–(4.9) with (4.10)–(4.11) is of mean-square order 3/2. We have thus proved the following theorem.

THEOREM 4.3. *Under conditions* (4.10)–(4.11)*, the explicit PRK method* (4.8)–(4.9) *for system* (4.2) *preserves symplectic structure and has mean-square order* 3/2.

REMARK 4.3. *Attracting other explicit deterministic second-order PRK methods from* [13, 16]*, it is possible to construct other explicit symplectic methods of order* 3/2 *for system* (4.2)*. For instance, by swapping the roles of p and q in method* (4.8)–(4.9)*, we can obtain another* 3/2*-order symplectic PRK method.*

5. Symplectic methods in the case of Hamiltonian $\boldsymbol{H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q)}$. Here we propose symplectic methods for the Hamiltonian system (4.2), when $\gamma_r(t) = 0$ and the separable Hamiltonian has the special form

$$H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q), \tag{5.1}$$

with M a constant, symmetric, invertible matrix (i.e., the kinetic energy $V(p)$ in (4.1) is equal to $\frac{1}{2}p^\top M^{-1}p$). In this case, system (4.2) reads

$$\begin{aligned} dP &= f(t,Q)dt + \sum_{r=1}^{m} \sigma_r(t)dw_r(t), \qquad P(t_0) = p, \\ dQ &= M^{-1}Pdt, \qquad Q(t_0) = q, \end{aligned} \tag{5.2}$$

$$f^i = -\partial U/\partial q^i, \qquad i = 1, \dots, n. \tag{5.3}$$

This system can be written as a second-order differential equation with additive noise

$$\frac{d^2Q}{dt^2} = M^{-1}f(t,Q) + M^{-1}\sum_{r=1}^{m}\sigma_r(t)\dot{w}_r(t). \tag{5.4}$$

Clearly, the symplectic methods from sections 3 and 4 can be applied to (5.2)–(5.3). Due to specific features of this system, these methods have a more simple form here. Moreover, one can prove that method (4.8)–(4.9), when applied to (5.2)–(5.3), is of mean-square order 2. In this section we restrict ourselves to new explicit methods of orders 2 and 3.

5.1. Explicit methods of order 2. On the basis of the Störmer–Verlet method [15, 16, 13] (the deterministic second-order symplectic method), we construct the method for system (5.2)–(5.3) as follows:

$$\begin{aligned} \mathcal{Q} &= Q_k + \frac{h}{2}M^{-1}P_k, \\ P_{k+1} &= P_k + \sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + hf\left(t_k + \frac{h}{2}, \mathcal{Q}\right) + \sum_{r=1}^{m}\sigma_r'(t_k)(I_{0r})_k, \\ Q_{k+1} &= Q_k + hM^{-1}P_k + \sum_{r=1}^{m}M^{-1}\sigma_r(t_k)(I_{r0})_k + \frac{h^2}{2}M^{-1}f\left(t_k + \frac{h}{2}, \mathcal{Q}\right), \\ & k = 0, \dots, N-1. \end{aligned} \tag{5.5}$$

THEOREM 5.1. *The explicit method* (5.5) *for system* (5.2)–(5.3) *is symplectic and of mean-square order* 2.

Other methods of order 2 are given in [10]. In [14] a symplectic method of mean-square order 1 for (5.2)–(5.3) is proposed on the basis of the Störmer–Verlet method. There, some physical applications of stochastic symplectic integrators are also discussed.

5.2. Explicit methods of order 3. Introduce the integrals

$$(I_{0r})_k = \int_{t_k}^{t_{k+1}} (\vartheta - t_k)\ dw_r(\vartheta), \qquad (I_{r0})_k = \int_{t_k}^{t_{k+1}} (w_r(\vartheta) - w_r(t_k))\ d\vartheta,$$

