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Review

Reaction–diffusion waves in biology

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Abstract

The theory of reaction–diffusion waves begins in the 1930s with the works in population dynamics, combustion theory and chemical kinetics. At the present time, it is a well developed area of research which includes qualitative properties of travelling waves for the scalar reaction–diffusion equation and for system of equations, complex nonlinear dynamics, numerous applications in physics, chemistry, biology, medicine. This paper reviews biological applications of reaction–diffusion waves.

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1. Introduction to the theory of reaction–diffusion waves

Reaction–diffusion equations are conventionally used in chemical physics in order to describe concentration and temperature distributions. In this case, heat and mass transfer are described by the diffusion term while the reaction term describes the rate of heat and mass production. These nonlinear terms are often considered in the form of mass-action law.

Similar models can also be used in other applications where transport phenomena are determined by a random motion and where production or consumption terms should be taken into account. A classical example is given by a system of reacting chemicals. In population dynamics, where we need to describe the density of some population, diffusion terms correspond to a random motion of individuals and reaction terms describe their reproduction. The choice of the reaction forms may vary. In the original works by Lotka and Volterra, they were considered as a product of the concentrations or densities of the reacting components or interacting populations. Nowadays, the mass-action law is still used in some applications, although somewhat more complicated functional forms such as Monod functional response in biochemistry or Holling types II and III in population dynamics are often considered as more realistic alternatives.

Reaction–diffusion systems are often characterized by the existence of homogeneous in space equilibria where the reaction terms vanish. If there are more than one such equilibrium, then we can expect a possible transition between them. These transitions are provided by reaction–diffusion waves. Propagation of flames, migration of biological species or tumor growth are among many examples of such phenomena. In this work we will present a review of some biological applications of reaction–diffusion waves. We will begin with a short introduction to the mathematical theory of reaction–diffusion waves (see [131] and the references therein).

1.1. Main definitions

The theory of reaction–diffusion waves begins in the 1930s with the works by Fisher [40] and Kolmogorov, Petrovskii, Piskunov [68] on propagation of dominant gene and by Zeldovich, Frank-Kamenetskii in combustion theory [140]. They have introduced the scalar reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \quad (1)$$

defined travelling wave solutions and studied their existence, stability, speed of propagation. These works were followed by many others where reaction–diffusion waves were studied in connection with various problems in chemistry, physics, biology, medicine. Throughout this paper, we will refer to (1) as the KPP–Fisher equation.

Travelling wave solutions of this equation are the solutions of the specific form, $u(x, t) = w(x - ct)$. They have a constant profile and propagate with a constant speed c . These solutions are defined for all x from $-\infty$ to $+\infty$ and satisfy the equation

$$w'' + cw' + F(w) = 0. \tag{2}$$

Existence and properties of such solutions are determined by the function $F(u)$. In population dynamics it describes the rate of the reproduction of the population. It is often considered in the form

$$F(u) = au^n(1 - u) - bu. \tag{3}$$

If $n = 1$, then the reproduction rate is proportional to the density u of the population and to available resources $(1 - u)$. The last term describes mortality of the population. If $a > b$, that is mortality is small enough, then the function $F(u)$ in (3) has two zeros, $w_+ = 0$ and $w_- = 1 - b/a$. In population dynamics, function $F(u) = au(1 - u/K)$ (where parameter $K = w_-$ is called the carrying capacity) is said to describe a population with the *logistic growth*. These two points are stationary solutions of the ordinary differential equation

$$\frac{du}{dt} = F(u). \tag{4}$$

It can be easily verified that the point w_+ is unstable with respect to this equation, while the point w_- is stable.

The case $n = 2$ takes into account collective effects, the simplest example of such effect is sexual reproduction when the reproduction rate is proportional to the square of the density. The function $F(u)$ in (3) can have from one to three zeros depending on the parameters. When there are three of them, $w_+ = 0$, w_0 and w_- , where $w_0 < w_+$, then the points w_+ and w_- are stable with respect to Eq. (4) while the intermediate point w_0 is unstable. In population dynamics, this case is often referred to as a strong *Allee effect*.

These two examples bring us to the following classification of nonlinearities. Let us look for the solution of Eq. (2) having limits at infinity,

$$w(\pm\infty) = w_{\pm}. \tag{5}$$

The existence of such limits implies that $F(w_{\pm}) = 0$. If both points w_+ and w_- are stable with respect to Eq. (4), that is $F(u) \leq 0$ in some right neighborhood of the point w_+ and $F(u) \geq 0$ in some left neighborhood of the point w_- , then it is the *bistable* case. If the point w_+ is unstable, that is $F(u) > 0$ in some its right neighborhood, and the point w_- is stable, then it is the *monostable* case. Finally, if the both points are unstable, then it is the *unstable* case.

1.2. Basic results for the scalar equation

1.2.1. Existence

The question about existence of travelling waves has not only mathematical interest. We will see below in Section 1.2.5 that even for the scalar equation the wave may not exist. In this case dynamics of the solution is determined by systems of waves. Moreover, wave existence is related to its uniqueness or non-uniqueness and to its stability. In practical applications, a reaction–diffusion system or a similar model is often solved numerically without paying much attention to the question whether it actually has a solution. That may sometimes lead to an embarrassing situation when a numerical “solution” is found, as a result of numerical artifacts, for a problem where it does not exist. A specific example can be found in [101], Chapter 1. A complete and consistent study therefore should include the issue of solution existence, ideally as a starting point.

Existence of solutions of problem (2), (5) can be studied by the phase space analysis. We reduce Eq. (2) to the system of two first order equation

$$w' = p, \quad p' = -cp - F(w)$$

and look for a trajectory connecting two stationary points, $(w_-, 0)$ and $(w_+, 0)$.

Consider first the monostable case assuming that $F(w) > 0$ for $w_+ < w < w_-$. The point $(w_-, 0)$ is a saddle, $(w_+, 0)$ is a stable node. Monotone with respect to x travelling waves exist for all values of the speed c greater or equal to some minimal speed c_0 . In the case of Eq. (2) with $n = 1, b = 0$ (the case of logistic growth), $c_0 = 2\sqrt{a}$. In the general case, $c_0 \geq 2\sqrt{F'(w_+)}$. Non-monotone waves can exist even for $c < c_0$. However, as we will see below, they are unstable and not interesting for applications. If we do not assume that the function $F(w)$ is positive in the interval (w_+, w_-) , then the waves with the given limits at infinity may not exist. If they exist, then the values of speed are bounded from above, $c_0 \leq c < c_1$ with some positive finite c_1 .

Next, let us consider the bistable case assuming that $F(w)$ has a single zero w_0 in the interval (w_+, w_-) such that

$$F(w) < 0, \quad w_+ < w < w_0, \quad F(w) > 0, \quad w_0 < w < w_-.$$

In this case, travelling wave solution exists for a unique value c of the speed. For general bistable nonlinearities, travelling wave with the given limits at infinity may not exist. If it exists, then the value of the speed is unique. Finally, in the unstable case problem (3), (5) does not have solutions.

1.2.2. Stability

Stability of stationary solutions of evolution problems is determined by the location of the spectrum of the equation linearized about the stationary solution. Stability of travelling waves has some specific features because they are not isolated solutions. Each solution $w(x)$ of problem (2), (5) generates the whole family of solutions $w(x + h)$ for any real h . From the point of view of the structure of the spectrum of the linearized operator

$$Lu = u'' + cu' + F'(w(x))u,$$

this results in the existence of the zero eigenvalue with the eigenfunction $u(x) = w'(x)$. Therefore, conventional results about the relation between the location of the spectrum and stability with respect to small perturbations cannot be directly applied in the case of travelling waves. In this case we should use the notion of stability with shift introduced by Barenblatt and Zeldovich [12].

Consider Eq. (1) written in the moving coordinates attached to the wave:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x} + F(u). \tag{6}$$

Function $w(x)$ is its stationary solution. Let us take the initial condition in the form

$$u(x, 0) = w(x) + \epsilon(x), \tag{7}$$

where $\epsilon(x)$ is a small perturbation. If the solution $u(x, t)$ converges to the wave $w(x)$ as t goes to infinity for any small perturbation, then the wave is asymptotically stable. If the convergence can occur to the wave $w(x + h)$ with some h , then this is the asymptotic stability with shift. The main result on stability of waves in the bistable case is given by the following theorem.

Theorem 1. Suppose that $F'(w_{\pm}) < 0$. There exist positive constants ϵ_0, M and λ such that for any piece-wise continuous function $\epsilon(x)$ satisfying the inequality $\sup_x |\epsilon(x)| \leq \epsilon_0$ the solution $u(x, t)$ of problem (6), (7) exponentially converges to the wave $w(x + h)$ with some h :

$$\sup_x |u(x, t) - w(x + h)| \leq Me^{-\lambda t} \quad \text{as } t \rightarrow \infty.$$

Stability with respect to small perturbations and the maximum principle allow one to prove the global stability of monotone waves (see, e.g., [39,60]). Non-monotone waves are unstable.

In the monostable case, since the waves exist for an interval of values of the speed, the choice of the initial condition determines to which wave the solution converges. We discuss this question in the next section.

1.2.3. Convergence of the initial conditions

In a rigorous mathematical sense, a travelling wave solution of a reaction–diffusion equation is a special solution, i.e. a solution corresponding to the initial condition of a special form. Indeed, Eq. (2) does not contain time and it

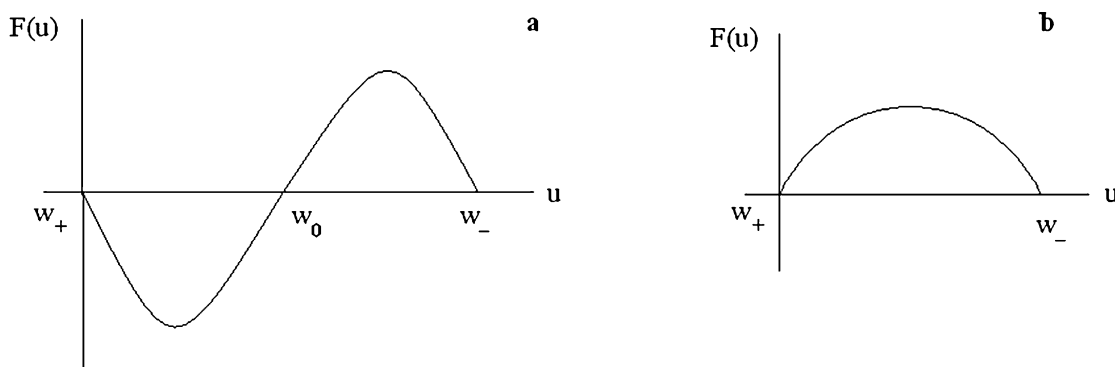


Fig. 1. Examples of nonlinearity in Eq. (1): (a) bistable case, (b) monostable case.

means that the shape of the profile does not change with time and hence should be given as it is at the very beginning of the process, e.g. at $t = 0$. However, the actual meaning of travelling wave solutions is of course much wider and is not restricted to a special choice of the initial condition. The matter is that the travelling wave profile emerges in the course of time as a result of *convergence* of initial conditions from a wide class. The first mathematical result of this kind was proved by Kolmogorov et al. in their seminal paper [68] where they showed that, for a concave-down monostable function $F(u)$ (cf. Fig. 1b), the following “transitional” initial conditions,

$$u(x, 0) = w_- \quad \text{for } -\infty < x < x_1, \tag{8}$$

$$u(x, 0) = w_+ \quad \text{for } x_2 < x < \infty, \tag{9}$$

$$u(x, 0) = \phi(x) \quad \text{for } x_1 < x < x_2 \tag{10}$$

(where x_1 and x_2 are certain numbers and $\phi(x)$ is a bounded piece-wise continuous function) converge in the large-time limit to a travelling wave propagating to the right with the speed $c_0 = 2\sqrt{F'(w_+)}$. The wave profile is a solution of Eq. (2) with the corresponding conditions at infinity. Therefore, strictly speaking the travelling wave solutions describe the dynamics of a reaction–diffusion system in the large-time limit while an early, transient stage of the dynamics depends on the details of the initial conditions. However, we want to emphasize it here that the rate of convergence is usually fast, so that the travelling wave solution describes with high precision the transient solution of Eq. (1) already after a short interval Δt .

The speed of the travelling wave appearing in the large-time limit can depend on the type of the initial conditions. In the above example, $u(x, 0)$ is semi-finite, i.e. it equals to w_+ identically for any $x > x_2$. Now, consider a situation when $u(x, 0) \rightarrow w_+$ for $x \rightarrow \infty$ but the difference $(u(x, 0) - w_+)$ remains positive. Depending on the rate of decay in $u(x, 0)$ at large x , the initial conditions of this type can converge to a travelling wave propagating with speed $c > c_0$ [73,89,110]. More specifically, if $u(x, 0)$ decays exponentially,

$$u(x, 0) - w_+ \sim e^{-sx}, \tag{11}$$

where $s > 0$ is a parameter, then

$$c = 2 \quad \text{if } s \geq 1 \quad \text{and} \quad c = s + \frac{1}{s} \quad \text{if } s < 1. \tag{12}$$

For a rate of decay slower than exponential, e.g. algebraic, instead of a travelling wave propagating with a constant speed, the initial conditions of Eq. (1) may converge to an accelerating solution propagating with a permanently increasing speed [63].

1.2.4. Speed of propagation

For some particular nonlinearities, the speed of wave propagation can be found explicitly (see [101] and the references therein). The two most well-known cases are the logistic population, see above, and the population with the growth described by a cubic polynomial.¹ In the general case it is possible to represent it in the minimax form. The

¹ I.e. when $F(u) = -a(u - w_+)(u - w_0)(u - w_-)$, then $c_0 = \sqrt{a/2}(w_+ + w_- - 2w_0)$.

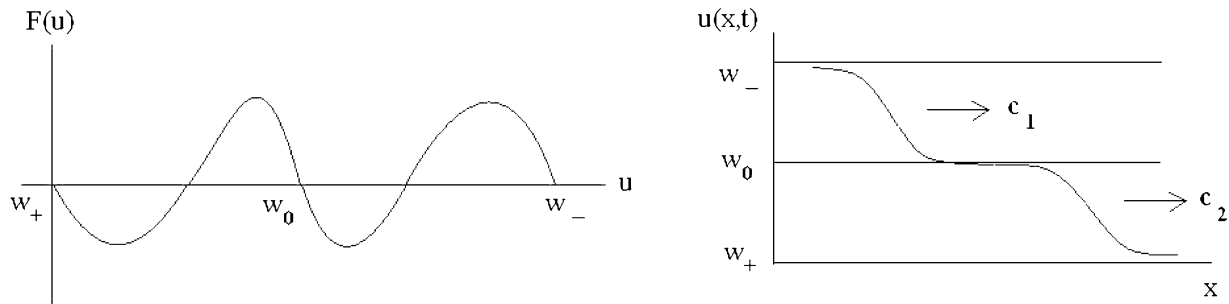


Fig. 2. Bistable nonlinearity with a stable intermediate zero w_0 (left). System of two waves, the speed c_2 of the lower wave is greater than the speed c_1 of the upper one (right).

first such representation was obtained in [53]. We will present here another representation which can be generalized for monotone systems (Section 1.3) [131].

In the bistable case the wave speed is unique. The following formula for the wave speed holds:

$$c = \inf_{\rho} \sup_x \frac{\rho'' + F(\rho)}{-\rho'} = \sup_{\rho} \inf_x \frac{\rho'' + F(\rho)}{-\rho'}. \tag{13}$$

The infimum in the first equality and the supremum in the second one is taken with respect to all monotonically decreasing functions ρ continuous together with their second derivatives and having the limits $\rho(\pm\infty) = w_{\pm}$. The proof of this formula is based on the global stability of travelling waves. In the monostable case, the waves are not unique. The first equality in (13) gives the value of the minimal speed.

1.2.5. Systems of waves

If the nonlinearity has a more complex form than in Fig. 1, then the wave with the limits (5) may not exist. Consider the function $F(u)$ shown in Fig. 2. According to the results presented above, there is a wave with the limits $w(+\infty) = w_+$, $w(-\infty) = w_0$ and another one with the limits $w(+\infty) = w_0$, $w(-\infty) = w_-$. Denote the speed of the former by c_2 and of the latter by c_1 . It is shown that the wave with the limits $w(\pm\infty) = w_{\pm}$ exists if $c_1 > c_2$ and does not exist otherwise. This result has a clear physical meaning: the second wave should move faster and merge with the first one in order to produce the wave with the limits (5). If $c_1 \leq c_2$, then the solution of Eq. (1) with the corresponding limits at infinity will converge to the structure shown in Fig. 2 and called system of waves or minimal decomposition [39]. Systems of waves exist for any nonlinearity of bistable, monostable or unstable types and they describe the asymptotic behavior of solutions for large time [131].

1.3. Reaction–diffusion systems

1.3.1. Monotone systems

The reaction–diffusion system

$$\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i}{\partial x^2} + F_i(u, \dots, u_n), \quad i = 1, \dots, n, \tag{14}$$

is called monotone system if the inequality

$$\frac{\partial F_i}{\partial u_j} > 0, \quad i, j = 1, \dots, n, \quad i \neq j, \tag{15}$$

is satisfied for all u . Such systems arise in numerous application in chemical kinetics and populations dynamics. Though the maximum principle is not in general applicable for systems of equations, it appears to be valid for monotone systems. Its applicability allows us to formulate the results on wave existence, stability and velocity similar to those for the scalar equation [129,131].

Theorem 2. Suppose that $F(w_{\pm}) = 0$, where w_+ and w_- are two constant vectors such that $w_+ < w_-$. (The inequality here is understood component-wise.) Let the matrices $F'(w_{\pm})$ have all eigenvalues in the left-half plane and for any

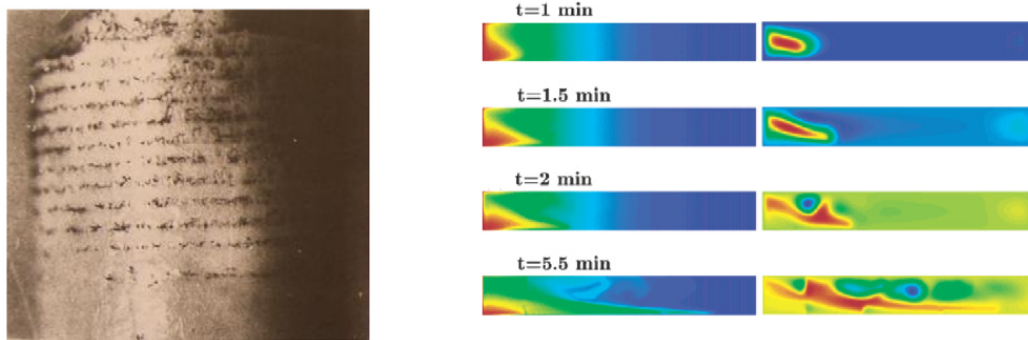


Fig. 3. Propagation of a polymerization front (left) [13]. High-temperature spots rotating near the surface leave spiral traces. Propagation of reaction front with convection (right) [14]. Level lines of the temperature distribution in consecutive moments of time (left column). The front propagates from the left to the right. High temperature zone (red) is located more at the bottom because the heavier reaction product descends. The stream function distribution in consecutive moments of time (right column). (For interpretation of colours in this figure, the reader is referred to the web version of this article.)

other zero w_0 of the function F such that $w_+ < w_0 < w_-$, the matrix $F'(w_0)$ has an eigenvalue with a positive real part. Then there exists a monotonically decreasing with respect to x travelling wave solution $w(x)$ of system (14) with the limits $w(\pm\infty) = w_{\pm}$. This wave is unique up to translation in space and asymptotically stable. Its speed admits a minimax representation similar to (13).

