

## NUMERICAL METHODS FOR STOCHASTIC SYSTEMS PRESERVING SYMPLECTIC STRUCTURE\*

G. N. MILSTEIN<sup>†‡</sup>, YU. M. REPIN<sup>‡</sup>, AND M. V. TRETYAKOV<sup>§</sup>

**Abstract.** Stochastic Hamiltonian systems with multiplicative noise, phase flows of which preserve symplectic structure, are considered. To construct symplectic methods for such systems, sufficiently general fully implicit schemes, i.e., schemes with implicitness both in deterministic and stochastic terms, are needed. A new class of fully implicit methods for stochastic systems is proposed. Increments of Wiener processes in these fully implicit schemes are substituted by some truncated random variables. A number of symplectic integrators is constructed. Special attention is paid to systems with separable Hamiltonians. Some results of numerical experiments are presented. They demonstrate superiority of the proposed symplectic methods over very long times in comparison with nonsymplectic ones.

**Key words.** stochastic Hamiltonian systems, symplectic integration, implicit methods, mean-square convergence

**AMS subject classifications.** 60H10, 65C30, 65P10

**PII.** S0036142901395588

**1. Introduction.** Consider the following Cauchy problem for the system of stochastic differential equations (SDEs) in the sense of Stratonovich:

$$(1.1) \quad \begin{aligned} dP &= f(t, P, Q)dt + \sum_{r=1}^m \sigma_r(t, P, Q) \circ dw_r(t), & P(t_0) &= p, \\ dQ &= g(t, P, Q)dt + \sum_{r=1}^m \gamma_r(t, P, Q) \circ dw_r(t), & Q(t_0) &= q, \end{aligned}$$

where  $P, Q, f, g, \sigma_r, \gamma_r$  are  $n$ -dimensional column vectors with the components  $P^i, Q^i, f^i, g^i, \sigma_r^i, \gamma_r^i, i = 1, \dots, n$ , and  $w_r(t), r = 1, \dots, m$ , are independent standard Wiener processes. The diffusion coefficients  $\sigma_r, \gamma_r$  depend on  $P, Q$  (i.e., (1.1) is a system with multiplicative noise), in contrast to [3], where stochastic systems with additive noise are treated.

We suppose that all the coefficients of considered systems are sufficiently smooth functions defined for  $(t, p, q) \in [t_0, t_0 + T] \times R^d, d = 2n$ , and they provide the property of extendability of solutions to the interval  $[t_0, t_0 + T]$ . (Additional conditions in connection with considered methods consist of appropriate behavior of partial derivatives of the coefficients on infinity.)

We denote by  $X(t; t_0, x) = (P^\top(t; t_0, p, q), Q^\top(t; t_0, p, q))^\top, t_0 \leq t \leq t_0 + T$ , the solution of problem (1.1). A more detailed notation is  $X(t; t_0, x; \omega)$ , where  $\omega$  is an

---

\*Received by the editors September 24, 2001; accepted for publication (in revised form) April 17, 2002; published electronically October 23, 2002. The authors were partially supported by Russian Foundation for Basic Research project 99-01-00134.

<http://www.siam.org/journals/sinum/40-4/39558.html>

<sup>†</sup>Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, D-10117 Berlin, Germany (milstein@wias-berlin.de).

<sup>‡</sup>Department of Mathematics, Ural State University, Lenin Str. 51, 620083 Ekaterinburg, Russia (Grigori.Milstein@usu.ru, Yuri.Repin@usu.ru).

<sup>§</sup>Department of Mathematics, University of Wales Swansea, Swansea SA2 8PP, UK. Current address: Department of Mathematics and Computer Science, University of Leicester, Leicester LE1 7RH, UK (Michael.Tretyakov@usu.ru, M.Tretyakov@le.ac.uk).

elementary event. It is known that  $X(t; t_0, x; \omega)$  is a phase flow (diffeomorphism) for almost every  $\omega$ . See its properties in, e.g., [1, 2].

If there are functions  $H_r(t, p, q)$ ,  $r = 0, \dots, m$ , such that (see [1] and [3])

$$(1.2) \quad \begin{aligned} f^i(t, p, q) &= -\partial H_0 / \partial q^i, & g^i(t, p, q) &= \partial H_0 / \partial p^i, \\ \sigma_r^i(t, p, q) &= -\partial H_r / \partial q^i, & \gamma_r^i(t, p, q) &= \partial H_r / \partial p^i, \quad i = 1, \dots, n, \quad r = 1, \dots, m, \end{aligned}$$

then the phase flow of (1.1) preserves the following symplectic structure:

$$(1.3) \quad dP \wedge dQ = dp \wedge dq;$$

i.e., the sum of the oriented areas of projections onto the coordinate planes  $(p^1, q^1), \dots, (p^n, q^n)$  is an integral invariant [4]. To avoid confusion, we note that the differentials in (1.1) and (1.3) have different meanings. In (1.1)  $P, Q$  are treated as functions of time and  $p, q$  are fixed parameters, while differentiation in (1.3) is made with respect to the initial data  $p, q$ .

Let  $P_k, Q_k, k = 0, \dots, N, t_{k+1} - t_k = h_{k+1}, t_N = t_0 + T$ , be a method for (1.1) based on the one-step approximation  $\bar{P} = \bar{P}(t + h; t, p, q), \bar{Q} = \bar{Q}(t + h; t, p, q)$ . We say that the method preserves symplectic structure if

$$(1.4) \quad d\bar{P} \wedge d\bar{Q} = dp \wedge dq.$$

The present paper deals with symplectic integration of the Hamiltonian system with multiplicative noise (1.1), (1.3). It is a continuation of [3], where symplectic methods for Hamiltonian systems with additive noise were proposed. For symplectic integration of deterministic Hamiltonian systems see, e.g., [5, 6, 7, 8, 9] and references therein.

As is known [5], in the case of deterministic general Hamiltonian systems symplectic Runge–Kutta (RK) methods are all implicit. Hence it is natural to expect that to construct symplectic methods for general Hamiltonian systems with multiplicative noise fully implicit methods are needed. The known implicit methods for stochastic systems with multiplicative noise (see [10, 11]) contain implicitness in deterministic terms only. In [12] an implicitness is introduced in stochastic terms as well. However, the methods of [12] are of a very special form. In section 2 a new class of fully implicit methods for general stochastic systems is proposed. Increments of Wiener processes in these implicit schemes are substituted by some truncated random variables. They are important for both theory and practice of numerical integration of SDEs. We use these fully implicit methods in section 3 to construct symplectic methods for general Hamiltonian systems with multiplicative noise. Section 4 is devoted to a special case of separable Hamiltonians. Explicit symplectic integrators are constructed for such systems. In addition, symplectic methods for Hamiltonian systems with small multiplicative noise can be found in the preprint [13]. There one can also find some Liouvillian methods for stochastic systems preserving phase volume. Let us recall that the mean-square methods of higher order contain repeated Ito integrals which are difficult for simulation. In this paper, we prefer to derive methods which are efficient with respect to simulation of the used random variables. That is why orders of the methods derived are not too high. In the last section of the paper we present numerical tests. They clearly demonstrate superiority of the proposed symplectic methods over very long times in comparison with nonsymplectic ones.

**2. Fully implicit methods.** Construction of implicit methods for stochastic systems with additive noise does not cause any difficulties in principle. However, all

is much more intricate in the case of stochastic systems with multiplicative noise. The known implicit methods for such systems (see [10, 11]) contain implicitness restricted to deterministic terms, e.g., to the drift terms in the implicit Euler scheme. In [12], an implicitness is introduced in stochastic terms as well. However, methods of [12] are of a very special form. In particular, this form does not allow us to construct symplectic methods for stochastic Hamiltonian systems with multiplicative noise. In this section we construct a sufficiently large class of fully implicit methods of mean-square order 1/2 for general stochastic systems. These results are of independent and general interest. That is why in this section we consider SDEs in the Ito sense, following the traditional way of developing numerical methods. At the same time we should note that the Stratonovich form is preferable for SDEs preserving integral invariants.

**2.1. The main idea and an example.** Let us start with an example. Consider the Ito scalar equation

$$(2.1) \quad dX = \sigma X dw(t).$$

The one-step approximation of the Euler method  $\hat{X}$  for (2.1) is

$$(2.2) \quad \hat{X} = x + \sigma x \Delta w(h).$$

We can represent this approximation in the form

$$\hat{X} = x + \sigma \hat{X} \Delta w + \sigma(x - \hat{X}) \Delta w = x - \sigma^2 x (\Delta w)^2 + \sigma \hat{X} \Delta w.$$

As  $h$  is small,  $(\Delta w)^2 \sim h$ , and we obtain the following “natural” implicit method:

$$(2.3) \quad \tilde{X} = x - \sigma^2 x h + \sigma \tilde{X} \Delta w(h).$$

However, this method cannot be realized since  $1 - \sigma \Delta w(h)$  can vanish for any small  $h$ . Further, for the formal value of  $\tilde{X}$  from (2.3)

$$\tilde{X} = \frac{x(1 - \sigma^2 h)}{1 - \sigma \Delta w(h)},$$

we have  $E|\tilde{X}| = \infty$ . (See [10].) Clearly, the method (2.3) is not suitable. The reason for this is the unboundedness of the random variable  $\Delta w(h)$  for any arbitrarily small  $h$ .

Our basic idea consists of replacement of  $\Delta w(h) = \xi \sqrt{h}$ , where  $\xi$  is an  $\mathcal{N}(0, 1)$ -distributed random variable, by another random variable  $\zeta \sqrt{h} = \zeta_h \sqrt{h}$  such that  $\zeta \sqrt{h}$  is bounded and the Euler type method

$$(2.4) \quad \check{X} = x + \sigma x \zeta \sqrt{h}$$

is of the mean-square order 1/2 as well. To achieve this, it is sufficient to require

$$(2.5) \quad E(\check{X} - \hat{X}) = O(h^{3/2}), \quad E(\check{X} - \hat{X})^2 = O(h^2).$$

We take a symmetric  $\zeta$ . Then  $E(\check{X} - \hat{X}) = 0$ . To satisfy the second equation in (2.5), the condition  $E(\zeta_h - \xi)^2 = O(h)$  is sufficient.