(5.6)

$$(I_{00r})_k := \frac{1}{2}\int_{t_k}^{t_{k+1}} (\vartheta - t_k)^2\ dw_r(\vartheta), \qquad (I_{0r0})_k := \int_{t_k}^{t_{k+1}}\int_{t_k}^{\vartheta_1} (\vartheta_2 - t_k)\ dw_r(\vartheta_2)d\vartheta_1,$$

$$(I_{r00})_k := \int_{t_k}^{t_{k+1}}\int_{t_k}^{\vartheta_1} (w_r(\vartheta_2) - w_r(t_k))\ d\vartheta_2 d\vartheta_1,\ (J_r)_k = \int_{t_k}^{t_{k+1}} (\vartheta - t_k)(w_r(\vartheta) - w_r(t_k))d\vartheta.$$

Joint distribution of the random variables $\Delta_k w_r(h)$, $(I_{0r})_k$, $(I_{r0})_k$, $(I_{0r0})_k$, $(I_{r00})_k$, $(I_{00r})_k$ is Gaussian. They can be simulated at each step by $3m$ independent $N(0,1)$-distributed random variables ξ_{rk}, η_{rk}, and ζ_{rk}, $r = 1, \ldots, m$ as follows:

(5.7)

$$\Delta_k w_r = h^{1/2}\xi_{rk}, \quad (I_{r0})_k = h^{3/2}(\eta_{rk}/\sqrt{3} + \xi_{rk})/2, \quad (I_{0r})_k = h\Delta_k w_r - (I_{r0})_k,$$
$$(J_r)_k = h^{5/2}(\xi_{rk}/3 + \eta_{rk}/(4\sqrt{3}) + \zeta_{rk}/(12\sqrt{5})),$$
$$(I_{r00})_k = h(I_{r0})_k - (J_r)_k, \quad (I_{0r0})_k = 2(J_r)_k - h(I_{r0})_k, \quad (I_{00r})_k = h^2\Delta_k w_r/2 - (J_r)_k.$$

Clearly, for $\sigma_r = 0$, $r = 1, \ldots, m$, stochastic system (5.2) is reduced to the deterministic system

$$\frac{dp}{dt} = f(t,q), \qquad \frac{dq}{dt} = M^{-1}p\ . \tag{5.8}$$

The following lemma is true for system (5.2) with an arbitrary f (i.e., f may not obey condition (5.3)). Its proof is available in [10].

LEMMA 5.2. *Let $\bar{q} = q + G(t+h;t,p,q)$, $\bar{p} = p + F(t+h;t,p,q)$ be a one-step approximation of the third-order explicit method for the deterministic system* (5.8). *Suppose an n-dimensional* (*deterministic*) *variable $\mathcal{Q} = \mathcal{Q}(t+h;t,p,q)$ is such that*

$$|\mathcal{Q} - q| = O(h).$$

Then, the following method for system (5.2) *is of mean-square order* 3:

$$P_{k+1} = P_k + F(t+h;t,P_k,Q_k) + \sum_{r=1}^m \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^m \sigma_r'(t_k)(I_{0r})_k \tag{5.9}$$
$$+ \sum_{r=1}^m \sigma_r''(t_k)(I_{00r})_k + \sum_{r=1}^m\sum_{i=1}^n (M^{-1}\sigma_r(t_k))^i \frac{\partial f}{\partial q^i}(t_k,\mathcal{Q}_k)(I_{r00})_k,$$
$$Q_{k+1} = Q_k + G(t+h;t,P_k,Q_k) + \sum_{r=1}^m M^{-1}\sigma_r(t_k)(I_{r0})_k + \sum_{r=1}^m M^{-1}\sigma_r'(t_k)(I_{0r0})_k.$$

REMARK 5.1. *Lemma* 5.2 *can be generalized to the system*

$$\frac{d^2Q}{dt^2} = M^{-1}f(t,Q) + \Gamma\frac{dQ}{dt} + M^{-1}\sum_{r=1}^m \sigma_r(t)\dot{w}_r(t),$$

where Γ is a constant matrix. Effective numerical solution of such stochastic systems will be considered in a separate publication.

