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NUMERICAL ALGORITHMS FOR FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

G. N. MILSTEIN† AND M. V. TRETYAKOV‡

Abstract. Efficient numerical algorithms are proposed for a class of forward-backward stochastic differential equations (FBSDEs) connected with semilinear parabolic partial differential equations. As in [J. Douglas, Jr., J. Ma, and P. Protter, *Ann. Appl. Probab.*, 6 (1996), pp. 940–968], the algorithms are based on the known four-step scheme for solving FBSDEs. The corresponding semilinear parabolic equation is solved by layer methods which are constructed by means of a probabilistic approach. The derivatives of the solution u of the semilinear equation are found by finite differences. The forward equation is simulated by mean-square methods of order 1/2 and 1. Corresponding convergence theorems are proved. Along with the algorithms for FBSDEs on a fixed finite time interval, we also construct algorithms for FBSDEs with random terminal time. The results obtained are supported by numerical experiments.

Key words. forward-backward stochastic differential equations, numerical integration, mean-square convergence, semilinear partial differential equations of parabolic type

AMS subject classifications. Primary, 60H35; Secondary, 65C30, 60H10, 62P05

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1. Introduction. Forward-backward stochastic differential equations (FBSDEs) have numerous applications in stochastic control theory and mathematical finance (see, e.g., [7, 5, 9, 25, 12] and references therein). Consider an FBSDE of the form

$$dX = a(t,X,Y)dt + \sigma(t,X,Y)dw(t), \quad X(t_0) = x, \tag{1.1}$$

$$dY = -g(t,X,Y)dt - f^{\intercal}(t,X,Y)Zdt + Z^{\intercal}dw(t), \quad Y(T) = \varphi(X(T)). \tag{1.2}$$

Here $X = X(t)$ and $a = a(t,x,y)$ are d-dimensional vectors; $\sigma = \sigma(t,x,y)$ is a $d \times n$-matrix; $Y = Y(t)$, $g = g(t,x,y)$, and $\varphi = \varphi(x)$ are scalars; $Z = Z(t)$ and $f = f(t,x,y)$ are n-dimensional vectors; and $w(t)$ is an n-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Wiener process, where $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $t_0 \leq t \leq T$, is a filtered probability space. It is known (see, e.g., [1, 11, 19, 25, 12] and also references therein) that there exists a unique $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solution $(X(t), Y(t), Z(t))$ of the system (1.1)–(1.2) under some appropriate smoothness and boundedness conditions on its coefficients.

Due to the four-step scheme from [11] (we recall it in the next section), the solution of (1.1)–(1.2) is connected with the Cauchy problem for the semilinear partial

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†Department of Mathematics, Ural State University, Lenin Str. 51, 620083 Ekaterinburg, Russia (Grigori.Milstein@usu.ru).

‡Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK (M.Tretiakov@le.ac.uk).

differential equation (PDE):

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} a^i(t,x,u)\frac{\partial u}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} \tag{1.3}$$
$$= -g(t,x,u) - \sum_{k=1}^{n} f^k(t,x,u)\sum_{i=1}^{d}\sigma^{ik}(t,x,u)\frac{\partial u}{\partial x^i}, \quad t < T, \quad x \in \mathbf{R}^d,$$

$$u(T,x) = \varphi(x), \tag{1.4}$$

where

$$a^{ij} := \sum_{k=1}^{d} \sigma^{ik}\sigma^{jk}.$$

In turn the corresponding solution of the semilinear PDE has a probabilistic representation using the FBSDE (1.1)–(1.2), which is a generalization of the Feynman–Kac formula (see, e.g., [22, 20, 19, 21, 25, 12]).

Our aim is to find an effective numerical algorithm for solving (1.1)–(1.2) in the mean-square sense. Not many works have been devoted to numerical integration of FBSDEs, but let us mention [4, 6, 2, 3]. Among these works, the present paper is most closely connected with [6]. As in [6], we exploit the four-step scheme for numerical solution of (1.1)–(1.2). Unlike [6], we use another approach for numerical solution of the corresponding semilinear PDE (1.3)–(1.4), which is developed in [14] (see also [16, 17, 18]).

The most significant distinction between our paper and [6] consists of numerical evaluation of Z. According to the four-step scheme, Z is expressed through first derivatives of the solution $u(t,x)$ to (1.3)–(1.4) with respect to x^i, $i = 1,\dots,d$, while finding X and Y of (1.1)–(1.2) requires knowledge of u only. In [6] the authors write a system of semilinear PDEs for u and $\partial u/\partial x^i$, $i = 1,\dots,d$, and solve it numerically in order to simulate then X, Y, and Z. This approach is rather expensive from the computational point of view. We approximate the derivatives by finite differences; thus we need to simulate u only. As it is proved in section 3, this approximation gives quite accurate results.

Along with the algorithms for FBSDEs on a fixed finite time interval, we also construct algorithms for FBSDEs with random terminal time. To the authors' best knowledge, numerical solution of FBSDEs with random terminal time is considered for the first time. Let us note that in this case the approach of [6] leads to a complicated system of boundary value problems whose numerical solution is less effective than the algorithms proposed here.

For clarity of exposition, we state and prove our results for the one-dimensional case ($d = 1$, $n = 1$) although it is not difficult to generalize them to any dimension. However, since the obtained algorithms require simulation of nonlinear PDEs, they can realistically be used in practice for solving FBSDEs with forward equation of dimension three or lower ($d \leq 3$). At the same time, the variable Y can be of a high dimension; in such a case we shall deal with a system of semilinear parabolic PDEs instead of (1.3)–(1.4).

Let us note that in [6] a more general FBSDE than (1.1)–(1.2) is considered; it is connected with quasi-linear parabolic equations. We will consider this case in a separate publication.

In section 2 we recall the four-step scheme due to [11] and the probabilistic approach to numerical solution of the Cauchy problem (1.3)–(1.4) from [14] (see also [18]). In section 3 we obtain some results concerning accuracy of approximating derivatives $\partial u/\partial x^i$ by finite differences. In section 4 we prove the mean-square convergence of the Euler method for FBSDEs when using the approximations of u and $\partial u/\partial x$ given in the previous two sections. Sections 5 and 6 are devoted to FBSDEs with random terminal time (see e.g., [19, 25] and references therein). In these sections we have to consider the Dirichlet boundary value problem for semilinear parabolic PDEs instead of (1.3)–(1.4). For solving the Dirichlet problem we use the probabilistic approach again [16, 18], and for simulating solutions of FBSDEs with random terminal time we use the approximations of SDEs in space-time bounded domains [15, 18]. We also consider the case of unbounded random terminal time, which is connected with the Dirichlet boundary value problem for semilinear elliptic PDEs. The results obtained are supported by numerical experiments which are presented in section 7.

2. Preliminaries.

2.1. Four-step scheme for solving FBSDEs. First we recall the four-step scheme for solving FBSDE (1.1)–(1.2) [11]. Assume that the solution $u(t,x)$ of the Cauchy problem (1.3)–(1.4) is known. Consider the following SDE:

$$dX = a(t,X,u(t,X))dt + \sigma(t,X,u(t,X))dw(t), \quad X(t_0) = x. \tag{2.1}$$

Let $X(t) = X_{t_0,x}(t)$ be a solution of the Cauchy problem (2.1). Introduce

$$\begin{aligned} &Y(t) = u(t, X_{t_0,x}(t)), \\ &Z^j(t) = \sum_{i=1}^{d} \sigma^{ij}(t, X_{t_0,x}(t), Y(t)) \frac{\partial u}{\partial x^i}(t, X_{t_0,x}(t)), \qquad j = 1, \dots, n. \end{aligned} \tag{2.2}$$

It turns out that $(X(t), Y(t), Z(t))$ defined by (2.1)–(2.2) is the solution of the FBSDE (1.1)–(1.2). Indeed, it is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and (1.1) is evidently satisfied. To verify (1.2), it suffices to apply Ito's formula to $Y(t) = u(t, X_{t_0,x}(t))$.

2.2. Layer methods for PDEs. Now we recall layer methods for solving the problem (1.3)–(1.4) due to [14] (see also [18]). For simplicity in writing, we restrict ourselves to a one-dimensional version of the problem ($d = 1$, $n = 1$). Introducing

$$b(t,x,y) := a(t,x,y) + f(t,x,y)\sigma(t,x,y),$$

we get

$$\frac{\partial u}{\partial t} + b(t,x,u)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u}{\partial x^2} + g(t,x,u) = 0, \quad t < T,\ x \in \mathbf{R}, \tag{2.3}$$

$$u(T,x) = \varphi(x)\,. \tag{2.4}$$

The solution to this problem is supposed to exist, be unique, be sufficiently smooth, and satisfy some conditions on boundedness. One can find theoretical results on this topic in [10, 23, 8, 24] (see also references therein). For convenience, we shall assume throughout the paper that the standing assumptions of [6] are fulfilled. We prefer to state them here in a less specific way since different numerical methods will require, e.g., different levels of smoothness of the coefficients.