We shall require a stronger inequality,

$$(2.6) \quad E(\zeta_h - \xi)^2 \leq h^k, \quad k \geq 1.$$

For  $A_h > 0$  let

$$(2.7) \quad \zeta_h = \begin{cases} \xi, & |\xi| \leq A_h, \\ A_h, & \xi > A_h, \\ -A_h, & \xi < -A_h. \end{cases}$$

Since

$$\begin{aligned} E(\zeta_h - \xi)^2 &= \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} (x - A_h)^2 e^{-x^2/2} dx \\ &= \frac{2}{\sqrt{2\pi}} e^{-A_h^2/2} \int_{A_h}^{\infty} y^2 e^{-y^2/2} e^{-A_h y} dy < e^{-A_h^2/2}, \end{aligned}$$

the inequality (2.6) is fulfilled if  $e^{-A_h^2/2} \leq h^k$ , i.e.,  $A_h^2 \geq 2k|\ln h|$ . Thus, if

$$A_h = \sqrt{2k|\ln h|}, \quad k \geq 1,$$

then the method based on the one-step approximation (2.4) has the mean-square order  $1/2$ .

LEMMA 2.1. *Let  $A_h = \sqrt{2k|\ln h|}$ ,  $k \geq 1$ , and  $\zeta_h$  be defined by (2.7). Then the following inequality holds:*

$$(2.8) \quad 0 \leq E(\xi^2 - \zeta_h^2) = 1 - E\zeta_h^2 \leq \left(1 + 2\sqrt{2k|\ln h|}\right) h^k.$$

*Proof.* We have

$$\begin{aligned} 1 - E\zeta_h^2 &= \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} (x^2 - A_h^2) e^{-x^2/2} dx \\ &= \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} [(x - A_h)^2 + 2A_h(x - A_h)] e^{-x^2/2} dx \\ &\leq e^{-A_h^2/2} + \frac{4A_h}{\sqrt{2\pi}} \int_{A_h}^{\infty} x e^{-x^2/2} dx = e^{-A_h^2/2} \left(1 + \frac{4A_h}{\sqrt{2\pi}}\right) \leq (1 + 2A_h) e^{-A_h^2/2}, \end{aligned}$$

whence (2.8) follows.  $\square$

Now consider the following implicit method (for definiteness we put  $k = 1$  and  $A_h = \sqrt{2|\ln h|}$ ):

$$(2.9) \quad \begin{aligned} \bar{X} &= x - \sigma^2 x h + \sigma \bar{X} \zeta_h \sqrt{h}, \\ \bar{X} &= \frac{x(1 - \sigma^2 h)}{1 - \sigma \zeta_h \sqrt{h}}. \end{aligned}$$

Since  $|\zeta_h| \leq \sqrt{2|\ln h|}$ , this method is realizable for all  $h$  satisfying the inequality

$$(2.10) \quad 2h|\ln h| < \frac{1}{\sigma^2}.$$

PROPOSITION 2.2. *The method (2.9) is of the mean-square order  $1/2$ .*

*Proof.* Let us compare method (2.9) with the Euler method (2.2). We get

$$E\bar{X} = x(1 - \sigma^2 h) E \sum_{m=0}^{\infty} \sigma^m \zeta_h^m h^{m/2} = x(1 - \sigma^2 h) E \sum_{m=0}^{\infty} \sigma^{2m} \zeta_h^{2m} h^m.$$

It is obvious from here that the principal term in the expansion of  $E(\bar{X} - \hat{X})$  is equal to  $x\sigma^2h(E\zeta_h^2 - 1)$ . Due to Lemma 2.1, we obtain for all sufficiently small  $h$

$$(2.11) \quad |E(\bar{X} - \hat{X})| \leq C|x|\sigma^2 \left(1 + 2\sqrt{2|\ln h|}\right) h^2,$$

where  $C$  is a positive constant.

Further,

$$(2.12) \quad \begin{aligned} E(\bar{X} - \hat{X})^2 &= E\left(-\sigma^2xh + \sigma\bar{X}\zeta_h\sqrt{h} - \sigma x\xi\sqrt{h}\right)^2 \\ &\leq 2\sigma^4x^2h^2 + 2E\left(\sigma\bar{X}\zeta_h\sqrt{h} - \sigma x\xi\sqrt{h}\right)^2 \\ &= 2\sigma^4x^2h^2 + 2E\left(\sigma \cdot \left(x - \sigma^2xh + \sigma\bar{X}\zeta_h\sqrt{h}\right)\zeta_h\sqrt{h} - \sigma x\xi\sqrt{h}\right)^2 \\ &\leq 2\sigma^4x^2h^2 + 2\sigma^2x^2hE(\zeta_h - \xi)^2 + C_1x^2h^2 \leq C_2x^2h^2 \end{aligned}$$

for all sufficiently small  $h$  and some positive constants  $C_1$  and  $C_2$ . The inequalities (2.11) and (2.12) imply the mean-square convergence of implicit method (2.9) with order 1/2.  $\square$

Introduction of implicitness in the stochastic term leads to the appearance of the compensating term  $-\sigma^2xh$  in (2.9). This can be explained in the following way. Since  $\bar{X}$  must be close to  $x + \sigma x\zeta_h\sqrt{h}$ , the expression  $x + \sigma\bar{X}\zeta_h\sqrt{h}$  is close to  $x + \sigma x\zeta_h\sqrt{h} + \sigma^2x\zeta_h^2h$ . Consequently, making use of the compensating term results in  $x + \sigma\bar{X}\zeta_h\sqrt{h} - \sigma^2xh = x + \sigma x\zeta_h\sqrt{h} + \sigma^2x(\zeta_h^2 - 1)h \approx x + \sigma x\zeta_h\sqrt{h}$ ; i.e., we get the correct result.

Now let us consider the expression  $\sigma((1-\beta)x + \beta\bar{X})\zeta_h\sqrt{h}$  which introduces implicitness in the stochastic term with the parameter  $0 \leq \beta \leq 1$ . Clearly, the compensating term in this case is equal to  $-\sigma^2\beta xh$ . Thus, we derive the following method:

$$(2.13) \quad \bar{X} = x - \sigma^2\beta xh + \sigma((1-\beta)x + \beta\bar{X})\zeta_h\sqrt{h}, \quad 0 \leq \beta \leq 1.$$

The following proposition can be proved analogously to Proposition 2.2.

PROPOSITION 2.3. *The method (2.13), as well as the methods*

$$(2.14) \quad \bar{X} = x - \sigma^2\beta x\zeta_h^2h + \sigma((1-\beta)x + \beta\bar{X})\zeta_h\sqrt{h}, \quad 0 \leq \beta \leq 1,$$

$$(2.15) \quad \bar{X} = x - \sigma^2\beta((1-\alpha)x + \alpha\bar{X})h + \sigma((1-\beta)x + \beta\bar{X})\zeta_h\sqrt{h}, \quad 0 \leq \alpha, \beta \leq 1,$$

are of the mean-square order 1/2.

**2.2. Convergence theorem.** Now we are in position to introduce fully implicit methods for general systems of SDEs. For simplicity in writing we deal here with the following scalar Ito SDE:

$$(2.16) \quad dX = a(t, X)dt + b(t, X)dw(t).$$

We suppose that  $a(t, x)$ ,  $b(t, x)$ ,  $\frac{\partial b}{\partial x}(t, x)$  are continuous for  $t_0 \leq t \leq T$ ,  $x \in \mathbf{R}$ , and there exists a positive constant  $L$  such that

$$(2.17) \quad |a(t, y) - a(t, x)| \leq L|y - x|, \quad \left|\frac{\partial b}{\partial x}(t, x)\right| \leq L, \quad t_0 \leq t \leq T, \quad x, y \in \mathbf{R}.$$

Note that below the same letter  $L$  (or  $K$ , or  $C$ ) is used for various constants.

Consider the implicit one-step approximation (cf. (2.9))

$$(2.18) \quad \bar{X} = x + a(t, \bar{X})h - b(t, x) \frac{\partial b}{\partial x}(t, x)h + b(t, \bar{X})\zeta_h \sqrt{h},$$

where  $\zeta_h$  is defined by (2.7) with  $A_h = \sqrt{2|\ln h|}$  for definiteness.

LEMMA 2.4. *There exist constants  $K > 0$  and  $h_0 > 0$  such that for any  $h \leq h_0$ ,  $t_0 \leq t \leq T$ ,  $x \in \mathbf{R}$  (2.18) has a unique solution  $\bar{X}$  which satisfies the inequality*

$$(2.19) \quad |\bar{X} - x| \leq K(1 + |x|) \left( |\zeta_h| \sqrt{h} + h \right).$$

The solution  $\bar{X}$  of (2.18) can be found by the method of simple iteration with  $x$  as the initial approximation.

*Proof.* For any fixed  $t$ ,  $x$ , and  $h$ , let us introduce the function

$$\varphi(z) = x + a(t, z)h - b(t, x) \frac{\partial b}{\partial x}(t, x)h + b(t, z)\zeta_h \sqrt{h}.$$

Then (2.18) can be written as

$$\bar{X} = \varphi(\bar{X}).$$

There is a positive constant  $C$  such that for any  $z \in \mathbf{R}$

$$\begin{aligned} |\varphi(z) - x| &\leq |a(t, x)h| + |a(t, z) - a(t, x)|h + |b(t, x)| |\zeta_h| \sqrt{h} + |b(t, z) - b(t, x)| |\zeta_h| \sqrt{h} \\ &\quad + \left| b(t, x) \frac{\partial b}{\partial x}(t, x) \right| h \leq C(1 + |x|) \left( |\zeta_h| \sqrt{h} + h \right) + L|z - x| \left( |\zeta_h| \sqrt{h} + h \right). \end{aligned}$$

Further, for any  $z_1, z_2 \in \mathbf{R}$

$$|\varphi(z_2) - \varphi(z_1)| \leq L|z_2 - z_1| \left( |\zeta_h| \sqrt{h} + h \right).$$

Clearly, there exist positive constants  $K$  and  $h_0$  such that for any  $h \leq h_0$ ,  $x \in \mathbf{R}$

$$L \left( |\zeta_h| \sqrt{h} + h \right) < 1,$$

and if

$$|z - x| \leq K(1 + |x|) \left( |\zeta_h| \sqrt{h} + h \right),$$

then

$$|\varphi(z) - x| \leq K(1 + |x|) \left( |\zeta_h| \sqrt{h} + h \right).$$

Let us note that the constants  $K$  in the last two inequalities are the same. Now the lemma follows from the contraction mapping principle.  $\square$

In addition to (2.17) suppose that there exist continuous  $\partial a/\partial t$ ,  $\partial b/\partial t$ , and  $\partial^2 b/\partial x^2$  and the inequalities

$$(2.20) \quad \left| \frac{\partial a}{\partial t}(t, x) \right| \leq L(1 + |x|), \quad \left| \frac{\partial b}{\partial t}(t, x) \right| \leq L(1 + |x|), \quad t_0 \leq t \leq T, \quad x \in \mathbf{R}$$

hold.

