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Mean velocity of noise-induced transport in the limit of weak periodic forcing

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Abstract. An analytical expression of the mean velocity for forced thermal ratchets is obtained under small amplitude of the periodic forcing. It gives quite accurate approximation of the mean velocity, in particular for fast periodic forcing, and reproduces the current reversal. Diffusion ratchets and forced ratchets with state-dependent noise are also considered.

1. Introduction

In recent years, one of the most intensively studied noise-induced phenomena is directed transport in Brownian ratchets (see [1–5] and references therein). The interest in this phenomenon is caused by its possible biological applications relating to movement of muscles or the operation of molecular combustion motors. The ratchet mechanism is also particularly interesting for novel separation techniques for particles of mesoscopic, micro- and nanoscales.

Analytical and numerical studies of the phenomenon have mainly dealt with evaluating the mean velocity of the noise-induced transport. An analytical expression of the mean velocity for forced thermal ratchets in the case of a sufficiently long period of the periodic forcing (adiabatic regime) was given in [1]. For clarity of exposition, we derive the expression in section 2. In accordance with this formula, given the ratchet potential, the sign of the mean velocity does not depend on parameters of the system. But this formula does not work for relatively fast periodic forcing (non-adiabatic regime). In section 3, we derive an analytical expression of the mean velocity for forced thermal ratchets under small amplitude of periodic forcing in the general case. Our tests demonstrate that the obtained formula gives quite good results.

In [6] it is found numerically that the direction of mean current can be reversed in the case of a short period of the periodic forcing. We obtain a formula which reproduces the current reversal.

In section 4, we extend the procedure of section 3 to evaluate the mean velocity for forced ratchets with multiplicative noise (see also [7, 8]) and for diffusion ratchets [9].

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2. Preliminaries

For clarity of exposition, we first derive some properties of solutions to the Ito equation (see also [1, 3, 7, 8])

$$dX = f(X)dt + \sigma(X)\, dw(t) \tag{2.1}$$

where $f(x)$ and $\sigma(x)$ are L-periodic functions and $w(t)$ is a standard Wiener process.

Introduce the process $\Phi(t) = X(t)$ (mod L) on a circle of radius $L/2\pi$. It is continuous on the circle. Due to the periodicity of f and σ, we can write (2.1) in the form

$$dX = f(\Phi)\, dt + \sigma(\Phi)\, dw(t). \tag{2.2}$$

Under sufficiently wide assumptions (e.g., $\sigma(x) \neq 0,\ x \in R$), $\Phi(t)$ is an ergodic process (see, e.g., [10]). Its invariant density $p(\varphi)$, $0 \leqslant \varphi \leqslant L$, is L-periodic and satisfies the stationary Fokker–Planck equation

$$\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}(\sigma^2 p) - \frac{\partial}{\partial\varphi}(fp) = 0$$
$$p(0) = p(L) \qquad \int_0^L p(\varphi)\, d\varphi = 1.$$

Solving this problem, we get

$$p(\varphi) = \frac{Cr(\varphi)}{\sigma^2(\varphi)}\left[r(L)\int_\varphi^L r^{-1}(\xi)\, d\xi + \int_0^\varphi r^{-1}(\xi)\, d\xi\right] \tag{2.3}$$

where

$$r(\varphi) = \exp\left(2\int_0^\varphi \frac{f(\xi)}{\sigma^2(\xi)}\, d\xi\right) \tag{2.4}$$

and $C > 0$ is found from the condition

$$\int_0^L p(\varphi)\, d\varphi = 1.$$

Let $EX(0) < \infty$. Due to the ergodicity of $\Phi(t)$, we have for the mean velocity $\bar{v}$ of $X(t)$:

$$\begin{aligned}\bar{v} &:= \lim_{t\to\infty}\frac{EX(t)}{t} = \lim_{t\to\infty}\frac{EX(0)}{t} + \lim_{t\to\infty}\frac{1}{t}\int_0^t Ef(\Phi(s))\, ds \\ &= \int_0^L f(\varphi)p(\varphi)\, d\varphi = \frac{LC}{2}[r(L) - 1]. \end{aligned}\tag{2.5}$$

The sign of $\bar{v}$ depends on the sign of $r(L) - 1$ only. Evidently, the necessary and sufficient condition for zero mean velocity consists of the equality (cf [7, 8])

$$\int_0^L \frac{f(\varphi)}{\sigma^2(\varphi)}\, d\varphi = 0. \tag{2.6}$$

For instance, if $\sigma \equiv$ const and the potential

$$F(x) = -\int f(x)\, dx$$

is an L-periodic function (e.g., a ratchet potential), we get the well known fact of thermodynamics [11] that $\bar{v} = 0$. Clearly, for an L-periodic potential $F(x)$, one can find an L-periodic state-dependent $\sigma(x)$ such that there is a noise-induced transport, i.e. $\bar{v} \neq 0$ (see, e.g., [7, 8]).