ASSUMPTION 2.1. *It is assumed that the coefficients b, σ, g and the function φ are sufficiently smooth and that all these functions together with their derivatives up to some order are bounded on $[t_0, T] \times \mathbf{R} \times \mathbf{R}$. In addition, it is supposed that σ is bounded away from zero: $\sigma \geq \sigma_0$, where σ_0 is a positive constant.*

Assumption 2.1 ensures the existence of a unique bounded solution $u(t,x)$ with bounded derivatives up to some order. We note that these assumptions are more than necessary (for instance, other types of assumptions are given in [14] (see also [18, pp. 419, 422])), and the methods constructed in this paper can be used under broader conditions.

We introduce a time discretization; to be definite let us take the equidistant one:

$$T = t_N > t_{N-1} > \cdots > t_0, \quad h := \frac{T - t_0}{N} .$$

Layer methods proposed in [14] (see also [18]) give an approximation $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$, $k = N, \ldots, 0$, to (2.3)–(2.4). These methods are based on the local probabilistic representation of the solution to (2.3)–(2.4):

$$u(t_k, x) = E\left(u(t_{k+1}, X_{t_k,x}(t_{k+1})) + \int_{t_k}^{t_{k+1}} g(s, X_{t_k,x}(s), u(s, X_{t_k,x}(s)))ds\right), \tag{2.5}$$

where $X_{t_k,x}(s)$ is the solution of the Cauchy problem for the SDE

$$dX = b(s, X, u(s,X))ds + \sigma(s, X, u(s,X))dw(s), \quad X(t_k) = x. \tag{2.6}$$

Exploiting the weak Euler scheme, the following layer method is constructed [14, 18]:

$$\tilde{u}(t_N, x) = \varphi(x), \tag{2.7}$$

$$\begin{aligned}\tilde{u}(t_k, x) = {} & \frac{1}{2}\tilde{u}(t_{k+1}, x + hb(t_k, x, \tilde{u}(t_{k+1}, x)) - \sqrt{h}\sigma(t_k, x, \tilde{u}(t_{k+1}, x))) \\ & + \frac{1}{2}\tilde{u}(t_{k+1}, x + hb(t_k, x, \tilde{u}(t_{k+1}, x)) + \sqrt{h}\sigma(t_k, x, \tilde{u}(t_{k+1}, x))) \\ & + hg(t_k, x, \tilde{u}(t_{k+1}, x)), \quad k = N-1, \ldots, 1, 0.\end{aligned}$$

It is proved (see either [14] or [18, p. 420]) that this method is of order one; i.e.,

$$|\tilde{u}(t_k, x) - u(t_k, x)| \leq Kh , \tag{2.8}$$

where K does not depend on x, h, k.

To obtain a numerical algorithm, we need to discretize (2.7) in the variable x. Consider the equidistant space discretization

$$x_j = x_0 + j\varkappa h, \quad j = 0, \pm 1, \pm 2, \ldots ,$$

where x_0 is a point on $\mathbf{R}$ and $\varkappa$ is a positive number; i.e., the step h_x of space discretization is equal to $\varkappa h$, where $h = h_t$ is the step of time discretization. Using, for instance, linear interpolation, we construct the following algorithm on the basis of the layer method (2.7).

ALGORITHM 2.2.

$$\bar{u}(t_N, x) = \varphi(x), \tag{2.9}$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + hb(t_k, x_j, \bar{u}(t_{k+1}, x_j)) - \sqrt{h}\sigma(t_k, x_j, \bar{u}(t_{k+1}, x_j)))$$
$$+ \frac{1}{2}\bar{u}(t_{k+1}, x_j + hb(t_k, x_j, \bar{u}(t_{k+1}, x_j)) + \sqrt{h}\sigma(t_k, x_j, \bar{u}(t_{k+1}, x_j)))$$
$$+ hg(t_k, x_j, \bar{u}(t_{k+1}, x_j)), \quad j = 0, \pm 1, \pm 2, \dots,$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{\varkappa h}\bar{u}(t_k, x_j) + \frac{x - x_j}{\varkappa h}\bar{u}(t_k, x_{j+1}), \quad x_j \le x \le x_{j+1}, \tag{2.10}$$

$$k = N-1, \dots, 1, 0.$$

Algorithm 2.2 is of order one; i.e.,

$$|\bar{u}(t_k, x) - u(t_k, x)| \le Kh\,, \tag{2.11}$$

where K does not depend on x, h, k (see either [14] or [18, p. 423] for a proof).

Along with linear interpolation, a spline approximation is also considered and the cubic interpolation with step $h_x = \varkappa\sqrt{h}$ is used to reduce the number of nodes x_j (see [14, 18]). Clearly, both the method and algorithm can be considered with variable time and space steps. Algorithms for the multidimensional parabolic problems (such as (1.3)–(1.4) with $d > 1$) are available in [14, 18] as well.

3. Approximation of the derivative $\partial u/\partial x$ by finite differences. Consider the solution $u(t,x)$ of the Cauchy problem for semilinear parabolic equation (2.3)–(2.4) and its approximations $\tilde{u}(t_k, x)$ by the layer method (2.7) and $\bar{u}(t_k, x)$ by Algorithm 2.2.

PROPOSITION 3.1. *The following formula holds:*

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\bar{u}(t_k, x + \gamma\sqrt{h}) - \bar{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h^{1/2}), \tag{3.1}$$

where γ is a positive number and h is the time step which is the same as in (2.9)–(2.10).

The analogous formula is valid if the function $\bar{u}$ is substituted by $\tilde{u}$.

Proof. Since the solution u has bounded third derivatives with respect to x (we assume that the functions from Assumption 2.1 have continuous bounded first derivative with respect to t and second derivatives with respect to x and u including the mixed ones), we have

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{u(t_k, x + \gamma\sqrt{h}) - u(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h).$$

Now (3.1) immediately follows from the inequality (2.11). ∎

Remark 3.2. Analogous to (3.1), we also get

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\bar{u}(t_k, x + \gamma h^{1/3}) - \bar{u}(t_k, x - \gamma h^{1/3})}{2\gamma h^{1/3}} + O(h^{2/3}). \tag{3.2}$$

In fact, it is possible to prove a more accurate result than (3.1) or (3.2) for the layer approximation $\tilde{u}$ from (2.7).

THEOREM 3.3. *The following formula holds:*

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\tilde{u}(t_k, x + \gamma\sqrt{h}) - \tilde{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h), \tag{3.3}$$

where γ is a positive number and h is the time step which is the same as in (2.7).

Proof. Clearly, the pair of functions $u(t,x)$ and $v(t,x) := \frac{\partial u}{\partial x}(t,x)$ satisfy the Cauchy problem for two parabolic equations consisting of (2.3)–(2.4) and

$$\begin{aligned} &\frac{\partial v}{\partial t} + b(t,x,u)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 v}{\partial x^2} + \left(\sigma\frac{\partial \sigma}{\partial x} + \sigma\frac{\partial \sigma}{\partial u}v\right)\frac{\partial v}{\partial x} \\ &\quad + \left(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u}v\right)v + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}v = 0, \quad t < T,\ x \in \mathbf{R}, \end{aligned} \tag{3.4}$$

$$v(T,x) = \varphi'(x)\,. \tag{3.5}$$

To solve the problem (2.3)–(2.4), (3.4)–(3.5), we use a layer method based on a local probabilistic representation. To this aim, introduce the system of SDEs with respect to X and scalars P, Q, R :

$$\begin{aligned} &dX = b(s,X,u(s,X))ds + \sigma(s,X,u(s,X))dw(s), \quad X(t_k) = x, \\ &dP = g(s,X,u(s,X))ds, \quad P(t_k) = 0, \\ &dQ = \left(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u}v(s,X)\right)Qds + \left(\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial u}v(s,X)\right)Qdw(s), \quad Q(t_k) = 1, \\ &dR = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}v(s,X)\right)Qds, \quad R(t_k) = 0, \end{aligned} \tag{3.6}$$

where $\partial b/\partial x$, $\partial b/\partial u$, and the other derivatives are known functions of s, X, $u(s,X)$. One can verify that the following local probabilistic representation holds (cf. (2.5)–(2.6)):

$$\begin{aligned} &u(t_k,x) = E\left[u(t_{k+1}, X_{t_k,x}(t_{k+1})) + P_{t_k,x,0}(t_{k+1})\right], \\ &v(t_k,x) = E\left[v(t_{k+1}, X_{t_k,x}(t_{k+1}))Q_{t_k,x,1}(t_{k+1}) + R_{t_k,x,1,0}(t_{k+1})\right]. \end{aligned} \tag{3.7}$$