THEOREM 2.5. Assume (2.17) and (2.20). Let there exist  $\delta > 0$  such that if  $|y - x| \leq \delta(1 + |x|)$ , then the inequality

$$(2.21) \quad \left| b(t, x) \frac{\partial^2 b}{\partial x^2}(t, y) \right| \leq L, \quad t_0 \leq t \leq T$$

holds.

Then the implicit method based on the one-step approximation (2.18) converges in mean-square with the order  $1/2$ .

Proof. Let  $\hat{X}$  be the Euler approximation for (2.16):

$$\hat{X} = x + a(t, x)h + b(t, x)\Delta w(h).$$

Using the condition (2.17) only, we get

$$\begin{aligned} E|\bar{X} - \hat{X}|^2 &\leq E \left| a(t, \bar{X})h - a(t, x)h + b(t, \bar{X})\zeta_h\sqrt{h} - b(t, x)\Delta w(h) - b(t, x)\frac{\partial b}{\partial x}(t, x)h \right|^2 \\ &\leq LE|a(t, \bar{X}) - a(t, x)|^2h^2 + LE|b(t, \bar{X}) - b(t, x)|^2\zeta_h^2h \\ &\quad + Lb^2(t, x)E(\zeta_h - \xi)^2h + L \left| b(t, x)\frac{\partial b}{\partial x}(t, x) \right|^2 h^2 \\ &\leq LE|\bar{X} - x|^2h^2 + LE|\bar{X} - x|^2\zeta_h^2h + L(1 + |x|)^2E(\zeta_h - \xi)^2h + L(1 + |x|)^2h^2. \end{aligned}$$

It follows from here, Lemma 2.4, the inequality  $E\zeta^4 < E\xi^4 = 3$ , and (2.6) that

$$(2.22) \quad E|\bar{X} - \hat{X}|^2 \leq L(1 + |x|)^2h^2.$$

Now let us proceed to the evaluation of  $E(\bar{X} - \hat{X})$ . We have

$$(2.23) \quad |E(\bar{X} - \hat{X})| \leq |Ea(t, \bar{X}) - a(t, x)|h + \left| E(b(t, \bar{X}) - b(t, x))\zeta_h\sqrt{h} - b(t, x)\frac{\partial b}{\partial x}(t, x)h \right|.$$

Due to Lemma 2.4,  $E|\bar{X} - x| \leq K(1 + |x|)(E|\zeta_h|\sqrt{h} + h)$ . Then

$$(2.24) \quad |Ea(t, \bar{X}) - a(t, x)|h \leq C(1 + |x|)h^{3/2}.$$

We have

$$\begin{aligned} (2.25) \quad &(b(t, \bar{X}) - b(t, x))\zeta_h\sqrt{h} - b(t, x)\frac{\partial b}{\partial x}(t, x)h \\ &= \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot (\bar{X} - x)\zeta_h\sqrt{h} - b(t, x)\frac{\partial b}{\partial x}(t, x)h \\ &= \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot \left( a(t, \bar{X})h + b(t, \bar{X})\zeta_h\sqrt{h} - b(t, x)\frac{\partial b}{\partial x}(t, x)h \right) \zeta_h\sqrt{h} \\ &\quad - b(t, x)\frac{\partial b}{\partial x}(t, x)h \\ &= \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot \left( a(t, \bar{X}) - b(t, x)\frac{\partial b}{\partial x}(t, x)h \right) \zeta_h h^{3/2} \\ &\quad + \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot b(t, \bar{X})\zeta_h^2h - b(t, x)\frac{\partial b}{\partial x}(t, x)h, \end{aligned}$$

where  $0 \leq \theta \leq 1$ .

Since  $|\bar{X} - x| \leq \rho(1 + |x|)$ , where  $\rho \rightarrow 0$  as  $h \rightarrow 0$ , we get  $|\bar{X}| \leq |x| + |\bar{X} - x| \leq K(1 + |x|)$  for all sufficiently small  $h$ . Therefore

$$(2.26) \quad \left| E \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot a(t, \bar{X}) \zeta_h h^{3/2} \right| \leq KE|a(t, \bar{X}) \zeta_h| h^{3/2} \\ \leq KE(1 + |\bar{X}|) |\zeta_h| h^{3/2} \leq K(1 + |x|) h^{3/2}.$$

Clearly,

$$\left| E \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot b(t, x) \frac{\partial b}{\partial x}(t, x) \zeta_h h^{3/2} \right| \leq K(1 + |x|) h^{3/2}.$$

Let us estimate the last two terms in (2.25). We obtain

$$\begin{aligned} & \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot b(t, \bar{X}) \zeta_h^2 h - b(t, x) \frac{\partial b}{\partial x}(t, x) h \\ &= \left( \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) - \frac{\partial b}{\partial x}(t, x) \right) b(t, \bar{X}) \zeta_h^2 h \\ &+ \frac{\partial b}{\partial x}(t, x) (b(t, \bar{X}) - b(t, x)) \zeta_h^2 h + \frac{\partial b}{\partial x}(t, x) b(t, x) (\zeta_h^2 - 1) h \\ &= \frac{\partial^2 b}{\partial x^2}(t, x + \theta_1(\bar{X} - x)) \cdot \theta(\bar{X} - x) \cdot b(t, \bar{X}) \zeta_h^2 h \\ &+ \frac{\partial b}{\partial x}(t, x) \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot (\bar{X} - x) \zeta_h^2 h + \frac{\partial b}{\partial x}(t, x) b(t, x) (\zeta_h^2 - 1) h, \end{aligned}$$

where  $0 \leq \theta, \theta_1 \leq 1$ . Due to Lemma 2.4, we get  $|x + \theta_1(\bar{X} - x) - \bar{X}| \leq |\bar{X} - x| \leq K(|\zeta_h| \sqrt{h} + h)(1 + |x|)$ . For all sufficiently small  $h$  we have  $K(|\zeta_h| \sqrt{h} + h) < \delta$  and consequently due to (2.21)

$$(2.27) \quad \left| \frac{\partial^2 b}{\partial x^2}(t, x + \theta_1(\bar{X} - x)) \cdot b(t, \bar{X}) \right| \leq L.$$

Using (2.27), the conditions (2.17), and Lemmas 2.1 and 2.4, we obtain for the last two terms in (2.25)

$$(2.28) \quad \left| E \frac{\partial b}{\partial x}(t, x + \theta(\bar{X} - x)) \cdot b(t, \bar{X}) \zeta_h^2 h - b(t, x) \frac{\partial b}{\partial x}(t, x) h \right| \leq K(1 + |x|) h^{3/2}.$$

Thus, (2.23)–(2.28) give

$$(2.29) \quad |E(\bar{X} - \hat{X})| \leq K(1 + |x|) h^{3/2}.$$

It follows from (2.22) and (2.29) (see [10]) that the method based on (2.18) is of the mean-square order 1/2.  $\square$

REMARK 2.1. *The condition (2.21) is satisfied if, for instance,*

$$(2.30) \quad |b(t, x)| \leq L, \quad \left| \frac{\partial^2 b}{\partial x^2}(t, x) \right| \leq L, \quad t_0 \leq t \leq T, \quad x \in \mathbf{R}$$

or

$$(2.31) \quad \left| \frac{\partial^2 b}{\partial x^2}(t, x) \right| \leq \frac{L}{1 + |x|}, \quad t_0 \leq t \leq T, \quad x \in \mathbf{R}$$



holds.

Let us underline that the conditions of Theorem 2.5 are not necessary and the method is applicable more widely. This is true for other methods proposed in the paper as well.

REMARK 2.2. Let the function  $c(t, x) := b(t, x) \frac{\partial b}{\partial x}(t, x)$  satisfy the condition

$$(2.32) \quad |c(t, y) - c(t, x)| \leq L|y - x|.$$

Consider the implicit one-step approximation

$$(2.33) \quad \bar{X} = x + a(t, \bar{X})h - b(t, \bar{X}) \frac{\partial b}{\partial x}(t, \bar{X})h + b(t, \bar{X})\zeta_h \sqrt{h}.$$

It is not difficult to prove that Theorem 2.5 is true for the implicit method based on (2.33) provided (2.32) is fulfilled.

