Abstract

Constructive sufficient conditions for regular oscillations in systems with stochastic resonance are given. Using these conditions, a numerical procedure for indicating domains of parameters corresponding to the regular oscillations are proposed. The regular oscillations in systems with additive and multiplicative noise are considered. Results of numerical experiments are presented. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 05.40.−j; 02.50.−r; 02.70.−c

Keywords: Noise-driven systems; Stochastic resonance; Boundary value problems of parabolic type; Numerical integration of stochastic differential equations

1. Introduction

Let us consider a bistable system with noise and periodic forcing (see, e.g., Eq. (4)). Let \( p_{ro} \) be a probability that a point transits from one well to another during the half-period of the periodic forcing and subsequently stays there during the rest of the half-period. Then we say that the noise-induced oscillations take place with the probability \( p_{ro} \). And we say that the regular oscillations (transitions) occur if \( p_{ro} \) is close to 1. A typical regular behavior of some bistable systems with noise and periodic forcing is presented, in particular, in Figs. 5, 7 and 8. In previous papers, such a regularity is named a fluctuation-mediated periodic modulation of the populations of the coexisting stable states or as statistical synchronization with periodic forcing (see, e.g. [1,2] and references therein).

The regular oscillations are connected with the phenomenon of amplifying the response to a periodic forcing which is commonly referred as stochastic resonance (SR). SR was first considered in the context of a model concerning climate dynamics [3–5]. In these initial works a fairly simple and robust mechanism of regular oscillations was explained. Then SR has been observed in a large variety of systems including lasers, noise-driven electronic circuits, superconducting quantum interference devices, chemical reactions, etc. Some theoretical investigations of SR have been done as well. For instance, the signal-to-noise ratio in the case of a sufficiently small amplitude of the periodic forcing has been studied by standard linear response theory. SR is also simulated numerically for...
Various physical and neurobiological problems modeled by stochastic differential equations (SDEs). For a review and extended list of references on SR see, e.g., [1,2,6–8].

Some conditions for regular oscillations, based on Kramers’ theory of diffusion over a potential barrier, are introduced in [3,4,9,10]. The subject of our paper is to give alternative constructive sufficient conditions for the presence of noise-induced regular oscillations. Using these conditions, we propose a numerical procedure for indicating domains of parameters under which regular oscillations exist. The approach proposed here is universal, it can be applied to any system with SR. At the same time, we should mention that Kramers-like approaches are analytically tractable in some limit cases while our approach is numerical.

A typical system, for which the SR phenomenon is observed, has the form of the Ito equation
\[ dX = \alpha(X) \, dt + b(t) \, dt + \sigma(t, X) \, dw(t), \]  
where \( b \) and \( \sigma \) are periodic in \( t \), \( w(t) \) is a standard Wiener process. For instance, the system
\[ dX = (\alpha X - X^3) \, dt + A \sin \omega t \, dt + \sigma \, dw(t) \]  
has the form (1). The following system in the sense of Stratonovich [11]:
\[ dX = \left( \alpha - X - 2c \frac{X}{1 + X^2} \right) \, dt + A \sin \omega t \, dt + \sigma \frac{X}{1 + X^2} \, dw(t) \]  
can be presented in the form (1) as well.

Let us investigate the conditions of arising regular oscillations for a specific system (see [3,4]) which is similar to Eq. (2)
\[ dX = (X - X^3) \, dt + A\chi(t; \theta) \, dt + \sigma \, dw(t), \]  
where \( \chi(t; \theta) \) is the following \( \theta \)-periodic function:
\[ \chi(t; \theta) = \begin{cases} 1, & 0 \leq t < \theta/2, \\ -1, & \theta/2 \leq t < \theta. \end{cases} \]  
Thus, \( \theta \) is a period and \( A \) is an amplitude of the constraining oscillations.

For clarity of exposition, let us give an explanation of the mechanism of arising regular oscillations. It differs from [3,4] (see [1,2,6–8] as well) in form only. In the absence of noise (\( \sigma = 0 \)) and periodic forcing (\( A = 0 \)) Eq. (4) has the stationary points \( x = -1, x = 0, x = 1 \). The points \( x = -1 \) and \( x = 1 \) are stable and \( x = 0 \) is unstable (see Fig. 1a, where \( f(x) \) is the right-hand side of the Eq. (4) for \( \sigma = 0, A = 0 \)). For \( \sigma = 0 \) and not large \( A > 0 \), the stationary points are displaced as shown in Fig. 1b during the first half-period and as in Fig. 1c during the second half-period. Clearly, for \( \sigma = 0 \) a point from a neighborhood of \( x = -1 \) (from the left well) cannot get into a neighborhood of \( x = 1 \) (into the right well) and vice versa. Such transitions become possible for \( \sigma \neq 0 \). Regular transitions (oscillations) arise if a point from the left well attains the point \( x = 1 \) with probability close to 1 at a random time \( \tau < \theta/2 \) and after that it remains in the right well with probability close to 1 during the time \( \theta/2 - \tau \). Indeed, the system acts by virtue of Fig. 1c after the half-period and, due to the symmetry, the situation repeats with changing the left well for the right one.