Using the known deterministic third-order symplectic method (see [15, 16, 13]), we obtain the following method for system (5.2)–(5.3):

$$\mathcal{Q}_1 = Q_k + \frac{7}{24} h M^{-1} P_k, \qquad \mathcal{P}_1 = P_k + \frac{2}{3} h f\left(t_k + \frac{7h}{24}, \mathcal{Q}_1\right),$$
$$\mathcal{Q}_2 = \mathcal{Q}_1 + \frac{3}{4} h M^{-1} \mathcal{P}_1, \qquad \mathcal{P}_2 = \mathcal{P}_1 - \frac{2}{3} h f\left(t_k + \frac{25h}{24}, \mathcal{Q}_2\right),$$
$$\mathcal{Q}_3 = \mathcal{Q}_2 - \frac{1}{24} h M^{-1} \mathcal{P}_2, \qquad \mathcal{P}_3 = \mathcal{P}_2 + h f(t_k + h, \mathcal{Q}_3), \tag{5.10}$$

$$P_{k+1} = \mathcal{P}_3 + \sum_{r=1}^{m} \sigma_r(t_k) \Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k)(I_{0r})_k$$
$$+ \sum_{r=1}^{m} \sigma_r''(t_k)(I_{00r})_k + \sum_{r=1}^{m} \sum_{i=1}^{n} (M^{-1}\sigma_r(t_k))^i \frac{\partial f}{\partial q^i}(t_k, \mathcal{Q}_3)(I_{r00})_k,$$
$$Q_{k+1} = \mathcal{Q}_3 + \sum_{r=1}^{m} M^{-1}\sigma_r(t_k)(I_{r0})_k + \sum_{r=1}^{m} M^{-1}\sigma_r'(t_k)(I_{0r0})_k, \qquad k = 0, \dots, N-1. \tag{5.11}$$

THEOREM 5.3. *The explicit method* (5.10)–(5.11) *for system* (5.2)–(5.3) *is symplectic and of mean-square order* 3.

Proof. It is not difficult to check that $dP_{k+1} \wedge dQ_{k+1} = d\mathcal{P}_3 \wedge d\mathcal{Q}_3$. The expression for $d\mathcal{P}_3 \wedge d\mathcal{Q}_3$ coincides with that corresponding to the deterministic third-order symplectic method from [15, 16, 13]. This implies that method (5.10)–(5.11) is symplectic. By Lemma 5.2 we get that the method has mean-square order 3. ◻

6. Numerical tests. In this section we consider the following Hamiltonian system with additive noise:

$$dX^1 = X^2 dt + \sigma dw_1(t), \qquad X^1(0) = X_0^1,$$
$$dX^2 = -X^1 dt + \gamma dw_2(t), \qquad X^2(0) = X_0^2. \tag{6.1}$$

Introduce the discretization $0 = t_0 < t_1 < \cdots < t_N = T$, $t_{k+1} - t_k = h$; $h > 0$ is a small number. We have, for the solution $X = (X^1, X^2)^\top$ of (6.1),

$$X(t_{k+1}) = F X(t_k) + u_k, \quad X(0) = X_0, \quad k = 0, 1, \dots, N-1, \tag{6.2}$$

where

$$F = \begin{bmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{bmatrix},$$
$$u_k = \begin{bmatrix} \sigma \int_{t_k}^{t_{k+1}} \cos(t_{k+1} - s) dw_1(s) + \gamma \int_{t_k}^{t_{k+1}} \sin(t_{k+1} - s) dw_2(s) \\ -\sigma \int_{t_k}^{t_{k+1}} \sin(t_{k+1} - s) dw_1(s) + \gamma \int_{t_k}^{t_{k+1}} \cos(t_{k+1} - s) dw_2(s) \end{bmatrix}.$$

When applied to (6.1), the explicit symplectic method (4.4) with $\alpha = 1$ takes the form

$$X_{k+1}^2 = X_k^2 - h X_k^1 + \gamma \Delta_k w_2, \qquad X_{k+1}^1 = X_k^1 + h X_{k+1}^2 + \sigma \Delta_k w_1. \tag{6.3}$$

Method (6.3) can be written as

$$X_{k+1} = H X_k + v_k, \qquad k = 0, 1, \dots, N-1, \tag{6.4}$$

where $X_k = (X_k^1, X_k^2)^\top$,

$$H = \begin{bmatrix} 1-h^2 & h \\ -h & 1 \end{bmatrix}, \qquad v_k = \begin{bmatrix} \sigma\Delta_k w_1 + \gamma h \Delta_k w_2 \\ \gamma \Delta_k w_2 \end{bmatrix}.$$