**2.3. General construction.** Let

$$(2.34) \quad dX^i = a^i(t, X)dt + \sum_{r=1}^m b_r^i(t, X)dw_r(t), \quad i = 1, \dots, d.$$

Introduce the following one-step approximation:

$$(2.35) \quad \begin{aligned} \bar{X}^i = & x^i + \sum_{k=1}^l \lambda_k^i a^i(t + \nu_k^i h, (1 - \alpha_{k1}^i)x^1 + \alpha_{k1}^i \bar{X}^1, \dots, (1 - \alpha_{kd}^i)x^d + \alpha_{kd}^i \bar{X}^d)h \\ & + \sum_{r=1}^m \sum_{k=1}^l \mu_{rk}^i b_r^i(t + \nu_{rk}^i h, (1 - \beta_{rk1}^i)x^1 + \beta_{rk1}^i \bar{X}^1, \dots, (1 - \beta_{rkd}^i)x^d + \beta_{rkd}^i \bar{X}^d)\zeta_{rh} \sqrt{h} \\ & + A^i, \end{aligned}$$

where  $0 \leq \nu, \alpha, \beta \leq 1$ ,  $\lambda, \mu \geq 0$ ,  $\sum_{k=1}^l \lambda_k^i = 1$ ,  $\sum_{k=1}^l \mu_{rk}^i = 1$ ,  $i = 1, \dots, d$ ,  $l$  is a positive integer, and  $A^i$  are some expressions to be found. Substituting the Euler-type approximation

$$\hat{X}^j = x^j + a^j(t, x)h + \sum_{s=1}^m b_s^j(t, x)\zeta_{sh} \sqrt{h}$$

instead of  $\bar{X}^j$ ,  $j = 1, \dots, d$ , in  $b_r^i$ , we obtain

$$\begin{aligned} & b_r^i(t + \nu_{rk}^i h, (1 - \beta_{rk1}^i)x^1 + \beta_{rk1}^i \bar{X}^1, \dots, (1 - \beta_{rkd}^i)x^d + \beta_{rkd}^i \bar{X}^d) \\ & \approx b_r^i(t, x) + \sum_{j=1}^d \frac{\partial b_r^i}{\partial x^j}(t, x)\beta_{rkj}^i \sum_{s=1}^m b_s^j(t, x)\zeta_{sh} \sqrt{h}. \end{aligned}$$

It is clear from here that either

$$(2.36) \quad A^i = - \sum_{r=1}^m \sum_{k=1}^l \mu_{rk}^i \sum_{j=1}^d \frac{\partial b_r^i}{\partial x^j}(t, x)\beta_{rkj}^i \sum_{s=1}^m b_s^j(t, x)\zeta_{sh} \sqrt{h}\zeta_{rh} \sqrt{h}$$

or

$$(2.37) \quad A^i = - \sum_{r=1}^m \sum_{k=1}^l \mu_{rk}^i \sum_{j=1}^d \frac{\partial b_r^i}{\partial x^j}(t, x)\beta_{rkj}^i b_r^j(t, x)h$$

can be put in (2.35).

Substituting one of these expressions in (2.35), we obtain a multiparameter family of implicit methods. It is also possible to introduce implicitness in  $A^i$  by changing  $t, x$  as it was done in the terms connecting with  $a^i$ . Moreover, the family can be extended if some  $a^i$  or  $b_r^i$  are represented as sums of terms. In this case the coefficients  $\lambda, \nu, \alpha, \mu, \beta$  can differ for different terms.

It can be proved that under appropriate conditions of smoothness and boundedness on the coefficients of (2.34) the method based on the one-step approximation (2.35) with  $A^i$  as in (2.36) or (2.37) is of the mean-square order 1/2. The proof is analogous to the proof of Theorem 2.5.

Here and below we will not precisely indicate conditions on the coefficients  $a$  and  $b_r$  assuming that appropriate conditions on the coefficients hold. These conditions can be restored using the general theory [10] and Theorem 2.5. (See also Remarks 2.1 and 2.2.)

Let us give an example of fully implicit methods:

$$\bar{X} = x + a(t, \bar{X})h - \sum_{r=1}^m \sum_{j=1}^d \frac{\partial b_r}{\partial x^j}(t, \bar{X}) b_r^j(t, \bar{X})h + \sum_{r=1}^m b_r(t, \bar{X}) \zeta_{rh} \sqrt{h}.$$

Further, in the case of SDEs in the sense of Stratonovich,

$$(2.38) \quad dX = a(t, X)dt + \sum_{r=1}^m b_r(t, X) \circ dw_r(t),$$

we construct the following derivative-free fully implicit method (midpoint method):

$$(2.39) \quad X_{k+1} = X_k + a\left(t_k + \frac{h}{2}, \frac{X_k + X_{k+1}}{2}\right)h + \sum_{r=1}^m b_r\left(t_k, \frac{X_k + X_{k+1}}{2}\right) (\zeta_{rh})_k \sqrt{h}.$$

For  $b_r^i = 0$ , this method coincides with the well-known deterministic midpoint scheme, which has the second order of convergence.

In the general case the method (2.39) is of the mean-square order 1/2. In the commutative case, i.e., when  $\Lambda_i b_r = \Lambda_r b_i$  (here the operator  $\Lambda_r := (b_r, \partial/\partial x)$ ), or in the case of a system with one noise (i.e.,  $m = 1$ ), the midpoint method (2.39) has the first mean-square order of convergence which is stated in the next theorem. (Its proof is available in the preprint [13].)

**THEOREM 2.6.** *Suppose that the commutative conditions  $\Lambda_i b_r = \Lambda_r b_i$ ,  $i, r = 1, \dots, m$ , are fulfilled. Let  $\zeta_{rh}$  be defined by (2.7) with  $A_h = \sqrt{4|\ln h|}$ . Then the method (2.39) for the system (2.38) has the first mean-square order of convergence.*

**3. Symplectic methods for the general Hamiltonian system.** Here, using the results of the previous section, we construct symplectic methods for the general Hamiltonian system with multiplicative noise (1.1), (1.3). Its Ito form reads

$$(3.1) \quad \begin{aligned} dP &= fdt + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r}{\partial p^j} \sigma_r^j dt + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r}{\partial q^j} \gamma_r^j dt + \sum_{r=1}^m \sigma_r dw_r(t), \\ dQ &= gdt + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \gamma_r}{\partial p^j} \sigma_r^j dt + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \gamma_r}{\partial q^j} \gamma_r^j dt + \sum_{r=1}^m \gamma_r dw_r(t). \end{aligned}$$

Introduce the following implicit method:

$$(3.2) \quad \begin{aligned} P_{k+1} &= P_k + fh - \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \left( \frac{\partial \sigma_r}{\partial p^j} \sigma_r^j - \frac{\partial \sigma_r}{\partial q^j} \gamma_r^j \right) h + \sum_{r=1}^m \sigma_r \cdot (\zeta_{rh})_k \sqrt{h}, \\ Q_{k+1} &= Q_k + gh - \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \left( \frac{\partial \gamma_r}{\partial p^j} \sigma_r^j - \frac{\partial \gamma_r}{\partial q^j} \gamma_r^j \right) h + \sum_{r=1}^m \gamma_r \cdot (\zeta_{rh})_k \sqrt{h}, \end{aligned}$$

where all the functions have  $t, P_{k+1}, Q_k$  as their arguments.

**THEOREM 3.1.** *The implicit method (3.2) for the system (3.1), (1.3) (or for system (1.1), (1.3)) is symplectic and of the mean-square order 1/2.*

*Proof.* The method (3.2) belongs to the family (2.35) and, consequently, the assertion about its order of convergence follows from the previous section. Let us prove symplecticness of the method. It is convenient to write the one-step approximation corresponding to (3.2) in the form

$$(3.3) \quad \begin{aligned} \bar{P}^i &= p^i - \frac{\partial H_0}{\partial q^i} h - \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial^2 H_r}{\partial q^i \partial p^j} \frac{\partial H_r}{\partial q^j} h - \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial^2 H_r}{\partial q^i \partial q^j} \frac{\partial H_r}{\partial p^j} h - \sum_{r=1}^m \frac{\partial H_r}{\partial q^i} \zeta_{rh} \sqrt{h}, \\ \bar{Q}^i &= q^i + \frac{\partial H_0}{\partial p^i} h + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial^2 H_r}{\partial p^i \partial p^j} \frac{\partial H_r}{\partial q^j} h + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial^2 H_r}{\partial p^i \partial q^j} \frac{\partial H_r}{\partial p^j} h + \sum_{r=1}^m \frac{\partial H_r}{\partial p^i} \zeta_{rh} \sqrt{h}, \end{aligned}$$

where  $i = 1, \dots, n$  and all the functions have  $t, \bar{P}, q$  as their arguments.

Introduce the following function  $F(t, p, q)$  ( $h, \zeta_{rh}$  are fixed here):

$$F(t, p, q) = H_0(t, p, q)h + \frac{1}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial H_r}{\partial q^j} (t, p, q) \frac{\partial H_r}{\partial p^j} (t, p, q)h + \sum_{r=1}^m H_r(t, p, q) \zeta_{rh} \sqrt{h}.$$

Then (3.3) can be written as

$$(3.4) \quad \begin{aligned} \bar{P}^i &= p^i - \frac{\partial F}{\partial q^i} (t, \bar{P}, q), \\ \bar{Q}^i &= q^i + \frac{\partial F}{\partial p^i} (t, \bar{P}, q). \end{aligned}$$

We have (the arguments everywhere are  $t, \bar{P}, q$ )

$$\begin{aligned} \sum_{i=1}^n d\bar{P}^i \wedge d\bar{Q}^i &= \sum_{i=1}^n d\bar{P}^i \wedge \left( dq^i + \sum_{j=1}^n F''_{p^i p^j} d\bar{P}^j + \sum_{j=1}^n F''_{p^i q^j} dq^j \right) \\ &= \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \sum_{j=1}^n F''_{p^i p^j} d\bar{P}^i \wedge d\bar{P}^j + \sum_{i=1}^n \sum_{j=1}^n F''_{p^i q^j} d\bar{P}^i \wedge dq^j. \end{aligned}$$

Since  $d\bar{P}^i \wedge d\bar{P}^j = -d\bar{P}^j \wedge d\bar{P}^i$ , we get

$$(3.5) \quad \begin{aligned} \sum_{i=1}^n d\bar{P}^i \wedge d\bar{Q}^i &= \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \sum_{j=1}^n F''_{p^i q^j} d\bar{P}^i \wedge dq^j \\ &= \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j \wedge dq^i. \end{aligned}$$

Further,

$$d\bar{P}^i = dp^i - \sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j - \sum_{j=1}^n F''_{q^i q^j} dq^j.$$

Substituting  $\sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j$  from here in (3.5), we obtain

$$\begin{aligned} \sum_{i=1}^n d\bar{P}^i \wedge d\bar{Q}^i &= \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \left( dp^i - d\bar{P}^i - \sum_{j=1}^n F''_{q^i q^j} dq^j \right) \wedge dq^i \\ &= \sum_{i=1}^n dp^i \wedge dq^i - \sum_{i=1}^n \sum_{j=1}^n F''_{q^i q^j} dq^j \wedge dq^i = \sum_{i=1}^n dp^i \wedge dq^i. \quad \square \end{aligned}$$

A more general symplectic method for the Hamiltonian system (1.1), (1.3) has the form

$$\begin{aligned} (3.6) \quad P_{k+1} &= P_k + f(t_k + \beta h, \alpha P_{k+1} + (1 - \alpha)P_k, (1 - \alpha)Q_{k+1} + \alpha Q_k)h \\ &\quad + \left( \frac{1}{2} - \alpha \right) \sum_{r=1}^m \sum_{j=1}^n \left( \frac{\partial \sigma_r}{\partial p^j} \sigma_r^j - \frac{\partial \sigma_r}{\partial q^j} \gamma_r^j \right) h + \sum_{r=1}^m \sigma_r \cdot (\zeta_{rh})_k \sqrt{h}, \\ Q_{k+1} &= Q_k + g(t_k + \beta h, \alpha P_{k+1} + (1 - \alpha)P_k, (1 - \alpha)Q_{k+1} + \alpha Q_k)h \\ &\quad + \left( \frac{1}{2} - \alpha \right) \sum_{r=1}^m \sum_{j=1}^n \left( \frac{\partial \gamma_r}{\partial p^j} \sigma_r^j - \frac{\partial \gamma_r}{\partial q^j} \gamma_r^j \right) h + \sum_{r=1}^m \gamma_r \cdot (\zeta_{rh})_k \sqrt{h}, \end{aligned}$$

where  $\sigma_r, \gamma_r, r = 1, \dots, m$ , and their derivatives are calculated at  $(t_k, \alpha P_{k+1} + (1 - \alpha)P_k, (1 - \alpha)Q_{k+1} + \alpha Q_k)$ , and  $\alpha, \beta \in [0, 1]$  are parameters.