In Section 2 we investigate two probabilities in conjunction with Fig. 1b: the probability of attainability of the point \( x = 1 \) from \( x = -1 \) for a time less than \( \theta/2 \) (which can be considered as the probability of getting into the right well from the left one) and the probability of unattainability of the point \( x = 0 \) from \( x = 1 \) during the first half-period \( \theta/2 \). It is clear that the closeness of the product \( p \) of these probabilities to 1 is a sufficient condition for the presence of regular oscillations. We observe that at the same time some fluctuations of such a regular behavior are unavoidable: it always remains a positive probability of unattainability from the left well into the right one, sometimes more than two transitions may occur during one period and so on. In other words, we assign a probability \( p_{ro} \) to the very phenomenon of regular oscillations and the above-mentioned product \( p \) bounds this probability from below. The magnitude of \( p \) is found by numerical solution of two boundary value problems of parabolic type in Section 2. As a result, given a level of probability, a domain of parameters can be found such that the probability \( p_{ro} \) is above this level.
The approach proposed here and the approaches based on Kramers’ theory of diffusion over a potential barrier are compared in Section 3. Some other measures of SR are also shortly discussed. Implementation of the proposed approach for various systems with SR is considered in Section 4. We study a system of the form (4) with \( \sigma \) depending on \( t, X \) (or on \( X \) only). Due to the multiplicative noise, one can essentially extend the domain of system parameters guaranteeing regular oscillations. Particularly, we succeed to get high-frequency regular oscillations. Let us observe that the new models presented in Section 4 are obtained due to the essential use of our approach.

Some other results on noise-induced regular oscillations and a detailed description of numerical algorithms used in the experiments can be found in our preprint [12]. In particular, we consider a system of two coupled oscillators [13] and show that an increase of coupling leads to shift of the domain of parameters corresponding to the regular oscillations.

2. The sufficient conditions for regular oscillations

2.1. The main conception

Let \( X_{s,x}(t) \) be the solution of Eq. (4) which starts from the point \( x \) at the moment \( s \). If \( s = 0 \), we write \( X_x(t) \) instead of \( X_{0,x}(t) \). It is known [3,4] that for suitable \( A, \theta, \sigma \) a point from a neighborhood of the point \( x = -1 \) gets into a neighborhood of the point \( x = 1 \) during the first half-period \( \theta/2 \) with the probability close to 1 and remains there up to the end of the half-period. The same takes place in the time interval \( [\theta/2, \theta] \) in the reverse order. Then all the events are repeated.
Let us underline: under the regular oscillations we understand a behavior of the solution $X(t)$ such that $X_{-1}(t)$ reaches $x = 1$ at a random time moment $\tau$ less than $\theta/2$ and $X_{r,1}(t)$ remains greater than zero during the rest $\theta/2 - \tau$ of the half-period. We emphasize that the transitions occur at random time moments, i.e., the phase at which the transitions occur is random.

As it has been explained in Section 1, acceptable sufficient conditions for the regular oscillations are the following ones: the probability $p_{-1,1} = p_{-1,1}(A, \theta, \sigma) := P(X_{-1}(t) < 1, 0 \leq t \leq \theta/2)$ has to be small, and the probability $p_{1,0} = p_{1,0}(A, \theta, \sigma) := P(X_1(t) > 0, 0 \leq t \leq \theta/2)$ has to be close to 1. The oscillations will occur with the probability $p_{ro}$ which exceeds the product

$$p = p(A, \theta, \sigma) := q_{-1,1}(A, \theta, \sigma)p_{1,0}(A, \theta, \sigma), \quad (6)$$

where $q_{-1,1} = 1 - p_{-1,1}$. So, we conclude that the closeness of $p = p(A, \theta, \sigma)$ to 1 is a sufficient condition of regular oscillations. Hereafter this condition is referred to as (RO). Thus, in practice, we can indicate domains of parameters, for which the RO take place with the probability exceeding the given level, by evaluating the product $p$.