Our aim is to analyze propagation of the error $r_k := X_k - X(t_k)$. We get

$$X(t_k) = F^k X_0 + F^{k-1}u_0 + F^{k-2}u_1 + \cdots + u_{k-1}, \tag{6.5}$$

$$X_k = H^k X_0 + H^{k-1}v_0 + H^{k-2}v_1 + \cdots + v_{k-1}. \tag{6.6}$$

PROPOSITION 6.1. *Suppose T and h are such that Th^2 is sufficiently small. Then for $k = 0, 1, \ldots, N$, $T = Nh$, the following inequality holds:*

$$||H^k - F^k|| \le \frac{h}{2} + \frac{kh^3}{24} + O(h^2 + Th^3) \le \frac{h}{2} + \frac{Th^2}{24} + O(h^2 + Th^3). \tag{6.7}$$

Proof. Clearly,

$$F^k = \begin{bmatrix} \cos kh & \sin kh \\ -\sin kh & \cos kh \end{bmatrix}.$$

Let us represent H as $H = G\Lambda G^{-1}$ with Λ and G such that $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, $\lambda_{1,2} = 1 - \frac{h^2}{2} \pm ih\sqrt{1 - \frac{h^2}{4}}$, and the columns of the matrix G are eigenvectors of H corresponding to the eigenvalues λ_1, λ_2. We write the matrices Λ and G in the form

$$\Lambda = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 1 \\ e^{i\psi} & e^{-i\psi} \end{bmatrix},$$

where $0 < \varphi, \psi < \frac{\pi}{2}$, $\cos\varphi = 1 - \frac{h^2}{2}$, $\cos\psi = \frac{h}{2}$.

We obtain $H^k = G\Lambda^k G^{-1}$,

$$H^k - F^k = G(\Lambda^k - G^{-1}F^k G)G^{-1}, \tag{6.8}$$

$$\begin{aligned} &\Lambda^{\,k} - G^{-1}F^k G \\ &= \begin{bmatrix} e^{ki\varphi} - e^{kih} - i\sin kh \dfrac{1-\sin\psi}{\sin\psi} & -\dfrac{i\sin kh \cdot e^{-i\psi}}{\sin\psi}\cos\psi \\ \dfrac{i\sin kh \cdot e^{i\psi}}{\sin\psi}\cos\psi & e^{-ki\varphi} - e^{-kih} + i\sin kh\dfrac{1-\sin\psi}{\sin\psi} \end{bmatrix}. \end{aligned} \tag{6.9}$$

Let us represent this matrix $\Lambda^k - G^{-1}F^kG$ as the sum $D_1 + D_2$, where $D_2 = \mathrm{diag}(e^{ki\varphi} - e^{kih}, e^{-ki\varphi} - e^{-kih})$. It is not difficult to show that (the norms of matrices are Euclidean)

$$\begin{aligned} &||G|| = \sqrt{2}(1 + O(h)), \qquad ||G^{-1}|| = \frac{\sqrt{2}}{2}(1 + O(h)), \\ &||D_1|| \le \frac{h}{2}(1 + O(h)), \qquad ||D_2|| = 2\left|\sin\frac{k\varphi - kh}{2}\right|. \end{aligned} \tag{6.10}$$

Taking into account that $\varphi = \arcsin\left(h\sqrt{1-\frac{h^2}{4}}\right) = h + \frac{h^3}{24} + O(h^5)$, $kh \leq T$, $k = 0, 1, \ldots, N$, and the assumption on smallness of Th^2, we get

$$||D_2|| \leq \frac{kh^3}{24} + O(h^2) \leq \frac{Th^2}{24} + O(h^2), \qquad k = 0, 1, \ldots, N. \tag{6.11}$$

Inequality (6.7) follows from (6.8)–(6.11). $\square$

Using Proposition 6.1, we prove the following proposition.