The corresponding layer method $\tilde{u}(t_k,x)$, $\tilde{v}(t_k,x)$ is given by the formulas (2.7) for $\tilde{u}$ while $\tilde{v}$ is found from

$$\tilde{v}(t_N,x) = \varphi'(x), \tag{3.8}$$

$$\begin{aligned} \tilde{v}(t_k,x) = &\ \frac{1}{2}\tilde{v}(t_{k+1}, x + h\tilde{b}_k - \sqrt{h}\tilde{\sigma}_k) \\ &\times \left[1 + h\left(\frac{\partial \tilde{b}_k}{\partial x} + \frac{\partial \tilde{b}_k}{\partial u}\tilde{v}(t_{k+1},x)\right) - h^{1/2}\left(\frac{\partial \tilde{\sigma}_k}{\partial x} + \frac{\partial \tilde{\sigma}_k}{\partial u}\tilde{v}(t_{k+1},x)\right)\right] \\ &+ \frac{1}{2}\tilde{v}(t_{k+1}, x + h\tilde{b}_k + \sqrt{h}\tilde{\sigma}_k) \\ &\times \left[1 + h\left(\frac{\partial \tilde{b}_k}{\partial x} + \frac{\partial \tilde{b}_k}{\partial u}\tilde{v}(t_{k+1},x)\right) + h^{1/2}\left(\frac{\partial \tilde{\sigma}_k}{\partial x} + \frac{\partial \tilde{\sigma}_k}{\partial u}\tilde{v}(t_{k+1},x)\right)\right] \\ &+ h\left(\frac{\partial \tilde{g}_k}{\partial x} + \frac{\partial \tilde{g}_k}{\partial u}\tilde{v}(t_{k+1},x)\right), \quad k = N-1, \ldots, 1, 0, \end{aligned}$$

where $\tilde{b}_k := b(t_k, x, \tilde{u}(t_{k+1}, x))$, $\tilde{\sigma}_k := \sigma(t_k, x, \tilde{u}(t_{k+1}, x))$, and the notation $\partial \tilde{b}_k / \partial x$ means $\frac{\partial \tilde{b}_k}{\partial x} := \frac{\partial b}{\partial x}(t_k, x, \tilde{u}(t_{k+1}, x))$ and so on. The layer method (2.7), (3.8) for the system (2.3)–(2.4), (3.4)–(3.5) is of order one. The order of convergence can be proved due to [18] if we assume that the functions from Assumption 2.1 have continuous bounded second mixed derivatives with respect to t, x and t, u and third derivatives with respect to x and u including the mixed ones. This also implies that $\tilde{u}(t_k, x)$ has, in particular, a continuous bounded third derivative with respect to x.

Further, it is straightforward to verify

$$\frac{\partial \tilde{u}}{\partial x}(t_k, x) = \tilde{v}(t_k, x).$$

Thus we get

$$\frac{\partial u}{\partial x}(t_k, x) = v(t_k, x) = \tilde{v}(t_k, x) + O(h) = \frac{\partial \tilde{u}}{\partial x}(t_k, x) + O(h). \tag{3.9}$$

Since $\tilde{u}(t_k, x)$ has a bounded third derivative with respect to x, we obtain

$$\frac{\partial \tilde{u}}{\partial x}(t_k, x) = \frac{\tilde{u}(t_k, x + \gamma\sqrt{h}) - \tilde{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h). \tag{3.10}$$

The formulas (3.9) and (3.10) imply (3.3). ∎

Remark 3.4. We have not succeeded in a rigorous proof of the relation

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\bar{u}(t_k, x + \gamma\sqrt{h}) - \bar{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h) \tag{3.11}$$

for $\bar{u}(t_k, x)$ from Algorithm 2.2. At the same time, we have some heuristic arguments justifying (3.11). It is possible to obtain the following representation for $\bar{u}(t_0, x)$:

$$\bar{u}(t_0, x) = \tilde{u}(t_0, x) + R(x, h)h + \sum_{k=1}^{N} \zeta_k h^2 + O(h^{3/2}). \tag{3.12}$$

Moreover, it is reasonable to consider $R(x, h)$ a function which changes in x slowly or, more exactly, that R satisfies the inequality

$$R(x + \gamma\sqrt{h}, h) - R(x - \gamma\sqrt{h}, h) = O(\sqrt{h}). \tag{3.13}$$

In (3.12) ζ_k are related to the distances between $x_j + hb(t_k, x_j, \bar{u}(t_{k+1}, x_j)) \pm \sqrt{h}\sigma(t_k, x_j, \bar{u}(t_{k+1}, x_j))$ and the nearest node x_i. These distances are random, in a sense, and it is natural to assume that ζ_k are independent and identically distributed (i.i.d.) uniformly bounded random variables with zero mean and variance $Var\zeta$. Then, due to the central limit theorem, we get

$$\sum_{k=1}^{N} \zeta_k h^2 \doteq \eta \sqrt{Var\zeta} \cdot \sqrt{N}\, h^2, \tag{3.14}$$

where η is a standard Gaussian random variable. The relation (3.11) follows from (3.12)–(3.14). We also note here in passing that if in Algorithm 2.2 we would put $h_x = \varkappa h^{5/4}$ instead of $h_x = \varkappa h$, then it would not be difficult to prove (3.11) rigorously.

In the multidimensional case ((1.3)–(1.4) with $d > 1$) we use the approximation

(3.15)
$$\frac{\partial u}{\partial x^i}(t_k, x) \approx \frac{\bar{u}(t_k, x^1, \ldots, x^i + \gamma\sqrt{h}, \ldots, x^d) - \bar{u}(t_k, x^1, \ldots, x^i - \gamma\sqrt{h}, \ldots, x^d)}{2\gamma\sqrt{h}}, \quad i = 1, \ldots, d,$$

where $\bar{u}$ is found by a multidimensional algorithm analogous to Algorithm 2.2 [14, 18]. Proposition 3.1 and Remark 3.4 are valid for (3.15) as well.

4. Numerical integration of FBSDEs. Let $\bar{u}(t_k, x)$ be defined by Algorithm 2.2 and introduce the notation

$$\frac{\Delta \bar{u}}{\Delta x}(t_k, x) := \frac{\bar{u}(t_k, x + \gamma\sqrt{h}) - \bar{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} \quad \text{for some} \quad \gamma > 0 \tag{4.1}$$

and also $\Delta_k w := w(t_k + h) - w(t_k)$. In practice it is advisable to choose the parameter γ close to the diffusion σ at the point (t_k, x).

Consider two numerical schemes for the FBSDE (1.1)–(1.2) with $d = n = 1$: *the Euler scheme*

$$\begin{gathered} X_0 = x, \\ X_{k+1} = X_k + a(t_k, X_k, \bar{u}(t_k, X_k))h + \sigma(t_k, X_k, \bar{u}(t_k, X_k))\Delta_k w, \quad k = 0, \ldots, N-1, \end{gathered} \tag{4.2}$$

and *the first-order scheme*

$$\begin{aligned} X_0 &= x, \\ X_{k+1} &= X_k + a(t_k, X_k, \bar{u}(t_k, X_k))h + \sigma(t_k, X_k, \bar{u}(t_k, X_k))\Delta_k w \\ &\quad + \frac{1}{2}\sigma(t_k, X_k, \bar{u}(t_k, X_k)) \left(\frac{\partial \sigma}{\partial x}(t_k, X_k, \bar{u}(t_k, X_k)) \right. \\ &\quad \left. + \frac{\partial \sigma}{\partial u}(t_k, X_k, \bar{u}(t_k, X_k)) \frac{\Delta \bar{u}}{\Delta x}(t_k, X_k) \right) \times \left(\Delta_k^2 w - h\right), \quad k = 0, \ldots, N-1; \end{aligned} \tag{4.3}$$

the components Y and Z of the solution to (1.1)–(1.2) are approximated as

$$Y_k = \bar{u}(t_k, X_k), \quad Z_k = \sigma(t_k, X_k, Y_k)\frac{\Delta \bar{u}}{\Delta x}(t_k, X_k), \tag{4.4}$$

where X_k is either from (4.2) or (4.3).

Let us note that the first-order scheme (4.3) becomes the Euler scheme in the case of additive noise in (1.1).

THEOREM 4.1. (i) *The Euler scheme* (4.2), (4.4) *has the mean-square order of convergence* 1/2*; i.e.,*

(4.5)
$$\left[E\left[(X(t_k) - X_k)^2 + (Y(t_k) - Y_k)^2 + (Z(t_k) - Z_k)^2\right]\right]^{1/2} \leq K\,(1 + x^2)^{1/2}h^{1/2},$$

where K does not depend on x, k, and h.

(ii) *The scheme* (4.3), (4.4) *has the first mean-square order of convergence for X and Y; i.e.,*

$$\left[E\left[(X(t_k) - X_k)^2 + (Y(t_k) - Y_k)^2\right]\right]^{1/2} \leq K\,(1 + x^2)^{1/2}h, \tag{4.6}$$

and if

$$\left| \frac{\Delta \bar{u}}{\Delta x}(t_k, x) - \frac{\partial u}{\partial x}(t_k, x) \right| \leq Ch, \tag{4.7}$$

then

$$\left[E \left(Z(t_k) - Z_k \right)^2 \right]^{1/2} \leq K \left(1 + x^2\right)^{1/2} h, \tag{4.8}$$

where K does not depend on x, k, and h.

With respect to the assumption (4.7), see Remark 3.4.

Proof. (i) Let us prove that the Euler scheme satisfies

$$\left[E \left(X(t_k) - X_k \right)^2 \right]^{1/2} \leq K \left(1 + x^2\right)^{1/2} h^{1/2}. \tag{4.9}$$

Assume for a while that the solution $u(t,x)$ to (1.3)–(1.4) is known exactly. Then the coefficients in (1.1) are known functions of t and x and we can apply the standard mean-square Euler scheme

$$\hat{X}_{k+1} = \hat{X}_k + a(t_k, \hat{X}_k, u(t_k, \hat{X}_k))h + \sigma(t_k, \hat{X}_k, u(t_k, \hat{X}_k))\Delta_k w, \quad k = 0, \ldots, N-1, \tag{4.10}$$

which is of mean-square order 1/2; i.e., $\hat{X}_k$ from (4.10) satisfies a relation like (4.9).