Remark 1. *Let us note that the condition (2.6) remains true if we consider a SDE in the sense of Stratonovich:*

$$\mathrm{d}X = f(X)\,\mathrm{d}t + \sigma(X) * \mathrm{d}w(t).$$

This is equivalent to the Ito equation

$$\mathrm{d}X = f(X)\,\mathrm{d}t + \frac{1}{2}\sigma(X)\frac{\mathrm{d}\sigma}{\mathrm{d}x}(X) + \sigma(X)\,\mathrm{d}w(t). \tag{2.7}$$

Analogously to (2.5), we get for the mean velocity $\bar{v}_{str}$ of the solution $X(t)$ to (2.7):

$$\bar{v}_{str} = \frac{LC_{str}}{2}[r_{str}(L) - 1]$$

where

$$r_{str}(\varphi) = \frac{\sigma(\varphi)}{\sigma(0)} \exp\left(2\int_0^{\varphi} \frac{f(\xi)}{\sigma^2(\xi)}\,\mathrm{d}\xi\right).$$

Due to the periodicity of $\sigma(\varphi)$, we arrive at condition (2.6).

As is known [1] (see also [3–6]), forced thermal ratchets exhibit noise-induced transport. Here we take a periodically forced thermal ratchet of the form

$$\mathrm{d}X = f(X)\,\mathrm{d}t + A\chi(t;T)\,\mathrm{d}t + \sigma\,\mathrm{d}w(t) \tag{2.8}$$

where $F(x) = -\int f(x)\,\mathrm{d}x$ is an L-periodic ratchet potential, $F(x) = F(x+L)$, $x \in R$, possessing no reflection symmetry $F(x) \neq F(-x)$, $x \in (0, L/2)$; A, T, and σ are some positive constants;

$$\chi(t;T) = \begin{cases} 1 & 0 \leqslant t < T/2 \\ -1 & T/2 \leqslant t < T \end{cases} \tag{2.9}$$

and $\chi(t;T)$ is T-periodical.

The model (2.8) is similar to ones investigated in [1] (see also [3, 4, 6]).

In connection with (2.8), consider two SDEs

$$\mathrm{d}X^+ = f(X^+)\mathrm{d}t + A\,\mathrm{d}t + \sigma\,\mathrm{d}w(t)$$
$$\mathrm{d}X^- = f(X^-)\mathrm{d}t - A\,\mathrm{d}t + \sigma\,\mathrm{d}w(t).$$

Let $\Phi^+(t)$ and $\Phi^-(t)$ be continuous random processes on the circle with radius $L/2\pi$ obtained by mapping $X^+(t)$ and $X^-(t)$ on the circle: $\Phi^{\pm}(t) = X^{\pm}(t) \pmod{L}$.

Just as (2.5), we find expressions for the mean velocities

$$\bar{v}^{\pm} = \lim_{t\to\infty} EX^{\pm}(t)/t.$$

We have

$$\bar{v}^+ = \frac{L\sigma^2}{2}(1 - \mathrm{e}^{-2AL/\sigma^2})\left[\int_0^L \mathrm{e}^{2[A\varphi - F(\varphi)]/\sigma^2}\,\mathrm{d}\varphi \int_0^L \mathrm{e}^{-2[A\varphi - F(\varphi)]/\sigma^2}\,\mathrm{d}\varphi \right.$$
$$\left. + (\mathrm{e}^{-2AL/\sigma^2} - 1)\int_0^L \mathrm{e}^{2[A\varphi - F(\varphi)]/\sigma^2} \int_0^{\varphi} \mathrm{e}^{-2[A\xi - F(\xi)]/\sigma^2}\,\mathrm{d}\xi\,\mathrm{d}\varphi\right]^{-1}. \tag{2.10}$$

Putting $-A$ instead of A in (2.10), we get the expression for $\bar{v}^-$. The asymmetry of the ratchet potential $F(x)$ can result in $\bar{v}^+ \neq -\bar{v}^-$. Note that if $F(x)$ were symmetric, i.e. $F(x) = F(-x)$, then $\bar{v}^+ = -\bar{v}^-$.