PROPOSITION 6.2. *Let T and h be such that Th^2 is sufficiently small. Suppose $E\,|X_0|^2 \leq C$. Then the mean-square error is estimated as*

$$(E|r_k|^2)^{1/2} \leq K \cdot (T^{1/2}h + T^{3/2}h^2), \qquad k = 0, 1, \ldots, N. \tag{6.12}$$

When applied to (6.1), the Euler method can be written in the form

$$\bar{X}_{k+1} = \bar{H}\bar{X}_k + \bar{v}_k = \begin{bmatrix} 1 & h \\ -h & 1 \end{bmatrix} \bar{X}_k + \begin{bmatrix} \sigma\Delta_k w_1 \\ \gamma\Delta_k w_2 \end{bmatrix}. \tag{6.13}$$

Analogously to (6.8)–(6.9), we get $\bar{H}^k - F^k = \bar{G}(\bar{\Lambda}^k - \bar{G}^{-1}F^k\bar{G})\bar{G}^{-1} := \bar{G}\bar{D}\bar{G}^{-1}$ with

$$\bar{\Lambda} = \begin{bmatrix} 1+ih & 0 \\ 0 & 1-ih \end{bmatrix}, \qquad \bar{G} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},$$
$$\bar{D} = \begin{bmatrix} (1+ih)^k - e^{ihk} & 0 \\ 0 & (1-ih)^k - e^{-ihk} \end{bmatrix}.$$

Further, $||\bar{G}|| = \sqrt{2}$, $||\bar{G}^{-1}|| = \sqrt{2}/2$, and

$$||\bar{D}|| = \left[((1+h^2)^{k/2} - 1)^2 + 4(1+h^2)^{k/2}\sin^2\frac{k(\varphi - h)}{2}\right]^{1/2}$$
$$\leq \left[(e^{Th/2} - 1)^2 + 4e^{Th/2}\sin^2\frac{k(\varphi - h)}{2}\right]^{1/2},$$

where

$$\varphi = \arcsin\frac{h}{\sqrt{1+h^2}} \simeq \frac{h}{\sqrt{1+h^2}} + \frac{1}{6}\frac{h^3}{(1+h^2)^{3/2}}, \qquad \varphi - h \simeq -\frac{h^3}{3}.$$

Hence if Th is small, then

$$||\bar{D}|| \leq \left[(e^{Th/2} - 1)^2 + 4e^{Th/2}\sin^2\frac{k(\varphi - h)}{2}\right]^{1/2} \simeq e^{Th/2} - 1 \simeq Th/2,$$

and it is not difficult to show that the mean-square error of the Euler method is estimated as $O(T^{3/2}h)$.

Consequently, the Euler method can be used on the interval $[0, T_E]$ if $T_E^{3/2}h$ is sufficiently small. Due to Proposition 6.2, the error of the symplectic method (6.3) on $[0, T_S]$ with $T_S = T_E^2$ is equal to $O(T_E h + T_E^3 h^2)$; i.e., the symplectic method is applicable on longer time intervals than the Euler method. Of course, the Euler method possesses properties worse than the symplectic method since the absolute values of the eigenvalues of $\bar{H}$ are greater than 1.

Finally, consider the following optimal method from [9, p. 62] (this method also uses only the increments $\Delta_k w$ as the information regarding $w(t)$ but it uses this information optimally):

$$\hat{X}_{k+1} = \hat{H}\hat{X}_k + \hat{v}_k = \begin{bmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{bmatrix} \hat{X}_k + \frac{1}{h} \begin{bmatrix} \sigma \sin h \cdot \Delta_k w_1 + 2\gamma \sin^2 \dfrac{h}{2} \cdot \Delta_k w_2 \\ -2\sigma \sin^2 \dfrac{h}{2} \cdot \Delta_k w_1 + \gamma \sin h \cdot \Delta_k w_2 \end{bmatrix}. \tag{6.14}$$

Evidently, this method is symplectic. Also, as $\hat{H} = F$, it has no error in the absence of noise. We get for its error

$$E|\hat{r}_N|^2 = \sum_{m=0}^{N-1} E|\hat{v}_m - u_m|^2 = N(\sigma^2 + \gamma^2)\frac{h^3}{12} + N \cdot O(h^5) \simeq \frac{\sigma^2 + \gamma^2}{12} T h^2.$$

Consequently, the error of the optimal method is estimated as $O(T^{1/2}h)$. This implies that method (6.14) is more applicable on the longer time interval $[0, T_O] = [0, T_E^3]$ than the symplectic method (6.3).