Now we compare X_k and $\hat{X}_k$. To this end, we exploit the *fundamental convergence theorem* (see [13] or [18, p. 4]). It states that if a one-step approximation $\bar{X}_{t,x}(t+h)$ of the solution $X_{t,x}(t+h)$ satisfies the conditions

$$|E(X_{t,x}(t+h) - \bar{X}_{t,x}(t+h))| \leq K(1 + |x|^2)^{1/2} h^{p_1}, \tag{4.11}$$

$$\left[E|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)|^2 \right]^{1/2} \leq K(1 + |x|^2)^{1/2} h^{p_2} \tag{4.12}$$

with $p_2 > 1/2$ and $p_1 \geq p_2 + 1/2$, then the corresponding mean-square method X_k has order of convergence $p_2 - 1/2$; i.e.,

$$\left[E \left(X(t_k) - X_k \right)^2 \right]^{1/2} \leq K \left(1 + x^2\right)^{1/2} h^{p_2 - 1/2}.$$

Introduce the one-step approximations corresponding to X_k and $\hat{X}_k$:

$$X(t+h) \approx \bar{X} = x + a(t, x, \bar{u}(t,x))h + \sigma(t, x, \bar{u}(t,x))\Delta w \tag{4.13}$$

and

$$X(t+h) \approx \hat{X} = x + a(t, x, u(t,x))h + \sigma(t, x, u(t,x))\Delta w. \tag{4.14}$$

It is known [13, 18] that $\hat{X}$ from (4.14) satisfies (4.11) with $p_1 = 2$ and (4.12) with $p_2 = 1$. Due to Assumption 2.1 and the relation (2.11), we get

$$E(\hat{X} - \bar{X}) = a(t, x, u(t,x))h - a(t, x, \bar{u}(t,x))h = O(h^2)$$

and

$$E(\hat{X}-\bar{X})^2 = \left[a(t,x,u(t,x)) - a(t,x,\bar{u}(t,x))\right]^2 h^2 + \left[\sigma(t,x,u(t,x)) - \sigma(t,x,\bar{u}(t,x))\right]^2 h$$
$$= O(h^3),$$

whence it follows that $\bar{X}$ from (4.13) also satisfies (4.11) with $p_1 = 2$ and (4.12) with $p_2 = 1$. Then, applying the fundamental convergence theorem, we prove (4.9). Now the rest of (4.5) follows from (2.2), (4.9), (2.11), and (3.1).

(ii) To prove (4.6), we just repeat all the arguments as above. The only difference is that this time we compare the one-step approximation corresponding to (4.3) with the one corresponding to the standard first-order mean-square scheme [18] applied to (1.1), assuming again that $u(t,x)$ is known exactly. We note that the estimate (3.1) is enough to obtain (4.6) but it would imply the mean-square order 1/2 for Z instead of (4.8). At the same time, it is clear that by (2.2), (4.6) together with (4.7) we obtain (4.8). □

Remark 4.2. Theorem 4.1 is also valid for other Euler-type methods (e.g., for the implicit Euler method) as well as for other first-order mean-square methods. Further, in the case of additive noise this theorem can be extended to constructive mean-square methods of order 3/2 [18] using second-order methods from [14, 18] for the semilinear parabolic problem (2.3)–(2.4).

Remark 4.3. Let us consider the weak Euler scheme

$$X_0 = x, \tag{4.15}$$
$$X_{k+1} = X_k + a(t_k, X_k, \bar{u}(t_k, X_k))h + \sigma(t_k, X_k, \bar{u}(t_k, X_k))\xi_k, \quad k = 0, \ldots, N-1,$$

where ξ_k, $k = 0, \ldots, N-1$, are i.i.d. random variables with the distribution $P(\xi = \pm 1) = 1/2$ and $\bar{u}(t_k, x)$ is defined by Algorithm 2.2. It is possible to prove that for a sufficiently smooth function $F(x,y)$ satisfying some boundedness conditions, the Euler method (4.15) and (4.4) is of weak order one; i.e.,

$$EF(X(t_k), Y(t_k)) - EF(X_k, Y_k) = O(h).$$

The proof is based on the main theorem on convergence of weak approximations [18, p. 100].

Remark 4.4. It is not difficult to generalize both the numerical algorithm considered and Theorem 4.1 to the case $d > 1$ using the multidimensional version of the Euler method (4.2) and (4.4), a multidimensional algorithm analogous to Algorithm 2.2 (see [14, 18]) to solve (1.3)–(1.4), and (3.15) to approximate the derivatives.

5. FBSDEs with random terminal time.

5.1. The parabolic case. Let G be a bounded domain in $\mathbf{R}^d$, let $Q = [t_0, T) \times G$ be a cylinder in $\mathbf{R}^{d+1}$, and let $\Gamma = \overline{Q} \setminus Q$. The set Γ is a part of the boundary of the cylinder Q consisting of the upper base and the lateral surface. Let $\varphi(t,x)$ be a function defined on Γ.

Consider the FBSDE with random terminal time (see e.g., [19, 25]):

$$dX = a(t,X,Y)dt + \sigma(t,X,Y)dw(t), \quad X(t_0) = x \in G, \tag{5.1}$$
$$dY = -g(t,X,Y)dt - f^{\mathsf{T}}(t,X,Y)Zdt + Z^{\mathsf{T}}dw(t), \quad Y(\tau) = \varphi(\tau, X(\tau)), \tag{5.2}$$

where $\tau = \tau_{t_0,x}$ is the first exit time of the trajectory $(t, X_{t_0,x}(t))$ from the domain Q; i.e., the point $(\tau, X(\tau))$ belongs to Γ. A solution to (5.1)–(5.2) is defined as an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted vector $(X(t), Y(t), Z(t))$ together with the Markov moment τ, which satisfy

(5.1)–(5.2). This solution is connected with the Dirichlet boundary value problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d}a^{i}(t,x,u)\frac{\partial u}{\partial x^{i}}+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(t,x,u)\frac{\partial^{2}u}{\partial x^{i}\partial x^{j}} = -g(t,x,u)-\sum_{k=1}^{n}f^{k}(t,x,u)\sum_{i=1}^{d}\sigma^{ik}(t,x,u)\frac{\partial u}{\partial x^{i}},\quad (t,x)\in Q, \tag{5.3}$$

$$u(t,x)|_{\Gamma}=\varphi(t,x). \tag{5.4}$$

Let $u(t,x)$ be the solution of (5.3)-(5.4), which is supposed to exist, be unique, and be sufficiently smooth. One can find many theoretical results on this topic in [10] (see also references therein and in [16, 18]). To be definite, we assume here that the conditions like Assumption 2.1 together with sufficient smoothness of the boundary ∂G and of the function φ are fulfilled.

Consider the following SDE in Q:

$$dX=a(t,X,u(t,X))dt+\sigma(t,X,u(t,X))dw(t),\quad X(t_0)=x, \tag{5.5}$$

with random terminal time τ which is defined as the first exit time of the trajectory $(t,X_{t_0,x}(t))$ of (5.5) from the domain Q. Introduce

$$\begin{aligned} &Y(t)=u(t,X_{t_0,x}(t)),\quad t_0\le t\le\tau,\\ &Z^{j}(t)=\sum_{i=1}^{d}\sigma^{ij}(t,X_{t_0,x}(t),Y(t))\frac{\partial u}{\partial x^{i}}(t,X_{t_0,x}(t)),\quad j=1,\dots,n,\quad t_0\le t\le\tau. \end{aligned} \tag{5.6}$$

Clearly, the four-tuple $(X_{t_0,x}(t),Y(t),Z(t),\tau)$ is a solution of (5.1)–(5.2).

In what follows we restrict ourselves to the one-dimensional version of (5.1)–(5.2) ($d=1$, $n=1$). Introducing

$$b(t,x,y):=a(t,x,y)+f(t,x,y)\sigma(t,x,y),$$

we get

$$\frac{\partial u}{\partial t}+b(t,x,u)\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^{2}(t,x,u)\frac{\partial^{2}u}{\partial x^{2}}+g(t,x,u)=0,\quad (t,x)\in Q, \tag{5.7}$$

$$u(t,x)|_{\Gamma}=\varphi(t,x). \tag{5.8}$$

In this case Q is the partly open rectangle $Q=[t_0,T)\times(\alpha,\beta)$, and Γ consists of the upper base $\{T\}\times[\alpha,\beta]$ and two vertical intervals, $[t_0,T)\times\{\alpha\}$ and $[t_0,T)\times\{\beta\}$.

In [16] (see also [18]) we propose a number of algorithms for solving the problem (5.7)–(5.8). As an example, let us recall one of them. Consider an equidistant space discretization with a space step h_x (recall that the notation for time step is h): $x_j=\alpha+jh_x$, $j=0,1,2,\dots,M$, $h_x=(\beta-\alpha)/M$. The algorithm has the following form.