This theorem is applicable to the bistable case where the matrices $F'(w_{\pm})$ are stable. In the monostable case, where one of them has all eigenvalues in the left-half plane and the principle eigenvalue of another one is positive, monotone waves exist for all values of the velocity greater or equal to some minimal velocity c_0 . Similar to the scalar equation the minimal velocity admits a minimax representation. These results remain valid for multi-dimensional monotone systems in unbounded cylinders.

The wave existence can be generalized for locally monotone systems for which condition (15) is satisfied not everywhere but only at the zero surfaces of the functions F_i . Their stability may not take place in this case. There are some other reaction–diffusion systems for which wave existence is proved [131].

There are only few results concerning systems of wave for reaction–diffusion systems. They are studied for monotone systems [130] and for combustion waves [67]. Our conjecture is that systems of waves exist for general reaction–diffusion systems and describe the behavior of their solutions.

1.3.2. Nonlinear dynamics

Specific properties of monotone systems imply a relatively simple dynamics of their solutions. System (14) can have homogeneous in space stationary solutions, travelling waves and systems of waves which provide transitions between them. However, if condition (15) is not satisfied and, as a consequence, the maximum principle is not applicable, nonlinear dynamics of such systems can be much more complex and interesting. We briefly recall here some results on propagation of reaction fronts, excitable media and Turing structures. There is a vast literature devoted to these topics (see, e.g., [35,38,93,113,131,139,141] and the references therein).

Chemical kinetics and combustion. Stability and bifurcations of travelling waves are intensively studied in chemical kinetics and combustion. Loss of stability of a stationary reaction front can result in various one-dimensional and multi-dimensional modes of propagation. Among them are periodic and chaotic regimes, cellular flames and so on [81,87,120,121,131]. We mention as example spinning waves observed in the case of a circular cylinder [85]. The temperature distribution is characterized in this case by the presence of high-temperature spots which leave spiral traces at the surface of the cylinder (Fig. 3, left). Among numerous other studies of nonlinear dynamics of reaction–diffusion waves, let us mention propagation of reaction fronts in fluids. In this case, the action of gravity can lead to various phenomena related to convective motion [14,88]. Fig. 3 (right) shows an example of numerical simulations of a reaction front propagating in the horizontal direction.

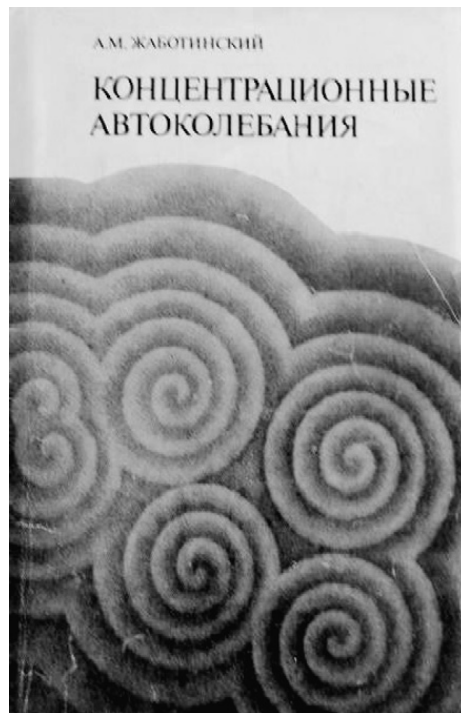


Fig. 4. The cover page of the original edition of the book “Concentrational auto-oscillations” by A. Zhabotinskii [141]. The image shows spiral waves in BZ reactions.

Excitable media. Oscillating chemical reactions discovered in the 1950s and named later Belousov–Zhabotinskii (BZ) reactions initiated investigation of reaction–diffusion waves in excitable media. Contrary to the systems with the periodic kinetics where a positive steady state is surrounded by a stable limit cycle, in an excitable system the steady state is stable locally but not globally. A sufficiently large ‘supercritical’ perturbation would therefore push the system out of equilibrium. The resulting system’s trajectory is a closed loop that would finally bring the system back to the steady state but not until after a long excursion over the phase space. The system would then rest in the equilibrium until another supercritical perturbation push it away, etc.

When considered in space and time, among various patterns and waves observed in excitable media there are famous spiral waves (Fig. 4). Similar waves were found later in other systems. In heart tissue they are related to fibrillation and can have dangerous consequences from the medical standpoint. Neural activity represent another important example of excitable media. Propagation of nerve pulses is intensively studied after the classical works by FitzHugh, Nagumo and Hodgkin, Huxley. These questions are beyond the scope of this paper and we will not return to them anymore.

Turing structures. As we discussed above, reaction–diffusion waves provide transition between homogeneous in space stationary solutions. If instead of homogeneous solutions we consider some other stationary structures, then transition between them can result in the appearance of generalized travelling waves. We discuss them in more detail in Section 4. Historically, the first mechanism which provides the appearance of spatial structures in reaction–diffusion systems was suggested by A. Turing [125]. Experimentally they were found in the 1980s [58]. In biological modelling Turing structures are widely used to describe animal skin patterns [93], sea shell ornament [90] and in other problems of morphogenesis [54,97,112].

Fig. 5 shows an example of such structures for the following system [46]:

$$\frac{\partial A}{\partial t} = d_A \Delta A + \alpha A^2(k - A) - \beta AB + \epsilon, \quad (16)$$

$$\frac{\partial B}{\partial t} = d_B \Delta B + \gamma A - \delta B. \quad (17)$$

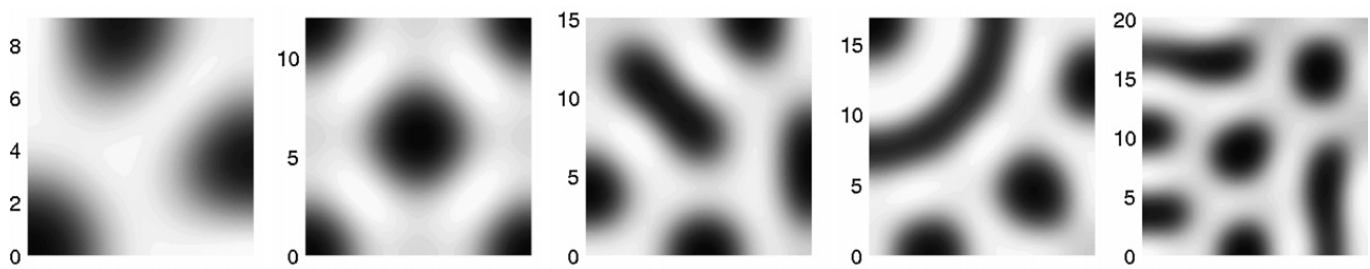


Fig. 5. Turing structure result from the instability of the homogeneous in space equilibrium. Stationary spatial structure for system (16), (17) for different sizes of the square domain.

It has a stationary solution stable with respect to the kinetic system (without diffusion), which can be unstable as a solution of system (16), (17). If it is unstable, then some spatial structure will emerge. These are Turing structures. For fixed values of the parameters, their shape and number may depend on the size of the square domain.

In a more general setting, dissipative structures are spatially inhomogeneous stationary structures which appear from a homogeneous in space equilibrium stable without diffusion but unstable with diffusion. In spite of numerous works devoted to them, many questions still wait their investigation. Transitions between homogeneous equilibria and spatial structures are not yet sufficiently well studied. In particular, a problem that has been attracting a considerable attention recently is the Turing instability and the consequent pattern formation in a growing domain; e.g. see [24,91]. Understanding of this issue is obviously very important for the understanding of the mechanisms of morphogenesis. Another question which requires further investigation concerns their interaction with mechanical phenomena, e.g., with natural convection [46]. One of the most challenging questions in application of Turing structures to biological pattern formation is to link them to realistic molecular mechanisms which determine the underlying biochemical processes.

2. Wave propagation in population dynamics

2.1. Single-species models

A population is a community of individuals of the same species living in the same area and sharing the same resources. In the case that the impact of space on the population structure and functioning can be neglected, e.g. when the inhabited area is relatively small and/or the individuals are distributed more or less homogeneously over it, the population can be described just by its ‘size’ U , i.e. the total number of the individuals. The population size can change in time, e.g. when the processes of birth and death are unbalanced, and hence U is a dynamical variable, $U = U(t)$.

The equations describing how the population size changes in time can be different depending on the biological/environmental conditions and the species traits. For populations with non-overlapping generations, a natural mathematical technique is a difference equation:

$$U(t + T) = F(U(t)) \quad (18)$$

(e.g. see [93]) where $U(t)$ and $U(t + T)$ are the population sizes in the previous and next generations, T is the time between the two subsequent generations and F is a function describing multiplication. Species with non-overlapping generations can be found in insects and plants.

In most cases, however, generations do overlap and then the population size can usually be regarded as a continuous function of time and an adequate mathematical tool is the ordinary differential equations:

$$\frac{dU(t)}{dt} = F(U), \quad (19)$$

where a generic form of function F is given by (3).

When space is included – i.e. when the heterogeneity of the spatial distribution of given population(s) cannot be neglected – the population size is not sufficiently informative any more and one must use the population density $u(\mathbf{r}, t)$ instead. Redistribution of the density in space takes place due to the movement of individuals. In case this movement can, on a certain spatial and temporal scale, be approximated as a random walk, then the dynamics of the population is

described by the diffusion equation [94,95,122]. Combining the diffusion with the population multiplication, instead of the non-spatial model (19) we now have a reaction–diffusion equation:

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = D\Delta u + F(u), \tag{20}$$

where D is the diffusion coefficient. In 1D case, it obviously coincides with (1) and hence its basic properties can be found in Section 1.2

One of the problems in spatial ecology where application of reaction–diffusion waves theory has been especially successful is biological invasion. Consider a situation when a number of organisms of an alien species² is brought, accidentally or deliberately into a given ecosystem. Mathematically, it means that the initial condition to Eq. (20) is described by a function of a finite support. In the large-time limit, it converges to a unique travelling front propagating with a speed c_0 . In particular, in the case of logistic growth $F(u) = \alpha u(1 - u/K)$, the value of the speed is $c_0 = 2\sqrt{\alpha D}$. Remarkably, prediction of this very schematic model of population dynamics appears to be in a good quantitative agreement with many field observations [117].

It should be mentioned here that, while application of reaction–diffusion equations to, for instance, chemical kinetics can often be reduced to a 1D case, the population dynamics typically takes place in two (sometimes, three) spatial dimensions (but see [79]). The spread of the population takes place through the growth of the invaded area or ‘patch.’ The dynamics of a reaction–diffusion system then brings the effects of curvature that makes the propagation of the front somewhat different from the planar case. It is intuitively clear that, when the radius of the patch becomes very large (i.e. for a sufficiently large time), any segment of the propagating patch boundary is close to a planar front; however, it is not immediately clear what happens at early stages of the species spread when the radius is not very large.

Here we focus on a 2D case. In order to make an insight into the impact of curvature, let us consider Eq. (20) in polar coordinates (ρ, ϕ) :

$$\frac{\partial u(\rho, \phi, t)}{\partial t} = D\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2}\right) + F(u). \tag{21}$$

For simplicity, we assume that the initial conditions do not depend on ϕ . The problem then possesses complete rotational symmetry, the population density depends only on ρ and t , and hence Eq. (21) takes a simpler form:

$$\frac{\partial u(\rho, t)}{\partial t} = D\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho}\right) + F(u). \tag{22}$$

Furthermore, we assume that the front width is small, which means that the gradient $\partial u/\partial \rho$ is large inside a front but approach zero promptly outside of the front.³ It means that the term $(1/\rho)\partial u/\partial \rho$ is non-vanishing only in a small vicinity of the front position. Then in Eq. (22) we can change ρ to R where $R = R(t)$ is the position of the front:

$$\frac{\partial u(\rho, t)}{\partial t} = D\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{R(t)} \frac{\partial u}{\partial \rho}\right) + F(u). \tag{23}$$

In order to reveal the relation to the travelling waves, we now apply the standard change of coordinate, $(\rho, t) \rightarrow \xi = \rho - c(R)t$ where $c(R)$ is the speed of the curvilinear ‘cylindrical’ front with radius R , so that Eq. (23) turns into

$$D \frac{d^2 u}{d\xi^2} + \left(c(R) + \frac{D}{R(t)}\right) \frac{du}{d\xi} + F(u) = 0. \tag{24}$$

Eq. (24) coincides with Eq. (2) for the planar case if the speeds of the propagation are related as $c = c(R) + D/R(t)$, so that

$$c(R) = c - \frac{D}{R(t)}. \tag{25}$$

² I.e. species that was not present in a given ecosystem until it is introduced.

³ Which assumes that the population density behind the front, i.e. in the middle part of the patch, has converged to the upper stable steady state K .

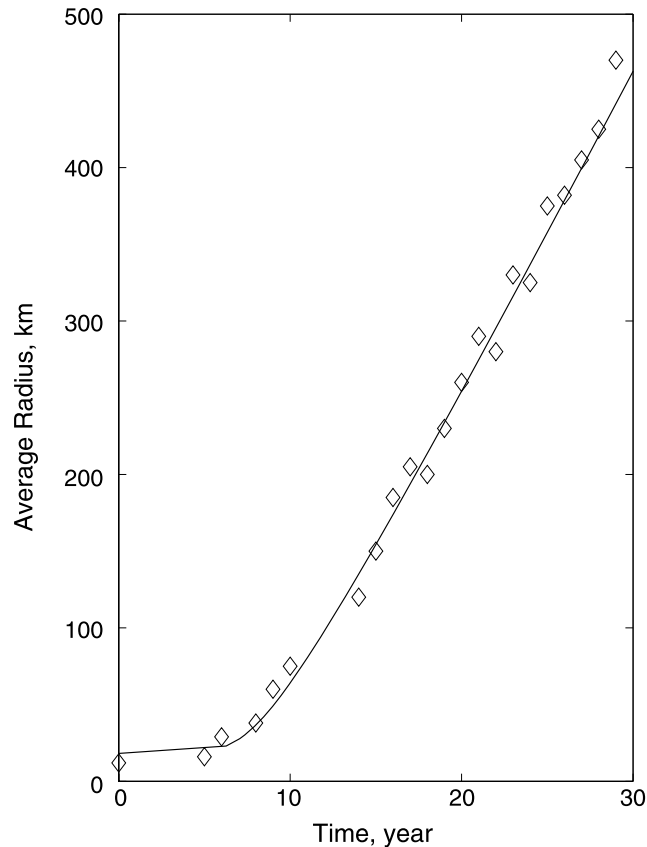


Fig. 6. Invasion front acceleration due to the effects of front’s curvature. Solid curve shows the position of the front predicted by Eqs. (25)–(26), the diamonds show the field data on the house finch spread in the USA. The data are taken from Shigesada and Kawasaki [117].

Eq. (25) thus predicts accelerating phase of the species spread from a finite support introduction. Taking into account that $c(R) = dR/dt$, Eq. (25) can be easily integrated resulting in

$$t = \frac{1}{c} \left[(R - R_0) + \frac{D}{c} \log \left(\frac{cR - D}{cR_0 - D} \right) \right], \tag{26}$$

where $R_0 = R(0)$ is the radius of the initially invaded area. Eq. (26) is valid for $R_0 > D/c$.

Interestingly, the data of field observations on invasive species often show the existence of an accelerating phase [117]. One example is shown in Fig. 6 where diamonds show an average position of the front of an invasive species as observed in the field. The solid curve shows $R(t)$ as given by (26). We want to emphasize that, although exact values of D and R_0 are not available, the very good agreement between the theory and data shown in Fig. 6 is obtained for hypothetical but biologically realistic parameter values.

Note that Eq. (25) seemingly predicts that the sign of the speed of the cylindrical front can become negative, which would mean that the patch shrinks rather than grows. This is, however, cannot be true. Indeed, for a population with logistic growth, it is straightforward to see that the total mass of the population cannot decrease (at least in case $0 < \max u(\rho, 0) < K$), and it is always bounded from zero. The matter is that Eq. (25) is only valid when the width of the front is much less than the radius R of the patch, which results in the term D/R being sufficiently small. The situation when the initial patch shrinks can indeed be observed when the population dynamics is hampered by a strong Allee effect (cf. “the problem of critical size” [101]) but this is a different mechanism that is not related to the effects of curvature.

2.1.1. Kernel-based models

Space- and time-continuous reaction–diffusion models of biological invasion therefore work very well for the populations with overlapping generations; however, they obviously do not apply to the case of non-overlapping generations, cf. Eq. (18). In order to describe species spread in a time-discrete system, another approach has been developed

[71,72]. It is based on the biological observation that, for many species (and especially for those with non-overlapping generations), the stage of dispersal and the stage of reproduction take place at distinctly different time.

Let the species first reproduce and then disperse. If the population density at the generation t (before reproduction) is $u(x, t)$, then after the reproduction it becomes $\tilde{u}(x, t) = F(u(x, t))$.

To take into account the dispersal stage, assume that we know the probability density $K(x, y)$ to find an offspring (e.g. a seed) that was released at a position y . Then the population density after dispersal is given as

$$u(x, t + T) = \int_{-\infty}^{\infty} K(x, y)\tilde{u}(y, t) dy, \tag{27}$$

where the integration is taken over all available space which we, for convenience, consider to be unbounded.

In case space is homogeneous and isotropic, the probability density K depends on the distance between x and y rather than on each of them separately. Then, combining it with the reproduction stage, from Eq. (27) we obtain:

$$u(x, t + T) = \int_{-\infty}^{\infty} K(x - y)F(u(y, t)) dy, \tag{28}$$

where $K(z) = K(-z)$.

Although Eq. (28) is obviously not a reaction–diffusion equation, it has a very similar meaning except for, instead of diffusion which is a local process, we now have dispersal that takes place non-locally. One then can expect that some properties of (28) may be similar to those of a reaction–diffusion equation. Indeed, Eq. (28) appears to have a stationary travelling front (i.e., a front with a profile of a constant shape) as its basic solution.

A stationary wave traveling with a speed c is a solution which exhibits invariance to translation, i.e., in the case of the time-discrete equation (28), possesses the following property:

$$u(x, t + T) = u(x - cT, t). \tag{29}$$

For the sake of simplicity, in the below we restrict our analysis to the travelling edge where the population density is small and therefore the growth function can be linearized, $F(u) \approx R_0u$ where $R_0 = F'(0)$. Eq. (28) then becomes linear,

$$u(x, t + T) = R_0 \int_{-\infty}^{\infty} K(x - y)u(y, t) dy, \tag{30}$$

and we can look for its solution as an exponential function, i.e.,

$$u(x, t) = \text{const} \cdot e^{-s(x-ct)}, \tag{31}$$

where $s > 0$ (assuming that the invasive species spreads from left to right) is a parameter giving the slope of the front.