**Remark 1.** The condition of closeness of the probability $q_{-1,1}(A, \theta, \sigma)$ to 1 is necessary, but closeness of $p_{1,0}(A, \theta, \sigma)$ to 1 is not necessary for the RO. Indeed, $X_{-1}(t)$ reaches $x = 1$ after some time $\tau > 0$ and in fact we need that $X_{r,1}(t)$ remains in the neighborhood of $x = 1$ during a time less than $\theta/2$. It is not difficult to get (see [12])

$$p_{ro}(A, \theta, \sigma) = \int_0^{\theta/2} p'_1(t; A, \sigma)p_2\left(\frac{\theta}{2} - t; A, \sigma\right) dt,$$

$$p'_1 = \frac{dp_1}{dt},$$

where $p_1(t; A, \sigma) := P(\tau_{-1}(1) \leq t)$, $p_2(t; A, \sigma) := P(\tau_1(0) > t)$, and $\tau_{-1}(1)$ and $\tau_1(0)$ are the first-passage times of $X_{-1}(t)$ to $x = 1$ and of $X_1(t)$ to $x = 0$, respectively. Note that $p_1(\theta/2; A, \sigma) = q_{-1,1}(A, \theta, \sigma)$ and $p_2(\theta/2; A, \sigma) = p_{1,0}(A, \theta, \sigma)$. One can see that $p(A, \theta, \sigma) \leq p_{ro}(A, \theta, \sigma) \leq q_{-1,1}(A, \theta, \sigma)$. Further, if $p_{1,0} \approx 1$ then $p \approx q_{-1,1}$ and $p_{ro} \approx p$. Closeness of $p_{ro}$ to 1 gives the necessary and sufficient condition of the RO. But this condition is less constructive than the given above sufficient condition (RO). Besides, the product $p$ from Eq. (6) approximates $p_{ro}$ quite accurately for a wide set of parameters according to our numerical experiments. For example, if we put the curves of both $p_{ro}$ and $p$ in Fig. 2, they coincide visually. At the same time we should mark that for some sets of parameters $p_{ro}$ and $p$ cannot be so close, e.g., for $A = 0.28$, $\theta = 30$, and $\sigma = 0.6$ we have $p_{ro} = 0.67$ and $p = 0.53$.

2.2. Evaluation of the product $p$

Our urgent aim is to evaluate $p(A, \theta, \sigma)$. Introduce the functions

$$u(s, x) = u(s, x; A, \theta, \sigma),$$

$$v(s, x) = v(s, x; A, \theta, \sigma),$$

$$u(s, x) = 1 - P\left(X_{s,x}(t) < 1, s \leq t \leq \frac{\theta}{2}\right),$$

$$0 \leq s \leq \frac{\theta}{2}, \quad x \leq 1,$n

$$v(s, x) = P\left(X_{s,x}(t) > 0, s \leq t \leq \frac{\theta}{2}\right),$$

$$0 \leq s \leq \frac{\theta}{2}, \quad x \geq 0.$$ We get

$$1 - p_{-1,1} = u(0, -1), \quad p_{1,0} = v(0, 1), \quad p = u(0, -1)v(0, 1).$$

It is well known that the probability $P(X_{s,x}(t) < 1)$ satisfies the corresponding Cauchy problem for the backward Kolmogorov equation (7). The probability $P(X_{s,x}(t) < 1, s \leq t \leq \theta/2)$ obeys the same backward Kolmogorov equation. However, unlike the previous probability, it satisfies a boundary value problem for this equation in half-band (see, e.g. [14–16]). As a consequence, the function $u(s, x)$ satisfies the following mixed problem:

$$\frac{\partial u}{\partial s} + \frac{\partial^2 u}{\partial x^2} + (x - x^3 + A) \frac{\partial u}{\partial x} = 0,$$

$$0 \leq s < \frac{\theta}{2}, \quad x < 1$$

(7)
Fig. 2. Dependence of the product $p(A; \theta, \sigma)$ in $\sigma$ for $A = 0.28$, $\theta = 10^{4}/3$ (left) and in $\theta$ for $A = 0.28$, $\sigma = 0.29$ (right); $\theta$ in the logarithmic scale.

with the initial and boundary conditions

$$u \left( \frac{\theta}{2}, x \right) = 0, \quad x < 1, \quad u(s, 1) = 1,$$

$$0 \leq s \leq \frac{\theta}{2}. \quad (8)$$

The solution of (7) and (8) has the probabilistic representation

$$u(s, x) = E \varphi \left( X_{s, x} \left( \tau_{s, x} \wedge \frac{\theta}{2} \right) \right), \quad (9)$$

where $\tau_{s, x}$ is the first (random) moment at which $X_{s, x}(t) = 1$ and

$$\varphi(x) = \begin{cases} 0, & x < 1, \\ 1, & x = 1. \end{cases}$$

We get analogously that the function $v(s, x)$ satisfies the mixed problem

$$\frac{\partial v}{\partial s} + \frac{1}{2} \sigma^{2} \frac{\partial^{2} v}{\partial x^{2}} + (x - x^{3} + A) \frac{\partial v}{\partial x} = 0,$$