ALGORITHM 5.1.

$$\bar{u}(t_N,x)=\varphi(t_N,x),\quad x\in[\alpha,\beta], \tag{5.9}$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h})$$
$$+ \bar{g}_{k,j} \cdot h \quad \textit{if } x_j + \bar{b}_{k,j} \cdot h \pm \bar{\sigma}_{k,j} \cdot \sqrt{h} \in [\alpha, \beta];$$

$$\bar{u}(t_k, x_j) = \frac{1}{1 + \sqrt{\bar{\lambda}_{k,j}}}\varphi(t_{k+\bar{\lambda}_{k,j}}, \alpha) + \frac{\sqrt{\bar{\lambda}_{k,j}}}{1 + \sqrt{\bar{\lambda}_{k,j}}}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h})$$
$$+ \bar{g}_{k,j} \cdot \sqrt{\bar{\lambda}_{k,j}}h \quad \textit{if } x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h} < \alpha;$$

$$\bar{u}(t_k, x_j) = \frac{1}{1 + \sqrt{\bar{\mu}_{k,j}}}\varphi(t_{k+\bar{\mu}_{k,j}}, \beta) + \frac{\sqrt{\bar{\mu}_{k,j}}}{1 + \sqrt{\bar{\mu}_{k,j}}}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h})$$
$$+ \bar{g}_{k,j} \cdot \sqrt{\bar{\mu}_{k,j}}h, \textit{ if } x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h} > \beta,$$
$$j = 1, 2, \dots, M - 1;$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\,\bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\,\bar{u}(t_k, x_{j+1}), \quad x_j < x < x_{j+1}, \tag{5.10}$$
$$j = 0, 1, 2, \dots, M - 1,\ k = N - 1, \dots, 1, 0,$$

where $\bar{b}_{k,j}$, $\bar{\sigma}_{k,j}$, $\bar{g}_{k,j}$ *are the coefficients* $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ *calculated at the point* $(t_k, x_j,\ \bar{u}(t_{k+1}, x_j))$, $t_{k+\bar{\lambda}_{k,j}} := t_k + h\bar{\lambda}_{k,j}$, $t_{k+\bar{\mu}_{k,j}} := t_k + h\bar{\mu}_{k,j}$, *and* $0 < \bar{\lambda}_{k,j}$, $\bar{\mu}_{k,j} \le 1$ *are unique roots of the quadratic equations*

$$\alpha = x_j + \bar{b}_{k,j} \cdot \bar{\lambda}_{k,j}h - \bar{\sigma}_{k,j} \cdot \sqrt{\bar{\lambda}_{k,j}h}, \qquad \beta = x_j + \bar{b}_{k,j} \cdot \bar{\mu}_{k,j}h + \bar{\sigma}_{k,j} \cdot \sqrt{\bar{\mu}_{k,j}h}.$$

It is proved in [16] (see also [18, p. 475]) that if the value of h_x is taken equal to $\varkappa h$ with $\varkappa$ being a positive constant, then

$$|\bar{u}(t_k, x) - u(t_k, x)| \le Kh\,, \tag{5.11}$$

where K does not depend on x, h, k.

To construct Z due to (5.6), we need an approximation of $\partial u/\partial x$. To this end, we propose to use the formulas (cf. (3.1))

(5.12)
$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\bar{u}(t_k, x + \gamma\sqrt{h}) - \bar{u}(t_k, x - \gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h^{1/2}) := \frac{\Delta\bar{u}}{\Delta x}(t_k, x) + O(h^{1/2})$$
$$\text{if } x \pm \gamma\sqrt{h} \in [\alpha, \beta],$$

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{4\bar{u}(t_k, x+\gamma\sqrt{h}) - 3\bar{u}(t_k, x) - \bar{u}(t_k, x+2\gamma\sqrt{h})}{2\gamma\sqrt{h}} + O(h^{1/2}) \tag{5.13}$$
$$:= \frac{\Delta\bar{u}}{\Delta x}(t_k, x) + O(h^{1/2})$$
$$\text{if } x - \gamma\sqrt{h} < \alpha,$$

and

$$\frac{\partial u}{\partial x}(t_k, x) = \frac{\bar{u}(t_k, x-2\gamma\sqrt{h}) - 4\bar{u}(t_k, x-\gamma\sqrt{h}) + 3\bar{u}(t_k, x)}{2\gamma\sqrt{h}} + O(h^{1/2}) \tag{5.14}$$
$$:= \frac{\Delta\bar{u}}{\Delta x}(t_k, x) + O(h^{1/2})$$
$$\text{if } x + \gamma\sqrt{h} > \beta.$$

Most probably, the accuracy in (5.12)–(5.14) is $O(h)$ rather than $O(h^{1/2})$, but we have not investigated this issue in detail.

Let us note that if we apply the method of differentiation to the boundary value problem (5.7)–(5.8), we obtain (3.4) for $v = \partial u/\partial x$ in Q and the Neumann boundary condition. Namely, this boundary condition is of the form

$$\text{on the upper base of } Q\text{:}\ \ v(T, x) = \frac{\partial\varphi}{\partial x}(T, x)$$

and, for example, on the vertical interval $[t_0, T] \times \{\alpha\}$:

$$b(t, \alpha, \varphi(t, \alpha))v(t, \alpha) + \frac{1}{2}\sigma^2(t, \alpha, \varphi(t, \alpha))\frac{\partial v}{\partial x}(t, \alpha) = -\frac{\partial\varphi}{\partial t}(t, \alpha) - g(t, \alpha, \varphi(t, \alpha)).$$

Thus, in the case of FBSDEs with random terminal time the approach of [6] leads to a complicated system of boundary value problems.

5.2. The elliptic case. The random terminal time in FBSDE (5.1)–(5.2) is bounded by the time T. Now we consider FBSDEs with unbounded random terminal time. Let G be a bounded domain in $\mathbf{R}^d$ and $Q = [0, \infty) \times G$ be a cylinder in $\mathbf{R}^{d+1}$.

Consider the FBSDE with random terminal time (see e.g., [19, 25]):

$$dX = a(X, Y)dt + \sigma(X, Y)dw(t), \ \ X(0) = x \in G, \tag{5.15}$$
$$dY = -g(X, Y)dt - f^{\mathsf{T}}(X, Y)Zdt + Z^{\mathsf{T}}dw(t), \ \ Y(\tau) = \varphi(X(\tau)), \tag{5.16}$$

where $\tau = \tau_x$ is the first exit time of the trajectory $X_x(t)$ from the domain G; i.e., the point $X(\tau)$ belongs to the boundary ∂G of G. A solution to (5.15)–(5.16) is defined as an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted vector $(X(t), Y(t), Z(t))$ together with the Markov moment τ, which satisfy (5.15)–(5.16). This solution is connected with the Dirichlet boundary value problem for the semilinear elliptic equation

$$\sum_{i=1}^{d} a^i(x, u)\frac{\partial u}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x, u)\frac{\partial^2 u}{\partial x^i \partial x^j} \tag{5.17}$$
$$= -g(x, u) - \sum_{k=1}^{n} f^k(x, u)\sum_{i=1}^{d}\sigma^{ik}(x, u)\frac{\partial u}{\partial x^i}, \quad x \in G,$$

$$u(x)|_{\partial G} = \varphi(x). \tag{5.18}$$

Let $u(x)$ be the solution of (5.17)–(5.18), which is supposed to exist, be unique, and be sufficiently smooth. Consider the following SDE in G:

$$dX = a(X,u(X))dt + \sigma(X,u(X))dw(t), \quad X(0) = x, \tag{5.19}$$

with random terminal time τ which is defined as the first exit time of the trajectory $X_x(t)$ of (5.19) from the domain G. Introduce

$$\begin{aligned} &Y(t) = u(X_x(t)), \quad 0 \le t \le \tau, \\ &Z^j(t) = \sum_{i=1}^{d} \sigma^{ij}(X_x(t), Y(t)) \frac{\partial u}{\partial x^i}(X_x(t)), \quad j = 1, \dots, n, \ 0 \le t \le \tau. \end{aligned} \tag{5.20}$$

Clearly, the four-tuple $(X_x(t), Y(t), Z(t), \tau)$ is a solution of (5.15)–(5.16). Note that in the one-dimensional case the problem (5.17)–(5.18) is the boundary value problem just for a second-order ordinary differential equation whose numerical solution does not cause any problem. To solve (5.17)–(5.18) for $d > 1$, one can use finite-difference methods or apply a multidimensional layer method analogous to (5.9)–(5.10) (see [16, 18]) using ideas of the relaxation method.

6. Numerical integration of FBSDEs with random terminal time. Trajectories $(t, X(t))$ of the SDE (5.1) belong to the space-time bounded domain $\bar{Q}$, and a corresponding approximation (ϑ_k, X_k) should possess the same property. It is obvious that, e.g., the standard Euler scheme (4.2) does not satisfy this requirement and specific methods are needed. Such approximations were proposed in [15] (see also [18]). In the one-dimensional case they take a simpler form, which is presented below. We note that here the notation differs partly from that used in [15, 18].