Having substituted (31) into (30), we obtain:

$$e^{-sx} e^{scT} = R_0 \int_{-\infty}^{\infty} K(x - y)e^{-sy} dy. \tag{32}$$

Introducing $z = x - y$, we arrive at

$$e^{scT} = R_0 M(s) \quad \text{where } M(s) = \int_{-\infty}^{\infty} K(z)e^{sz} dz. \tag{33}$$

From (33), we immediately obtain the equation for the speed of the population wave:

$$c(s) = \frac{1}{sT} \ln[R_0 M(s)]. \tag{34}$$

It is readily seen that Eq. (34) has multiple solutions corresponding to different s . Which value of speed is actually ‘chosen’ by the travelling wave depends on the initial spatial distribution of species density, in particular, on its asymptotics for large x . However, biological invasion usually starts with species local introduction so that before starting its geographical spread the exotic species is present only inside a certain area. Correspondingly, relevant initial conditions are described by functions of compact support. In this case, it can be proved [135] that the wave propagates with the minimum value given by Eq. (34). Thus, the relevant value of the invasion speed is given by the following equation:

$$c_0 = \min_s \frac{1}{sT} \ln[R_0 M(s)]. \quad (35)$$

The above analysis was based on the assumption that the moment-generating function $M(s)$ exists, at least, for certain positive values of s . That implies that, for large z , $K(z)$ decays exponentially or faster. In this case, it can be proved that, in the continuity limit (i.e. on a large time scale $t \gg T$) the integral-difference kernel-based equation (28) appears to be in some sense equivalent to a reaction–diffusion equation, e.g. giving the same speed of the front propagation [101]. However, in the case of fat-tailed kernels when $M(s)$ is infinite, Eq. (28) is not immediately reducible to a standard reaction–diffusion equation and possesses somewhat different properties. In particular, a more refined analysis [72] predicts that the invasive species can spread with increasing speed.⁴

2.2. Multi-species models

In a natural environment, populations of different species are not isolated from each other but, on the contrary, are involved in various inter-species interactions. A crude way to take the impact of other species into account is to adjust parameter values in Eq. (19), such as the birth and death rates contained in function F . However, this leaves completely out of scope the variations in the other species’ population size. A more consistent way to describe a community of interacting species is therefore a system of equations:

$$\frac{dU_i}{dt} = F_i(U_1, \dots, U_n), \quad i = 1, \dots, n, \quad (36)$$

where $U_i(t)$ is the population size of the i th species and n is the total number of species taken into account by the model.

Having included redistribution of population densities due to the random motion of individuals, system of ODEs (36) turns into a system of reaction–diffusion equations:

$$\frac{\partial u_i(\mathbf{r}, t)}{\partial t} = D_i \Delta u_i + F_i(u_1, \dots, u_n), \quad i = 1, \dots, n, \quad (37)$$

where D_i is the diffusivity of the i th species. Obviously, in 1D case system (37) falls into the general case considered in Section 1.3

When applying generic system (37) to a specific real-world problem, the choice of n is a subtle issue. Earlier modelling studies, especially those that were inspired by field data obtained for a specific community, e.g. a plankton system, tended to include as many species as possible in order to make the model “biologically more realistic.” This approach often ends up with dozens or even hundreds of equations. A trouble is that a model like that contain a very large number of parameters, their values are often just roughly estimated. The problem is that, due to the effects of nonlinearity, a multi-component model can have a complicated bifurcation structure, so that the prediction may vary significantly depending on the parameter values used.

Instead, a current tendency in theoretical ecology is to use conceptual few-component models in order to reveal the main driving mechanisms behind a given phenomenon and identify the trends in the system behaviour with changes in some controlling parameters. The most common examples are given by two-component prey–predator systems (with different types of the growth rates and predation), two- and three-species competition models, etc. Although the predictive power of these models is not high, they have proved to be very helpful for understanding the ecosystems dynamics. Below we revisit a few cases that have become paradigms of the population dynamics in space and time.

⁴ Accelerating travelling fronts can be observed in a modified reaction–diffusion equation either by introducing a scale-dependence into the diffusion coefficient [100] or by assuming singularity in growth function $F(u)$ at $u = 0$ [103].

2.2.1. Prey and predator

In this section, we consider the case when the species spread is affected by the direct trophic interaction with another species, i.e. through prey–predator interaction. A generic prey–predator model is given by the following equations:

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = D_u \Delta u(\mathbf{r}, t) + F(u) - E(u, v), \quad (38)$$

$$\frac{\partial v(\mathbf{r}, t)}{\partial t} = D_v \Delta v(\mathbf{r}, t) + \kappa E(u, v) - M(v), \quad (39)$$

where u and v are the population densities of prey and predator, respectively, $F(u)$ is the prey population growth, E describes predation, $M(v)$ is the predator mortality, and the coefficient κ is called the predation efficiency or conversion rate.

Depending on the biological traits of the involved species as well as on the goals of the study, functions F , E and M can possess different properties. A rather general parametrization for F is given by Eq. (3). The simplest yet biologically relevant choice for predator mortality is a linear function, $M(u) = \mu u$. As for predation, the simplest case is given by a bilinear function,

$$E(u, v) = Auv, \quad (40)$$

which is sometimes referred to as Holling type I predator response. A more general case, which is also more realistic from the biological point of view, is given by the following formula:

$$E(u, v) = \frac{Au^{\nu}v}{B^{\nu} + u^{\nu}}, \quad (41)$$

where A , B and ν are parameters. Note that (41) is linear with respect to v . The particular cases of $\nu = 1$ and $\nu = 2$ are known in population dynamics as Holling type II and III, respectively. There can also be more complicated responses (e.g. the ratio-dependent Holling type IV which is nonlinear with respect to both u and v) that will not be considered here.

We begin with 1D case. Numerical simulations show that, depending on the form of the nonlinear responses as well as on parameter values, system (38)–(39) exhibits a rich variety of dynamical regimes [83]. In particular, there can be a variety of travelling fronts.

The properties of the waves depend to a large extent on the properties of the local kinetics such as the type of the coexistence steady state. Although in general there is no one-to-one correspondence between the properties of the corresponding non-spatial system (i.e. system (38)–(39) where the solutions do not depend on space) and those of the spatial one, the properties of the non-spatial system provide a certain ‘skeleton’ that appears to be useful for understanding the richer dynamics of a heterogeneous system.

Another important factor is the type of the initial conditions. Consider the case when a predator invades into the space which is inhabited by a prey, i.e. the initial conditions to system (38)–(39) are given as

$$u(x, 0) = K \quad \forall x, \quad v(x, 0) = \psi(x), \quad (42)$$

where $u = K > 0$ is a stable steady state of the prey population in the absence of predator and $\psi(x)$ is a certain function of finite support.

In case the prey growth rate $F(u)$ is of the type shown in Fig. 1b, i.e. it is positive between $u = 0$ and $u = K$ and negative otherwise, the initial conditions evolve to two travelling waves propagating in the opposite directions from the initially inhabited domain. Then there can be two different patterns; see Fig. 7 where only one-half of the domain is shown (i.e. for $x \geq 0$).

Note that, since function E is assumed to be linear with respect to v , the non-spatial system possesses exactly one positive state (\bar{u}, \bar{v}) for any $0 < \bar{u} < K$. (For $\bar{u} > K$, predator is not viable and goes extinct quickly.) This state can be either stable or unstable depending whether it lies on the right or on the left of the hump. In case it is stable, then the travelling wave has a form of the front connecting the prey-only state $(K, 0)$ (which is stable without predator but becomes unstable in the two-dimensional phase space) to (\bar{u}, \bar{v}) . The profile can be either monotonous or having promptly decaying oscillating at the front (cf. Fig. 7, left) depending on whether (\bar{u}, \bar{v}) is a stable node or focus, respectively.

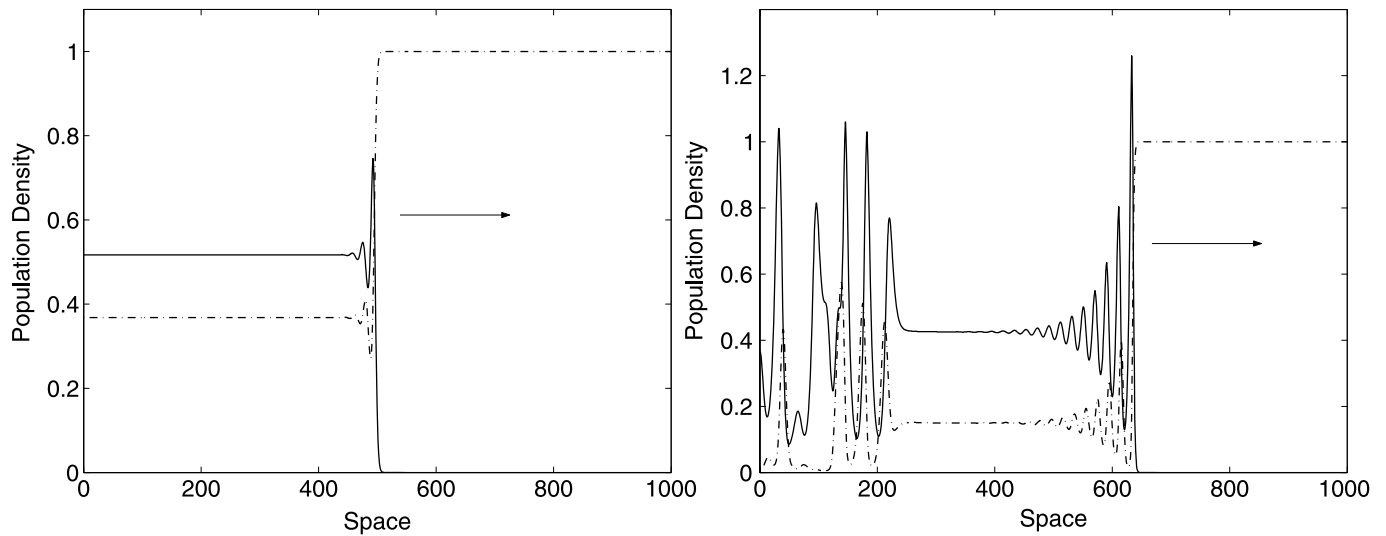


Fig. 7. Travelling front propagating from the place of the predator introduction (around the origin, not shown) into the space inhabited by prey, cf. (42). Left, the coexistence steady state is a stable focus; right, the local kinetics is oscillatory.

However, in case (\bar{u}, \bar{v}) loses its stability, usually through the Hopf bifurcation (which happens when it passes the hump from right to left), kinetics becomes oscillatory and the properties of the travelling wave changes dramatically. The oscillations at the front now decay more slowly. Eventually, at some distance behind the front they converge to a (quasi-)homogeneous spatial profile where $u(x, t) = \bar{u}$ and $v(x, t) = \bar{v}$. Remarkably, the coexistence state is now unstable, so this is convergence to an unstable equilibrium, which would of course be impossible in a non-spatial system. This phenomenon of a “dynamical stabilization” of an unstable steady state was studied in [82,102,106]; see also [83] for more details and further references. The unstable plateau can reach a considerable length but it appears to be bounded at the left end where it gives way to irregular spatiotemporal oscillations. These oscillations were proved to be chaotic [115,116] and are sometimes referred to as “biological turbulence” [83]. Note that, contrary to the case of periodic environmental forcing, e.g. see [11], the spatiotemporal oscillations shown in Fig. 7 are self-organized.

Note that the speed of the front can be obtained easily from heuristic arguments. Consider the travelling front at its leading edge, i.e. in the area where $v \approx 0$ and $u \approx K$. Eq. (39) then can be linearized resulting in

$$\frac{\partial v(x, t)}{\partial t} = D_2 \frac{\partial^2 v(x, t)}{\partial x^2} + \kappa \left(\frac{\partial E(K, 0)}{\partial v} \right) v - \mu v, \quad (43)$$

where $E(u, v)$ can be, for instance, either (40) or (41).

Eq. (43) coincides with the KPP–Fisher equation linearized at the leading edge and hence the speed of the travelling front is

$$c = 2 \left[D_v \left(\kappa \frac{\partial E(K, 0)}{\partial v} - \mu \right) \right]^{1/2}. \quad (44)$$

Therefore, the traveling front solution can exist only if

$$\kappa \frac{\partial E(K, 0)}{\partial v} > \mu. \quad (45)$$

Since the left-hand side of (45) is readily interpreted as the predator maximum reproduction rate, condition (45) has a clear biological meaning that the invasive predatory species must be viable, i.e. its reproduction rate must be greater than its mortality.

The system’s dynamics become slightly different if the initial conditions to Eqs. (38)–(39) are finite for both species,

$$u(x, 0) = \chi(x) \quad \text{and} \quad v(x, 0) = \psi(x), \quad (46)$$

where χ and ψ are certain function of finite support. Biologically, these initial conditions may correspond to biological control when a predatory species is deliberately introduced in the wake of the spreading pest with a hope that it will

slow down or stop the spread of the harmful invader. In this case, generally speaking there is a succession of travelling fronts; see Fig. 8. In case the front of the prey population is faster than the front of predator, there is a growing domain which is inhabited only by prey. The speed of the both waves can be readily calculated. At the position of the prey front, predator is absent and hence this is a problem effectively described by a single KPP–Fisher equation, its speed therefore being given as $c_0 = 2\sqrt{D_u F'(0)}$. In its turn, the predator front propagates into the area already inhabited by prey and hence its speed c is given by (44). What happens behind the front of predator depend on whether (\bar{u}, \bar{v}) is stable or unstable; correspondingly it can be either a stable homogeneous spatial distribution (Fig. 8, top) or spatiotemporal chaos (Fig. 8, middle).

For some parameter values, the predator front appears to be faster, it catches up with the prey front and then there is a single travelling interface separating the domain invaded by both species from the domain where both of them are still absent (Fig. 8, bottom). Interestingly, an insight into this matter shows that, in this system, the predator front can only catch up with the prey when the local kinetics is oscillatory; in case (\bar{u}, \bar{v}) is stable, the prey front is always faster [102].

The dynamics becomes richer if the prey growth is damped by the strong Allee effect, i.e. when the growth rate becomes negative for sufficiently small prey density, cf. Fig. 1a. The waves shown in Figs. 7 and 8 still can be observed but they are now complemented by some new phenomena. A very detailed numerical investigation of this issue in relation to the problem of biological control, i.e. with the initial conditions (46), can be found in [104]. Here we give just one example. Let $u = \beta_A$ be the threshold value of the population density so that the prey growth rate is negative for $0 < u < \beta_A$. For the parameter values when $\beta_A < \bar{u} < \hat{u} < u_0$ where u_0 is the position of the maximum of $F(u)$ and \hat{u} is a certain number, the limit cycle disappears giving way to a heteroclinic connection between (\bar{u}, \bar{v}) and $(\beta_A, 0)$ [83]. The only stable attractor of the corresponding non-spatial system is $(0, 0)$. The system's kinetics now becomes excitable, i.e. its actual behaviour depends significantly on the initial values of the population densities. In the diffusion–reaction system (38)–(39), the evolution of finite initial conditions (Fig. 9, top, left) lead to formation of a travelling solitary patch or pulse (Fig. 9, bottom, left). Note that the population densities are now vanishing both in front of and behind the wave.⁵ Biologically, it means that the species spread does not result in invasion and hence this situation can probably be regarded as a success of the biological control strategy.

Interestingly, in this parameter range the system appears to be sensitive to the initial conditions, so that, for the same parameter values, slightly different initial species distribution (cf. Fig. 9, top, left and right) results in a different system's dynamics. Now, in the wake of the travelling pulse a complicated irregular spatiotemporal pattern is formed. Long-term numerical simulations show that this pattern is self-sustained and persistent so that, in this case, biological control fails and the spread of prey does result in its invasion over the whole space.

Predator and prey in 2D space. An interesting and practically important question is how the travelling wave scenarios described above can look in case of two spatial dimensions. Indeed, in reality, biological invasions normally take place on the surface rather than along a line. Numerical simulations show that, in the case prey growth is not hampered by an Allee effect, there is one-to-one correspondence between the patterns of spread observed in 1D and 2D cases,⁶ although the corresponding parameter ranges can be slightly different. An example is given in Fig. 10 where the 3D plot shows what happens in the case of predator invading into the space already inhabited by prey if their coexistence state is monotonously stable. In case it is unstable and the local kinetics is oscillatory, chaotic spatiotemporal patterns are formed⁷ behind the propagating circular-shaped boundary; see the upper row of Fig. 11 where the white colour corresponds to the predator absence and the red colour to its presence at a high density. For other parameter values, the onset of chaos may be preceded by formation of an unstable plateau due to the dynamical stabilization effect, cf. Fig. 8, middle.

In case the prey growth is damped by the Allee effect, the population waves observed in the 1D system have their 2D analogues as well. In particular, chaos in the wake can be observed (cf. Fig. 11, top) if the local kinetics is determined by a stable limit cycle. The travelling pulses turn into growing rings; see Fig. 11, bottom. However, we

⁵ From the mathematical perspective, it means that, in the four-dimensional phase space of system (38)–(39), this profile corresponds to a heteroclinic loop starting in the origin.

⁶ Up to the effects of curvature that can make the rate of spread somewhat lower in two dimensions, see Eq. (25).

⁷ In some cases, the patterns in the wake can be periodical [114]; however, they appear to be unstable to a sufficiently large perturbation and would normally give way to chaos.

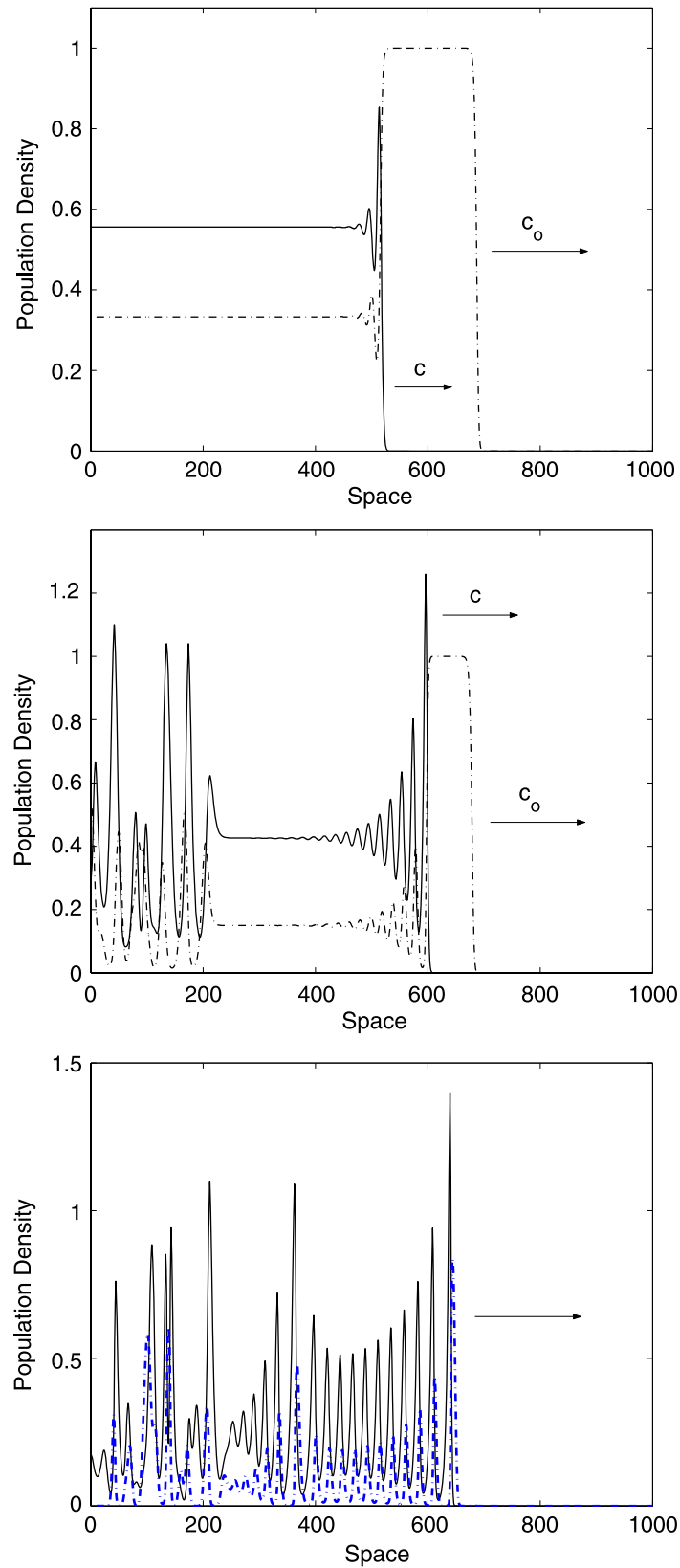


Fig. 8. Different types of travelling fronts propagating from the place of the simultaneous introduction of both species (around the origin, not shown) into an empty space, cf. (46).

