TEICHMÜLLER THEORY AND CLUSTER ALGEBRAS:

Cluster Algebras:

Cluster algebras were developed by S. Fomin and A. Zelevinsky in [6], [7], and [8] as an apparatus to study multiplicative properties of dual canonical bases — as defined by Lusztig [15] — for complex semi-simple algebraic groups. For this reason many cluster algebras arise naturally in Lie theory as the coordinate rings of varieties attached to algebraic groups. In such instances a *cluster* is a toric chart on the variety consisting of regular functions, which are referred to as *cluster variables*. The set of all clusters forms a toric atlas for the variey with the key feature — known as the *Laurent Phenomenon* — that the transition functions for any pair of charts (clusters) are always expressible as Laurent polynomials in the coordinates (cluster variables) of the source chart (cluster). Examples of cluster algebras include the coordinate ring of any complex semi-simple algebraic group G see [8], the coordinate ring of any of its *double Bruhat cells* $G^{u,v}$ see [8], and the homogeneous coordinate ring of the Grassmannian $G(k,n)$ see [18].

Teichmüller Theory

The work of M. Gekhtman, M. Shapiro, and A. Vainshtein in [12] is the basis underlying the recent developments relating Teichmüller theory and cluster algebras. In [12] they establish that to a surface $\Sigma_{g,s}$ of genus g with s marked points one can associate a cluster algebra $\mathcal{A}(\Sigma_{g,s})$ for which every *nice* ideal triangulations of the surface $\Sigma_{g,s}$ (whose vertices are the s marked points) gives rise to a cluster inside this algebra. These clusters serve as toric charts on the *cluster manifold* — as defined in [11] — associated to $\mathcal{A}(\Sigma_{g,s})$ and the decorated Teichmüller space $\tilde{T}_g^s$ itself can be realized as the positive part of this manifold (over which these clusters restrict to the charts introduced by Penner in [17]). This construction bears similarity to the approach used by [7] where clusters in the homogeneous coordinate ring of the Grassmannian $G(2,n)$ are parameterized by triangulations of a labelled regular n-gon. In both cases the edges of each triangulation are naturally identified with coordinate functions and the *exchange relations* — which govern the cluster algebra structure — are variations of the Ptolemy rule for inscribed quadrilaterals.

Interest in decorated Teichmüller spaces is in part motivated by the study of its quantization — determined in [13] by R. Kashaev — owing to its use as a model for quantum gravity in $2+1$ dimensions; in the classical setting the phase space of Einstein gravity in a 3 dimensional open manifold M is the Teichmüller space of its boundary ∂M (see [3]). The cluster variables associated to a nice ideal triangulation of $\Sigma_{g,s}$ are known to quasi-commute in Kashaev's quantization of $\tilde{T}_g^s$. This leaves open the possibility of quantizing the cluster algebra $\mathcal{A}(\Sigma_{g,s})$ — in accordance with the technique introduced by A. Berenstein and Zelevinsky in [1] — and using this quantum cluster structure to examine the Kashaev quantization.

In a line of inquiry related to [12], V. Fock and A. Goncharov study in [5] generalizations of the decorated Teichmüller space associated to split semi-simple algebraic groups G. The moduli spaces $\mathcal{X}_{G,\Sigma_g}$ and $\mathcal{A}_{G,\Sigma_g}$ of, respectively, framed and decorated G-local systems on a surface Σ_g with s punctures are shown to be *cluster ensembles* — as defined in their earlier work [4]. Their works relates dilogarithms associated to the classifying space BG with analogs of the Weil-Peterson form for $\mathcal{A}_{G,\Sigma_g}$. In other directions, E. Frenkel and A. Szenes in [10] — and independently F. Chapoton in [2] — have studied a certain monomial degenerations of cluster algebras — so called Y-*systems* — in connection with functional identities of the Roger's dilogarithm.

References

[1] A. Berenstein, A. Zelevinsky, Quantum cluster algebras. Preprint posted April 2004 on the ArXiv server: **math.QA/0404446**. To appear in *Advances in Math.*

[2] F. Chapoton, Functional identities for the Roger's dilogarithm associated to cluster Y-systems. Private communication (2005).

[3] L. Chekhov, R. Penner, On quantizing Teichmüller and Thurston theories. Preprint posted March 2004 on the ArXiv server: **math.AG/0403247**.

[4] V.V. Fock, A.B. Goncharov, Cluster ensembles, quantization, and the dilogarithm. Preprint posted November 2003 on the ArXiv server: **math.AG/0311245**.

[5] V.V. Fock, A.B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory. Preprint posted November 2003 on the ArXiv server: **math.AG/0311149**.

[6] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.*, **15** (2002), 497-529.

[7] S. Fomin and A. Zelevinsky, Cluster algebras II: Finite type classification, *Invent. Math.*, **154** (2003), 63-121.

[8] S. Fomin and A. Zelevinsky, Cluster algebras III: Upper bounds and double Bruhat cells, *Duke Math. Journal*, **126** (2005), 1-52.

[9] S. Fomin and A. Zelevinsky, Y-systems and generalized associahedra, *Annals of Mathematics*, **158** (2003), 977-1018.

[10] E. Frenkel, A. Szenes, Thermodynamic Bethe Ansatz and dilogarithm identities I, *I. Math. Res. Lett.*, **2** (1995), 677-693.

[11] M. Gekhtman, M. Shapiro, A. Vainshtein, Cluster algebras and Poisson geometry, *Moscow Math. Journal*, **3** (2003), 899-934.

[12] M. Gekhtman, M. Shapiro, A. Vainshtein, Cluster algebras and Weil-Petersson forms. Preprint posted September 2003 on the ArXiv server: **math.QA/0309138**.

[13] R. M. Kashaev, Quantization of Teichmüller spaces and the quantum dilogarithm, *Lett. Math. Phys.*, **43** (1998), 105-115.

[14] A. N. Kirillov, Dilogarithm identities, *Progress of Theor. Phys. Suppl.*, **118** (1995), 61-142.

[15] G. Lusztig, Canonical bases arising from quantized enveloping algebras, *J. Amer. Math. Soc.* **3** no. 2 (1990), 447-498.

[16] R. Marsh, M. Reineke, A. Zelevinsky, Generalized associahedra via quiver representations, *Trans. Amer. Math. Soc.*, **355** (2003), 4171-4186.

[17] R. Penner, The decorated Teichmüller space of punctured surfaces, *Comm. Math. Phys.*, **113** (1987), 299-339.

[18] J. Scott, Grassmannians and cluster algebras, Preprint posted August 2003 on the ArXiv server: **math.CO/0311148**. Accepted February 2005 to *London Mathematical Society.*

[19] D. Thurston, From dominoes to hexagons. Preprint posted May 2004 on the ArXiv server: **math.CO/0405482**.