Consider the one-dimensional SDE

$$dX = \chi_{\tau_{t,x}>s} b(s,X)ds + \chi_{\tau_{t,x}>s}\sigma(s,X)dw(s), \quad X(t) = X_{t,x}(t) = x, \tag{6.1}$$

in a space-time bounded domain $Q = [t_0, T) \times (\alpha, \beta)$; the Markov moment $\tau_{t,x}$ is the first-passage time of the process $(s, X_{t,x}(s))$, $s \ge t$, to $\Gamma = \overline{Q} \diagdown Q$.

Let $I_r := [-r, r]$, $r > 0$, $\Pi := [0, l) \times I_1$ for some $l > 0$, and $\Pi_h := [0, lh) \times I_{\sqrt{h}}$. Take a point $(s, y) \in Q$ and introduce another interval $I(s, y; h) := [x + hb(s, y) - \sigma(s, y)\sqrt{h}, x + hb(s, y) + \sigma(s, y)\sqrt{h}]$ and also the space-time rectangle $\Pi(s, y; h) = [s, s + lh) \times I(s, y; h)$. Let Γ_δ be an intersection of a δ-neighborhood of the set Γ with the domain Q. Below we take δ equal to $\lambda h^{(1-\varepsilon)/2}$ with $0 < \varepsilon \le 1$ and $\lambda = 2 \max_{(s,y)\in\overline{Q}} |\sigma(s, y)|$. Now we construct a random walk over small space-time rectangles.

Algorithm 6.1 (random walk over small space-time rectangles). Choose a time step $h > 0$ and numbers $0 < \varepsilon \le 1$ and $L > 0$.

Step 0. $X_0 = x$, $\vartheta_0 = t$, $(t, x) \in Q$, $k = 0$.

Step 1. If $(\vartheta_k, X_k) \in \Gamma_{\lambda h^{(1-\varepsilon)/2}}$ or $k \ge L/h$, then Stop and
(i) put $\nu = k$, $(\vartheta_\nu, X_\nu) = (\vartheta_k, X_k)$;
(ii) if $\vartheta_\nu \ge T - \lambda h^{(1-\varepsilon)/2}$, then $\bar{\tau}_{t,x} = T$ and $\xi_{t,x} = X_\nu \in (\alpha, \beta)$; otherwise $\bar{\tau}_{t,x} = \vartheta_\nu$ and $\xi_{t,x}$ is the end of the interval $[\alpha, \beta]$ nearest to X_ν.

Step 2. Put $k := k + 1$. Simulate the first exit point $(\theta_k, w(\vartheta_{k-1} + \theta_k) - w(\vartheta_{k-1}))$ of the process $(s - \vartheta_{k-1}, w(s) - w(\vartheta_{k-1}))$, $s > \vartheta_{k-1}$, from the rectangle Π_h. Put

$$\vartheta_k = \vartheta_{k-1} + \theta_k, \tag{6.2}$$

$$X_k = X_{k-1} + b(\vartheta_{k-1}, X_{k-1})\theta_k + \sigma(\vartheta_{k-1}, X_{k-1})(w(\vartheta_k) - w(\vartheta_{k-1})). \tag{6.3}$$

Go to Step 1.

The sequence (ϑ_k, X_k) obtained by Algorithm 6.1 is a Markov chain stopping at the Markov moment ν in the neighborhood $\Gamma_{\lambda h^{(1-\varepsilon)/2}}$ of the boundary Γ. At each step $(\vartheta_k, X_k) \in \partial\Pi(\vartheta_{k-1}, X_{k-1}; h)$ and $\overline{\Pi}(\vartheta_{k-1}, X_{k-1}; h) \subset Q$; i.e., the chain belongs to the space-time bounded domain Q with probability one. The simulated points (ϑ_k, X_k) are close in the mean-square sense to $(\vartheta_k, X(\vartheta_k))$, and the point $(\bar{\tau}_{t,x}, \xi_{t,x})$ is close to $(\tau_{t,x}, X(\tau_{t,x}))$. It is proved in [15, 18] that

$$\begin{gathered}\left(E\,|X(\vartheta_k) - X_k|^2\right)^{1/2} \le K\,(\sqrt{h} + e^{-c_h L}), \quad k = 1, \dots, \nu, \\ E|\tau_{t,x} - \bar{\tau}_{t,x}| \le K\,(h^{(1-\varepsilon)/2} + e^{-c_h L}), \\ \left(E\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2\right)^{1/2} \le K\,(\,h^{(1-\varepsilon)/4} + e^{-c_h L/2}),\end{gathered} \tag{6.4}$$

where the constant K is independent of h, k, t, x and c_h tends to a positive constant independent of L as $h \to 0$. We note that the accuracy of the algorithm depends on the choice of h, ε, and L. Clearly, we reach higher accuracy by decreasing h and/or ε and increasing L.

Algorithm 6.1 in its turn requires an algorithm, which is considered below, for simulating the first exit point $(\theta, w(\theta))$ of the process $(s, w(s))$, $s > 0$, from the rectangle Π_h.

Let $W(s)$ be a one-dimensional standard Wiener process and let τ be the first exit time of $W(s)$ from the interval $I_1 = [-1, 1]$. Then the following formulas for the distribution and density of τ take place:

$$\mathcal{P}(t) = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cdot\exp\left(-\frac{1}{8}\pi^2(2k+1)^2 t\right), \quad t > 0, \tag{6.5}$$

and

$$\mathcal{P}(t) = 2\sum_{k=0}^{\infty}(-1)^k \operatorname{erfc}\frac{2k+1}{\sqrt{2t}}, \quad t > 0, \qquad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp(-y^2)dy\,; \tag{6.6}$$

$$\mathcal{P}'(t) = \frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k(2k+1)\exp\left(-\frac{1}{8}\pi^2(2k+1)^2 t\right), \quad t > 0, \tag{6.7}$$

and

$$\mathcal{P}'(t) = \frac{2}{\sqrt{2\pi t^3}}\sum_{k=0}^{\infty}(-1)^k(2k+1)\exp\left(-\frac{1}{2t}(2k+1)^2\right), \quad t > 0. \tag{6.8}$$

The formulas (6.5) and (6.7) are suitable for calculations under big t, and the formulas (6.6) and (6.8) are suitable for small t. See further computational details in [15, 18].

For the conditional probability

$$\mathcal{Q}(\mu; t) := P(W(t) < \mu \,/\, |W(s)| < 1,\ 0 < s < t)\,,$$

where $-1 < \mu \le 1$, the following equalities hold [15, 18]:

$$\begin{aligned}\mathcal{Q}(\mu; t) &= \frac{P(W(t) < \mu\,,\ \tau \ge t)}{P(\tau \ge t)} \\ &= \frac{1}{1-\mathcal{P}(t)}\cdot\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left((-1)^k + \sin\frac{\pi(2k+1)\mu}{2}\right)\exp\left(-\frac{1}{8}\pi^2(2k+1)^2 t\right)\end{aligned} \tag{6.9}$$

and

$$\mathcal{Q}(\mu;t)=\frac{1}{1-\mathcal{P}(t)} \tag{6.10}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{2}\left(\operatorname{erfc}\frac{2k-1}{\sqrt{2t}}-\operatorname{erfc}\frac{2k+\mu}{\sqrt{2t}}-\operatorname{erfc}\frac{2k+2-\mu}{\sqrt{2t}}+\operatorname{erfc}\frac{2k+3}{\sqrt{2t}}\right).$$

Note that the series (6.9) and (6.10) are of the Leibniz type, the formula (6.9) is convenient for calculations under big t, and the formula (6.10) is convenient for small t. We draw attention to the denominator $(1-\mathcal{P}(t))$ in (6.9) which is close to zero for $t\gg 1$. But it is not difficult to transform (6.9) to a form suitable for calculations. See further computational details in [15, 18].

ALGORITHM 6.2 (simulating exit point of $(t,W(t))$ from space-time rectangle Π [15, 18]). Let ι, ν, and γ be independent random variables. Let ι be simulated by the law $P(\iota=-1)=\mathcal{P}(l)$, $P(\iota=1)=1-\mathcal{P}(l)$, let ν be simulated by the law $P(\nu=\pm 1)=\frac{1}{2}$, and let γ be uniformly distributed on $[0,1]$.

Then a random point (τ,ξ), distributed as the exit point $(\tau,W(\tau))$, is simulated as follows. If the simulated value of ι is equal to -1, then the point (τ,ξ) belongs to the lateral sides of Π and

$$\tau=\mathcal{P}^{-1}(\gamma\mathcal{P}(l)),\quad \xi=\nu;$$

otherwise, when $\iota=1$, the point (τ,ξ) belongs to the upper base of Π and

$$\tau=l,\quad \xi=\mathcal{Q}^{-1}(\gamma;l).$$

COROLLARY 6.3. *Let θ be the first-passage time of the process $(s,w(s))$, $s>0$, to the boundary $\partial\Pi_h$. Then the point*

$$(\theta,w)=(h\tau,\sqrt{h}\,\xi),$$

where (τ,ξ) is simulated by Algorithm 6.2, has the same distribution as $(\theta,w(\theta))$.

Algorithm 6.1 together with Algorithm 6.2 and Corollary 6.3 gives the constructive procedure for modeling the Markov chain (ϑ_k,X_k) which approximates trajectories $(t,X(t))$ of the SDE (6.1) in the space-time bounded domain Q.