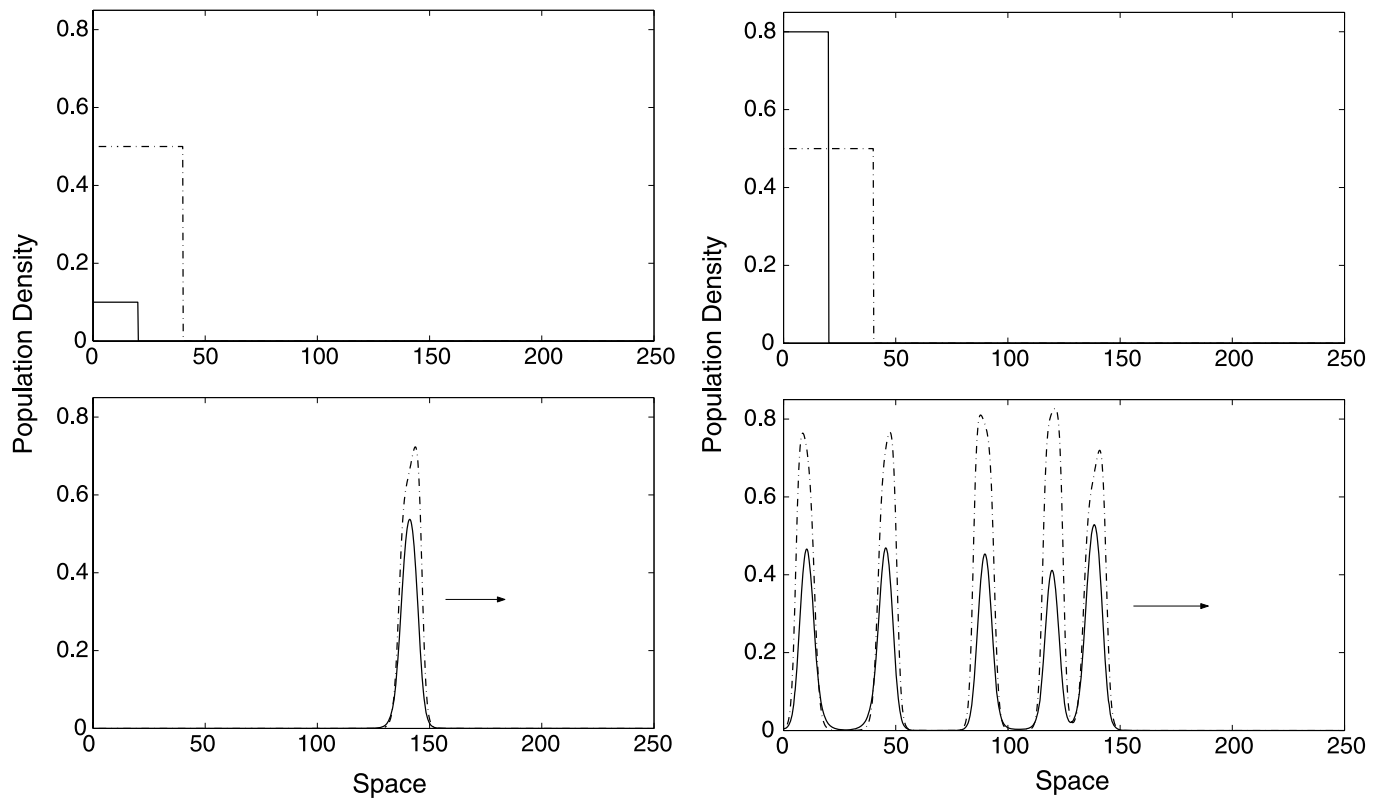


Fig. 9. Travelling waves in a predator–prey system with a strong Allee effect: evolution of somewhat different initial conditions (top row) results in an essentially different system’s dynamics (bottom row); all other parameters are the same.

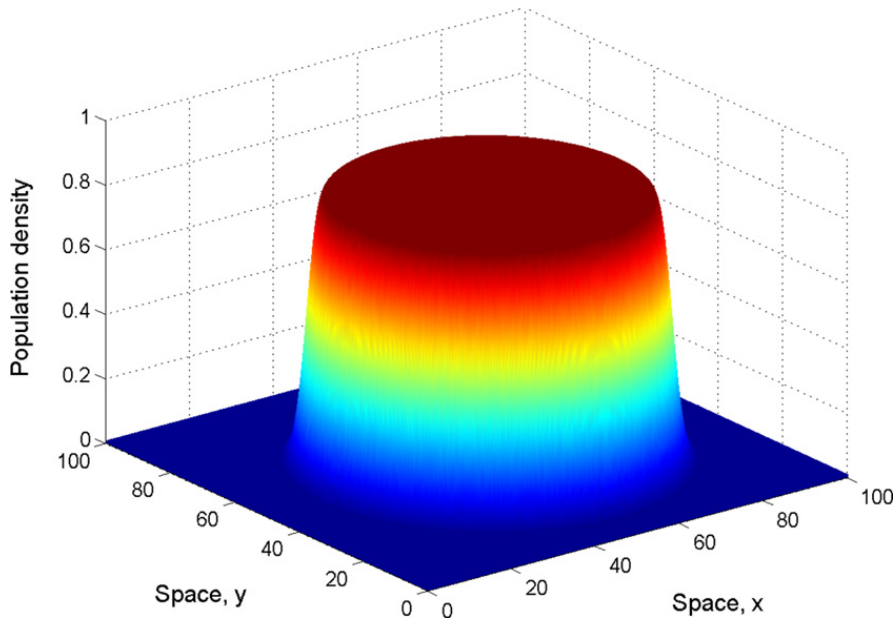


Fig. 10. 3D graph of a growing patch of an invasive predator. Behind the propagating front (i.e. in the middle of the patch) a steady stable spatially homogeneous distribution of the population density is formed with $v(x, t) = \bar{v}$.

want to emphasize that the dynamics of a 2D predator–prey system with the Allee effect attains some new features which cannot be observed in a 1D case. We will discuss it in Section 4.3.

2.2.2. Competition of species

Another type of inter-species interaction is given by competition. For instance, species may have to share some resources and that can affect their reproduction rate. The simplest yet instructive case is given by a system of just

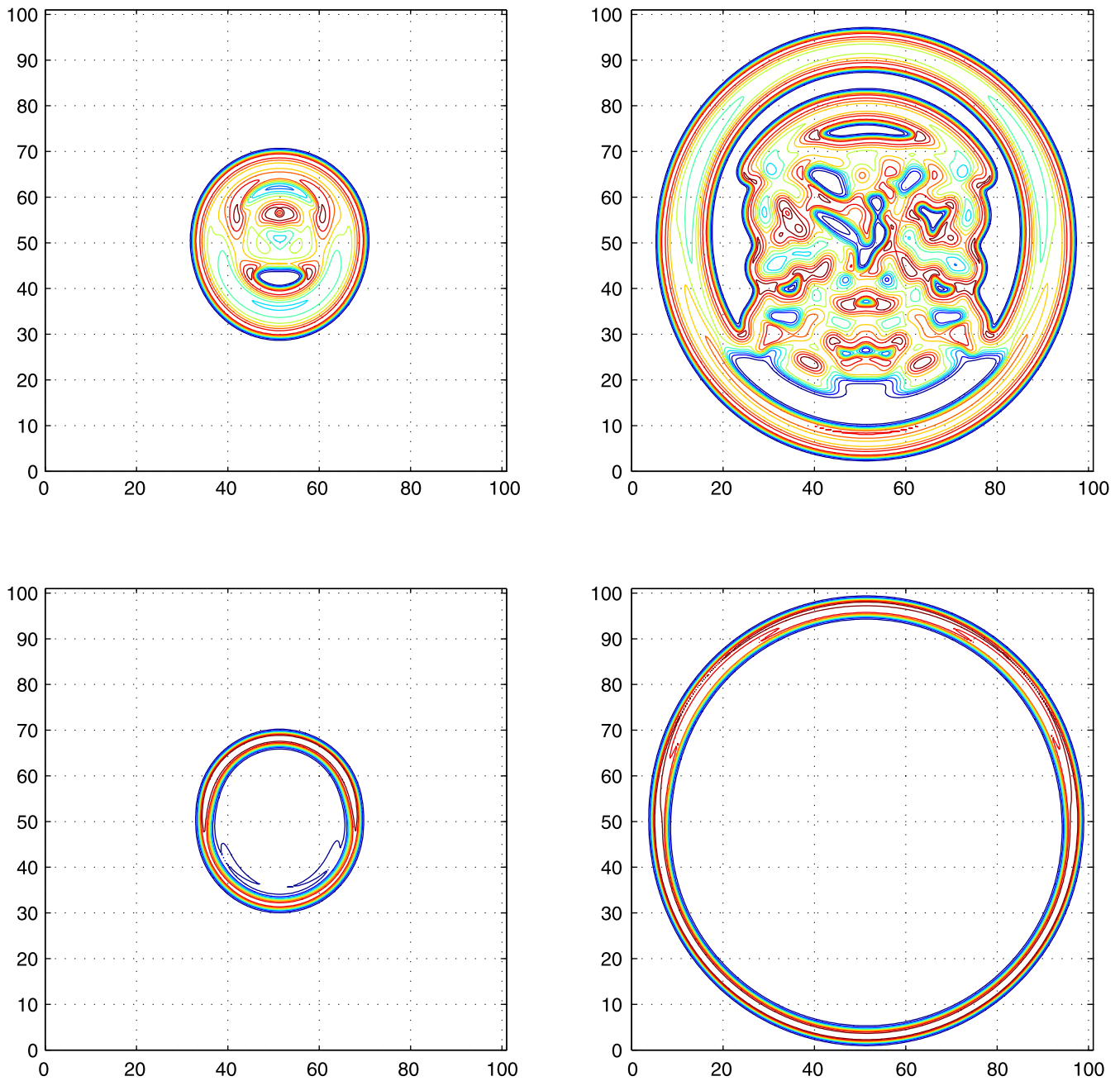


Fig. 11. Travelling waves of predator invasion in a 2D space inhabited by its prey; only the distribution of the predator density is shown. (For interpretation of colours in this figure, the reader is referred to the web version of this article.)

two competing species. Mathematically, this system can be described by two coupled reaction–diffusion equations; however, contrary to (38)–(39) the interaction term now has the same sign in both equations because competition is mutually damaging, even that some species may be stronger competitors than others. Specific parametrization of the functional responses can be somewhat different; here we consider it with a logistic growth for both species and the mass-action law for the competition terms:

$$\frac{\partial u_1(x, t)}{\partial t} = D_1 \frac{\partial^2 u_1(x, t)}{\partial x^2} + a_1 u_1 (1 - b_{11} u_1 - b_{12} u_2), \quad (47)$$

$$\frac{\partial u_2(x, t)}{\partial t} = D_2 \frac{\partial^2 u_2(x, t)}{\partial x^2} + a_2 u_2 (1 - b_{21} u_1 - b_{22} u_2), \quad (48)$$

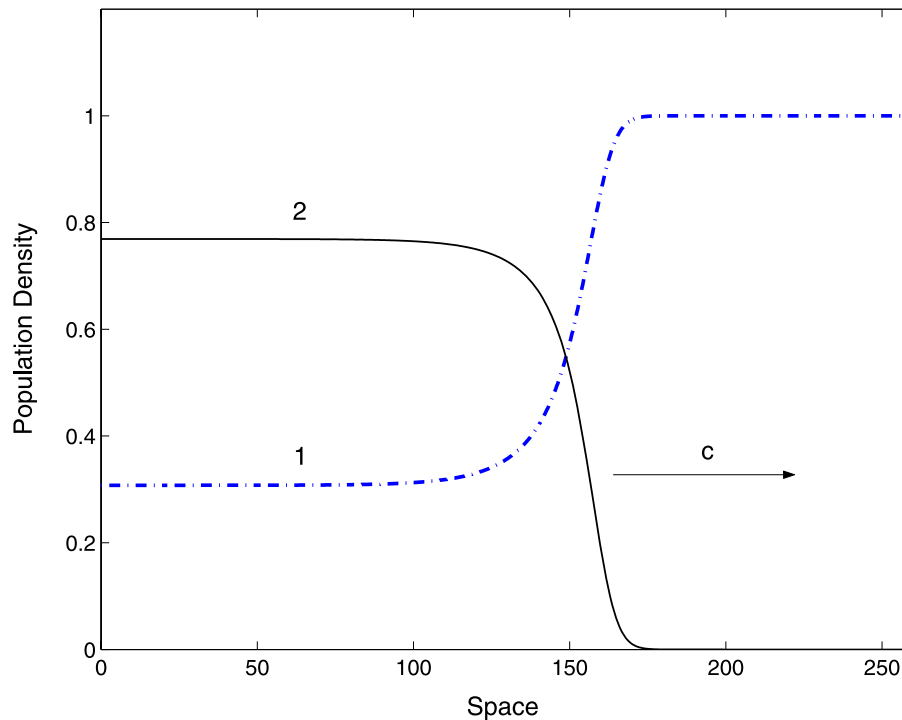


Fig. 12. Travelling waves of competing species, cf. Eqs. (47)–(48); blue dashed-and-dotted curve is for retreating species u_1 , black solid curve is for invading species u_2 . (For interpretation of colours in this figure, the reader is referred to the web version of this article.)

where parameters b_{ij} quantifies the intensity of inter-species (for $i \neq j$) and intra-species (for $i = j$) competition. Here we restrict our attention to a 1D case. By a change of variables, this system of equations can be reduced to a monotone system (Section 1.3.1).

It is readily seen [70] that the corresponding non-spatial system always have three boundary states such as extinction $(0, 0)$ and two single species states $(1, 0)$ and $(0, 1)$. There also can be one positive state (\bar{u}_1, \bar{u}_2) which can be either monotonously stable (node) or unstable (a saddle), and there can be no limit cycles. In case (\bar{u}_1, \bar{u}_2) is a saddle then both $(1, 0)$ and $(0, 1)$ are stable, and in case (\bar{u}_1, \bar{u}_2) is a stable node then both $(1, 0)$ and $(0, 1)$ are unstable. In case (\bar{u}_1, \bar{u}_2) does not exist, only one of the two single species states is stable while the other is unstable, and which is which depends on the parameter values. The extinction state is always unstable.

System (47)–(48) exhibits a variety of travelling wave regimes, their details depending on the initial conditions. A biologically interesting situation is an invasion of one species into the space already inhabited by the other; the corresponding population densities are given as

$$u_1(x, 0) = 1 \quad \forall x \quad \text{and} \quad u_2(x, 0) = \psi(x), \tag{49}$$

where $\psi(x)$ is a certain function of compact support. The type of system's dynamics depends on the properties of the non-spatial system. In case (\bar{u}_1, \bar{u}_2) exists and is stable, initial conditions (49) evolve to two monotonous travelling fronts connecting the state $(1, 0)$ to (\bar{u}_1, \bar{u}_2) and propagating from the place of the second species introduction so that, as a result of the invasion, both species coexist; see Fig. 12.

In case (\bar{u}_1, \bar{u}_2) does not exist, the outcome of the species introduction depends on the stability of the single species states. If $(1, 0)$ is stable, then the second species – i.e. the invader – will go extinct and no invasion occurs. If $(0, 1)$ is stable, then a travelling front is formed connecting $(1, 0)$ to $(0, 1)$, i.e. invasion is successful and leads to the eradication of the host species.

Finally, in case (\bar{u}_1, \bar{u}_2) exists but is not stable, the evolution of initial conditions (49) depend on the properties of $\psi(x)$. The invasion of the second species can only be successful if both $\max_x \psi(x)$ and the support of function ψ are large enough⁸; otherwise the invader goes extinct.

⁸ A situation similar to “the problem of critical size” [101].

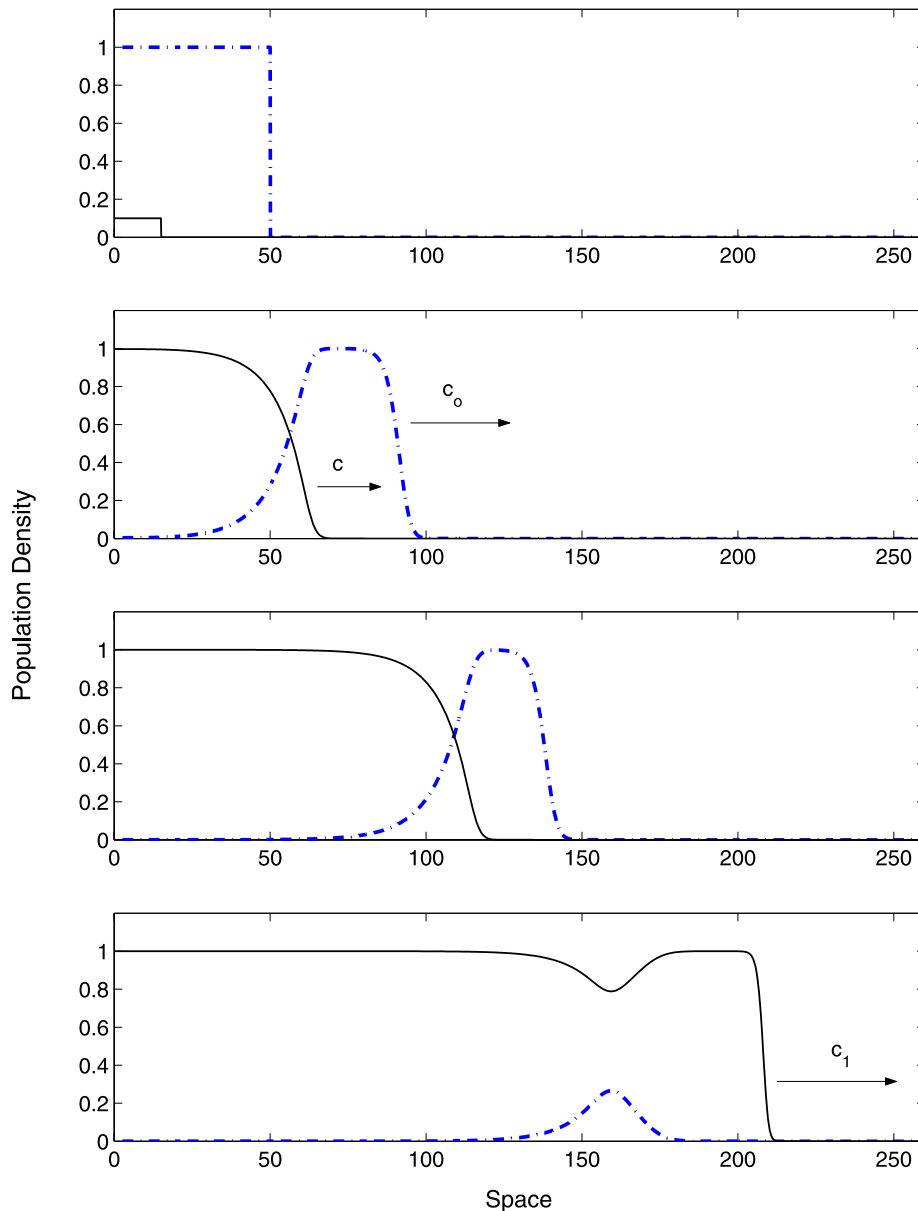


Fig. 13. Travelling waves of competing species. The population densities (blue dashed-and-dotted curves are for species u_1 , black solid curves are for species u_2) are calculated at equidistant moments, top to bottom. (For interpretation of colours in this figure, the reader is referred to the web version of this article.)