Suppose that the period T of $\chi(t;T)$ is sufficiently large so that stationary regimes of $\Phi^+(t)$ and $\Phi^-(t)$ are established in a time essentially less than $T/2$. In this case the mean velocity $\bar{v}$ of the solution $X(t)$ to (2.8) can be approximated by

$$\bar{v} \doteq \frac{\bar{v}^+ + \bar{v}^-}{2}. \tag{2.11}$$

As far as we know, this formula was first obtained in [1]. Due to numerical experiments (see, e.g., [6]), the mean velocity $\bar{v}$ is fairly well approximated by the expression (2.11) in the limit of large T.

Expanding $\bar{v}^+$ in powers of small amplitude A, we obtain

$$\bar{v}^+ \doteq AL^2 N_F M_F + \frac{2}{\sigma^2} A^2 L^2 N_F M_F J_F \tag{2.12}$$

where

$$\begin{aligned} N_F &= N_F(\sigma) = \left[\int_0^L \mathrm{e}^{-2F(\varphi)/\sigma^2} \, \mathrm{d}\varphi \right]^{-1} \\ M_F &= M_F(\sigma) = \left[\int_0^L \mathrm{e}^{2F(\varphi)/\sigma^2} \, \mathrm{d}\varphi \right]^{-1} \\ J_F &= J_F(\sigma) = L N_F M_F \int_0^L \mathrm{e}^{-2F(\varphi)/\sigma^2} \int_0^\varphi \mathrm{e}^{2F(\xi)/\sigma^2} \, \mathrm{d}\xi \, \mathrm{d}\varphi + M_F \int_0^L \varphi \mathrm{e}^{2F(\varphi)/\sigma^2} \, \mathrm{d}\varphi \\ &\quad - N_F \int_0^L \varphi \mathrm{e}^{-2F(\varphi)/\sigma^2} \, \mathrm{d}\varphi - \frac{L}{2}. \end{aligned} \tag{2.13}$$

Analogously

$$\bar{v}^- \doteq -AL^2 N_F M_F + \frac{2}{\sigma^2} A^2 L^2 N_F M_F J_F.$$

Then

$$\bar{v} \doteq \frac{2}{\sigma^2} A^2 L^2 N_F M_F J_F. \tag{2.14}$$

Clearly, if the period T is not sufficiently large such that stationary regimes of the processes $\Phi^-(t)$ and $\Phi^+(t)$ are established in a time less than $T/2$, the formula (2.11) does not work. In section 3 we obtain an approximate expression for the mean velocity in the case of a small amplitude A without any assumption on the length of the period T.

3. The mean velocity for thermal ratchets in the limit of small amplitude of the periodic forcing

In this section we consider systems with small-amplitude periodic forcing of the form

$$\mathrm{d}X = f(X)\,\mathrm{d}t + A\beta(t;T)\,\mathrm{d}t + \sigma\,\mathrm{d}w(t) \tag{3.1}$$

where the potential $F(x) = -\int f(x)\,\mathrm{d}x$ is an L-periodic function, the force $\beta(t;T)$ is a T-periodic function, $A > 0$ is a sufficiently small number and L, T, σ are some positive constants.

At first we demonstrate the procedure of evaluating the mean velocity $\bar{v} = \lim_{t\to\infty} EX(t)/t$ in the case of $\beta(t;T) = \sin 2\pi t/T$. Because we can expand a periodic function in the Fourier series, we are able to generalize the procedure for an arbitrary force $\beta(t;T)$. As an example, we evaluate the mean velocity $\bar{v}$ in the case of $\beta(t;T) = \chi(t;T)$ defined in (2.9) (see remark 4 below).

One can associate an autonomous system with the system (3.1) in a standard way. In the case of $\beta(t;T) = \sin 2\pi t/T$, we have

$$\begin{aligned} \mathrm{d}X &= f(X)\,\mathrm{d}t + A\sin\left(\frac{2\pi}{T}S\right)\mathrm{d}t + \sigma\,\mathrm{d}w(t) \\ \mathrm{d}S &= \mathrm{d}t. \end{aligned}$$

Consider the random process $(\Phi(t), \Theta(t))$: $\Phi(t) = X(t) \pmod L$, $\Theta(t) = S(t) \pmod T$ on a torus traced by a circle with radius $L/2\pi$ whose centre runs along a circle of radius $T/2\pi$. Due to the periodicity of the coefficients, we get

$$\mathrm{d}X = f(\Phi)\,\mathrm{d}t + A\sin\left(\frac{2\pi}{T}\Theta\right)\mathrm{d}t + \sigma\,\mathrm{d}w(t). \tag{3.2}$$