Remark 6.4. In the one-dimensional case one can construct a random walk which terminates on the boundary Γ rather than in a boundary layer. Indeed, fix a sufficiently small $h>0$ and define the function $\rho(t,x;h)$, $(t,x)\in Q$, in the following way. If $\Pi(t,x;h)\in Q$, set $\rho\equiv\rho(t,x;h)=h$. Otherwise, find $\rho(t,x;h)<h$ such that $\Pi(t,x;\rho)$ touches the boundary Γ; i.e., either $t+\rho=T$ or one of the ends of the interval $I(t,x;\rho)$ coincides with α or β. At each iteration of the algorithm we find $h_k=\rho(\vartheta_{k-1},X_{k-1};h)$ and simulate the first exit point $(\theta_k,w(\vartheta_{k-1}+\theta_k)-w(\vartheta_{k-1}))$ of the process $(s-\vartheta_{k-1},w(s)-w(\vartheta_{k-1}))$, $s>\vartheta_{k-1}$, from the rectangle Π_{h_k}. Then we evaluate (ϑ_k,X_k) due to (6.2)–(6.3). We stop the algorithm when $(\vartheta_k,X_k)\in\Gamma$ and put $\nu=k$, $(\vartheta_\nu,X_\nu)=(\vartheta_k,X_k)$, $\bar{\tau}_{t,x}=\vartheta_\nu$, $\xi_{t,x}=X_\nu$. In comparison with Algorithm 6.1, the algorithm of this remark allows us to simulate a one-dimensional space-time Brownian motion $(s,w(s))$ exactly. We note that this algorithm cannot be generalized even to the two-dimensional case, while Algorithm 6.1 is available for any dimension [15, 18].

Now we are in position to propose a numerical algorithm for solving the FBSDE with random terminal time (5.1)–(5.2). Let $\bar{u}(t_k,x)$ be defined by Algorithm 5.1 and

$\frac{\Delta \bar{u}}{\Delta x}(t_k, x)$ by (5.12)–(5.14). Further, we define $\bar{u}(t,x)$ by linear interpolation as

$$\bar{u}(t,x) = \frac{t_k - t}{h}\bar{u}(t_{k-1},x) + \frac{t - t_{k-1}}{h}\bar{u}(t_k,x), \quad t_{k-1} \le x \le t_k, \tag{6.11}$$

and analogously we define $\frac{\Delta \bar{u}}{\Delta x}(t,x)$. It is clear (see (5.11) and (5.12)–(5.14)) that

$$|u(t,x) - \bar{u}(t,x)| \le Kh \tag{6.12}$$

and

$$\left|\frac{\partial u}{\partial x}(t,x) - \frac{\Delta \bar{u}}{\Delta x}(t,x)\right| \le K\sqrt{h}\,. \tag{6.13}$$

We approximate $X(t)$ from (5.1)–(5.2) by Algorithm 6.1 in which (6.3) is replaced by

$$\begin{aligned} X_k = X_{k-1} &+ b(\vartheta_{k-1}, X_{k-1}, \bar{u}(\vartheta_{k-1}, X_{k-1}))\theta_k \\ &+ \sigma(\vartheta_{k-1}, X_{k-1}, \bar{u}(\vartheta_{k-1}, X_{k-1}))(w(\vartheta_k) - w(\vartheta_{k-1})). \end{aligned} \tag{6.14}$$

The algorithm also gives us the approximation $(\bar{\tau}_{t,x}, \xi_{t,x})$ for the first exit point $(\tau_{t,x}, X_{t,x}(\tau_{t,x}))$ of the trajectory $(s, X_{t,x}(s))$ from Q. Further, we compute the components Y and Z as

$$\begin{aligned} &Y_k = \bar{u}(\vartheta_k, X_k), \quad Z_k = \sigma(\vartheta_k, X_k, Y_k)\, \frac{\Delta \bar{u}}{\Delta x}(t_k, X_k), \quad k = 1, \dots, \nu, \\ &\bar{Y}_\nu = \bar{u}(\bar{\tau}_{t,x}, \xi_{t,x}), \quad \bar{Z}_\nu = \sigma(\bar{\tau}_{t,x}, \xi_{t,x}, \bar{Y}_\nu)\, \frac{\Delta \bar{u}}{\Delta x}(\bar{\tau}_{t,x}, \xi_{t,x})\,. \end{aligned} \tag{6.15}$$

It is possible to prove (cf. (6.4) and (6.12)–(6.13)) that

$$\begin{gathered} \left[E\left[(X(\vartheta_k) - X_k)^2 + (Y(\vartheta_k) - Y_k)^2 + (Z(\vartheta_k) - Z_k)^2\right]\right]^{1/2} \le K\ (\sqrt{h} + e^{-c_h L}), \\ k = 1, \dots, \nu, \\ \left[E\left[\left(X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right)^2 + \left(Y(\tau_{t,x}) - \bar{Y}_\nu\right)^2 + \left(Z(\tau_{t,x}) - \bar{Z}_\nu\right)^2\right]\right]^{1/2} \\ \le K\ (h^{(1-\varepsilon)/4} + e^{-c_h L/2}), \\ E|\tau_{t,x} - \bar{\tau}_{t,x}| \le K\ (h^{(1-\varepsilon)/2} + e^{-c_h L}). \end{gathered} \tag{6.16}$$

Using Algorithm 6.1, we can also simulate the FBSDE with unbounded terminal time (5.15)–(5.16) analogously to the approximation of the FBSDE (5.1)–(5.2) considered in this section.

7. Numerical tests.

7.1. Description of the test problems. Consider the FBSDE

$$\begin{gathered} dX = \frac{X\left(1 + X^2\right)}{\left(2 + X^2\right)^3}dt + \frac{1 + X^2}{2 + X^2}\sqrt{\frac{1 + 2Y^2}{1 + Y^2 + \exp\left(-\dfrac{2X^2}{t+1}\right)}}\,dw(t), \\ X(0) = x, \end{gathered} \tag{7.1}$$

$$\begin{gathered} dY = -g(t,X,Y)dt - f(t,X,Y)Z dt + Z\,dw(t), \\ Y(T) = \exp\left(-\frac{X^2(T)}{T+1}\right), \end{gathered} \tag{7.2}$$

where

$$g(t,x,u) = \frac{1}{t+1}\exp\left(-\frac{x^2}{t+1}\right) \times\left[\frac{4x^2\left(1+x^2\right)}{\left(2+x^2\right)^3} + \left(\frac{1+x^2}{2+x^2}\right)^2\left(1-\frac{2x^2}{t+1}\right) - \frac{x^2}{t+1}\right], \tag{7.3}$$

$$f(t,x,u) = \frac{x}{\left(2+x^2\right)^2}\sqrt{\frac{1+u^2+\exp\left(-\frac{2x^2}{t+1}\right)}{1+2u^2}}.$$

Note that Assumption 2.1 is satisfied.

The corresponding Cauchy problem (see (2.3)–(2.4)) has the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{1+x^2}{2+x^2}\right)^2 \frac{1+2u^2}{1+u^2+\exp\left(-\frac{2x^2}{t+1}\right)}\frac{\partial^2 u}{\partial x^2} + \frac{2x\left(1+x^2\right)}{\left(2+x^2\right)^3}\frac{\partial u}{\partial x} \tag{7.4}$$

$$= \frac{1}{t+1}\exp\left(-\frac{x^2}{t+1}\right)\left[\frac{x^2}{t+1} - \frac{4x^2\left(1+x^2\right)}{\left(2+x^2\right)^3} - \left(\frac{1+x^2}{2+x^2}\right)^2\left(1-\frac{2x^2}{t+1}\right)\right],$$

$$t<T,\ x\in\mathbf{R},$$

$$u(T,x) = \exp\left(-\frac{x^2}{T+1}\right). \tag{7.5}$$

We use the problem (7.1)–(7.2) to test the numerical algorithms proposed in section 4. To this end, we need to know the exact solution of this problem. First, it can easily be verified that the solution of the problem (7.4)–(7.5) is the function

$$u(t,x) = \exp\left(-\frac{x^2}{t+1}\right). \tag{7.6}$$

Now we find the solution of (7.1)–(7.2). Substituting

$$Y(t) = u(t,X(t)) = \exp\left(-\frac{X(t)^2}{t+1}\right)$$

in (7.1), we get

$$dX = \frac{X\left(1+X^2\right)}{\left(2+X^2\right)^3}dt + \frac{1+X^2}{2+X^2}\,dw(t), \quad X(0)=x, \tag{7.7}$$

whose solution can be expressed by the formula

$$X(t) = \Lambda(x+\arctan x + w(t)), \tag{7.8}$$

where the function $\Lambda(z)$ is defined by the equation

$$\Lambda + \arctan\Lambda = z\,. \tag{7.9}$$

Indeed, $X(0) = \Lambda(x+\arctan x) = x$. Further, due to the Ito formula, we have

$$dX = \Lambda'(x+\arctan x + w(t))\,dw + \frac{1}{2}\Lambda''(x+\arctan x+w(t))\,dt$$

and by (7.9) we get

$$\Lambda' = \frac{1+\Lambda^2}{2+\Lambda^2}, \quad \Lambda'' = \frac{2\Lambda(1+\Lambda^2)}{(2+\Lambda^2)^3},$$

whence it follows that (7.8) satisfies (7.7).