To guarantee the same sample paths for the Wiener processes in realization of the exact, symplectic, and Euler methods, we simulate six independent $\mathcal{N}(0,1)$-distributed random variables $\xi_{1,k+1},\ \eta_{1,k+1},\ \zeta_{1,k+1},\ \xi_{2,k+1},\ \eta_{2,k+1},\ \zeta_{2,k+1}$ at every step $k+1 = 1, \dots, N-1$. It is not difficult to show that the needed random variables can be evaluated as

$$\Delta_k w_i = \sqrt{h}\xi_{i,k+1}, \qquad \int_{t_k}^{t_{k+1}} \cos(t_{k+1} - s) dw_i(s) = \frac{1}{\sqrt{h}} \sin h \cdot \xi_{i,k+1} + c_1 \eta_{i,k+1},$$
$$\int_{t_k}^{t_{k+1}} \sin(t_{k+1} - s) dw_i(s) = \frac{2}{\sqrt{h}} \sin^2 \frac{h}{2} \cdot \xi_{i,k+1} + c_2 \eta_{i,k+1} + c_3 \zeta_{i,k+1}, \qquad i = 1, 2,$$

where

$$c_1 = \left(\frac{1}{2}h + \frac{1}{4}\sin 2h - \frac{\sin^2 h}{h} \right)^{1/2}, \qquad c_2 = \frac{1}{c_1}\left(\frac{1}{2}\sin^2 h - \frac{2}{h}\sin^2 \frac{h}{2} \sin h \right),$$
$$c_3 = \left(\frac{1}{2}h - \frac{1}{4}\sin 2h - \frac{4}{h}\sin^4 \frac{h}{2} - c_2^2 \right)^{1/2}.$$

In the numerical tests we simulate system (6.1) by (i) the exact formulae (6.2), (ii) the symplectic method (6.3), and (iii) the Euler method (6.13). Figure 1 corresponds to the time interval $[0, 128]$ which contains approximately 20 oscillations of (6.1) (note that the period of free oscillations of (6.1) is equal to 2π).

The results clearly demonstrate that the Euler method is unacceptable for simulation of the Hamiltonian system (6.1) on a long time interval. After 10 oscillations (Figure 1) the norm of its error is already half the norm of the solution, and after 200 oscillations (see Figure 2) the amplitude of oscillations simulated by the Euler method is 50,000 times greater than the exact amplitude.

In contrast to the Euler method, the symplectic method reproduces oscillations of the system (6.1) quite accurately. After 10 oscillations (Figure 1) the norm of its error is approximately 2% of the norm of the solution. But it is more astonishing that

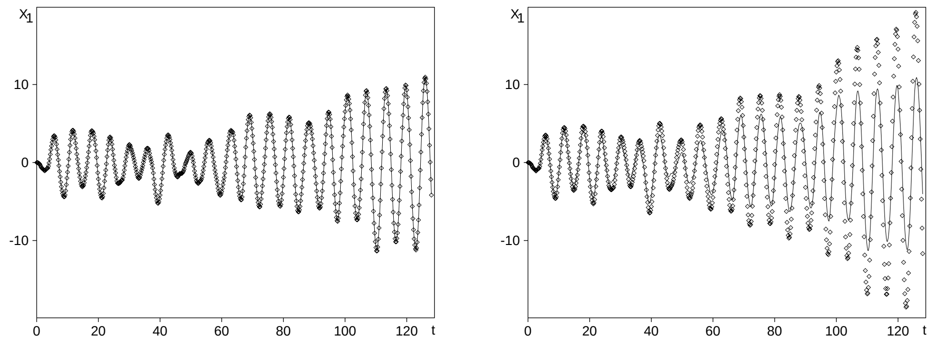

FIG. 1. *A sample trajectory of the solution to* (6.1) *for* $\sigma = 0$*,* $\gamma = 1$*,* $X_1(0) = X_2(0) = 0$ *obtained by the exact formulae* (6.2) (*solid line*)*, the symplectic method* (6.3) *with* $h = 0.02$ (*points in the left figure*)*, and the Euler method* (6.13) *with* $h = 0.02$ (*points in the right figure*)*. The points of the symplectic and Euler methods are plotted once per* 10 *steps, i.e., once per each interval* 0.2*.*