Because of $\sigma \neq 0$, the process (Φ, Θ) is ergodic. Its stationary density $p(\varphi, \vartheta)$, $0 \leqslant \varphi \leqslant L$, $0 \leqslant \vartheta \leqslant T$, is L-periodic in φ and T-periodic in ϑ. The density $p(\varphi, \vartheta)$ satisfies the stationary Fokker–Planck equation

$$\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial\varphi^2} - \frac{\partial p}{\partial\vartheta} - \frac{\partial}{\partial\varphi}\left[\left(-F'(\varphi) + A\sin\frac{2\pi}{T}\vartheta\right)p\right] = 0. \tag{3.3}$$

One can see that

$$p(\vartheta) := \int_0^L p(\varphi, \vartheta)\,\mathrm{d}\varphi = \frac{1}{T} \qquad 0 \leqslant \vartheta \leqslant T \tag{3.4}$$

i.e., $p(\vartheta)$ is the uniform distribution on the interval $[0, T]$.

Due to ergodicity of the process (Φ, Θ), the mean velocity $\bar{v} := \lim_{t\to\infty} EX(t)/t$ is equal to

$$\begin{aligned}\bar{v} &= -\int_0^T\int_0^L F'(\varphi)p(\varphi, \vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta + \int_0^T\int_0^L A\sin\left(\frac{2\pi}{T}\vartheta\right)p(\varphi, \vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta \\ &= -\int_0^T\int_0^L F'(\varphi)p(\varphi, \vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\end{aligned}$$

where the last equality follows from (3.4).

Expand the density $p(\varphi, \vartheta)$ in powers of A:

$$p(\varphi, \vartheta) = \frac{1}{T}p_0(\varphi) + Ap_1(\varphi, \vartheta) + \frac{A^2}{2}p_2(\varphi, \vartheta) + \cdots. \tag{3.5}$$

Therefore,

$$\begin{aligned}\bar{v} = -\int_0^L F'(\varphi)p_0(\varphi)\,\mathrm{d}\varphi - A\int_0^T\int_0^L F'(\varphi)p_1(\varphi, \vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta \\ -\frac{A^2}{2}\int_0^T\int_0^L F'(\varphi)p_2(\varphi, \vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta - \cdots.\end{aligned} \tag{3.6}$$

By substituting (3.5) in (3.3) and collecting terms with the same factors A^k, $k = 0, 1, 2, \ldots$, we get a system of equations for p_0, p_1, p_2, etc.

The function $p_0(\varphi)$ satisfies the problem

$$\frac{\sigma^2}{2}\frac{\partial^2}{\partial\varphi^2}p_0 + \frac{\partial}{\partial\varphi}(F'(\varphi)p_0) = 0$$

$$p_0(0) = p_0(L) \qquad \int_0^L p_0(\varphi)\,\mathrm{d}\varphi = 1$$

whence

$$p_0(\varphi) = N_F \cdot \mathrm{e}^{-2F(\varphi)/\sigma^2} \tag{3.7}$$

where N_F is defined in (2.13).

Due to (3.7) and the periodicity of $F(x)$, the first term in (3.6) is equal to zero.

In accordance with (3.4), we obtain

$$\int_0^L p_k(\varphi, \vartheta)\,\mathrm{d}\varphi = 0 \qquad k = 1, 2, \ldots. \tag{3.8}$$

The function $p_1(\varphi, \vartheta)$ satisfies the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 p_1}{\partial\varphi^2} - \frac{\partial p_1}{\partial\vartheta} + \frac{\partial}{\partial\varphi}(F'(\varphi)p_1) - \frac{1}{T}\sin\left(\frac{2\pi}{T}\vartheta\right)p_0'(\varphi) = 0.$$

Its solution has the form

$$p_1(\varphi, \vartheta) = b_1(\varphi)\cos\frac{2\pi}{T}\vartheta + c_1(\varphi)\sin\frac{2\pi}{T}\vartheta \tag{3.9}$$

where $b_1(\varphi)$ and $c_1(\varphi)$ satisfy the boundary value problem

$$\begin{aligned}
&\frac{\sigma^2}{2}b_1'' + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'(\varphi)b_1) - \frac{2\pi}{T}c_1 = 0\\
&\frac{\sigma^2}{2}c_1'' + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'(\varphi)c_1) + \frac{2\pi}{T}b_1 - \frac{1}{T}p_0'(\varphi) = 0\\
&b_1(0) = b_1(L) \qquad b_1'(0) = b_1'(L) \qquad c_1(0) = c_1(L) \qquad c_1'(0) = c_1'(L).
\end{aligned} \tag{3.10}$$

Apparently, there exists a unique solution to the problem (3.10) for any periodic potential $F(\varphi)$. We have proved this for a sufficiently small T and for a sufficiently big T, but we have not succeeded in giving a general proof. At the same time it is not difficult to detect this fact numerically for any concrete $F(\varphi)$.