Thus, the solution of (7.1)–(7.2) is

$$X(t) = \Lambda(x + \arctan x + w(t)), \quad Y(t) = \exp\left(-\frac{X(t)^2}{t+1}\right), \tag{7.10}$$
$$Z(t) = -\frac{2X(t)\left(1+X^2(t)\right)}{(t+1)\left(2+X^2(t)\right)} \exp\left(-\frac{X(t)^2}{t+1}\right),$$

where $\Lambda(z)$ is defined by (7.9).

Now consider the test problem for numerical algorithms for FBSDEs with random terminal time (cf. (7.1)–(7.2)):

$$dX = \frac{X\left(1+X^2\right)}{\left(2+X^2\right)^3}dt + \frac{1+X^2}{2+X^2}\sqrt{\frac{1+2Y^2}{1+Y^2+\exp\left(-\frac{2X^2}{t+1}\right)}}\,dw(t), \tag{7.11}$$
$$X(t_0) = x \in (0,\beta),$$

$$dY = -g(t,X,Y)dt - f(t,X,Y)Z dt + Z\,dw(t), \tag{7.12}$$
$$Y(\tau_x) = \exp\left(-\frac{X^2(\tau_{t_0,x})}{\tau_{t_0,x}+1}\right),$$

where $g(t,x,y)$ and $f(t,x,y)$ are from (7.3) and $\tau_{t_0,x}$ is the first exit time of the trajectory $(t, X_{t_0,x}(t))$, $t > t_0 > -1$, from the space-time rectangle $[t_0, T) \times (0,\beta)$; i.e., either $\tau_{t_0,x} = T$ or $X(\tau_{t_0,x})$ is equal to 0 or β. The corresponding Dirichlet problem (see (5.3)–(5.4) and also (7.4)–(7.5)) has the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{1+x^2}{2+x^2}\right)^2 \frac{1+2u^2}{1+u^2+\exp\left(-\frac{2x^2}{t+1}\right)}\frac{\partial^2 u}{\partial x^2} + \frac{2x\left(1+x^2\right)}{\left(2+x^2\right)^3}\frac{\partial u}{\partial x} \tag{7.13}$$
$$= \frac{1}{t+1}\exp\left(-\frac{x^2}{t+1}\right)\left[\frac{x^2}{t+1} - \frac{4x^2\left(1+x^2\right)}{\left(2+x^2\right)^3} - \left(\frac{1+x^2}{2+x^2}\right)^2\left(1-\frac{2x^2}{t+1}\right)\right],$$
$$t<T,\ x \in (0,\beta)\,,$$

$$u(t,0) = 1, \quad u(t,\beta) = \exp\left(-\frac{\beta^2}{T+1}\right), \tag{7.14}$$

$$u(T,x) = \exp\left(-\frac{x^2}{T+1}\right). \tag{7.15}$$

Obviously, the solution of this problem is given by (7.6) again. The exact solution of (7.11)–(7.12) can be simulated using formulas (7.10).

7.2. Numerical experiments. We simulate (7.1)–(7.2) using the Euler scheme (4.2) and (4.4), where the solution u of (7.4)–(7.5) is approximated by Algorithm 2.2. Of course, practical realization of such algorithms always requires a truncation of the infinite space domain using the knowledge of behavior of solutions at infinity. In

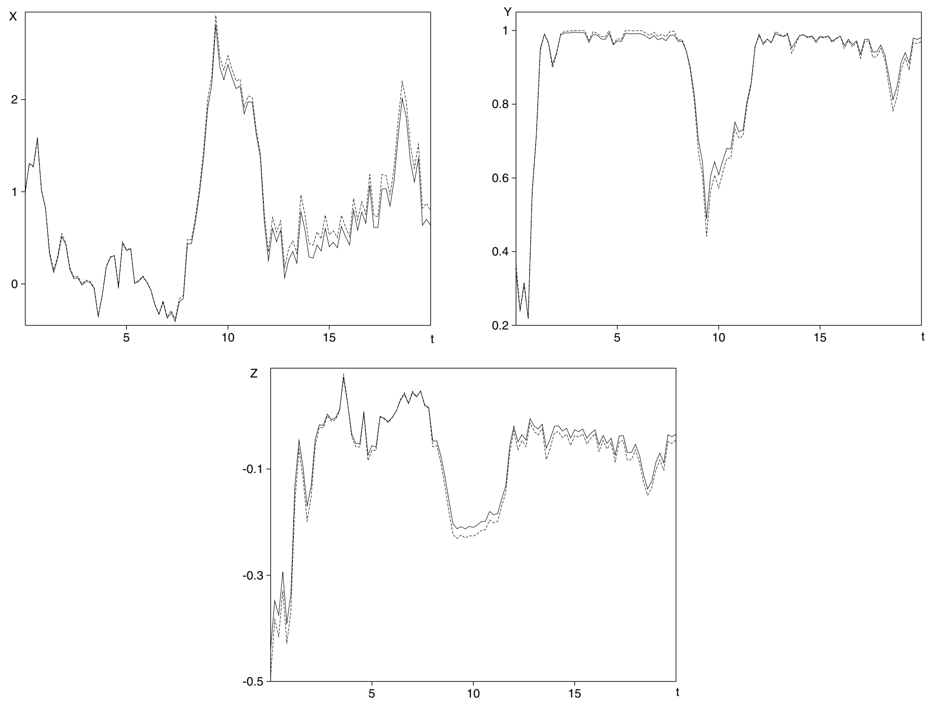

FIG. 1. *Simulation of the FBSDE* (7.1)–(7.2) *using the layer method from Algorithm* 2.2 *and the Euler scheme* (4.2) *and* (4.4) *(solid lines) with $h = 0.2$, $\varkappa = 1$, and $x = 1$. The corresponding exact trajectory (dashed lines) is found due to* (7.10)*. The upper left figure gives the sample trajectories for $X(t)$, the upper right figure for $Y(t)$, and the lower figure for $Z(t)$.*

this example we restrict simulation to the space interval $[-20, 20]$. To check that this truncation does not affect accuracy, we performed control simulation for the interval $[-30, 30]$.

Figure 1 presents a comparison of the exact sample trajectories $X(t)$, $Y(t)$, $Z(t)$ found due to (7.10) and the approximate trajectories obtained by the Euler scheme (4.2) and (4.4). Table 1 gives errors in simulation of the test problem (7.1)–(7.2) by the Euler scheme (4.2) and (4.4). The "±" reflects the Monte Carlo error only; it does not reflect the error of the method. More precisely, the averages presented in the table are computed in the following way:

$$E\left(X(T) - X_N\right)^2 \doteq \frac{1}{M}\sum_{m=1}^{M}\left(X^{(m)}(T) - X_N^{(m)}\right)^2 \pm 2\sqrt{\frac{\bar{D}_M}{M}},$$

where

$$\bar{D}_M = \frac{1}{M}\sum_{m=1}^{M}\left(X^{(m)}(T) - X_N^{(m)}\right)^4 - \left[\frac{1}{M}\sum_{m=1}^{M}\left(X^{(m)}(T) - X_N^{(m)}\right)^2\right]^2$$

and $X^{(m)}(T)$ and $X_N^{(m)}$ are independent realizations of $X(T)$ and X_N, respectively. The numerical results are in good agreement with the theoretical ones proved for the Euler method: Convergence of X_N, Y_N, Z_N is of mean-square order $1/2$. We also see that $\bar{u}$ has the first-order convergence.

TABLE 1

Errors in simulation of the FBSDE (7.1)–(7.2) *by the Euler scheme* (4.2) *and* (4.4) *with* $\varkappa = 1$ *and various time steps* h. *The corresponding exact solution is found due to* (7.10). *Here* $T = 20$ *and* $x = 1$. *The expectations are computed by the Monte Carlo technique simulating* $M = 1000$ *independent realizations of* $X(T)$ *and* X_N. *The "±"reflects the Monte Carlo error only; it does not reflect the error of the method.*

h	$\max_{k,j} \lvert u(t_k, x_j) - \bar{u}(t_k, x_j)\rvert$	$\left[E\,(X(T) - X_N)^2\right]^{1/2}$	$\left[E\,(Y(T) - Y_N)^2\right]^{1/2}$	$\left[E\,(Z(T) - Z_N)^2\right]^{1/2}$
0.5	0.15×10^0	0.249 ± 0.014	0.0330 ± 0.0021	0.0127 ± 0.0008
0.2	0.58×10^{-1}	0.162 ± 0.010	0.0220 ± 0.0014	0.0080 ± 0.0005
0.05	0.14×10^{-1}	0.080 ± 0.005	0.0109 ± 0.0007	0.0041 ± 0.0003
0.02	0.53×10^{-2}	0.051 ± 0.003	0.0069 ± 0.0004	0.0024 ± 0.0002
0.005	0.12×10^{-2}	0.025 ± 0.002	0.0034 ± 0.0002	0.0012 ± 0.0001

The algorithms from section 6 were tested on the FBSDE with random terminal time (7.11)–(7.12). The tests supported the obtained theoretical results.

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