$$0 \leq s < \frac{\theta}{2}, \quad x > 0, \quad (10)$$

$$v \left( \frac{\theta}{2}, x \right) = 1, \quad x > 0, \quad v(s, 0) = 0,$$

$$0 \leq s \leq \frac{\theta}{2}. \quad (11)$$

The solution of (10) and (11) has the probabilistic representation

$$u(s, x) = E \psi \left( X_{s, x} \left( \tau_{s, x} \wedge \frac{\theta}{2} \right) \right), \quad (12)$$

where $\tau_{s, x}$ is the first (random) moment at which $X_{s, x}(t) = 0$ and

$$\psi(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0. \end{cases}$$

One can prove that the function $u(s, x; A, \theta, \sigma)$ is increasing and the function $v(s, x; A, \theta, \sigma)$ is decreasing with respect to $\theta$. And typically there is a fairly wide range of $\theta \in (\theta_{a}(A, \sigma), \theta^{\ast}(A, \sigma))$, where under fixed $A$ and $\sigma$ the product $p(A, \theta, \sigma)$ is close to its maximum. Analogously, the product $p(A, \theta, \sigma)$ is close to its maximum in the range of the noise intensity $\sigma \in (\sigma_{a}(A, \theta), \sigma^{\ast}(A, \theta))$. We take the amplitude $A$ of the constraining oscillations less than $A^{\ast} = 2\sqrt{3}/9$ so that the system (4) with $A < A^{\ast}$ and $\sigma = 0$ has three stationary points for each of the half-periods. Evidently, the product $p(A, \theta, \sigma)$ is an increasing function with respect to $A$ under fixed $\theta$ and $\sigma$.

As is known and has already been mentioned, there is a domain of parameters such that the RO are observed. Due to the sufficient condition of RO, we are able to get estimates of this domain in terms of the product $p(A, \theta, \sigma)$.

To find the probabilities $q_{-1, 1}$ and $p_{1, 0}$, we have to solve the problems (7)–(8) and (10)–(11) numerically. In a number of tests we have used both probability
methods from [17] and finite-difference schemes and have seen ourselves that they give coincident results.

Probabilistic methods for boundary value problems are based on probabilistic representations of their solutions. The representations are connected with systems of SDEs. To realize them, Markov chains which weakly approximate the solutions of these systems are constructed. Unlike usual approximations of SDEs (see, e.g. [18]), when a time discretization is exploited, space–time discretizations are recommended in the case of parabolic boundary value problems [17]. If the value of $\sigma$ is small, it is preferable to attract weak numerical methods from [19]. These methods are specially intended to approximate solutions of SDEs with small noise and are highly efficient. In [20] the special methods of [19] were effectively applied to evaluation of the signal-to-noise ratio in systems with SR.

Due to the fact that the absolute value of the term $x - x^3$ becomes big at $|x| \gg 1$, there are difficulties in implementation of finite-difference schemes for solving the boundary value problems (7)–(8) and (10)–(11). The difficulties do not arise in simulating the problems by the probabilistic methods. Moreover, we need the individual values $u_0, -1$ and $v_0, 1$ only, and in such a case the probabilistic approach with the Monte Carlo technique is the most relevant. That is why we mainly use the probabilistic methods in our experiments and attract finite-difference ones from time to time to control the obtained results. Some further details on the used probabilistic methods are available in [12].

2.3. Numerical results

Figs. 2 and 3 show typical behavior of the product $p(A, \theta, \sigma)$. The remarkable feature is that there is a range of parameters where the product $p$ is close to 1 that corresponds to the RO. Given the level of the product $p$, the domains of parameters are indicated in Fig. 4. Let us emphasize that the range of parameters for the RO is fairly large. For some fixed $A$ and $\sigma$, the system (4) can be turned to a regular behavior ($p$ becomes close to 1) by choosing the period $\theta$ (cf. Fig. 2 (right), Fig. 4, and properties of the functions $u$ and $v$ in Section 2.2). It remembers a “bona-fide” SR discussed in [21,22]. We should also mention that the product $p$ can have maximum in $\theta$ for fixed $A$ and $\sigma$ which is far from 1, and, evidently, in this case the system is far from the regular behavior. Influence of the amplitude $A$ on the effect of synchronization in

![Fig. 3. Dependence of the product $p(A, \theta, \sigma)$ in $A$ for $\theta = 10^4/3$ and various $\sigma$.](image-url)
Fig. 4. Level curves of the product $p(A, \theta, \sigma)$ in the plane $(\theta, \sigma)$ for $A = 0.28$; $\theta$ in the logarithmic scale.

systems with SR was investigated in [23] (see also [8] and references therein). It was stated that increase of $A$ leads to an extension of the range of $\sigma$ (for fixed $\theta$), where the RO occur. This follows from our analysis as well (see Fig. 3 and properties of the functions $u$ and $v$ in Section 2.2). Let us also observe that in the case of model (4) the RO are realized for sufficiently large periods $\theta$ only. This corresponds to common knowledge on SR [1,2,6–8]. To get high-frequency RO (i.e., for rather small $\theta$), we involve into consideration models with specific multiplicative noises in Section 4.