An interesting question is a possibility of biological control of the invading species by means of introducing a strong competitor into the wake of its spread. Let species u_1 be the spreading pest that needs to be controlled. The corresponding initial conditions are given by two functions of compact support:

$$u_1(x, 0) = \chi(x) \quad \text{and} \quad u_2(x, 0) = \psi(x), \tag{50}$$

where the support of function χ is assumed to be much larger than the support of ψ . Mathematically, in the two-species system the competitor can be regarded as sufficiently strong when it can lead to a competitive exclusion of the other species, i.e. when $(0, 1)$ is stable and $(1, 0)$ is unstable. In terms of system (47)–(48), it takes place when $b_{12} > b_{22}$ and $b_{21} < b_{11}$. What is then happening is shown in the upper part of Fig. 13, i.e. the evolution of the initial distribution (Fig. 13, top) leads to formation of the travelling front of species u_2 following the front of the invasive species u_1 (Fig. 13, second row), so that in the wake of the second front species u_1 disappears.

Further dynamics of the system depend on the relation between the speed of the first and the second fronts, i.e. c_0 and c respectively. In case $c < c_0$, the area where species u_1 is present grows with time, in case $c > c_0$, the second

front catches up with the first one and species u_1 is eradicated completely; the latter situation is shown in Fig. 13, bottom.

Note that the front speeds can be easily calculated using the same heuristic approach as above, see Eqs. (43)–(44). The first front is obviously a solution of the standard KPP–Fisher equation (and arising from initial conditions of compact support), hence its speed is given as $c_0 = 2\sqrt{D_1 a_1}$. The second front propagates in the area where the first species is already present at its carrying capacity $u_1 = 1$; therefore, at the leading edge of the front, the equation for the second species can be linearized,

$$\frac{\partial u_2(\mathbf{r}, t)}{\partial t} = D_2 \frac{\partial^2 u_2(x, t)}{\partial x^2} + a_1 u_2(1 - b_{21}), \quad (51)$$

so that the speed of the front is given as

$$c = 2(D_2 a_2 (1 - b_{21}))^{1/2}. \quad (52)$$

Thus, we arrive at the following condition showing when the spread of species u_1 is blocked by its competitor u_2 :

$$D_2 a_2 > \frac{D_1 a_1}{1 - b_{21}}. \quad (53)$$

Note that, especially in case b_{21} is not small compare to 1, condition (53) can only hold if the diffusion coefficient and/or the growth rate of species u_2 exceed those of species u_1 considerably. (For instance, if $b_{21} = 0.75$ and $D_1 = D_2$, then relation (53) is equivalent to $a_2 > 4a_1$.) Correspondingly, at a later stage of the invasion species u_2 spreads very fast: After the propagating front of the second species catches up with the first front and brings species u_1 down, its speed increases by the factor $(1 - b_{21})^{-1/2}$, cf. Eq. (52).

We also want to mention here that, although the situation shown in Fig. 13 can perhaps be treated as a certain success of the controlling efforts because species u_1 is eradicated, its competitor u_2 spreads over the space successfully and at a rather high rate. This is still a biological invasion, even that species u_2 may be less harmful than species u_1 . Therefore, any possible practical application of this strategy should be used with a great care and take into account the biological traits of the species, in particular, assessing the extent of a possible damage that invasion of species u_2 can bring to a given ecosystem.

Competition in a multi-species community. The system of two competing species considered above is relatively simple and, being a focus of theoretical/mathematical studies for a few decades, has been by now investigated almost completely.⁹ However, a general case of n competing species is much less studied and is still poorly understood. In fact, there has been just a handful of papers considering spatiotemporal dynamics of a competition system with $n > 2$. In particular, a community of three competing species was studied in [106]. The phase space of a three-species system can contain not only equilibrium points but also limit cycles. Correspondingly, apart from monotonous travelling fronts connecting different steady states of the system, there can also be oscillating fronts, effects of the dynamical stabilization of an unstable equilibrium and spatiotemporal chaos. A typical system dynamics evolving from initial conditions of compact support may exhibit a few propagating fronts that are followed by formation of a chaotic pattern (which looks somewhat similar to what has been observed in a predator–prey system with an oscillatory kinetics, cf. Fig. 8). For the sake of brevity, we will not go into details here; more information can be found in [83].

2.2.3. Epidemiology

Mathematical epidemiology studies how an infectious disease circulates in a given population, in particular trying to identify the factors that control its spread, with the final goals to predict its development and to prevent disease outbreaks. According to a standard approach, the population is split into a few classes with different properties with regards to a possibility to catch up the infection or to the stage in the disease development. A simplest model therefore consists of just two classes, i.e. susceptibles and infected (Fig. 14). As a result of an encounter between any two individuals from these two different classes, a susceptible catches up, with some probability, the disease and hence becomes infected. This is described as a *disease transmission*; qualitatively, the number of individuals per unit time

⁹ Even after a few decades, some subtle mathematical aspects of this system have been addressed only recently, e.g. see [25,55].

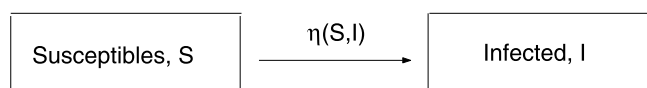


Fig. 14. The simplest conceptual epidemics model.

that leaves class S to enter class I is given by a transmission term $\eta(S, I)$ where S and I are the population densities of susceptibles and infected, respectively.

Mathematically, this situation is described by the following equations:

$$\frac{dS(t)}{dt} = F(S, I) - \mu_0 S - \eta(S, I), \quad \frac{dI(t)}{dt} = \eta(S, I) - (\mu_0 + \mu_1)I, \quad (54)$$

where term F describes reproduction and μ_0 is the natural mortality rate. It is assumed that both classes contribute to population reproduction so that $F = F(S, I)$. Generally speaking, a disease results in a higher mortality rate of the infected, which is called *virulence*; this is taken into account by term $\mu_1 I$. The functional form of the transmission term can be different; the two parameterizations that are used more often are

$$(a) \eta(S, I) = aSI \quad \text{and} \quad (b) \eta(S, I) = \frac{aSI}{S + I} \quad (55)$$

(that are called the mass-action law and the proportionate [frequency-dependent] law, respectively), although more complicated situations can also occur [43]. Coefficient a is a product of the encounter rate and the probability to catch up the disease.

What is shown in Fig. 14 and described by Eqs. (54) is usually known as an SI model. A more elaborated model would normally include more details and therefore more than just two classes. This also depends on the features of the disease under study. For instance, a fraction of infected may recover from the disease and then they either enter again the susceptible class or they may become immune and hence create a new class R of ‘recovered’ individuals. Having included that into consideration would result in the so-called SIR model. The immune individuals then can stay immune for life or they can become susceptible again after some time. Also, the disease can have several stages or several strains, each of them can be taken into account by introducing a special class of infected. As a result, instead of I , one would have I_1, I_2, \dots, I_n . A rather general mathematical model of infectious disease dynamics would therefore look as follows:

$$\frac{dS(t)}{dt} = F(S, I, R) - \mu_0 S - \sum_{i=1}^n \eta_i(S, I_i) + k_R R, \quad (56)$$

$$\frac{dI_i(t)}{dt} = \eta_i(S, I_i) - (\mu_0 + \mu_i)I_i - k_i I_i, \quad i = 1, \dots, n, \quad (57)$$

$$\frac{dR(t)}{dt} = \sum_{i=1}^n k_i I_i - k_R R - \mu_0 R, \quad (58)$$

where k_R and k_i are nonnegative coefficients describing the transitions between different classes. Especially for a large n (say, $n > 3$), the structure of the system’s phase space and the corresponding properties of its solutions can be rather complicated and are still poorly understood; in fact, we are not aware about any work where a more or less general case would have been addressed. However, in a special case when function F and all functions η_i are bilinear with respect to their arguments, system (56)–(58) becomes equivalent to a classical multi-component Lotka–Volterra system and is studied relatively well [57]; in particular, the conditions of existence of a positive steady state were obtained [118].

When space is taken into account, under the usual assumption of individuals moving randomly, system (56)–(58) turns into a system of reaction–diffusion equations. In a general case its properties can be complicated and exotic. In the case that the corresponding non-spatial system possesses only equilibrium states, the general mathematical theory, cf. Section 1, predicts (for appropriately chosen initial and boundary conditions) existence of a generic solution describing a travelling front. Remarkably, this is in a good agreement with observations. Historical data of the Black Death spread over Europe, as well as more recent data of the spread of rabies [93] suggest that a travelling front of the infection can be the case. Travelling waves and patterns in the Hantavirus epidemics are studied in [1,2,66].

Although in theory a system like (56)–(58) can take into account a lot of details of specific diseases, in practice the studies available from literature rarely address a system with more than three equations. The system's dynamics then exhibits the features similar to that of a predator–prey system. Indeed, it is readily seen that an SI model with mass-action transmission rate and with the reproduction rate depending only on S (which may happen if a disease affects individual reproductive abilities) coincides with a predator–prey system with Holling type I predator response. In case the population growth rate is hampered by the Allee effect (i.e. $F(S)$ is a sigmoid-shaped function, e.g. a cubic polynomial), this system can have oscillatory kinetics and the scenarios of disease spread appear to be similar to those observed for a predator–prey system, see Figs. 7 to 11, i.e. a succession of travelling fronts is followed by the onset of spatiotemporal chaotic oscillations [107].

In a less specific SI system where the reproduction rate depends on the total population density $S + I$, and the transmission rate is frequency-dependent, see (55b), there can be no limit cycle in the phase space and oscillatory kinetics becomes impossible [56]. Initial conditions of compact support (which corresponds to a local introduction of the disease) then generate a variety of travelling wave connecting the equilibrium states. Interaction between fronts can change the direction of their propagation. In particular, a possibility of biological control was shown by means of introducing a disease in the wake of an invading species when the invasion front is reversed due to the impact of the infection.

However, when other ecological interactions are included and the total number of the system components become large than two, a possibility of oscillatory kinetics reappears even in an SI model with the frequency-dependent transmission rate. Malchow et al. [84] considered a predator–prey model where prey is infected and hence is split into S and I classes. The corresponding model consists of three reaction–diffusion equations and can possess oscillatory regimes, which makes possible the onset of chaos in the wake of the travelling front of the spreading infection.

2.3. Non-local consumption of resources and evolutionary branching

Let us return to the scalar reaction–diffusion equation (1) with the nonlinearity given by (3). In the context of population dynamics, u is the dimensionless population density, the first term in the right-hand side of Eq. (1) describes random displacement of the individuals of this population, the second term their reproduction and mortality. The reproduction term is proportional to the density in the first or second power and to available resources $(1 - u)$. Here 1 is the amount of resources available at some space point x , and their consumption is proportional to the density of the population $u(x, t)$ at this point.

Assume now that the individual located at the point x can consume resources not only at this point but also in some area around it. Since the characteristic time of reproduction is much longer than the displacement for consumption of resources, then individuals become distributed around their average positions with some probability density function $\phi(x - y)$. We assume here that it depends on the distance from the average point x to the actual point y . Under these assumptions we arrive to the integro-differential equation

$$\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + au^n \left(1 - \int_{-\infty}^{\infty} \phi(x - y)u(y, t) dy \right) - bu. \quad (59)$$

Here a and b are some positive parameters, and the function ϕ is normalized in such a way that $\int_{-\infty}^{\infty} \phi(y) dy = 1$. If it is a Dirac δ -function, then Eq. (59) reduces to Eq. (1). For other functions ϕ Eq. (59) can have some qualitatively new properties in comparison with (1). We begin to discuss them with stability of homogeneous in space stationary solutions.

2.3.1. Stability of homogeneous solutions

Suppose for simplicity that $b = 0$ and $n = 1$. Similar to Eq. (1), $u = 1$ is a stationary solution of Eq. (59). In the case of the reaction–diffusion equation it is stable. Let us discuss its stability for the integro-differential equation. Linearizing Eq. (59) around this solution, we obtain the eigenvalue problem

$$dv'' - a \int_{-\infty}^{\infty} \phi(x - y)v(y) dy = \lambda v. \quad (60)$$

Applying the Fourier transform, we obtain

$$\lambda = -a\tilde{\phi}(\xi) - d\xi^2,$$

where $\tilde{\phi}(\xi)$ denotes the Fourier transform of the function ϕ . If ϕ is a δ -function, then λ is negative for all real ξ and the solution $u = 1$ of Eq. (59) is stable. If the function $\tilde{\phi}(\xi)$ has a variable sign, then λ can become positive for some values of ξ providing a possible instability of the solution. This was noticed probably for the first time in [20], and then in other works [44,45,47,51]. Let us take for example

$$\phi(x) = \begin{cases} 1/(2N), & |x| \leq N, \\ 0, & |x| > N. \end{cases}$$

Then

$$\lambda = -\frac{a}{\xi N} \sin(\xi N) - d\xi^2$$

and the instability conditions can be easily found explicitly [47,48]. It appears if the support N of the kernel is sufficiently large, i.e., if the competition involves a large range of individuals. The piece-wise constant form of the function ϕ has various biological interpretations. Let us consider the example of herbivorous animals taken by a shepherd to a pasture. Suppose that they leave at 6 a.m. and return at 6 p.m., they can move with a speed 4 km/hour, and the animals need 8 hours of feeding. Then they have two hours to go to the pasture and two hours to return back. Therefore, with an equal probability they can be anywhere at the distance less than 8 km from the village. We will consider some other examples below.

Thus, non-local consumption of resources can result in the instability of homogeneous equilibria. We will see below that this will have some important consequences both from the point of view of nonlinear dynamics and of biological applications. The mechanism of the instability is different in comparison with Turing structures. In particular, we can note that it can occur for a scalar integro-differential equation while Turing structure can be observed only for systems of equations.

2.3.2. Speciation

Suppose now that the space variable x in the model discussed above corresponds to a morphological parameter. It can be for example the size of some plants or animals, their weight, the size of bird's beak, and so on. Then $u(x, t)$ is the density distribution with respect to this parameter. The reproduction term means that the descendants have the same value of the parameter as their parents. The diffusion term shows that there is a small random perturbation of its value from one generation to another. Finally, non-local consumption of resources signifies that individuals with different values of the parameter can have access to the same resources.

Consider again our herbivores who now eat leaves from bushes or trees at some height. Clearly this height depends on the height of the individuals and represent some interval $[x - h, x + h]$ around an average value x . Similarly, the size of grains consumed by birds depend on the size of their beak, and the amount of sun available for plant depends on their size.

Two individuals with the values of the morphological parameter, respectively, x_1 and x_2 consume resources from the intervals $[x_1 - h, x_1 + h]$ and $[x_2 - h, x_2 + h]$, which can overlap. Thus, non-local consumption of resources describes intra-specific competition of individuals in the population.

Let us now consider behavior of the solution of Eq. (59) in the case $n = 2$, $b > 0$ studied in [7]. Fig. 15 shows the level lines of the function $u(x, t)$. The t -axis is vertical, the x -axis horizontal. The support of the initial condition is located at the center of the interval. From the biological point of view this means that all individuals of the population have approximately the same value of the morphological parameter.

After some time, the solution changes its structure. It has two maxima and a minimum between them. Hence, the population splits into two sub-populations, which are characterized by different values of the parameter. We can interpret them as two different biological species. Some time later, each of the two sub-population split again. After that, the sub-populations at the border continue splitting while those inside do not change any more. Mathematically, we observe propagation of a periodic travelling wave which connects the stable equilibrium $u = 0$ with a stable periodic in space stationary solution.

We can compare the numerical simulations in Fig. 15 with the diagram in Fig. 16 adapted from Darwin's book [26]. Each line in the diagram corresponds to a biological species, the horizontal axis is a morphological parameter,

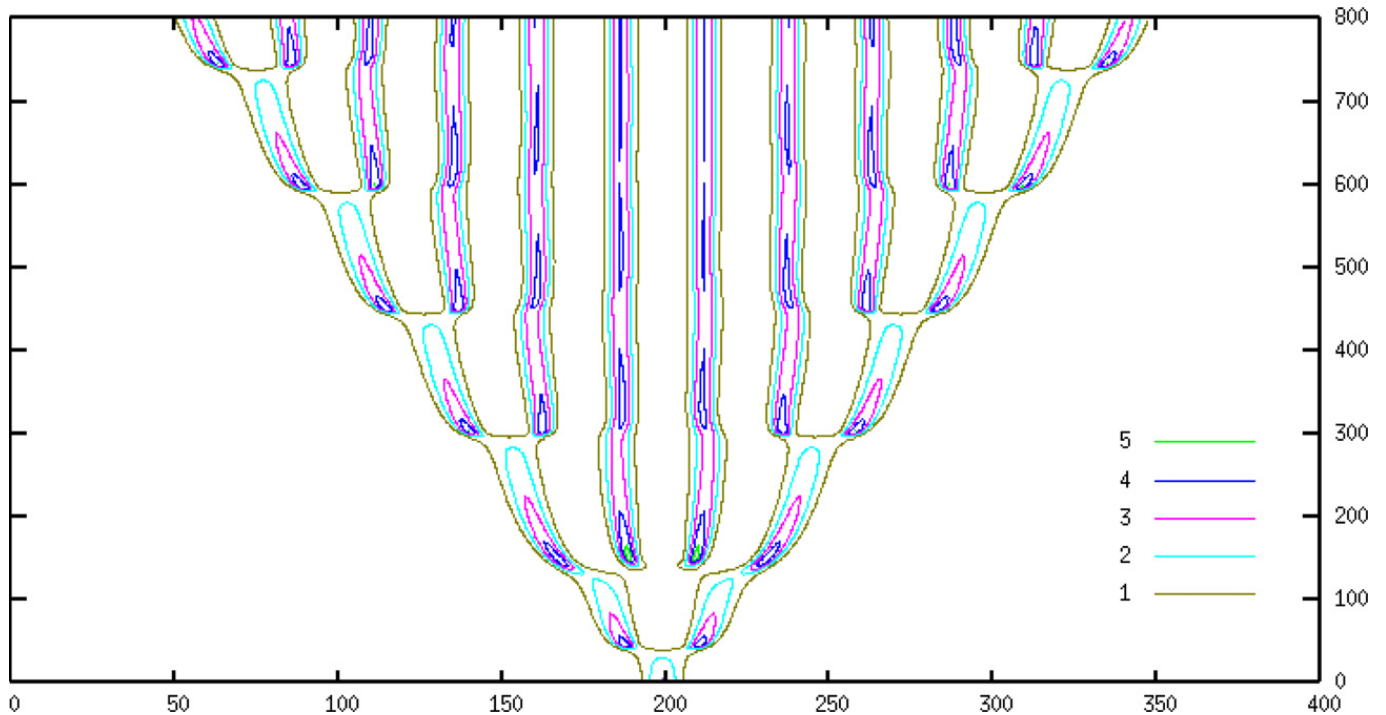


Fig. 15. Level lines of the solution of Eq. (59) with $n = 2$. Horizontal axis – space variable, vertical axis – time. The support of the initial condition is at the center of the interval. Solution propagates to the left and to the right as a periodic travelling wave. A stationary periodic structure is formed behind the wave.