Using arguments similar to ones in the proof of Theorem 3.1, we obtain the following theorem.

**THEOREM 3.2.** *The implicit method (3.6) for the system (1.1), (1.3) (or for system (3.1), (1.3)) is symplectic and of the mean-square order 1/2.*

The method (3.2) is a particular case of (3.6) when  $\alpha = 1, \beta = 0$ . If  $\alpha = \beta = 1/2$  the method (3.6) becomes the midpoint method (cf. (2.39)):

$$\begin{aligned} (3.7) \quad P_{k+1} &= P_k + f \left( t_k + \frac{h}{2}, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2} \right) h \\ &\quad + \sum_{r=1}^m \sigma_r \left( t_k, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2} \right) (\zeta_{rh})_k \sqrt{h}, \\ Q_{k+1} &= Q_k + g \left( t_k + \frac{h}{2}, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2} \right) h \\ &\quad + \sum_{r=1}^m \gamma_r \left( t_k, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2} \right) (\zeta_{rh})_k \sqrt{h}. \end{aligned}$$

**REMARK 3.1.** *In the commutative case, i.e., when  $\Lambda_i b_r = \Lambda_r b_i$ , or in the case of a system with one noise (i.e.,  $m = 1$ ), the symplectic method (3.7) for (1.1), (1.3) has the first mean-square order of convergence.*

**REMARK 3.2.** *In the case of Hamiltonians that are separable in the noise part, i.e., when  $H_r(t, p, q) = U_r(t, q) + V_r(t, p)$ ,  $r = 1, \dots, m$ , we can obtain symplectic*

methods for (1.1), (1.3) which are explicit in stochastic terms and do not need truncated random variables. For instance, (3.2) acquires the form

$$\begin{aligned}
 (3.8) \quad P_{k+1} &= P_k + f(t_k, P_{k+1}, Q_k)h \\
 &+ \frac{h}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r}{\partial q^j}(t_k, Q_k) \cdot \gamma_r^j(P_{k+1}) + \sum_{r=1}^m \sigma_r(t_k, Q_k) \Delta_k w_r, \\
 Q_{k+1} &= Q_k + g(t_k, P_{k+1}, Q_k)h \\
 &- \frac{h}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \gamma_r}{\partial p^j}(P_{k+1}) \cdot \sigma_r^j(t_k, Q_k) + \sum_{r=1}^m \gamma_r(t_k, P_{k+1}) \Delta_k w_r.
 \end{aligned}$$

Of course, if it is necessary, fully implicit methods which require truncated random variables can be used in the case of separable Hamiltonians as well.

REMARK 3.3. It is possible to construct fully explicit symplectic methods for the following partitioned system:

$$\begin{aligned}
 (3.9) \quad dP &= f(t, Q)dt + \sum_{r=1}^m \sigma_r(t, Q) \circ dw_r(t), \quad P(t_0) = p, \\
 dQ &= g(P)dt + \sum_{r=1}^m \gamma_r(t)dw_r(t), \quad Q(t_0) = q,
 \end{aligned}$$

with  $f^i = -\partial U_0 / \partial q^i$ ,  $g^i = \partial V_0 / \partial p^i$ ,  $\sigma_r^i = -\partial U_r / \partial q^i$ ,  $r = 1, \dots, m$ ,  $i = 1, \dots, n$ .

For instance, the explicit partitioned Runge-Kutta (PRK) method (cf. (4.5)–(4.6))

$$\begin{aligned}
 (3.10) \quad Q_1 &= Q_k + \alpha hg(P_k), \\
 P_1 &= P_k + hf(t_k + \alpha h, Q_1) + \frac{h}{2} \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r}{\partial q^j}(t_k, Q_1) \cdot \gamma_r^j(t_k), \\
 Q_2 &= Q_1 + (1 - \alpha)hg(P_1),
 \end{aligned}$$

$$\begin{aligned}
 (3.11) \quad P_{k+1} &= P_1 + \sum_{r=1}^m \sigma_r(t_k, Q_2) \Delta_k w_r, \\
 Q_{k+1} &= Q_2 + \sum_{r=1}^m \gamma_r(t_k) \Delta_k w_r, \quad k = 0, \dots, N - 1,
 \end{aligned}$$

with the parameter  $0 \leq \alpha \leq 1$ , is symplectic and of the mean-square order 1/2.

A particular case of the system (3.9) is considered in the next section, where explicit symplectic methods of a higher order are proposed.

**4. Explicit symplectic methods in the case of separable Hamiltonians.**

Consider a special case of the Hamiltonian system (1.1), (1.3) such that

$$(4.1) \quad H_0(t, p, q) = V_0(p) + U_0(t, q), \quad H_r(t, p, q) = U_r(t, q), \quad r = 1, \dots, m.$$

In this case we get the following system in the sense of Stratonovich:

$$\begin{aligned}
 (4.2) \quad dP &= f(t, Q)dt + \sum_{r=1}^m \sigma_r(t, Q) \circ dw_r(t), \quad P(t_0) = p, \\
 dQ &= g(P)dt, \quad Q(t_0) = q,
 \end{aligned}$$

with

(4.3)

$$f^i = -\partial U_0/\partial q^i, \quad g^i = \partial V_0/\partial p^i, \quad \sigma_r^i = -\partial U_r/\partial q^i, \quad r = 1, \dots, m, \quad i = 1, \dots, n.$$

We note that it is not difficult to consider a slightly more general separable Hamiltonian  $H_0(t, p, q) = V_0(t, p) + U_0(t, q)$ , but we restrict ourselves to  $H_0$  from (4.1).

It is obvious that the system (4.2) has the same form in the sense of Ito.

For  $V_0(p) = \frac{1}{2}(M^{-1}p, p)$  with  $M$  a constant, symmetric, invertible matrix, the system (4.2) takes the form

$$(4.4) \quad \begin{aligned} dP &= f(t, Q)dt + \sum_{r=1}^m \sigma_r(t, Q)dw_r(t), & P(t_0) &= p, \\ dQ &= M^{-1}Pdt, & Q(t_0) &= q. \end{aligned}$$

This system can be written as a second-order differential equation with multiplicative noise. Some physical applications of stochastic symplectic integration for such systems are discussed in [14].

Due to specific features of the system (4.2), (4.3) we have succeeded in construction of explicit partitioned Runge-Kutta (PRK) methods of a higher order.

**4.1. First-order methods.** A PRK method for (4.2) has the form (cf. (3.10)–(3.11)):

$$(4.5) \quad \begin{aligned} \mathcal{Q}_1 &= Q_k + \alpha hg(P_k), & \mathcal{P}_1 &= P_k + hf(t_k + \alpha h, \mathcal{Q}_1), \\ \mathcal{Q}_2 &= \mathcal{Q}_1 + (1 - \alpha)hg(\mathcal{P}_1), \end{aligned}$$

$$(4.6) \quad P_{k+1} = \mathcal{P}_1 + \sum_{r=1}^m \sigma_r(t_k, \mathcal{Q}_2)\Delta_k w_r, \quad Q_{k+1} = \mathcal{Q}_2, \quad k = 0, \dots, N - 1,$$

where  $0 \leq \alpha \leq 1$  is a parameter.

**THEOREM 4.1.** *The explicit method (4.5)–(4.6) for the system (4.2) with (4.3) is symplectic and of the first mean-square order.*

*Proof.* In the case of the system (4.2) the operators  $\Lambda_r$  take the form  $\Lambda_r = (\sigma_r, \partial/\partial p)$ . Since  $\sigma_r$  do not depend on  $p$ , we get  $\Lambda_i \sigma_j = 0$ . It is known [10] that in such a case the Euler method has the first mean-square order of accuracy. Comparing the method (4.5)–(4.6) with the Euler method, it is not difficult to get that the method (4.5)–(4.6) is of the first mean-square order as well.