Fig. 5 presents typical sample trajectories of the solution to Eq. (4). We take values of parameters corresponding to Fig. 2. For the parameters $A, \theta, \sigma$ such that the product $p(A, \theta, \sigma)$ is close to 1, i.e., the sufficient condition (RO) takes place, we observe the RO (see Fig. 5b). A sample trajectory in the case when $q_{-1,1} \approx 0.8$ and $p_{1,0} \approx 1$, i.e., when the necessary condition does not fulfill, is given in Fig. 5a. One can see that transitions between two wells during $\theta/2$ occur with the probability close to 0.8. Fig. 5c demonstrates a typical trajectory in the case of $q_{-1,1} \approx 1$ and $p_{1,0} \approx 0.8$. After reaching $x = 1$ ($x = -1$), the trajectory remains in the corresponding well during the rest of the half-period with the probability close to 0.8.

To simulate trajectories we use the mean-square Euler method. This simplest mean-square method is usually exploited for trajectory analysis of SR models. Let us mention that more accurate mean-square methods can be applied to the SDE with additive noise (4) (see, e.g. [18] for the modern theory of numerical integration of SDEs). Besides, if the noise intensity $\sigma$ is small, SR is observed for large $\theta$ and one should simulate the system on long time intervals. In this case the most preferable methods are ones of [24], where efficient high-exactness mean-square methods for SDEs with small noise are proposed. To use these special methods is essentially important if a system under consideration (e.g., an array of coupled oscillators [25]) is of high-dimension.

Remark 2. We also implement the approach for the model with sinusoidal forcing (2). Some numerical experiments are performed. In particular, they approve the fact that the domain of parameters corresponding to the RO in the case of sinusoidal forcing is narrower than in the case of periodic rectangular pulses forcing.

Remark 3. The approach proposed here can be carried over to the problem of noise-induced transport in Brownian ratchets [26] (for a review on this topic see, e.g. [27,28]). By our approach we are able to indicate the domains of parameters, where the transport is unidirectional. A separate paper will be devoted to this subject.

3. Comparison with the approach based on Kramers’ theory of diffusion over a potential barrier

Let us consider system (4) when it acts by virtue of Fig. 1b, i.e.,
dX = a(X) dt + σ dw(t),
\[ a(x) = x - x^3 + A, \quad A < 2\sqrt{3}/9. \]

Evaluate some mean characteristics of \( \tau_{-1}(1) \) (the first-passage time of \( X_{-1}(t) \) to \( x = 1 \)) and \( \tau_{1}(0) \) (the first-passage time of \( X_{1}(t) \) to \( x = 0 \)).

The mean value \( E\tau_{-1}(1) \) can be found in the following way. Consider the boundary value problem

\[ \frac{1}{2}\sigma^2\Psi'' + a(x)\Psi' + 1 = 0, \quad \Psi(C;C) = 0, \quad \psi(1;C) = 0, \quad C < -1 \]

for the function \( \psi(x;C) \), where \( C \) is a parameter. It is known [14–16] that \( \psi(-1;C) \) is equal to the mean value of the first-exit time of the process \( X_{-1}(t) \) from the interval \( (C,1) \). Clearly, \( E\tau_{-1}(1) = \lim_{C \to -\infty} \Psi(-1;C) \). In addition, one can prove that \( E\tau_{-1}(1) = \Psi(-1) \), where \( \Psi(x) \) is a solution to the problem

\[ \frac{1}{2}\sigma^2\Psi'' + a(x)\Psi' + 1 = 0, \quad \Psi'(-\infty) = 0, \quad \Psi(1) = 0. \quad (13) \]

The second moment \( E\tau_{-1}^2(1) \) is equal to \( \Psi_1(-1) \), where \( \Psi_1(x) \) is a solution to the problem

\[ \frac{1}{2}\sigma^2\Psi_1'' + a(x)\Psi_1' + 2\Psi(x) = 0, \quad \Psi_1'(-\infty) = 0, \quad \Psi_1(1) = 0, \]

and \( \Psi(x) \) is the solution of problem (13) (see [14–16]).

The mean \( E\tau_{1}(0) \) and the second moment \( E\tau_{1}^2(0) \) can be found analogously.