In accordance with (3.8) (this is also clear from (3.10))

$$\int_0^L b_1(\varphi)\,\mathrm{d}\varphi = 0 \qquad \int_0^L c_1(\varphi)\,\mathrm{d}\varphi = 0. \tag{3.11}$$

Substituting (3.9) in (3.6), we obtain that the second term of (3.6) is equal to zero.

Now consider the function $p_2(\varphi, \vartheta)$. It satisfies the equation

$$\frac{\sigma^2}{4}\frac{\partial^2 p_2}{\partial\varphi^2} - \frac{1}{2}\frac{\partial p_2}{\partial\vartheta} + \frac{1}{2}\frac{\partial}{\partial\varphi}(F'(\varphi)p_2) - \sin\left(\frac{2\pi}{T}\vartheta\right)\frac{\partial p_1}{\partial\varphi} = 0. \tag{3.12}$$

Its solution has the form (the form is distinguished from (3.9) because p_1 depends on φ and ϑ)

$$p_2(\varphi, \vartheta) = a_2(\varphi) + b_2(\varphi)\cos\frac{4\pi}{T}\vartheta + c_2(\varphi)\sin\frac{4\pi}{T}\vartheta$$

where the functions $a_2(\varphi)$, $b_2(\varphi)$, and $c_2(\varphi)$ are found by solving the corresponding boundary value problem for three linear ordinary differential equations of the second order with periodic coefficients. A nonzero contribution of p_2 to $\bar{v}$ (see (3.6)) is given by the term with $a_2(\varphi)$ only. That is why we need not consider $b_2(\varphi)$ and $c_2(\varphi)$ but are interested in $a_2(\varphi)$ only, which satisfies the equation

$$\begin{aligned}
&\frac{\sigma^2}{2}a_2'' + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'(\varphi)a_2) - c_1' = 0\\
&a_2(0) = a_2(L) \qquad a_2'(0) = a_2'(L).
\end{aligned} \tag{3.13}$$

The function

$$a_2(\varphi) = C_1\mathrm{e}^{-2F(\varphi)/\sigma^2} + \frac{2}{\sigma^2}\mathrm{e}^{-2F(\varphi)/\sigma^2}\int_0^\varphi (c_1(\xi) + C_0)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi \tag{3.14}$$

with (M_F is defined in (2.13))

$$C_0 = -M_F\int_0^L c_1(\xi)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi$$

is the solution to the problem (3.13) under any C_1. But due to (3.8)

$$\int_0^L a_2(\varphi)\,\mathrm{d}\varphi = 0$$

whence the constant C_1 can be found uniquely.

Substituting (3.14) in (3.6) and using the periodicity of $F(\varphi)$, the second equality of (3.11), and the expression for C_0, we come to the formula

$$\bar{v} \doteq -\frac{A^2T}{2}\int_0^L \left[F'(\varphi)\frac{2}{\sigma^2}\mathrm{e}^{-2F(\varphi)/\sigma^2}\int_0^\varphi (c_1(\xi)+C_0)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi\right]\mathrm{d}\varphi$$
$$= -\frac{A^2TL}{2}C_0 = \frac{1}{2}A^2TLM_F\int_0^L c_1(\xi)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi \qquad (3.15)$$

where $c_1(\varphi)$ is from (3.10). Note that $c_1(\varphi)$ depends on the parameters T and σ (see (3.10)).

Continuing the procedure and finding p_3, p_4, etc, it is possible to get other terms of the expansion (3.6). For instance, it is not difficult to see that the next nonzero term in the expansion (3.6) is a term with factor A^4.

Let us state the obtained result.

Theorem 1. *For small amplitude A of the periodic forcing the mean velocity $\bar{v}$ is evaluated by the formula*

$$\bar{v} = \tfrac{1}{2}A^2TLM_F\int_0^L c_1(\xi)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi + \mathrm{O}(A^4) \qquad (3.16)$$

where $c_1(\varphi)$ is from (3.10) and M_F is defined in (2.13).