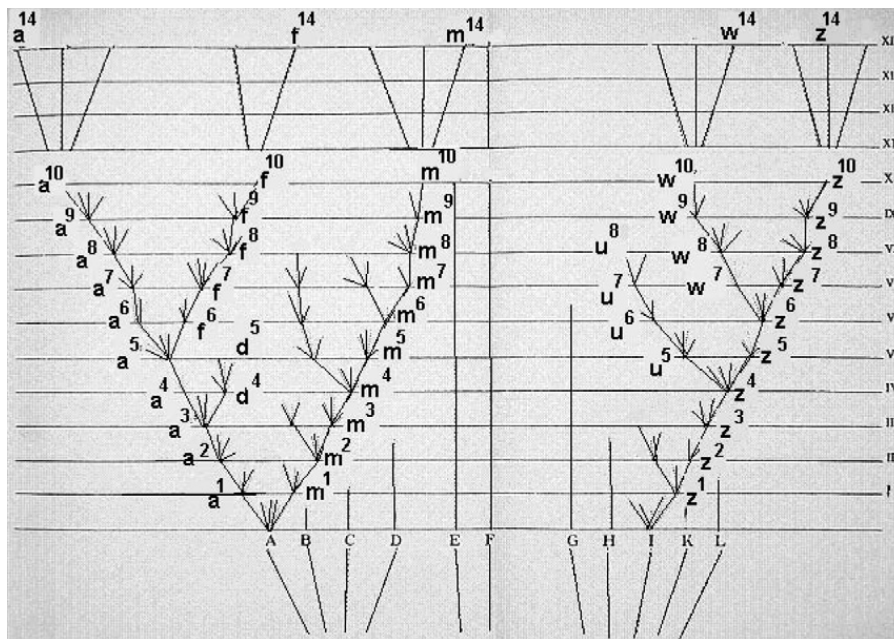


Fig. 16. Emergence of biological species in the process of evolution according to Darwin [26]. Horizontal axis – values of some morphological characteristics, vertical axis – generations. Each line corresponds to a species. They can bifurcate giving new species, disappear or remain unchanged.

the vertical axis is time measured in generations. With this diagram Darwin explains the emergence of new species in the process of evolution. According to him, due to intra-specific variability, individuals of the same population can have slightly different values of the morphological parameter. The individuals located at the tails of this distribution can have a selective advantage because intra-specific competition there is weaker than in the center. This can result

in the splitting of the population into sub-populations. This description is in a good agreement with the results of the modelling.

Thus, in the framework of this approach, the minimal model which describes the process of speciation is based on three conditions: small random variation of the phenotype (diffusion), reproduction with the same phenotype, intra-specific competition due to non-local consumption of resources.

This seems to be a rather general law applicable not only for biological species but also in many other situations. For each specific example we need to specify in what sense we understand variability, reproduction and non-local consumption of resources. We will briefly discuss here two examples among many others: speciation in science and in manufactured products. In the middle ages science was mainly englobed by philosophy. In the process of its evolution it becomes more and more specialized. A good representation of the structure of mathematical sciences is given by the AMS subject classification with numerous divisions and subdivisions, e.g., 35 – partial differential equations (PDE), 35K – parabolic PDE, 35K57 – reaction–diffusion equations. Reproduction in science is due to pupils, the space variable x corresponds to the area of research, resources correspond to the scientific knowledge in this area.

Production of consumer goods also provides numerous examples of speciation. Evolution of automobiles from the first one constructed by the French engineer Fardier de Cugnot in 1771 to modern trucks, buses, passenger cars is one of such examples. In the context of the modelling discussed in this section, we can say that automobiles consume those who buy them and reproduce themselves due to the demand from the market. Thus, market competition leads to its diversification with a certain discrete distribution of the properties of the products. It should be noticed that this is not competition between different producers of similar products but competition between different products for consumers.

To conclude this short discussion of the speciation theory, we can remark that biological evolution has many complicating factors, and other example of speciation can be easier to investigate. Finally, we note that nonuniform distribution of animals or humans in the physical space can also be related to non-local consumption of resources.

2.3.3. Nonlinear dynamics and biological interpretations

Stable pulses. Numerical simulations described in the previous section are obtained for the bistable case where $n = 2$ and $b > 0$. The corresponding function $F(u)$ given by (3) has three zeros (Fig. 1, left). Two of them, w_+ and w_- are stable.

If $\int_{-\infty}^{\infty} F(u) du > 0$, then reaction–diffusion equation (1) has stationary pulses. These are travelling waves with the zero speed and with the same limits at $\pm\infty$. They are solutions of the problem

$$w'' + F(w) = 0, \quad w(\pm\infty) = w_{\pm}.$$

Similar to all other solutions of the scalar reaction–diffusion equation, which are not monotone with respect to the space variable, such solutions are unstable with respect to small perturbations.

It appears that integro-differential equation (59) can have stable stationary pulses if the support of the function ϕ is sufficiently large. Fig. 17 shows the solution $u(x, t)$ as a function of two variables (top, left) and level lines of this function (top, right). If we consider the width of the support of the function ϕ as a parameter, then we can observe the transition from the periodic wave shown in Fig. 15 and the standing wave in Fig. 17. This transition occurs by the increase of the time period between the consecutive splitting (Fig. 17, bottom, left). In the case of asymmetric support of the function ϕ the pulse is not stationary. It moves with a constant speed (Fig. 17, bottom, right). Stability of pulses is observed numerically but it is not yet proved analytically.

Coming back to Darwin's diagram in Fig. 16, we note that some lines, e.g., E and F do not bifurcate and remain constant till possible disappearance. This corresponds to stable stationary pulses described above. It is interesting to note that such solutions exist in the bistable case ($n = 2$) but not in the monostable case ($n = 1$). From the point of view of population dynamics, the former corresponds to sexual reproduction where the population can either expand or remain stable and the latter to asexual reproduction where the population will necessarily expand. Another point to be noted is that most of the lines in the diagram are inclined, that is the corresponding biological species gradually changes the value of the morphological parameter. This can be due to the evolutionary pressure from the neighboring species (cf. Fig. 15) or possibly to asymmetry of the kernel (cf. Fig. 17, bottom, right). Finally, we remark that when two lines approach each other one of them or both disappear. This can be related to the competition of species

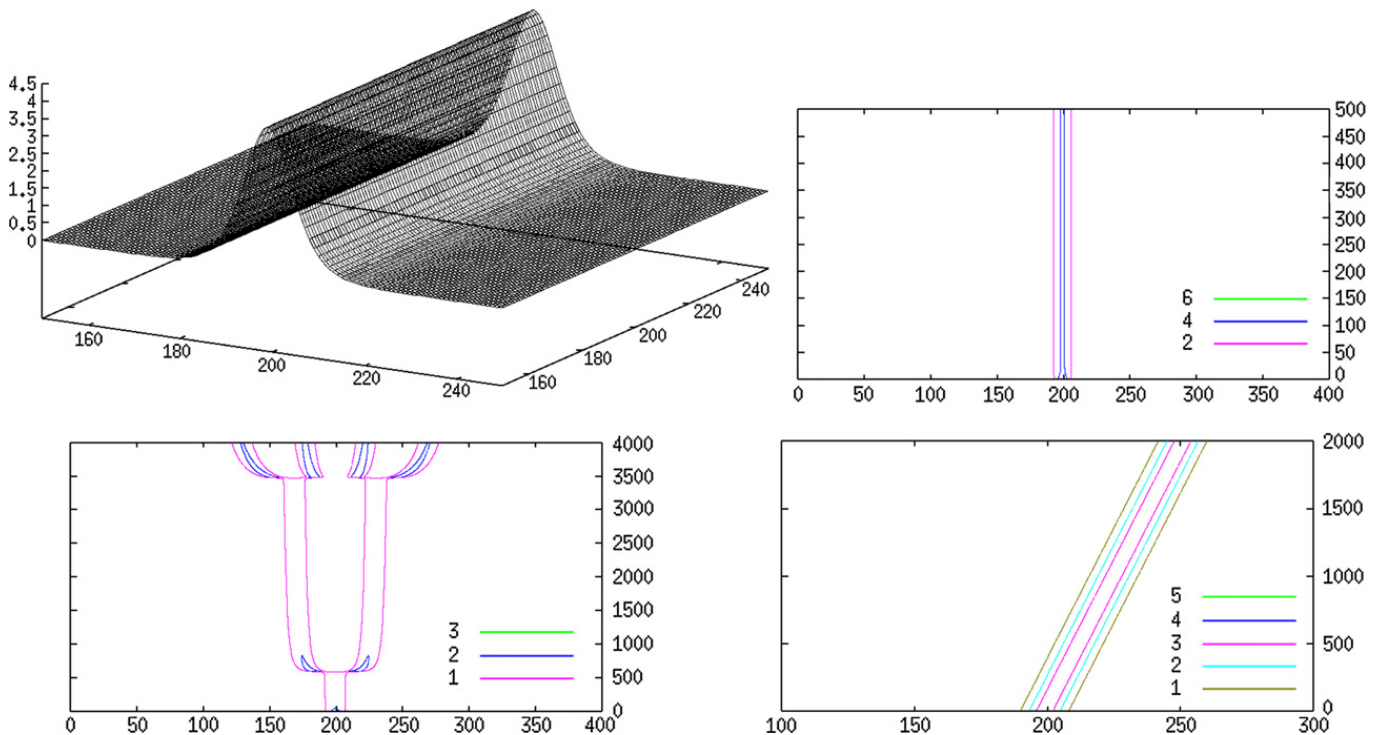


Fig. 17. Stable stationary pulse in the bistable case (top). Solution $u(x, t)$ of Eq. (59) as a function of two variables (top, left), level lines of the solution (top, right). It is observed if the support of the kernel of the integral is sufficiently large. Moving stable pulse in the case of asymmetric kernel (bottom, right). Level lines of the solution which shows the transition from the solution shown in Fig. 15 and the stable stationary pulse (bottom, left).

(Section 2.2.2). Competition of species with intra-specific competition is studied in [8]. Another model of extinction, due to the environmental changes, will be discussed below. Other approaches and models of evolutionary dynamics can be found in [21,28,29,98].

Monostable case. Various types of waves in the monostable stable case ($n = 1$) are studied in [9,22,47,48] (Fig. 18). A wave with a constant speed non-monotone with respect to x (top, left), periodic wave (top, right), spatiotemporal structures in the case where the kernel ϕ is not symmetric (middle). Existence, speed of propagation and some other properties of waves in the monostable case are studied in [5,30,80,99,136].

We have discussed in Section 1.2.1 the difference between the bistable and the monostable case for the reaction–diffusion equation. In the bistable case the wave is unique while in the monostable case they exist for all speeds greater or equal to the minimal speed. Non-uniqueness of waves remains valid for the integro-differential equation. The structures in Fig. 18 (bottom) are obtained for the same values of parameters but for different initial conditions. We will return to this question in Section 4 where we discuss generalized travelling waves.

Extinction. There are different mechanisms of extinction: competition of species, environmental changes independent of the population or caused by the population itself [7,8]. We illustrate the evolution of the population when the quantity of available resources decreases. We continue the simulation shown in Fig. 15 after changing the parameters of the equation. The value of available resources is now smaller. Then some of the species can disappear (Fig. 19). The species with extremal values of the morphological parameter appear to be more resistant.

Multi-dimensional equations and systems, other non-local equations. In the 2D case, the first term in the right-hand side of Eq. (59) is replaced by the Laplace operator. Additional factor in comparison with the 1D case is related to the form of the support of the kernel ϕ . Fig. 20 shows numerical simulations in the monostable case with the circular (left) and square (right) supports (adapted from [9,49]). In both cases there is circular wave followed by some structure. The form of this structure depends on the form of the support. Reaction–diffusion systems of equations with non-local consumption of resources are studied in [8,49].

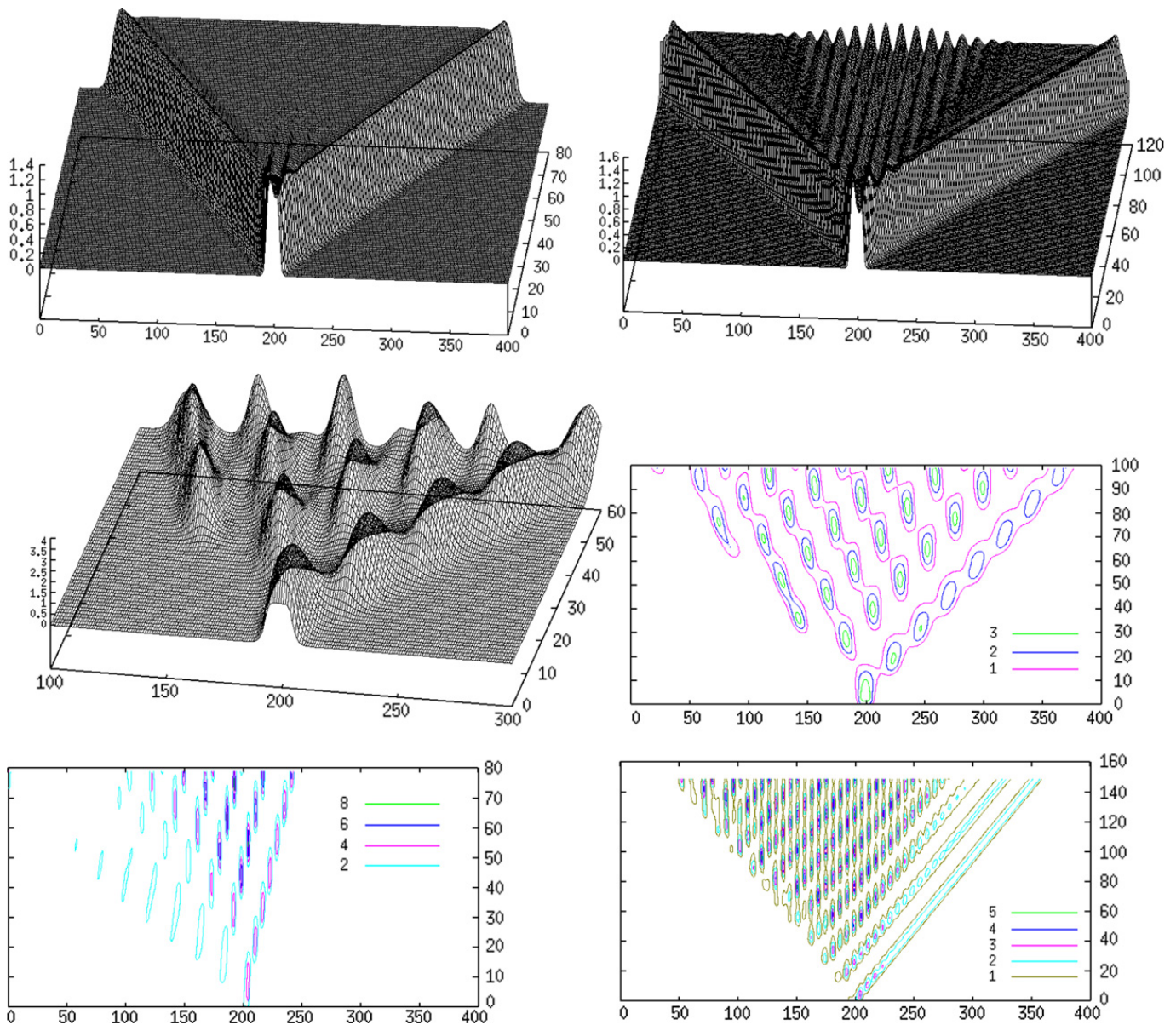


Fig. 18. Wave propagation in the monostable case. Solution $u(x, t)$ of Eq. (59) with $n = 1$ in the case of a simple wave (top, left). The initial condition is localized in the center of the interval. The wave propagates to the left and to the right with the same speed. The profile of the wave is not monotone with respect to x , there is a maximum at the wave front. Such waves are not stable for the reaction–diffusion equation but apparently can be stable for the integro-differential equation. Periodic wave (top, right). A periodic spatial structure emerges at the center of the interval. Asymmetric wave propagation where the kernel $\phi(x)$ is not an even function (middle). This results in the emergence of spatiotemporal structures. Wave propagation with exponentially decaying initial conditions (bottom). The decay rate is different at the left- and at the right-half of the interval. The waves propagating to the left and to the right have different speeds and structures.

Individuals in the population can not only compete non-locally for resources and, by this, decrease the reproduction of other individuals but also stimulate non-locally this reproduction. This is the case for example of some plants, which distribute their pollen in some area around their location, or biological cells, which can produce some signaling molecules diffusing in the extracellular matrix. Instead of Eq. (59) in this case we have

$$\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + auJ(1 - u) - bu, \tag{61}$$

where $J = \int_{-\infty}^{\infty} \phi(x - y)u(y, t) dy$. The difference between these two equations does not seem very important: the integral enters in a different way. In the case of the δ -function both equations are reduced to the same reaction–diffusion equation. It appears however that properties of Eq. (61) are quite different. They are more close to the scalar

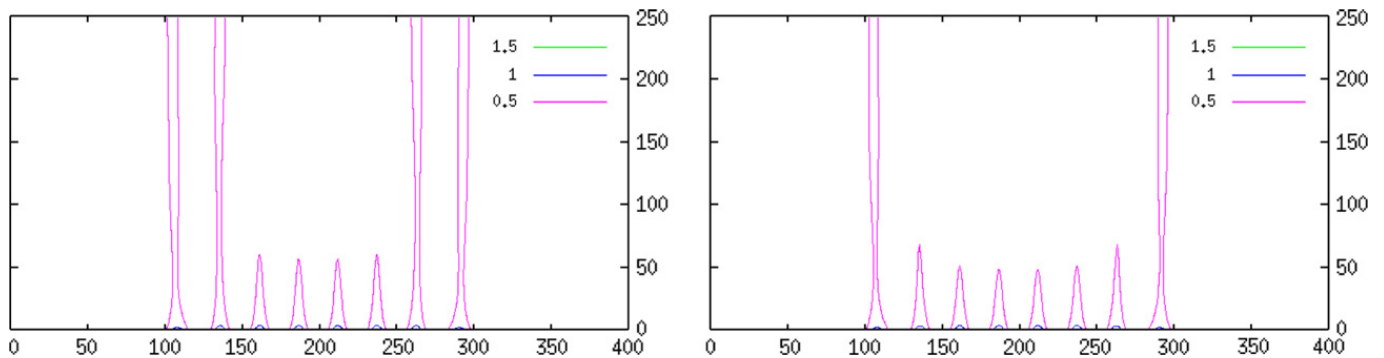


Fig. 19. Extinction. Continuation of the simulation in Fig. 15 with a smaller value of resources. The species with extremal values of the morphological parameter appear to be more resistant.

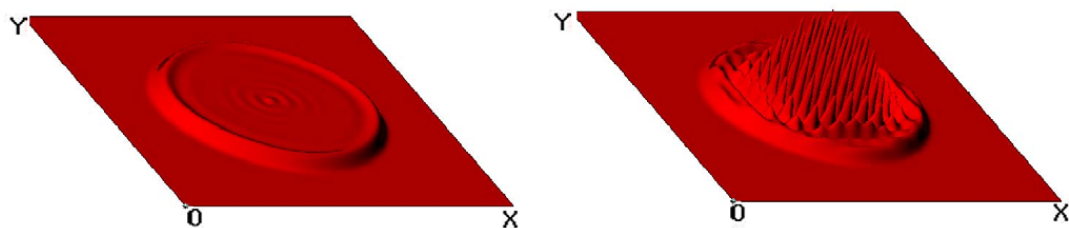


Fig. 20. Wave propagation in the 2D case. Snapshot of the solution of Eq. (59) with $n = 1$ and with circular (left) or square (right) support of the kernel ϕ . The wave front initiated at the center of the domain. It propagates outwards with radial symmetry. After the front propagation a circular or a square structure with sharp picks emerges.

reaction–diffusion equation due to applicability of the maximum principle. Existence and stability of travelling waves for this equation are studied in [27,31]. Other reaction–diffusion equations with space–time delay are studied in [75, 96,111,133,134].