Due to (4.3),  $\partial \sigma_r^i/\partial q^j = \partial \sigma_r^j/\partial q^i$ . Using this, we obtain  $dP_{k+1} \wedge dQ_{k+1} = d\mathcal{P}_1 \wedge d\mathcal{Q}_2$ . It is easy to prove that  $d\mathcal{P}_1 \wedge d\mathcal{Q}_2 = d\mathcal{P}_1 \wedge d\mathcal{Q}_1 = dP_k \wedge dQ_k$ . Therefore the method (4.5)–(4.6) is symplectic.  $\square$

**REMARK 4.1.** *By swapping the roles of  $p$  and  $q$ , we can propose the following symplectic method of the first mean-square order for the system (4.2) with (4.3):*

$$(4.7) \quad \mathcal{P} = P_k + \alpha hf(t_k, Q_k), \quad \mathcal{Q} = Q_k + hg(\mathcal{P}),$$

$$(4.8) \quad P_{k+1} = \mathcal{P} + (1 - \alpha)hf(t_{k+1}, \mathcal{Q}) + \sum_{r=1}^m \sigma_r(t_k, \mathcal{Q})\Delta_k w_r, \quad Q_{k+1} = \mathcal{Q}.$$

**4.2. Methods of order 3/2.** Consider the relations

$$(4.9) \quad P_i = p + h \sum_{j=1}^s \alpha_{ij} f(t + c_j h, Q_j) + \sum_{j=1}^s \sum_{r=1}^m \sigma_r(t + d_j h, Q_j) (\lambda_{ij} \varphi_r + \mu_{ij} \psi_r),$$

$$Q_i = q + h \sum_{j=1}^s \hat{\alpha}_{ij} g(P_j), \quad i = 1, \dots, s,$$

$$(4.10) \quad \bar{P} = p + h \sum_{i=1}^s \beta_i f(t + c_i h, Q_i) + \sum_{i=1}^s \sum_{r=1}^m \sigma_r(t + d_i h, Q_i) (\nu_i \varphi_r + \varkappa_i \psi_r),$$

$$\bar{Q} = q + h \sum_{i=1}^s \hat{\beta}_i g(P_i),$$

where  $\varphi_r, \psi_r$  do not depend on  $p$  and  $q$ , the parameters  $\alpha_{ij}, \hat{\alpha}_{ij}, \beta_i, \hat{\beta}_i, \lambda_{ij}, \mu_{ij}, \nu_i, \varkappa_i$  satisfy the conditions

$$(4.11) \quad \begin{aligned} \beta_i \hat{\alpha}_{ij} + \hat{\beta}_j \alpha_{ji} - \beta_i \hat{\beta}_j &= 0, \\ \nu_i \hat{\alpha}_{ij} + \hat{\beta}_j \lambda_{ji} - \nu_i \hat{\beta}_j &= 0, \quad \varkappa_i \hat{\alpha}_{ij} + \hat{\beta}_j \mu_{ji} - \varkappa_i \hat{\beta}_j = 0, \quad i, j = 1, \dots, s, \end{aligned}$$

and  $c_i, d_i$  are arbitrary parameters.

If  $\sigma_r \equiv 0$  the relations (4.9)–(4.10) coincide with a general form of  $s$ -stage PRK methods for deterministic differential equations. (See, e.g., [5, p. 34].) It is known [9, 5] that the symplectic condition holds for  $\bar{P}, \bar{Q}$  from (4.9)–(4.10) with (4.11) in the case of  $\sigma_r \equiv 0$ . By a generalization of the proof of Theorem 6.2 from [5], we prove the following lemma. (Another generalization is given in [3].)

LEMMA 4.2. *The relations (4.9)–(4.10) with conditions (4.11) preserve symplectic structure, i.e.,  $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$ .*

*Proof.* Denote for awhile:  $f_i = f(t + c_i h, Q_i), g_i = g(P_i), \sigma_{ri} = \sigma_r(t + d_i h, Q_i)$ . We get

$$(4.12) \quad \begin{aligned} d\bar{P} \wedge d\bar{Q} &= dp \wedge dq + h \sum_{j=1}^s \hat{\beta}_j dp \wedge dg_j + h \sum_{i=1}^s \beta_i df_i \wedge dq + h^2 \sum_{i=1}^s \sum_{j=1}^s \beta_i \hat{\beta}_j df_i \wedge dg_j \\ &+ \sum_{i=1}^s \sum_{r=1}^m (\nu_i \varphi_r + \varkappa_i \psi_r) d\sigma_{ri} \wedge dq + h \sum_{i=1}^s \sum_{j=1}^s \sum_{r=1}^m (\nu_i \varphi_r + \varkappa_i \psi_r) \hat{\beta}_j d\sigma_{ri} \wedge dg_j. \end{aligned}$$

Then we express  $dp \wedge dg_j$  from

$$dP_j \wedge dg_j = dp \wedge dg_j + h \sum_{i=1}^s \alpha_{ji} df_i \wedge dg_j + \sum_{i=1}^s \sum_{r=1}^m (\lambda_{ji} \varphi_r + \mu_{ji} \psi_r) d\sigma_{ri} \wedge dg_j$$

and substitute it in (4.12). Analogously, we act with  $df_i \wedge dq$  and  $d\sigma_{ri} \wedge dq$  finding them from the expressions for  $df_i \wedge dQ_i$  and  $d\sigma_{ri} \wedge dQ_i$ . As a result, using (4.11), we obtain

$$\begin{aligned} d\bar{P} \wedge d\bar{Q} &= dp \wedge dq + h \sum_{i=1}^s \hat{\beta}_i dP_i \wedge dg_i + h \sum_{i=1}^s \beta_i df_i \wedge dQ_i \\ &+ \sum_{i=1}^s \sum_{r=1}^m (\nu_i \varphi_r + \varkappa_i \psi_r) d\sigma_{ri} \wedge dQ_i. \end{aligned}$$

Taking into account that the wedge product is skew-symmetric, the vector functions  $f, g, \sigma_r$  are gradients,  $f, \sigma_r$  do not depend on  $p$ , and  $g$  does not depend on  $q$ , it is not difficult to see that each of the terms  $d\mathcal{P}_i \wedge dg_i, df_i \wedge d\mathcal{Q}_i, d\sigma_{ri} \wedge d\mathcal{Q}_i$  vanishes. Therefore  $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$ .  $\square$

Introduce the following 2-stage explicit PRK method for the system (4.2), (4.3):

$$\begin{aligned}
 (4.13) \quad \mathcal{Q}_1 &= Q_k, \quad \mathcal{P}_1 = P_k + \frac{h}{4}f(t_k, \mathcal{Q}_1) + \frac{1}{2} \sum_{r=1}^m \sigma_r(t_k, \mathcal{Q}_1) (3(J_{r0})_k - \Delta_k w_r), \\
 \mathcal{Q}_2 &= \mathcal{Q}_1 + \frac{2}{3}hg(\mathcal{P}_1), \\
 \mathcal{P}_2 &= \mathcal{P}_1 + \frac{3}{4}hf\left(t_k + \frac{2}{3}h, \mathcal{Q}_2\right) + \frac{3}{2} \sum_{r=1}^m \sigma_r\left(t_k + \frac{2}{3}h, \mathcal{Q}_2\right) (-(J_{r0})_k + \Delta_k w_r), \\
 (4.14) \quad P_{k+1} &= \mathcal{P}_2, \quad Q_{k+1} = \mathcal{Q}_2 + \frac{h}{3}g(\mathcal{P}_2), \quad k = 0, \dots, N-1,
 \end{aligned}$$

where

$$(4.15) \quad J_{r0} := \frac{1}{h} \int_t^{t+h} (w_r(\vartheta) - w_r(t)) d\vartheta.$$

**THEOREM 4.3.** *The explicit PRK method (4.13)–(4.14) for system (4.2), (4.3) preserves symplectic structure and has the mean-square order 3/2.*

*Proof.* The method (4.13)–(4.14) has the form of (4.9)–(4.10), and its parameters satisfy the conditions (4.11). Then, Lemma 4.2 implies that this method preserves symplectic structure.

Now let us prove mean-square order of convergence of (4.13)–(4.14). To this end, introduce the one-step approximation for (4.2):

$$\begin{aligned}
 (4.16) \quad \tilde{P} &= p + \sum_{r=1}^m \sigma_r \Delta w_r + hf + \sum_{r=1}^m \left[ \frac{\partial \sigma_r}{\partial t} + \sum_{i=1}^n g^i \frac{\partial \sigma_r}{\partial q^i} \right] I_{0r} + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t} + \sum_{i=1}^n g^i \frac{\partial f}{\partial q^i} \right], \\
 \tilde{Q} &= q + hg + \sum_{r=1}^m \sum_{i=1}^n \sigma_r^i \frac{\partial g}{\partial p^i} I_{r0} + \frac{h^2}{2} \left[ \sum_{i=1}^n f^i \frac{\partial g}{\partial p^i} + \frac{1}{2} \sum_{r=1}^m \sum_{i,j=1}^n \sigma_r^i \sigma_r^j \frac{\partial^2 g}{\partial p^i \partial p^j} \right],
 \end{aligned}$$

where

$$(4.17) \quad I_{0r} = \int_t^{t+h} (\vartheta - t) dw_r(\vartheta), \quad I_{r0} = \int_t^{t+h} (w_r(\vartheta) - w_r(t)) d\vartheta = hJ_{r0},$$

and all the coefficients are calculated at  $(t, p, q)$ . We note that

$$(\Delta w_r - J_{r0}) h = I_{0r}.$$

Using the general theory of numerical integration of SDEs [10], it is not difficult to show that the method based on (4.16) is of the mean-square order 3/2. Our nearest aim is to prove that the one-step approximation  $\bar{P}, \bar{Q}$  corresponding to the method (4.13)–(4.14) is such that

$$(4.18) \quad \left| E \left[ \begin{array}{c} \bar{P} - \tilde{P} \\ \bar{Q} - \tilde{Q} \end{array} \right] \right| = O(h^3), \quad \left( E \left[ \begin{array}{c} \bar{P} - \tilde{P} \\ \bar{Q} - \tilde{Q} \end{array} \right]^2 \right)^{1/2} = O(h^2).$$



Expanding the right-hand sides of the approximation  $\bar{P}, \bar{Q}$  about  $(t, p, q)$ , we obtain

$$\begin{aligned}
 (4.19) \quad \bar{P} &= p + hf + \frac{h^2}{2} \frac{\partial f}{\partial t} + \frac{3}{4} h \sum_{i=1}^n \Delta Q_2^i \frac{\partial f}{\partial q^i} + \sum_{r=1}^m \sigma_r \Delta w_r \\
 &\quad + \frac{3}{2} \sum_{r=1}^m \sum_{i=1}^n \Delta Q_2^i \frac{\partial \sigma_r}{\partial q^i} (\Delta w_r - J_{r0}) + h \sum_{r=1}^m \frac{\partial \sigma_r}{\partial t} (\Delta w_r - J_{r0}) + \rho_1, \\
 \bar{Q} &= q + hg + \frac{h}{3} \sum_{i=1}^n (2\Delta P_1^i + \Delta P_2^i) \frac{\partial g}{\partial p^i} + \frac{h}{6} \sum_{i,j=1}^n (2\Delta P_1^i \Delta P_1^j + \Delta P_2^i \Delta P_2^j) \frac{\partial^2 g}{\partial p^i \partial p^j} + \rho_2, \\
 \Delta P_1 &:= P_1 - p = \frac{h}{4} f + \frac{1}{2} \sum_{r=1}^m \sigma_r (3J_{r0} - \Delta w_r), \\
 \Delta Q_2 &:= Q_2 - q = \frac{2}{3} hg + \frac{2}{3} h \sum_{i=1}^n \Delta P_1^i \frac{\partial g}{\partial p^i} + \frac{h}{3} \sum_{i,j=1}^n \Delta P_1^i \Delta P_1^j \frac{\partial^2 g}{\partial p^i \partial p^j} + r_1, \\
 \Delta P_2 &:= P_2 - p = hf + \sum_{r=1}^m \sigma_r \Delta w_r + r_2,
 \end{aligned}$$

where all the coefficients are calculated at  $(t, p, q)$ .