We perform some numerical experiments. We take the following ratchet potential $F(x)$:

$$F(x) = -\frac{L}{2\pi}\left(\sin\frac{2\pi x}{L} + \frac{1}{4}\sin\frac{4\pi x}{L}\right) \qquad L > 0 \qquad (3.17)$$

that is used for some tests, e.g. in [6]. Recall that the current reversal for just this potential was announced in [6]. For sufficiently big periods T, the mean velocity $\bar{v}$ is always positive for this potential. But for small T, the value of $\bar{v}$ can become negative, i.e. current reversal may occur.

In figure 1 we present a comparison of approximate values given by formula (3.15) and the mean velocity $\bar{v}$ evaluated by direct Monte Carlo simulations of the SDE (3.1) with $\beta(t;T) = \sin 2\pi t/T$. We apply a third-order weak scheme [12] to this SDE. To obtain a sample trajectory, we numerically integrate the SDE during 1000 periods of the periodic forcing $\beta(t;T)$ with the time step 0.02–0.01. In our tests we simulate 200 000–1000 000 sample trajectories. The Monte Carlo errors of the given points are no greater than 3×10^{-5}. Other errors are less than or comparable to the Monte Carlo ones.

According to figure 1, the mean velocity $\bar{v}$ of the noise-induced transport can be approximated quite accurately by (3.16) for a small amplitude of the periodic forcing. Let us underline that the expression (3.16) works, in particular, for small T (i.e., in the fast-forcing regime) and reproduces the current reversal. In passing, we note that the standard deviation $(E(X(t)-EX(t))^2)^{1/2}$ is essentially greater than the mean value $EX(t)$ when the current reversal is observed.

Remark 2. *Expanding the expression (3.16) in powers of $1/T$, we get in the case of large T:*

$$\bar{v} = \frac{1}{\sigma^2}A^2L^2N_FM_FJ_F + \mathrm{O}\left(\frac{A^2}{T^2}\right) + \mathrm{O}(A^4)$$

where N_F, M_F, and J_F are from (2.13).

Remark 3. *Expanding the expression (3.16) in powers of small T, we obtain*

$$\bar{v} = \frac{1}{16\pi^4}A^2T^4LN_FM_F\int_0^L (F'''(\xi))^2F'(\xi)\,\mathrm{d}\xi + \mathrm{O}(A^2T^6) + \mathrm{O}(A^4T^4). \qquad (3.18)$$

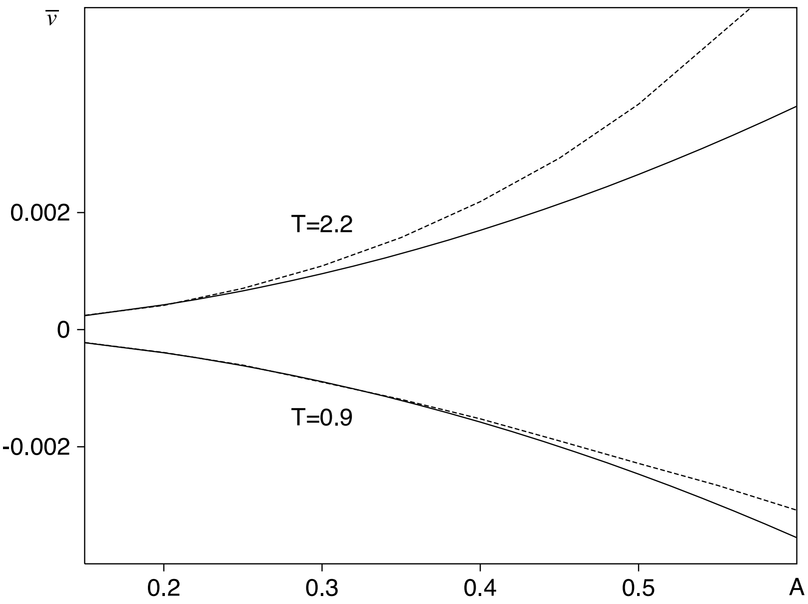

Figure 1. Solid curves are obtained by formula (3.15) and dashed curves by direct Monte Carlo simulations of the SDE (3.1) with $\beta(t;T) = \sin 2\pi t/T$, the potential $F(x)$ of (3.17), $L = 1$ and $\sigma = 0.4$.

Substituting the ratchet potential $F(\varphi)$ of the form (3.17) in (3.18), we get

$$\bar{v} \doteq -\frac{9}{8L^2}A^2T^4N_FM_F$$

which approves the possibility of the current reversal.