The approach of [3,4] based on Kramers’ theory employs the following conditions as sufficient ones for the existence of RO (from the principal point of view our exposition here only slightly differs from [3]): (i) \( E\tau_{-1}(1) \ll \theta/2 \), (ii) \( E\tau_{1}(0) \gg \theta/2 \), (iii) \( (D\tau_{-1}(1))^{1/2} = [E\tau_{-1}^2(1) - (E\tau_{-1}(1))]^{1/2} \ll \theta/2 \). These conditions are fairly constructive because all
the magnitudes \( E\tau_{-1}(1) \), \( E\tau_{1}(0) \), and \( E\tau^2_{-1}(1) \) can be found by quadratures. Moreover, for small \( \sigma \) they can be expressed by exponential Kramers’ formulas.

Other conditions of RO are given in [9,10]. To get them, exponential Kramers formulas are exploited.

In the previous section we proposed an alternative approach, which is sufficiently constructive as well and possessed more generality. Besides, one can get the more exhaustive answers using the sufficient condition (RO) from Section 2.1 in comparison to the conditions of [3,4,9,10] which are only qualitative in nature. Let us emphasize once more that the probability in Section 2 is assigned to the very phenomenon of RO. The universality and utility of the proposed approach are demonstrated in Section 4. At the same time, Kramers-like approaches are analytically tractable in some limit cases while our approach is numerical. We should also mention here about some measures commonly used for SR (signal-to-noise ratio, response amplitude, waiting time distributions) [1,2,6–8]. As is known [2,22], a maximum of signal-to-noise ratio does not directly reflect the RO (synchronization between the hopping and driving) in systems with SR. Such characteristics of SR as the response amplitude at the frequency of the periodic signal and waiting time distributions reflect the phenomenon of RO. Generally, waiting time distributions are less effective in computational sense than the measure \( p \) introduced in Section 2.1. And the \( p \) gives more information on the RO than the response amplitude. We restrict ourselves to this short comment on characteristics of SR. A more profound comparison analysis requires a special consideration.

4. High-frequency RO in systems with multiplicative noise

In the case of system (4) it is impossible to get high-frequency RO. Indeed, if we decrease the period length \( \theta \), we should increase the noise level \( \sigma \) to preserve the level of the probability \( q_{-1,1} \) of escape from the metastable state to absolutely stable state. But the probability \( p_{1,0} \) of return from the absolutely stable state to the metastable one decreases with an increase of \( \sigma \). Therefore, the product \( p \) becomes low and the RO disappear. In this section we consider some specific systems with multiplicative noise such that the probability \( p_{1,0} \) is always equal to 1 and due to this fact we are able to obtain the high-frequency RO.

Consider the model with multiplicative time-dependent noise

\[
\begin{align*}
\mathrm{d}X &= (X - X^3) \, \mathrm{d}t + A \chi(t; \theta) \, \mathrm{d}t \\
&+ \sigma \gamma(t, X; \theta) \, \mathrm{d}w(t),
\end{align*}
\]

(14)

where \( \chi(t; \theta) \) is the \( \theta \)-periodic function defined in (5) and \( \gamma(t, x; \theta) \) is the following \( \theta \)-periodic function:

\[
\gamma(t, x; \theta) = \begin{cases}
1, & 0 \leq t < \theta/2, \quad x < 1, \\
0, & 0 \leq t < \theta/2, \quad x \geq 1, \\
1, & \theta/2 \leq t < \theta, \quad x > -1, \\
0, & \theta/2 \leq t < \theta, \quad x \leq -1.
\end{cases}
\]

As was marked in [29] (where SR for periodically modulated noise intensity was considered), periodically modulated noise is not uncommon and it arises, for example, at the output of any amplifier whose gain varies periodically in time. And a system with diffusion coefficient which depends on its state plays an important role in a number of physical systems (see, e.g., [30–32] and references therein).

It is evident that \( p_{1,0} = 1 \) in the case of (14) and, consequently, the necessary and sufficient condition of RO consists in the closeness of the probability \( q_{-1,1}(A, \theta, \sigma) \) to 1. This probability can be close to 1 even for a fairly small \( \theta \) (i.e., for high-frequency \( \omega = 2\pi/\theta \)) and for very small \( A \) under an appropriate value of \( \sigma \). Thus, it is possible to organize the high-frequency RO in system (14) with small periodic forcing. Moreover, in the case of the model (14) the RO can be obtained under zero \( A \). Another system with high-frequency SR was considered in [33].