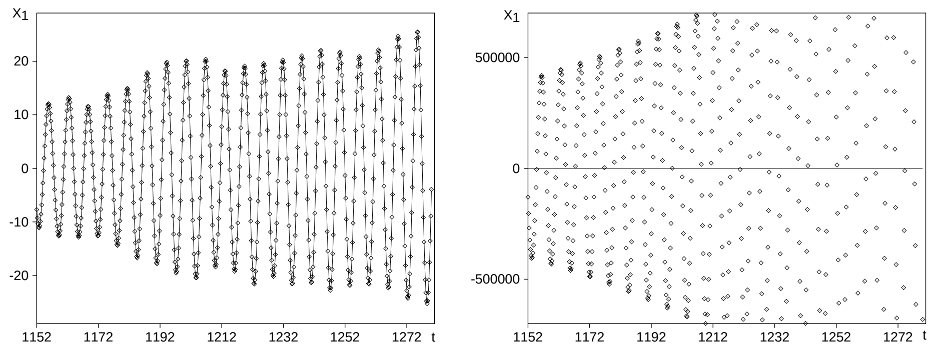

FIG. 2. *Another part of the same sample trajectory as in Figure* 1 *. Solid line: the exact solution. Points: the symplectic method* (*left*) *and the Euler method* (*right*)*.*

after 200 oscillations (see Figure 2) the relative error remains the same. The error of the amplitude of oscillations on the considered time intervals is also about 2%. As is known, a symplectic method in application to a deterministic oscillator preserves conservative properties of solutions, in particular their boundedness on infinite time intervals. One can say that the symplectic method generates a discrete conservative system ("discrete linear oscillator"). It turns out that behavior of this system affected by noise (which is also discrete) is qualitatively identical to the behavior of the continuous Hamiltonian system with noise. For instance, the approximate solution adequately reproduces an increase of the amplitude of the oscillations.

Figure 3 presents the evolution of domains in the phase plane of system (6.1). The initial domain is the circle with center at the origin and with the unit radius. We plot images of this circle that are obtained at three time moments by the exact mapping, by the mapping in the case of the symplectic method (6.3), and by the mapping in the case of the Euler method (6.13). For the considered system (6.1), exact images of the unit circle are circles of the unit radius shifted from the origin due to the action of noise. In the case of the Euler method, these images are also circles but with increasing radius. In the case of symplectic method (6.3), the images of the initial circle are ellipses. In spite of the fact that the symplectic method (6.3) and the Euler method (6.13) have the same mean-square order of accuracy, these ellipses approximate the exact images better than the circles obtained by the Euler method.

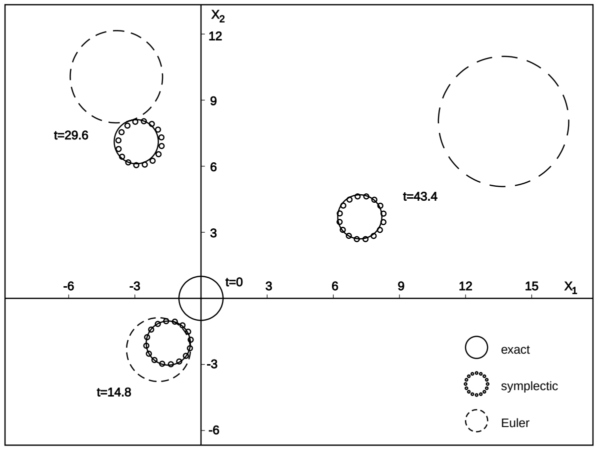

Fig. 3. *The evolution of domains in the phase plane of system* (6.1) *for* $\sigma = 0$, $\gamma = 1$. *Images of the initial unit circle are obtained at three time moments by the exact mapping, by the mapping in the case of the symplectic method* (6.3) *with* $h = 0.05$*, and by the mapping in the case of the Euler method* (6.13) *with* $h = 0.05$.

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