Remark 4. *As has already been mentioned, we are able to obtain an approximation like (3.15) for other systems with the directed noise-induced transport in the same way as above. For instance, we evaluate the mean velocity $\bar{v}$ in the case of the system (2.8) with $\beta(t;T) = \chi(t;T)$ defined in (2.9). To this end we expand $\chi(t;T)$ in the Fourier series and apply the procedure given above. As a result, we obtain*

$$\bar{v} = \frac{2}{\pi}A^2TLM_F\int_0^L \mathrm{e}^{2F(\xi)/\sigma^2}\sum_{k=1}^{\infty}\frac{c_1^k(\xi)}{2k-1}\,\mathrm{d}\xi + \mathrm{O}(A^4) \tag{3.19}$$

where $c_1^k(\varphi)$ satisfies the system

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2b_1^k}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'b_1^k) - \frac{2\pi(2k-1)}{T}c_1^k = 0$$

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2c_1^k}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'c_1^k) + \frac{2\pi(2k-1)}{T}b_1^k - \frac{4}{\pi}\frac{1}{2k-1}\frac{1}{T}p_0' = 0$$

$$b_1^k(0) = b_1^k(L) \qquad \frac{\mathrm{d}}{\mathrm{d}\varphi}b_1^k(0) = \frac{\mathrm{d}}{\mathrm{d}\varphi}b_1^k(L) \qquad c_1^k(0) = c_1^k(L) \qquad \frac{\mathrm{d}}{\mathrm{d}\varphi}c_1^k(0) = \frac{\mathrm{d}}{\mathrm{d}\varphi}c_1^k(L)$$

$k = 1, 2, \ldots.$

Here $p_0(\varphi)$ is the same as in (3.7).

Expanding the expression (3.19) in powers of $1/T$, we get for large T:

$$\bar{v} = \frac{2}{\sigma^2} A^2 L^2 N_F M_F J_F + \mathrm{O}\left(\frac{A^2}{T^2}\right) + \mathrm{O}(A^4)$$

which coincides with (2.14).

4. Some extensions

4.1. Forced ratchets with state-dependent noise

The procedure proposed above can be applied in the case of multiplicative (state-dependent) noise

$$\mathrm{d}X = f(X)\,\mathrm{d}t + A\beta(t;T)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}w(t) \tag{4.1}$$

where $f(x)$ and $\sigma(x)$ are L-periodic functions, the force $\beta(t;T)$ is a T-periodic function, $A > 0$ is a sufficiently small number, L and T are some positive constants.

Let $f(x)$ and $\sigma(x)$ be such that the condition (2.6) takes place. Then if $A = 0$, the mean velocity $\bar{v} = \lim_{t\to\infty} EX(t)/t$ is equal to zero (see section 2). In this case we get for small A and $\beta(t;T) = \sin 2\pi t/T$ that

$$\bar{v} = \frac{A^2 T L}{2} \int_0^L c_1(\xi) r^{-1}(\xi)\,\mathrm{d}\xi \left[\int_0^L r^{-1}(\xi)\,\mathrm{d}\xi\right]^{-1} + \mathrm{O}(A^4) \tag{4.2}$$

where $r(\varphi)$ is defined in (2.4) and $c_1(\varphi)$ satisfies the system

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}(\sigma^2(\varphi)b_1) - \frac{\mathrm{d}}{\mathrm{d}\varphi}(f(\varphi)b_1) - \frac{2\pi}{T}c_1 = 0 \\
&\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}(\sigma^2(\varphi)c_1) - \frac{\mathrm{d}}{\mathrm{d}\varphi}(f(\varphi)c_1) + \frac{2\pi}{T}b_1 - \frac{1}{T}p_0'(\varphi) = 0 \\
&b_1(0) = b_1(L) \qquad b_1'(0) = b_1'(L) \qquad c_1(0) = c_1(L) \qquad c_1'(0) = c_1'(L).
\end{aligned}$$

Here

$$p_0(\varphi) = \frac{r(\varphi)}{\sigma^2(\varphi)}\left[\int_0^L \frac{r(\xi)}{\sigma^2(\xi)}\,\mathrm{d}\xi\right]^{-1}.$$

The formula (3.16) is a special case of (4.2).

4.2. Diffusion ratchets

Using the procedure of section 2, we can also find the mean velocity of noise-induced transport in the case of diffusion ratchets with small periodic perturbation of the diffusion coefficient:

$$\mathrm{d}X = f(X)\,\mathrm{d}t + \sigma\cdot(1 + A\beta(t;T))\,\mathrm{d}w(t) \tag{4.3}$$

where the potential $F(x) = -\int f(x)\,\mathrm{d}x$ is an L-periodic function, the force $\beta(t;T)$ is a T-periodic function, $A > 0$ is a sufficiently small number and L, T, σ are some positive constants. The transport in diffusion ratchets was investigated analytically for small and large T and numerically for a wide set of parameters in [9] (see also [4]).