Fig. 6 demonstrates level lines of \( q_{-1,1}(A, \theta, \sigma) \) in the plane \((\theta, \sigma)\) for \( A = 0.28 \). A typical trajectory with the high-frequency oscillations are given in Fig. 7. Let us observe that rather long excursions of trajectories (up to \( x = \pm 5 \) or even more) are possible. Consider, for instance, a trajectory \( X_{s,1}(t), 0 \leq s < \theta/2, t \geq s \). When \( t < \theta/2 \), we have \( \chi(t; \theta) > 0 \) and the noise \( \sigma \gamma(t, x; \theta) = \sigma \) for \( x < 1 \) and \( \sigma \gamma(t, x; \theta) = 0 \) for
Fig. 6. Level curves of the probability $q_{-1,1}(A, \theta, \sigma)$ in the plane $(\theta, \sigma)$ for $A = 0.28$; \( \theta \) in the logarithmic scale.

Fig. 7. Sample trajectory of the solution to Eq. (14) for $A = 0.28$; \( \theta \approx 0.524\omega(\omega = 12) \), \( \sigma = 35 \).

Let the solution $X(t)$ to Eq. (15) start from $x = -1$. During the time $[0, \theta/2)$ the drift in system (15) corresponds to Fig. 1b. Clearly, the probability of attainability of the point $x = 1$ for the time less than $\theta/2$ is not less than $q_{-1,1}$ in model (4). After reaching the point $x = 1$ at a random moment, the trajectory moves deterministically in positive direction to a point $X(\theta/2) > 1$. Then the drift in system (15) becomes corresponding to Fig. 1c and the trajectory changes its movement direction. The trajectory comes back to the point $x = 1$ at a moment $\theta/2 + \tau$, where $\tau$ is random. It remains the time $\theta/2 - \tau$ for the trajectory to reach the point $x = -1$. The random moment $\tau$ is less than a quantity $s^*$ which can be evaluated in the following way. Let the solution $X(t)$ of the equation

\[
dX = (X - X^3) \, dt + A \chi(t) \, dt + \sigma \gamma(X) \, dw(t),
\]

(15)

where

\[
\gamma(x) = \begin{cases}
1, & -1 < x < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Let the solution $X(t)$ to Eq. (15) start from $x = -1$. During the time $[0, \theta/2)$ the drift in system (15) corresponds to Fig. 1b. Clearly, the probability of attainability of the point $x = 1$ for the time less than $\theta/2$ is not less than $q_{-1,1}$ in model (4). After reaching the point $x = 1$ at a random moment, the trajectory moves deterministically in positive direction to a point $X(\theta/2) > 1$. Then the drift in system (15) becomes corresponding to Fig. 1c and the trajectory changes its movement direction. The trajectory comes back to the point $x = 1$ at a moment $\theta/2 + \tau$, where $\tau$ is random. It remains the time $\theta/2 - \tau$ for the trajectory to reach the point $x = -1$. The random moment $\tau$ is less than a quantity $s^*$ which can be evaluated in the following way. Let the solution $X(t)$ of the equation

\[
X' = X - X^3 + A \chi(t; \theta)
\]

start from $x = 1$. Then the trajectory $X(t)$ moves in positive direction up to $t = \theta/2$, when the trajectory changes its movement direction, and comes back to the point $x = 1$ at the instance $t^* \in (\theta/2, \theta)$. The value of the desired $s^*$ is equal to $t^* - \theta/2$.

Introduce the probability

\[
p_{-1,1}^* = p_{1,-1}^* = p_{1,-1}(A, \theta, \sigma)
\]

where

\[
\gamma(x) = \begin{cases}
1, & -1 < x < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

The sufficient condition of RO of the solution to Eq. (15) consists in the closeness of the probability $q_{-1,1}^* = 1 - p_{1,-1}^*$ to 1. By the same arguments as in Section 2.2, it is not difficult to see that $q_{-1,1}^* = u(0, -1)$, where $u(s, x)$ is the solution to the following mixed problem

\[
\begin{align*}
\frac{\partial u}{\partial s} + \frac{\sigma^2}{2} \gamma(x) \frac{\partial^2 u}{\partial^2 x} + (x - x^3 + A) \frac{\partial u}{\partial x} &= 0, \\
0 & \leq s < \frac{\theta}{2} - s^*, \quad x < 1, \\
u\left(\frac{\theta}{2} - s^*, x\right) &= 0, \quad x < 1, \\
u(s, 1) &= 1, \quad 0 \leq s \leq \frac{\theta}{2} - s^*.
\end{align*}
\]

(17)

(18)
The RO in the case of Eq. (15) are observed under a more wide set of parameters than for Eq. (4) however under a more restricted set of parameters than for Eq. (14). Fig. 8 shows a typical trajectory of the solution to Eq. (15) under values of parameters such that they do not ensure the RO in the case of the model (4).