Due to properties of the Wiener process and Ito integrals, we have

$$\begin{aligned}
 (4.20) \quad E\Delta w_i &= 0, \quad E\Delta w_i \Delta w_j = \delta_{ij} h, \quad E\Delta w_i \Delta w_j \Delta w_k = 0, \quad E(\Delta w_i)^4 = 3h^2, \\
 EJ_{i0} &= 0, \quad EJ_{i0} J_{j0} = \delta_{ij} \frac{h}{3}, \quad EJ_{i0} J_{j0} J_{k0} = 0, \quad E(J_{i0})^4 = \frac{h^2}{3}, \\
 E\Delta w_i J_{j0} &= \delta_{ij} \frac{h}{2}, \quad E\Delta w_i \Delta w_j J_{k0} = 0, \quad E\Delta w_i J_{j0} J_{k0} = 0.
 \end{aligned}$$

Then, under appropriate conditions on smoothness and boundedness of the coefficients of (4.2), we get

$$\begin{aligned}
 (4.21) \quad |E\rho_i| &= O(h^3), \quad E(\rho_i)^2 = O(h^5), \quad i = 1, 2, \\
 |Er_1| &= O(h^3), \quad E(r_1)^2 = O(h^5), \quad |Er_2| = O(h^2), \quad E(r_2)^2 = O(h^3).
 \end{aligned}$$

In addition to (4.20) we note that

$$(4.22) \quad E(\Delta w_r - J_{r0})(3J_{r0} - \Delta w_r) = 0, \quad E(3J_{r0} - \Delta w_r)^2 = h.$$

Using (4.20)–(4.22), we obtain from (4.19)

$$\begin{aligned}
 \bar{P} &= p + \sum_{r=1}^m \sigma_r \Delta w_r + hf + \sum_{r=1}^m \left[ \frac{\partial \sigma_r}{\partial t} + \sum_{i=1}^n g^i \frac{\partial \sigma_r}{\partial q^i} \right] I_{0r} + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t} + \sum_{i=1}^n g^i \frac{\partial f}{\partial q^i} \right] + R_1, \\
 \bar{Q} &= q + hg + \sum_{r=1}^m \sum_{i=1}^n \sigma_r^i \frac{\partial g}{\partial p^i} I_{r0} + \frac{h^2}{2} \left[ \sum_{i=1}^n f^i \frac{\partial g}{\partial p^i} + \frac{1}{2} \sum_{r=1}^m \sum_{i,j=1}^n \sigma_r^i \sigma_r^j \frac{\partial^2 g}{\partial p^i \partial p^j} \right] + R_2
 \end{aligned}$$

with  $R_i$ ,  $i = 1, 2$ , such that

$$|ER_i| = O(h^3), \quad E(R_i)^2 = O(h^4), \quad i = 1, 2.$$

This implies (4.18). It follows from (4.18) that the method (4.13)–(4.14) is of the mean-square order 3/2.  $\square$

REMARK 4.2. *The random variables  $\Delta_k w_r(h)$ ,  $(J_{r0})_k$  have a Gaussian joint distribution, and they can be simulated at each step by  $2m$  independent  $\mathcal{N}(0, 1)$ -distributed random variables  $\xi_{rk}$  and  $\eta_{rk}$ ,  $r = 0, \dots, m$ :*

$$\Delta_k w_r(h) = \xi_{rk} \sqrt{h}, \quad (J_{r0})_k = \left( \xi_{rk}/2 + \eta_{rk}/\sqrt{12} \right) \sqrt{h}.$$

As a result, the method (4.13)–(4.14) takes the constructive form.

REMARK 4.3. *It is very unusual that the direct expansion of (4.13)–(4.14) does not contain the habitual term  $\frac{h^2}{4} \sum_{r=1}^m \sum_{i,j=1}^n \frac{\partial^2 g}{\partial p^i \partial p^j} \sigma_r^i \sigma_r^j$ . The similar term in the expansion contains some combinations of  $\Delta w_r$  and  $J_{r0}$  instead of  $h$ . (See a similar remark in [3].)*

REMARK 4.4. *In the case of  $\sigma_r = 0$ ,  $r = 1, \dots, m$ , the method (4.13)–(4.14) coincides with the well-known deterministic symplectic PRK method of the second order. Adapting other explicit deterministic second-order PRK methods from [5, 9], it is possible to construct other explicit symplectic methods of the order 3/2 for the system (4.2), (4.3).*

REMARK 4.5. *In the case of a more general system than (4.2) methods of the order 3/2 require simulation of repeated Ito integrals which is a laborious problem from the computational point of view. We do not consider such methods in the paper. (See also the introduction.)*

*Lemma 4.2 can be generalized for the general separable case, i.e., for the system (1.1), (1.3) with  $H_r = V_r(p) + U_r(t, q)$ ,  $r = 0, 1, \dots, m$ , and it can also be generalized for the general stochastic Hamiltonian system (1.1), (1.3). In the case of systems with one noise repeated Ito integrals can effectively be simulated, and generalizations of Lemma 4.2 can be used for constructing high-order symplectic methods for Hamiltonian systems with one noise (i.e., when  $m = 1$ ).*

### 5. Numerical tests.

**5.1. Kubo oscillator.** The system of SDEs in the sense of Stratonovich (Kubo oscillator)

$$(5.1) \quad \begin{aligned} dX^1 &= -aX^2 dt - \sigma X^2 \circ dw(t), & X^1(0) &= x^1, \\ dX^2 &= aX^1 dt + \sigma X^1 \circ dw(t), & X^2(0) &= x^2, \end{aligned}$$

is often used for testing numerical methods. (See, e.g., [15], where some nonsymplectic stochastic methods based on deterministic symplectic methods are used.) Here  $a$  and  $\sigma$  are constants, and  $w(t)$  is a one-dimensional standard Wiener process.

The phase flow of this system preserves symplectic structure. Moreover, the quantity  $\mathcal{H}(x^1, x^2) = (x^1)^2 + (x^2)^2$  is conservative for this system; i.e.,

$$\mathcal{H}(X^1(t), X^2(t)) = \mathcal{H}(x^1, x^2) \quad \text{for } t \geq 0.$$

This means that a phase trajectory of (5.1) belongs to the circle with the center at the origin and with the radius  $\sqrt{\mathcal{H}(x^1, x^2)}$ .

We test here three methods. In application to (5.1) the symplectic PRK method (3.8) takes the following form:

$$(5.2) \quad \begin{aligned} X_{k+1}^1 &= X_k^1 - aX_k^2h - \frac{\sigma^2}{2}X_{k+1}^1h - \sigma X_k^2\Delta_k w, \\ X_{k+1}^2 &= X_k^2 + aX_{k+1}^1h + \frac{\sigma^2}{2}X_k^2h + \sigma X_{k+1}^1\Delta_k w. \end{aligned}$$

This method is implicit in the deterministic part only.

The midpoint method (3.7) applied to the system with one noise (5.1) reads

$$(5.3) \quad \begin{aligned} X_{k+1}^1 &= X_k^1 - a\frac{X_k^2 + X_{k+1}^2}{2}h - \sigma\frac{X_k^2 + X_{k+1}^2}{2}(\zeta_h)_k\sqrt{h}, \\ X_{k+1}^2 &= X_k^2 + a\frac{X_k^1 + X_{k+1}^1}{2}h + \sigma\frac{X_k^1 + X_{k+1}^1}{2}(\zeta_h)_k\sqrt{h}. \end{aligned}$$

This is a fully implicit method. Note that due to specific features of the system (5.1), the formula (5.3) is valid (solvable) not only in the case of the truncated random variable  $\zeta_h$  but also if we put  $\Delta_k w$  instead of  $(\zeta_h)_k\sqrt{h}$ .

The method (5.3) is of the first mean-square order. The method (5.2) is of the mean-square order 1/2 as well as the Euler method:

$$(5.4) \quad \begin{aligned} X_{k+1}^1 &= X_k^1 - aX_k^2h - \frac{\sigma^2}{2}X_k^1h - \sigma X_k^2\Delta_k w, \\ X_{k+1}^2 &= X_k^2 + aX_k^1h - \frac{\sigma^2}{2}X_k^2h + \sigma X_k^1\Delta_k w, \end{aligned}$$

which, of course, is not symplectic.

Figure 1 gives approximations of a sample phase trajectory of (5.1) simulated by the symplectic methods (5.2) and (5.3) and by the Euler method (5.4). The initial condition is  $x^1 = 1, x^2 = 0$ . The corresponding exact phase trajectory belongs to the circle with the center at the origin and with the unit radius.

We see that the Euler method is not appropriate for simulation of the oscillator (5.1) on long time intervals, while the symplectic methods preserve conservative properties of the Kubo oscillator.

These experiments also demonstrate that the midpoint method is much more accurate than the other methods applied. It is not difficult to check that  $\mathcal{H}(x^1, x^2)$  is conserved by the midpoint method (5.3), but it is not conserved by the symplectic PRK method (5.2). This is similar to the deterministic case. Indeed, it is known [8, 5] that symplectic deterministic RK methods (e.g., the midpoint scheme) conserve all quadratic functions that are conserved by the Hamiltonian system being integrated, while deterministic PRK methods do not possess this property.