For definiteness, let us take $\beta(t;T) = \sin 2\pi t/T$. Then the mean velocity is approximately evaluated under small A as

$$\bar{v} = -\tfrac{1}{2}A^2\sigma^2 T L M_F \int_0^L c_1'(\xi)\mathrm{e}^{2F(\xi)/\sigma^2}\,\mathrm{d}\xi + \mathrm{O}(A^4) \tag{4.4}$$

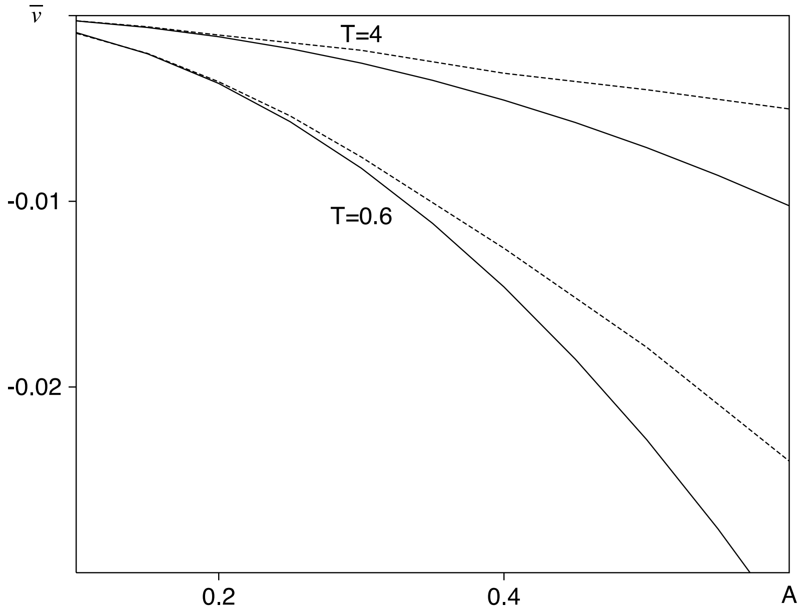

Figure 2. Solid curves are obtained by formula (4.4) and dashed curves by direct Monte Carlo simulations of the SDE (4.3) with $\beta(t;T) = \sin 2\pi t/T$, the potential $F(x)$ of (3.17), $L = 1$ and $\sigma = 0.4$.

where M_F is defined in (2.13) and $c_1(\varphi)$ satisfies the system

$$\frac{\sigma^2}{2}b_1'' + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'(\varphi)b_1) - \frac{2\pi}{T}c_1 = 0$$
$$\frac{\sigma^2}{2}c_1'' + \frac{\mathrm{d}}{\mathrm{d}\varphi}(F'(\varphi)c_1) + \frac{2\pi}{T}b_1 + \frac{\sigma^2}{T}p_0''(\varphi) = 0$$
$$b_1(0) = b_1(L) \qquad b_1'(0) = b_1'(L) \qquad c_1(0) = c_1(L) \qquad c_1'(0) = c_1'(L)$$

with $p_0(\varphi)$ from (3.7).

In figure 2 we present a comparison of approximate values given by the formula (4.4) and the mean velocity $\bar{v}$ evaluated by direct Monte Carlo simulations of the SDE (4.3) with the potential $F(x)$ of (3.17) and $\beta(t;T) = \sin 2\pi t/T$. We apply a third-order weak scheme [12] to this SDE. To obtain a sample trajectory, we numerically integrate the SDE during 1000 periods of $\beta(t;T)$ with the time step 0.02–0.01. In these tests we simulate 2×10^3–2×10^5 sample trajectories. The Monte Carlo errors of the given points are not greater than 5×10^{-4}. Other errors are less than or comparable to the Monte Carlo ones. According to figure 2, the mean velocity $\bar{v}$ of the noise-induced transport in diffusion ratchets can be approximated quite accurately by (4.4) in the case of a small periodic perturbation of the diffusion coefficient.

Remark 5. *Expanding the expression (4.4) in powers of small T, we get*

$$\bar{v} = \frac{2}{\pi^2}A^2T^2LM_FN_F\int_0^L F'(\xi)(F''(\xi))^2\,\mathrm{d}\xi + \mathrm{O}(A^2T^4) + \mathrm{O}(A^4T^2). \qquad (4.5)$$

An expression for the mean velocity $\bar{v}$ in the limit of small T is given in [9]. If we substitute $\beta(t;T) = \sin 2\pi t/T$ in that expression and expand it in powers of A, we also arrive at (4.5).

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