3. Cell dynamics

Dynamics of cell populations is important for many biological and medical questions, such as morphogenesis [97], growth of normal tissues and tumors [6,42] or hematopoiesis (blood cell production) [23]. It is determined by cell proliferation and apoptosis (programmed cell death) and by their mechanical and chemical interaction between each other and with the surrounding medium. Cell proliferation can lead to self-renewal, where the daughter cells are similar to the mother cell, or to differentiation, where the daughter cells are different.

Growing cell populations spread in space and can be described by reaction–diffusion waves. Consider a single cell population assuming that cells can either self-renew or die. Denote by $u(x, t)$ the dimensionless density of the population. Evolution of this population can be described by Eq. (1) with the function $F(u)$ given by formula (3) with $n = 1$. Random cell motion can be related to cell division [18] or to other mechanisms [76]. Cell proliferation is proportional to cell density multiplied by $(1 - u)$. This factor shows that the rate of cell proliferation decays when their density increases (see, e.g., [3]).

If the cell density becomes sufficiently large, cells push each other creating pressure difference and convective motion. In this case the reaction–diffusion equations should be completed by the equations of motion, e.g., the Navier–Stokes equations or the Darcy law [6,32,41].

Cell populations can have some specific features in comparison with the classical population dynamics, and the corresponding models can have some particular properties. We will discuss in this section two applications of cell population dynamics related to leukemia development and to atherosclerosis. In both cases the disease appears as a new equilibrium when the disease free equilibrium becomes unstable. Transition from the disease free state to the disease spreads in space as a reaction–diffusion wave.

3.1. Leukemia

Blood cell production in the bone marrow is a complex process which begins with hematopoietic stem cells. Consecutive cell proliferation and differentiation results in production of mature blood cells (erythrocytes, white blood cells, platelets) which leave the bone marrow into blood. This process is regulated by numerous hormones and growth factors internal for the bone marrow or coming from other organs. There are numerous works devoted to modelling of normal and pathological hematopoiesis (see [4] and the references therein). We discuss here leukemia development from the point of view of spatial cell population dynamics [18]. Cell population dynamics coupled with the motion in a porous medium is studied in [32].

Cell division is often accompanied by mutations. Mutated cells can self-renew more and differentiate less than normal cells. As a consequence, the number of immature cells or blasts will increase replacing other cells, possibly resulting in the development of acute leukemia. Let us describe this process with a simplest model which includes two types of cells, normal P and mutated Q :

$$\frac{\partial P}{\partial t} = d_1 \frac{\partial^2 P}{\partial x^2} + H + kP(P_0 - P - Q) - aP, \quad (62)$$

$$\frac{\partial Q}{\partial t} = d_2 \frac{\partial^2 Q}{\partial x^2} + k_m Q(P_0 - P - Q) - a_m Q. \quad (63)$$

Here H is cell influx from the stem cell compartment, parameters k and a characterize cell proliferation and apoptosis for normal cells, k_m and m for mutated cells.

We begin with the analysis of stationary points of this system. Zero lines of the source terms in (62), (63) are given by the equalities

$$H + kP(-b - P - Q) = 0, \quad k_m Q(b_m - P - Q) = 0, \quad (64)$$

where $b = -P_0 + \frac{a}{k}$, $b_m = P_0 - \frac{a_m}{k_m}$. According to the biological meaning of the parameters, all values k , k_m , b and b_m are positive. From the first equation in (64),

$$Q = -b - P + \frac{H}{kP}, \quad (65)$$

and from the second equation

$$Q = 0 \quad \text{or} \quad Q = b_m - P. \quad (66)$$

System (65)–(66) can have one or two stationary points (Fig. 21). If we put $Q = 0$ in (65), and denote by P^* the positive solution of this equation, then we find

$$P^* = \frac{b}{2} \left(\sqrt{1 + \frac{4H}{kb^2}} - 1 \right).$$

The stationary solution $(P, Q) = (P^*, 0)$ always exists. On the other hand, from (65) and the second equation in (66), we obtain

$$P_m^* + Q_m^* = -b + \frac{H}{kP_m^*} \quad \text{and} \quad P_m^* + Q_m^* = b_m.$$

If $b_m > P^*$, then we have the case 1 in Fig. 21. There exist two stationary points: A with the coordinates (P_m^*, Q_m^*) , and B with the coordinates $(P^*, 0)$. If $b_m < P^*$, then we have the case 3 in Fig. 21, there exists a unique stationary point $B = (P^*, 0)$. It follows that leukemia may develop only if $b_m > P^*$. This condition can be written as $b_m(b_m + b) > \frac{H}{k}$ or equivalently

$$\left(\mu - \frac{P_0 k}{a} \right) (\mu - 1) > \frac{Hk}{a^2}, \quad (67)$$

where

$$\mu = \frac{a_m k}{a k_m}. \quad (68)$$

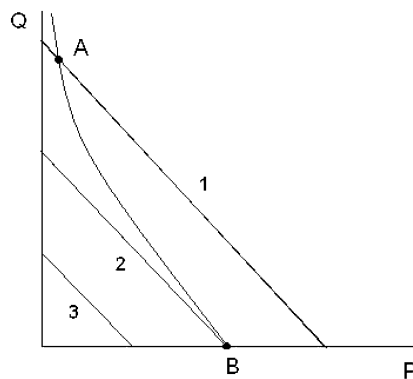


Fig. 21. Graphical solution of system (65)–(66). Case 1 corresponds to the existence of two stationary points, A and B , the case 3 to the existence of only one stationary point, B . In case 2, the points A and B have the same location.

We call the inverse parameter $1/\mu$ the *strength of mutation*. If a malignant mutation decreases the rate of apoptosis and differentiation and increases the rate of proliferation, then $\mu < 1$. This means that the strength of mutation is greater than 1.

If (67) is satisfied, that is the mutation is sufficiently strong and the influx is not strong enough, then there are two stationary points A and B (Fig. 21). It can be verified that A is stable while B is unstable. This means that the disease will develop. The values of the concentrations P_m^* and Q_m^* in the leukemic equilibrium depend on μ . In particular, the concentration Q_m^* of malignant cells can be rather low if μ is close to the critical value. If (67) is not satisfied (weak mutation or strong influx), then there exists a unique stable stationary point B , which corresponds to the disease free situation.

Thus, if mutation is sufficiently strong, system (62), (63) has two stationary points. It can be reduced to a monotone system (Section 1.3.1) by a change of variables. There exists a travelling wave connecting the two equilibria. Its propagation corresponds to spreading of malignant cells in the bone marrow.

3.2. Atherosclerosis

Cardio-vascular diseases represent one of the major death factors. They can be related to atherosclerosis, a chronic inflammation of artery walls which results in their remodelling, formation of a plaque and its possible rupture or spontaneous blood coagulation. Both cases can lead to stroke or to heart attack.

High plasma concentration of low density lipoprotein (LDL) or “bad” cholesterol is one of the principal risk factors for atherosclerosis [109,124]. The process of atherosclerosis begins when LDLs penetrate into the intima of the arterial wall where they are oxidized. Oxidized LDL (ox-LDL) in the intima is considered by the immune system as a dangerous substance, hence an immune response is launched: chemoattractants, which mediate the adhesion of the monocytes to the endothelium and the penetration of the monocytes through the endothelium, are released and endothelial cells are activated. As a result, monocytes circulating in the blood adhere to the endothelium and then they penetrate to the arterial intima. Once in the intima, these monocytes are converted into macrophages.

The macrophages phagocytose the ox-LDL and transform into foam cells (lipid-laden cells) which amplify the chronic inflammatory reaction secreting pro-inflammatory cytokines (e.g., $\text{TNF-}\alpha$, IL-1). The latter increase endothelial cell activation, promote the recruitment of new monocytes and support the production of new pro-inflammatory cytokines.

This auto-amplification phenomenon is compensated by an anti-inflammatory response mediated by the anti-inflammatory cytokines (e.g., IL-10) which inhibit the production of pro-inflammatory cytokines (biochemical anti-inflammation). Next, the inflammation process involves the proliferation and the migration of smooth muscle cells to create a fibrous cap over the lipid deposit which isolates this deposit center from the blood flow (mechanical anti-inflammation). This mechanical inhibition of the inflammation may become a part of the disease process. Indeed the fibrous cap changes the geometry of the vasculature and modifies the blood flow. The interaction between the flow and the cap may lead to a thrombus, or to the degradation and rupture of the plaque liberating dangerous solid parts into the flow [77,78].

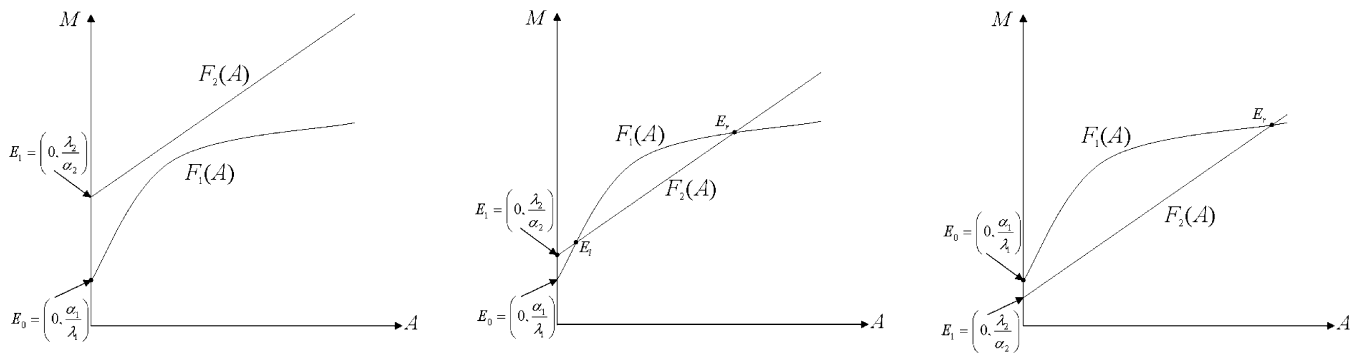


Fig. 22. Zero lines of the nonlinearities in (69) for different cholesterol concentrations: low (left), intermediate (middle), high(right).

3.2.1. Atherosclerosis is a reaction–diffusion wave

Following [33], we consider here a reaction–diffusion system of equations on an interval representing the arterial intima:

$$\begin{cases} \frac{\partial M}{\partial t} = d_1 \frac{\partial^2 M}{\partial x^2} + f_1(A) - \lambda_1 M, \\ \frac{\partial A}{\partial t} = d_2 \frac{\partial^2 A}{\partial x^2} + f_2(A)M - \lambda_2 A, \end{cases} \quad (69)$$

for $x \in [0, L]$. Here M is the density of immune cells (monocytes, macrophages) and A is the density of the cytokines secreted by the immune cells. The function $f_1(A)$ models the recruitment of the immune cells from the blood flow, promoted by the inflammatory cytokines:

$$f_1(A) = \frac{\alpha_1 + \beta_1 A}{1 + A/\tau_1},$$

where $\alpha_1 = f_1(0)$ corresponds to the beginning of the inflammation, that is the recruitment of monocytes due to the presence of ox-LDL activating endothelial adhesion molecules, chemoattractants and growth factors.

The factor β_1 represents the auto-amplification of the recruitment of monocytes due to the inflammatory cytokines secreted by the monocytes themselves. The factor $1 + A/\tau_1$ represents the mechanical saturation of the recruitment of M , that is the effect of the fibrous cap created by smooth muscle cells, with τ_1 being the characteristic time for the fibrous cap formation. Although this phenomenon is quite complex and involves proliferation and migration of smooth muscle cells, we only describe it as a saturation factor. Besides, for f_1 to be an increasing function of A , we impose the condition $\tau_1 > \frac{\alpha_1}{\beta_1}$.

The term $f_2(A)M$ models the cytokines production rate,

$$f_2(A) = \frac{\alpha_2 A}{1 + A/\tau_2},$$

where $\alpha_2 A$ represents the secretion of pro-inflammatory cytokines promoted by the pro-inflammatory cytokines themselves and $1 + A/\tau_2$ represents the inhibition of the pro-inflammatory cytokines secretion mediated by the anti-inflammatory cytokines, with τ_2 being the necessary time for this inhibition to act. The terms $-\lambda_1 M$ and $-\lambda_2 A$ represent the degradation of the immune cells M and the cytokines A respectively, the second derivatives describe their diffusion (or cell displacement) in the intima.

Let us first analyze the kinetic system (without diffusion). Equating zero the nonlinearities in (69), we can easily express M as a function of A . Denote the corresponding functions by $F_1(A)$ and $F_2(A)$, respectively. If the concentration of cholesterol is low, then there is a unique disease free stationary point E_0 (Fig. 22, left). It is globally stable and atherosclerosis cannot develop. For larger values of its concentration, that is for larger values of α_1 , the curve $M = F_1(A)$ goes up. In this case there are three stationary points (Fig. 22, middle). The two extremal points are stable while the intermediate is unstable. Therefore, for small perturbation, the system remain in the disease free equilibrium. However, if the perturbation is large enough, the system can move to another stable stationary point, which corresponds to development of atherosclerosis. Such perturbation can be provided by injury, inflammation or other

factors. Finally, if the LDL concentration is high, then there are two stationary points. The disease free equilibrium is unstable, another one is stable. In this case atherosclerosis will develop even for small perturbations which are always present in real systems.

Consider now the system with diffusion. We note first of all that it is a monotone system. Therefore we can apply the results on the existence and stability of travelling waves discussed in Section 1. According to the discussion above, if the cholesterol concentration is low, there exists a unique disease free stationary solution. It remains globally stable for the system with diffusion. For intermediate cholesterol concentrations, we have a bistable case where the travelling wave is unique and globally stable. If its velocity is positive and the initial condition is sufficiently large, then the atherosclerosis will develop. The inflammation will spread as a reaction–diffusion wave. For large cholesterol concentrations, we have the monostable case. The waves exist for all values of the velocity greater or equal to the minimal velocity. The inflammation will spread even for small initial perturbations of the disease free equilibrium.

3.2.2. 2D model

Though 1D model captures some essential features of atherosclerosis development, it does not take into account a finite width of the blood vessel wall. This approximation signifies that the vessel wall is very narrow and the concentrations across it are practically constant. In a more realistic situation, we should consider a multi-dimensional model and take into account the recruitment of monocytes from the blood flow. The flux of monocytes depends on the concentration of cytokines at the surface of endothelial cells which separate the blood flow and the intima. This should be described by nonlinear boundary conditions which change the mathematical formulation of the problem. We consider the system of equations

$$\frac{\partial M}{\partial t} = d_M \Delta M - \beta M, \tag{70}$$

$$\frac{\partial A}{\partial t} = d_A \Delta A + f(A)M - \gamma A + b, \tag{71}$$

in the two-dimensional strip $\Omega \subset \mathbb{R}^2$,

$$\Omega = \{(x, y), -\infty < x < \infty, 0 \leq y \leq h\}$$

with the boundary conditions

$$y = 0: \quad \frac{\partial M}{\partial y} = 0, \quad \frac{\partial A}{\partial y} = 0, \quad y = h: \quad \frac{\partial M}{\partial y} = g(A), \quad \frac{\partial A}{\partial y} = 0. \tag{72}$$

Here M is the concentration of white blood cells inside the intima, A is the concentration of cytokines, the constant b describes a constant source of the activator in the intima. It can be oxidized LDL coming from the blood stream. Assuming that its diffusion into the vessel wall is sufficiently fast we can describe it by means of the additional term in the equation and not as a flux through the boundary as in the case of monocytes. The functions f and g are sufficiently smooth and satisfy the following conditions:

$$\begin{aligned} f(A) > 0 \quad \text{for } A > 0, \quad f(0) = 0, \quad f(A) \rightarrow f_+ \quad \text{as } A \rightarrow \infty, \\ g(A) > 0 \quad \text{for } A > A_0, \quad g(A_0) = 0, \quad g(A) \rightarrow g_+ \quad \text{as } A \rightarrow \infty, \end{aligned}$$

and $g'(A) > 0$. We put $A_0 = \frac{b}{\gamma}$. This is a constant level of cytokines in the intima such that the corresponding concentration of monocytes is zero, and they are not recruited through the boundary. If the width h of the domain tends to zero, we can reduce problem (70)–(72) to the 1D problem considered in the previous section.

Consider stationary problem (70)–(72) in the section of the cylinder, that is the system

$$d_M M'' - \beta M = 0, \quad d_A A'' + f(A)M - \gamma A + b = 0 \tag{73}$$

in the interval $0 \leq y \leq h$ with the boundary conditions

$$y = 0: \quad M' = A' = 0, \quad y = h: \quad M' = g(A), \quad A' = 0. \tag{74}$$

Here prime denotes the derivative with respect to y . Problem (73), (74) has a solution $M = 0, A = A_0$ and, for certain values of parameters, another solution $M_s(y), A_s(y)$, which depends on the space variable y . We prove the existence

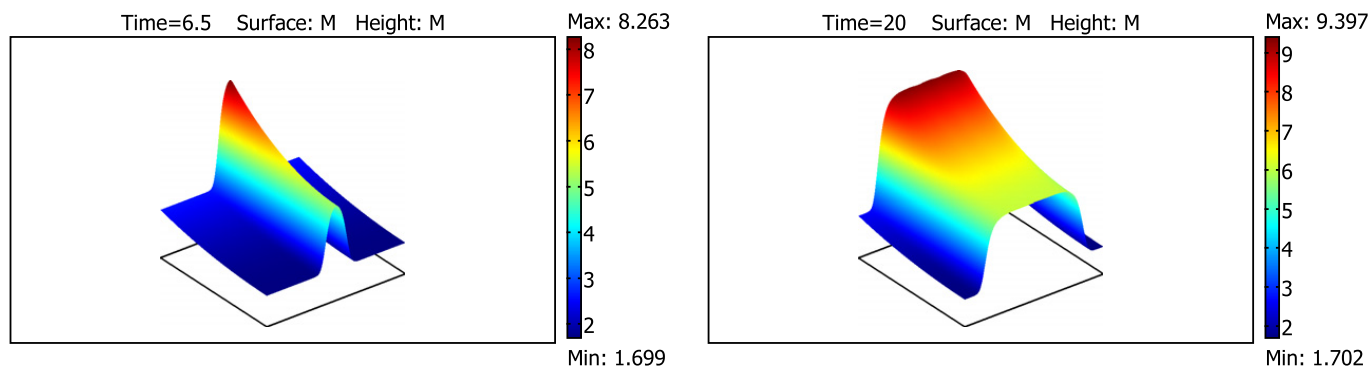


Fig. 23. Propagation of the inflammation in the vessel wall. Distribution of monocytes. Left: beginning of the wave development. Right: the wave propagation.

of travelling waves connecting these two stationary solutions [34] with the methods developed for monotone systems and adapted for the case of nonlinear boundary conditions. These waves describe the transition from the disease free state to atherosclerosis. Fig. 23 shows numerical solution of problem (70)–(72) with a compact initial condition. There are two wave propagating in the opposite directions. This dynamics corresponds to the development of chronic inflammation in blood vessel walls. Similar results are obtained in [108] by cellular automata modelling.