**5.2. A model for synchrotron oscillations of particles in storage rings.**

In [14] a model describing synchrotron oscillations of particles in storage rings under the influence of external fluctuating electromagnetic fields was considered. This model can be written in the following form:

$$(5.5) \quad \begin{aligned} dP &= -\omega^2 \sin(Q)dt - \sigma_1 \cos(Q)dw_1 - \sigma_2 \sin(Q)dw_2, \\ dQ &= Pdt. \end{aligned}$$

$P$  and  $Q$  are scalars here. The system (5.5) is of the form (4.2), and therefore its phase flow preserves symplectic structure.

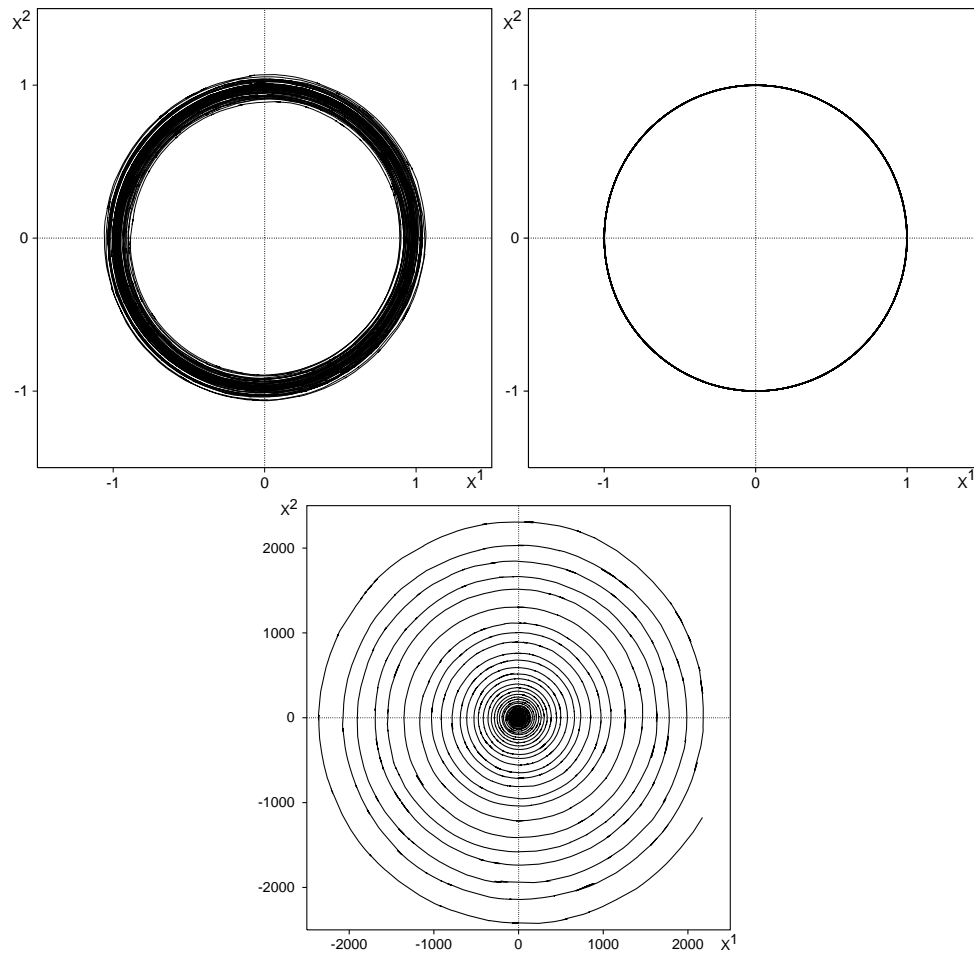


FIG. 1. A sample phase trajectory of (5.1) with  $X^1(0) = 1, X^2(0) = 0$  obtained by the symplectic method (5.2) (top left), the midpoint method (5.3) (top right), and by the Euler method (5.4) (bottom) for  $a = 2, \sigma = 0.3, h = 0.02$  on the time interval  $t \leq 200$ .

The Euler method for (5.5) takes the form

$$(5.6) \quad \begin{aligned} P_{k+1} &= P_k - h\omega^2 \sin(Q_k) - h^{1/2}(\sigma_1 \cos(Q_k)\Delta_k w_1 + \sigma_2 \sin(Q_k)\Delta_k w_2), \\ Q_{k+1} &= Q_k + hP_k. \end{aligned}$$

In application to (5.5) the explicit symplectic method (4.5)–(4.6) with  $\alpha = 1$  is written as

$$(5.7) \quad \begin{aligned} Q &= Q_k + hP_k, \\ P_{k+1} &= P_k - h\omega^2 \sin(Q) - h^{1/2}(\sigma_1 \cos(Q)\Delta_k w_1 + \sigma_2 \sin(Q)\Delta_k w_2), \quad Q_{k+1} = Q. \end{aligned}$$

Both methods are of the first mean-square order.

Approximations of a sample trajectory of (5.5) simulated by the symplectic method (5.7) and the Euler method (5.6) are plotted on Figure 2. The trajectory obtained by the symplectic method with  $h = 0.02$  (solid line) visually coincides with

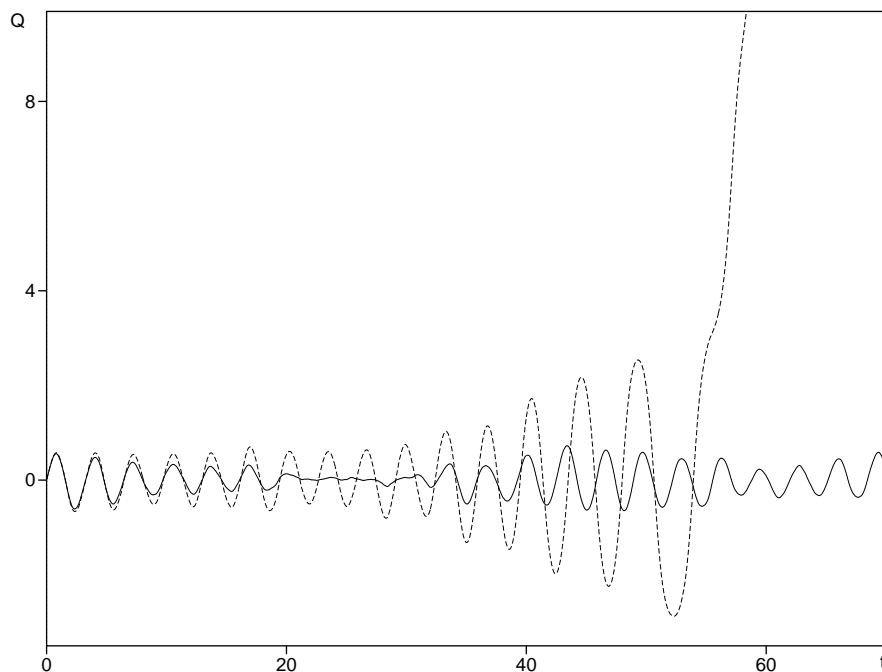


FIG. 2. A sample trajectory of (5.5) for  $\omega = 2$ ,  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.1$ ,  $h = 0.02$ . Solid line—the symplectic method (5.7), dashed line—the Euler method (5.6).

the one obtained for a smaller step, e.g., for  $h = 0.002$ , using the same sample paths for the Wiener processes; i.e., this trajectory visually coincides with the exact solution of (5.5). This figure clearly demonstrates that the Euler method (dashed line) is unacceptable for simulation of the solution to (5.5) on a long time interval, while the symplectic method (5.7) produces quite accurate results despite both methods having the same mean-square order of accuracy.

## REFERENCES

- [1] J.-M. BISMUT, *Mécanique Aléatoire*, Lecture Notes in Math. 866, Springer, Berlin, 1981.
- [2] N. IKEDA AND S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, 1981.
- [3] G.N. MILSTEIN, YU. M. REPIN, AND M.V. TRETYAKOV, *Symplectic integration of Hamiltonian systems with additive noise*, SIAM J. Numer. Anal., 39 (2002), pp. 2066–2088.
- [4] V.I. ARNOLD, *Mathematical Methods of Classical Mechanics*, Springer, Berlin, 1989.
- [5] J.M. SANZ-SERNA AND M.P. CALVO, *Numerical Hamiltonian Problems*, Chapman and Hall, London, 1994.
- [6] E. HAIRER, S.P. NØRSETT, AND G. WANNER, *Solving Ordinary Differential Equations. I. Non-stiff Problems*, Springer, Berlin, 1993.
- [7] P.J. CHANNEL AND C. SCOVEL, *Symplectic integration of Hamiltonian systems*, Nonlinearity, 3 (1990), pp. 231–259.
- [8] J.M. SANZ-SERNA, *Symplectic integrators for Hamiltonian problems: An overview*, Acta Numer. 1, Cambridge University Press, Cambridge, UK, 1992, pp. 243–286.
- [9] YU. B. SURIS, *Hamiltonian methods of Runge-Kutta type and their variational interpretation*, Mat. Model., 2 (1990), pp. 78–87.
- [10] G.N. MILSTEIN, *Numerical Integration of Stochastic Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.
- [11] P.E. KLOEDEN AND E. PLATEN, *Numerical Solution of Stochastic Differential Equations*,

- Springer, Berlin, 1992.
- [12] G.N. MILSTEIN, E. PLATEN, AND H. SCHURZ, *Balanced implicit methods for stiff stochastic systems*, SIAM J. Numer. Anal., 35 (1998), pp. 1010–1019.
  - [13] G.N. MILSTEIN, YU. M. REPIN, AND M.V. TRETYAKOV, *Mean-Square Symplectic Methods for Hamiltonian Systems with Multiplicative Noise*, Preprint 670, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, Germany, 2001.
  - [14] M. SEEßELBERG, H.P. BREUER, H. MAIS, F. PETRUCCIONE, AND J. HONERKAMP, *Simulation of one-dimensional noisy Hamiltonian systems and their application to particle storage rings*, Z. Phys. C, 62 (1994), pp. 62–73.
  - [15] M.V. TRETYAKOV AND S.V. TRET'JAKOV, *Numerical integration of Hamiltonian systems with external noise*, Phys. Lett. A, 194 (1994), pp. 371–374.