Now let us give a brief remark on simulation of SDEs (14) and (15). If a model has discontinuous in time and continuous in space coefficients, there are no serious problems in its simulation. Despite the diffusion coefficient in Eq. (14) is discontinuous in $t$ and $x$, principal difficulties do not arise as well. It is so because any trajectory of Eq. (14) feels the discontinuity of the diffusion coefficient in $x$ not more than once during the half-period $\theta/2$. As to Eq. (15), discontinuity in $x$ of the diffusion coefficient leads to some problems in numerical simulations. Indeed, if $X(t) \geq -1$ at a moment $t \in [n\theta, (n + 1/2)\theta]$, $n = 0, 1, 2, \ldots$, then $X(s) > -1$ for all $s \in (t, (n + 1/2)\theta)$ with probability 1. But due to the discretization error, the usual mean-square Euler approximation of $X(t_k)$ violates this property and can become less than $-1$. As a result, it gives a too distorted image of the real behavior. To overcome this difficulty, we propose a modified approximation (agreeing with the above-mentioned property of trajectories) [12], which is used for simulation of SDE (15) here.

**Remark 4.** Using the approach proposed in Section 2, one can obtain a sufficient condition for the RO (3) which has an asymmetrical bistable potential (see [11]). Let the system (3) have two stable points $x_-$ and $x_+$, and $x_+ < x_-$, and one unstable $x_\sigma$, $x_- < x_\sigma < x_+$, in the absence of periodic forcing and noise. To give a sufficient condition for the RO in the asymmetrical case, four probabilities have to be considered: the probability $q_{x_- x_\sigma}$ with which the trajectory starting from $x = x_-$ reaches the point $x = x_\sigma$ during the first half-period of the periodic forcing (i.e., when the periodic forcing is positive); the probability $p_{x_\sigma x_+}$ of unattainability of the point $x = x_\sigma$ during the first half-period by the trajectory starting from $x = x_+$; the probability $q_{x_+ x_-}$ with which the trajectory starting from $x = x_+$ reaches the point $x = x_-$ during the second half-period of the periodic forcing; the probability $p_{x_- x_\sigma}$ of unattainability of the point $x = x_\sigma$ during the second half-period by the trajectory starting from $x = x_-$. Due to the asymmetry, $q_{x_+ x_-} \neq q_{x_- x_+}$ and $p_{x_\sigma x_+} \neq p_{x_\sigma x_-}$. In this situation, the sufficient condition of RO consists in the closeness of the products $q_{x_- x_\sigma} \times p_{x_\sigma x_+}$ and $q_{x_\sigma x_-} \times p_{x_- x_\sigma}$ to 1. One can easily write down boundary value problems for these probabilities.

It follows from the analysis given in [11] that the model (3) can operate in the regime of high-frequency RO. Probably, this becomes possible owing to the specific type of noise which acts, in essence, in bounded range of states $x$ only (cf. models (14) and (15) which also have bounded-type noise).

**Remark 5.** The approach of Section 2 can also be applied to studying RO in the case of a monostable system. For instance, we consider SDE (1) with

$$a(x) = \begin{cases} -2\alpha(x + \beta), & x < -\beta, \\ -\gamma \pi \sin \pi x, & |x| < 1, \\ 0, & 1 \leq |x| \leq \beta, \\ -2\alpha(x - \beta), & x > \beta, \end{cases}$$

and $b(t) = A\chi(t; \theta)$, $\chi(t; \theta)$ is from (5). For $A = 0$ and $\sigma = 0$, the solution to this equation has the unique globally stable point $x = 0$. If $\sigma = 0$ and $A$ is not large, the equation has a $\theta$-periodic solution with
an amplitude less than 1. After adding the noise of a certain not large level, the system does not exhibit any RO. But an increase of the noise intensity leads to the RO with a large amplitude approximately equal to $\beta$. To find a set of parameters under which the RO with large amplitude are observed, we introduce the probabilities

$$q_{-\beta, \beta} := 1 - P\left(X_{-\beta}(t) < \beta, \ 0 \leq t \leq \frac{\theta}{2}\right),$$

$$p_{\beta, 1} := P\left(X_{\beta}(t) > 1, \ 0 \leq t \leq \frac{\theta}{2}\right).$$

Then the sufficient condition for the RO with the amplitude $\beta$ consists in the closeness of the product $q_{-\beta, \beta} \cdot p_{\beta, 1}$ to 1. The boundary value problems for calculating these probabilities can be written down just as in Section 2.2. Some details on this model can be found in [12].

Acknowledgements

This work is partially supported by the Russian Foundation for Basic Research (project 99-01-00134). The second author is also grateful to the Alexander von Humboldt Foundation for support of this work through a research fellowship.

References