4. Beyond the classical definitions

4.1. Generalized travelling waves

According to the definition given above, travelling wave is a solution propagating with a constant speed and a constant profile. If we consider a nonhomogeneous medium, for example flame propagation across a layered structure, then the speed of propagation and the temperature distribution can depend on time. Therefore, we need to introduce a more general notion of waves. We call them *generalized travelling waves* (GTW). The notion of GTW was first introduced in [126] for reaction–diffusion systems. In order to explain it, let us consider the perturbed reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) + \epsilon g(x, u), \tag{75}$$

where ϵ is a small parameter, $g(x, u)$ is some given function and $F(u)$ is a bistable nonlinearity, that is $F(w_{\pm}) = 0$ for some w_+ and w_- , and $F'(w_{\pm}) < 0$. If $\epsilon = 0$, then under certain conditions on the function F there exists a unique up to translation in space travelling wave solution $u(x, t) = w(x - ct)$ of this equation. If $\epsilon \neq 0$ and $g(x, u)$ depends on x , then travelling wave in the usual definition does not exist. However, we can expect that similar in some sense solutions exist at least for small values of ϵ . Such solutions can be characterized by two main properties:

- (1) They exist for all t from $-\infty$ to $+\infty$. The solution of Eq. (75) with some initial function $u(x, 0) = u_0(x)$ exists for positive time and, generally speaking, cannot be extended for all negative t . It appears that the existence of such global solutions, which exist for all $t \in \mathbb{R}$, can be proved. Moreover, under certain conditions such solution can be unique and stable.
- (2) These are propagating solutions. The property of propagation can be explained as follows. Let q be a constant, $w_+ < q < w_-$. For each t fixed consider the equation $u(x, t) = q$ with respect to x . Denote by $m_q^+(t)$ its maximal solution (if it exists) and by $m_q^-(t)$ its minimal solution. If $m_q^{\pm}(t)/t \rightarrow c$ as $t \rightarrow \infty$, then we say that this solution propagates with the speed c .

Thus, GTW are global propagating solutions. Their existence and structure are studied in [126–128] for reaction–diffusion systems of equations. We note that this definition applies also for autonomous equations. In particular, various oscillating solutions, which appear when a usual travelling wave loses its stability, are particular examples of GTW. Periodic travelling waves [15,86,119,137] and other propagating solutions [16,132] in a nonhomogeneous medium provide other examples.

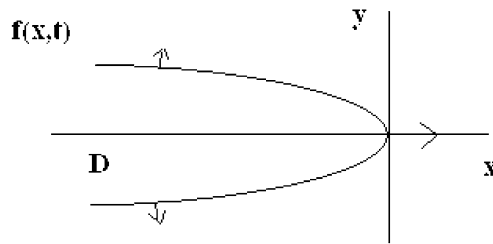


Fig. 24. Schematic representation of travelling wave for a free boundary problem. Domain D moves to the right with a constant speed and shape.

Let us return to the integro-differential equation (59) discussed in Section 2.3. Its principal difference with respect to Eq. (1) is that the homogeneous in space stationary solution w_- can lose its stability [20,48–51]. If this is the case, then some spatial or temporal–spatial structures can bifurcate from it. Therefore, instead of travelling waves connecting w_+ and w_- in the case of the reaction–diffusion equation we can expect the existence of some other solutions connecting w_+ at $+\infty$ with some structures at $-\infty$. These are generalized travelling waves. Some examples of numerical simulations of GTW are presented in Section 2.3. It appears to be possible to prove the existence of GTW in the monostable case [9]. Similar to the reaction–diffusion equation they exist for all values of the velocity greater or equal to the minimal velocity. This result is in agreement with numerical simulations (Section 2.3.3).

4.2. Free boundary problems

Many phenomena in developmental biology can be described by free boundary problems (see, e.g., [42,97]). In this section we will present some free boundary problems specific for growth of cellular structures.

Consider a domain D on the plane (x, y) with the boundary $y = f(x, t)$ which depends on time (Fig. 24). We suppose that it is symmetric with respect to the x -axis. This domain is filled by biological cells. The cells inside the domain do not divide, while the cells at the boundary can proliferate. Their division provides the motion of the boundary in the direction normal to it with some local speed $v_n = \phi(s)$, where ϕ is some give function and s is the length of the boundary from the origin, $s = \int_x^0 \sqrt{1 + (f'_x)^2}$. Then

$$v_n = f'_t \frac{1}{\sqrt{1 + (f'_x)^2}}.$$

We are looking for a travelling wave solution where the boundary of the domain moves with a constant speed c to the right and its shape does not change. Then

$$f(x, t) = h(x - ct),$$

for some function h . Substituting it into the previous equation, we obtain

$$\phi(s) = -\frac{ch'}{\sqrt{1 + (h')^2}}, \quad s = \int_x^0 \sqrt{1 + (h')^2}. \tag{76}$$

It is a closed equation with respect to the function $h(x)$. In order to simplify it, let us notice that $c = \phi(0)$ and introduce the function

$$g(s) = \int_0^s \frac{\phi(\xi)}{\phi(0)} d\xi.$$

Then from (76)

$$\frac{dg(s)}{dx} = h'(x)$$

or

$$g(s) = h(x), \quad s = \int_x^0 \sqrt{1 + (h')^2}. \tag{77}$$

Consider as example the function $\phi(s) \equiv \text{const}$. Then $g(s) = s$ and Eq. (77) cannot have finite solutions. In this case, the boundary of the domain D is vertical.

The function $\phi(s)$ should be specified for each particular problem. Consider now the problem of tip growth in the following formulation. Suppose that a point source of the growth factor ϕ is located at $s = 0$. Then its distribution is described by the equation

$$d\phi'' - a\phi + b\delta(0) = 0.$$

Here the first term in the left-hand side describes diffusion of the growth factor along the surface of the domain, the second term its degradation, b the intensity of the point source. Hence

$$\phi(s) = \frac{b}{2} \sqrt{\frac{d}{a}} e^{-\lambda s}, \quad s \geq 0,$$

where $\lambda = \sqrt{a/d}$. Therefore, $g(s) = \frac{1}{\lambda}(1 - e^{-\lambda s})$. From (76) we obtain the equation for h :

$$-\sqrt{1 + (h'(x))^2} = \frac{h'(x)}{1 - \lambda h(x)}.$$

Without solving it, we can easily conclude that

$$h(x) \sim \frac{1}{\lambda}(1 - e^{\lambda x}) \quad \text{as } x \rightarrow -\infty.$$

Thus, the width of the tip at infinity equal $2/\lambda = 2\sqrt{d/a}$ is independent of the intensity of the point source, the speed of tip growth $\phi(0) = b\sqrt{d}/(2\sqrt{a})$ is proportional to it.

Consider some other related questions.

1. Suppose that cell proliferation at the boundary of the domain is determined by the flux of nutrients coming from outside of the domain D . In the biological context this problem describes growth of solid tumors. It is also related to coral growth [59]. Similar mathematical formulation is given by the Stokes–Leibenzon problem of spreading of viscous fluid in a Hele–Shaw cell. Its travelling wave solutions are described by Saffman–Taylor fingers [52].
2. Consider the problem of plant growth. The domain D is filled by differentiated cells which do not proliferate. Meristem is a narrow layer of proliferating cells at the boundary of the domain. The apical meristem is located at the growing end of the branch, the secondary meristem or cambium at the lateral surface. Under some simplification, we can assert that cell proliferation in the meristem is determined by some biochemical substances produced in the meristem and diffused along this layer. Hence, the function $\phi(s)$ is determined as a solution of a reaction–diffusion system. In the simplest case this can be the scalar reaction–diffusion equation

$$\frac{\partial R}{\partial t} = d \frac{\partial^2 R}{\partial s^2} + cg(R) - \sigma R$$

for the mitosis and growth factor R which determines cell proliferation [19]. For appropriate values of the nutrient concentration c , the nonlinearity has the bistable type. This equation has a stationary solution $\phi(s)$, which can describe branch growth. In the example above we have constructed an explicit solution in the case where the nonlinearity is replaced by a point source.

3. We conclude this section with a model of sea shell growth. Growth occurs at the external section of the spiral tube called opening (Fig. 25). In the mathematical approximation it is a 1D curve moving in the 3D space. The reaction–diffusion system (16), (17) considered along this curve describes dynamics of calcium ions Ca^{2+} which determine cell contraction due to actin–myosin belt and, by this, the direction of growth. The homogeneous in space stationary solution of system (16), (17) can lose its stability either due to Hopf bifurcation or due to a simple eigenvalue crossing the origin and resulting in stationary Turing structures [69]. In the former case, periodic time

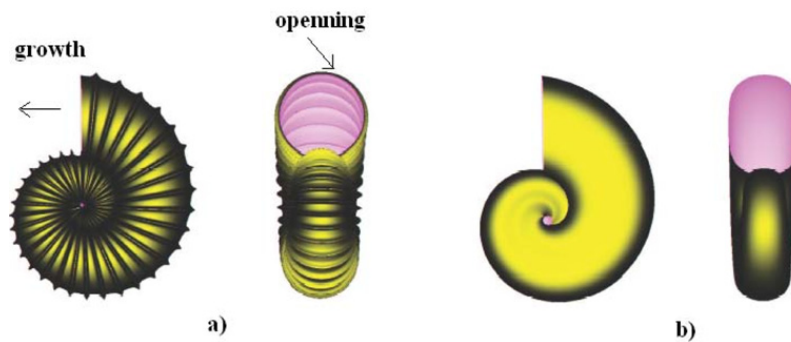


Fig. 25. Shells can have round opening with ribs or elongated opening without ribs. The former correspond to oscillating instability while the latter to stationary spatial structures of the reaction–diffusion system (16), (17). Numerical simulations by N. Bessonov.

oscillations lead to appearance of ribs on the shell surface while the opening is circular (Fig. 25a). In the latter case, the opening is compressed but there are no ribs (Fig. 25b). This corresponds to Buckman’s law which relates the form of the opening and the presence of ribs.

4.3. Patchy invasion

In Section 2.2, we revisited some typical scenarios of travelling wave propagation in a system of interacting species both in one and two spatial dimensions. In particular, we have shown that a 1D system may exhibit a rather wide variety of different regimes of species spread. The travelling wave can have a complicated structure, cf. Figs. 7 to 9, and, in case the local kinetics is oscillatory, front propagation often results in the formation of a chaotic spatiotemporal pattern in its wake. What is observed in the corresponding 2D system looks, for most of these regimes, like a natural extension of this dynamics, i.e. the propagation of a round-shaped travelling boundary which is followed by a pattern formation.

As a whole, however, the dynamics of a 2D system is not just a straightforward extension of a 1D case. There is a phenomenon called “patchy invasion” [105] that is not reducible to a standard travelling wave scenario [83,92,107]. An example is given in Fig. 26 that shows the spatial distribution of the prey density at different moments (time increases from left to right, top to bottom; white colour is for the species absence, red colour for its presence at a high density) obtained by means of numerical simulations in system (38)–(39) when the prey growth is damped by the strong Allee effect and the predation is of Holling type II. The distribution of the predator density is qualitatively similar and therefore is not shown here for the sake of brevity. At $t = 0$, the species were distributed over finite areas of rectangular shape close to the center of the domain. It is readily seen that, at an early stage of the system dynamics (Fig. 26, top, left), the invaded area is of a perfectly round shape. Its size grows with time and hence its boundary propagates as a curvilinear front, in a manner very similar to what has been considered above. However, at a somewhat later stage, the round-shaped patch breaks to pieces and a continuous boundary never reappears. For all later stages, the species spread takes place by means of movement of separate patches. The individual patches grow and interact with each other, some of them can disappear, they can split and produce new patches and, eventually, the patchy pattern invades over the whole domain. We want to emphasize that Eqs. (38)–(39) do not contain space or time explicitly, hence there is no any forcing or pre-defined heterogeneity, and the emerging pattern is completely self-organized.

Although in this case there is no any continuous travelling front that would separate invaded and non-invaded areas, numerical results show that at any given moment of time the patches are concentrated inside a certain area. Thus, one can introduce the diameter of the invaded area as

$$d(t) = \max_{\Omega} ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1/2}$$

where Ω is the invaded domain which can be defined as

$$\Omega = \{(x, y) \mid u(x, y) > \epsilon\}$$

where ϵ is a certain small number. The value of $R(t) = 0.5d(t)$ then gives the position of the imaginary boundary of the invaded area with respect to the place of the original species introduction. Thus, in terms of radius R , the patchy invasion falls into the concept of generalized travelling waves. Interestingly, due to the collapse of some patches

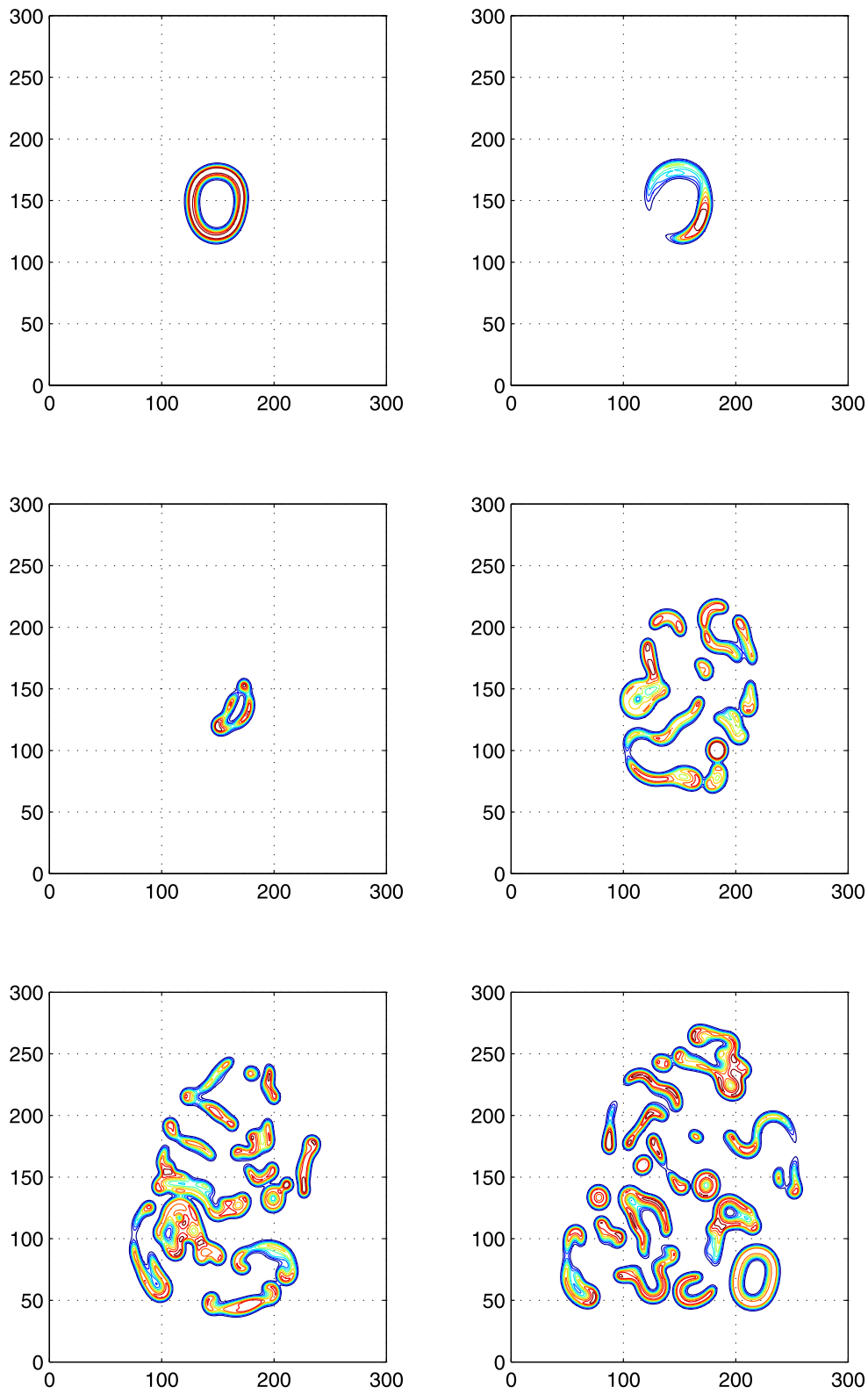


Fig. 26. Patchy invasion: species spread and formation of a chaotic spatiotemporal pattern is not preceded by a propagation of a continuous front. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

that takes place from time to time, both $d(t)$ and the total population size $M(t) = \iint_{\Omega} u(x, y, t) dx dy$ grow non-monotonically [92], e.g. compare Fig. 26, top left and middle left.

A question arises as to what is a possible succession of the different regimes of species spread when parameter values are changed. An extensive numerical study [92,107] shows that a change in the parameter values will either restore the standard regime of continuous travelling fronts or drive the species to extinction. A change in the form of the initial conditions has a similar effect. The patchy spread can therefore be described as an “invasion at the edge of extinction.” Note that the conditions of species extinction appear to be different in the 1D and 2D cases, and the patchy invasion is usually observed for the parameter values when the species go extinct in the corresponding 1D system.

In conclusion, we want to mention that, up to certain restrictions on the parameter values, the patchy invasion is a relatively common phenomenon in the sense that it can be observed in different systems (e.g. in models of population dynamics and/or epidemic spread [83]) when the system is close to extinction.

5. Trends and perspectives

The theory of reaction–diffusion waves develops since the 1930s. It was widely used for the description and explanation of various physical, chemical and biological phenomena. On the other hand, it stimulates mathematical analysis of ordinary and partial differential equations, nonlinear dynamics, bifurcation theory.

There is a number of engineering applications of reaction–diffusion waves, e.g., various combustion devices. In biology, they are more used for qualitative understanding of the processes and less for precise quantitative description and prediction. This situation is clearly related to the complexity of biological phenomena. On the other hand, serious limitation in application of mathematical modelling in biology and medicine is related to the lack of reproducible experiments. However, in some cases it should be possible to choose relatively simple and reproducible model systems. For example, cell populations, including tumors, spread as reaction–diffusion waves. Experiments with cell cultures can be compared with the theory. Experiments on bacterial population dynamics are proposed in [64,65].

Reaction–diffusion waves manifest interesting nonlinear dynamics. Some examples of such dynamics were discussed above. The structure of emerging patterns is often determined by succession of several bifurcations. Mathematical background of complex nonlinear dynamics, especially of chaotic behavior, is important for its understanding.

Over the last two decades, many new models and applications of reaction–diffusion waves have appeared. Some of them were discussed in this paper. Among them, non-local reaction–diffusion equations which have attracted much attention recently. They show rich dynamics, quite different in comparison with the classical reaction–diffusion models. Another example concerns reaction–diffusion systems with nonlinear boundary conditions. It is a new class of models which naturally arise in biological applications. They describe active transport specific for cell membranes or for the boundary between two tissues. Some other applications remained out of the scope of this review. Among them calcium waves involved in intra-cellular regulation [17,61,62,74,123] and blood coagulation [10,36,37,138]. Clot growth is a reaction–diffusion wave whose speed of propagation, when it is smaller than normal, can characterize hemophilia.

Free boundary problems were only briefly discussed here. They describe many problems in developmental biology such as, for instance, plant growth and embryogenesis. In fact, from the mathematical point of view, morphogenesis, that is appearance of biological forms in the process of growth of organism, can also be regarded as a free boundary problem. In particular, in the case of plants, cell proliferation occurs in a narrow cell layer, meristem, and free boundary problems provide the most appropriate approach. Mathematical studies of such problems and their biological applications are in the very beginning of their development.

The concept of generalized travelling waves seems to be important in order to describe wave propagation in inhomogeneous media and waves with nonconstant speed or structure, e.g., periodic or chaotic. We have already discussed above that reaction–diffusion waves provide transitions between various states of the system. If these states are homogeneous in space and time independent, then these are classical travelling waves. If they are space inhomogeneous and/or time-dependent, then such transitions can result in generalized travelling waves. We can mention Turing structures studied in many works devoted to biological pattern formation. However, transitions from the unstable homogeneous solution to the inhomogeneous in space structure are studied much less. The speed of such transition and its pattern can be important for biological applications.

The theory of classical reaction–diffusion waves is also far from being complete. The scalar reaction–diffusion equation and some classes of systems are studied in detail. However, many other systems arising in application, especially with more than two equations and in two or three space dimensions, still require their systematic investigation.

We expect that these topics will determine the development of the theory of reaction–diffusion waves in the context of mathematical biology.

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