From hyperbolic regularization to exact hydrodynamics via simple kinetic models

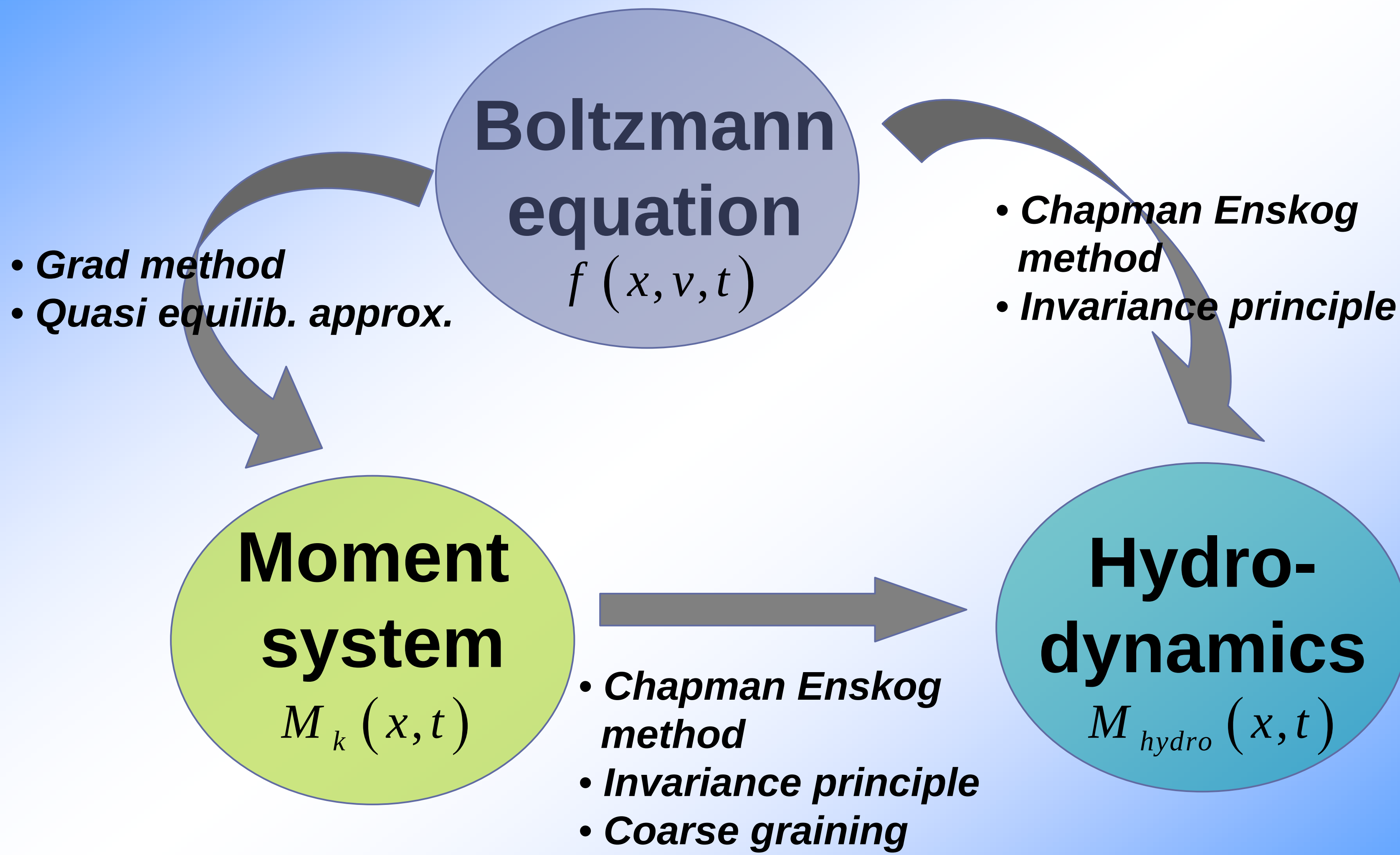
Matteo Colangeli
I.V.Karlin, M. Kroeger

ETH Zurich
Switzerland

Outline:

- Kinetic theory and methods of reduced description
- The meaning of stability: “*H-theorem*”
- The concept of hyperbolicity for O.D.E.’s
- Relation between hyperbolicity and stability (by Bobylev) for *Burnett equations*
- *Exact Hydrodynamics* derived from13M Grad System
- The message of hyperbolicity
- The goal: exact hydrodynamics from Boltzmann.
- What’s next?

1. Kinetic theory and methods of Reduced Description:



• Phase space representation

Phase space *E* consists of distribution functions $f\left(x,v,t\right)$:

$$fdxdv =$$ particles in the volume element $dxdv$ around $\left(x,v\right)$ at time t

with: $x \in \Omega_x^3 \subseteq \Re_x^3, v \in \Omega_v^3 \subseteq \Re_v^3$

Good definition of $f\left(x,v,t\right)$: $f\left(x,v,t\right) \in F$ are nonnegative functions and the following integrals are finite $\forall x \in \Omega_x$:

- $I_x^{(i_1,i_2,i_3)} = \int\limits_{\Re_v^3} v_1^{i_1} v_2^{i_2} v_3^{i_3} f\left(v,x,t\right) d^3v,\quad i_1 \geq 0, i_2 \geq 0, i_3 \geq 0$; (existence of the moments)

Lower order moments correspond to the hydrodynamic fields:

$$\int f\left(x,v\right) m dv = \rho\left(x,t\right)$$

$$\int f\left(x,v\right) m v dv = \rho u\left(x,t\right)$$

$$\int f\left(x,v\right) m \frac{\left(v-u\right)^2}{2} dv = e\left(x,t\right)$$

- $H_x\left(f\right) = \int\limits_{\Re_v^3} f\left(v,x,t\right)\left[\ln f\left(v,x,t\right) - 1\right] d^3v \;, H\left(f\right) = \int\limits_{\Re_x^3} H_x\left(f\right) d^3x;$

(existence of the entropy)

• Boltzmann Equation:

$$\frac{\partial f}{\partial t} = J(f) \quad J(f) = -v\frac{\partial f}{\partial x} + Q(f,f)$$

Properties:

- Additive invariants of the collision operator:

$$\int Q(f,f)\{1, v, v^2\}dv = 0$$

- Detailed balance ($Q(f,f) = 0$) :

$$f(v', x, t)f(w', x, t) = f(v, x, t)f(w, x, t)$$

with f given by: $f = f_{\substack{local \\ Maxwellian}} = \frac{\rho}{m}\left(\frac{2\pi k_B T}{m}\right)^{-3/2} \exp\left(\frac{-m(v-u)^2}{2K_B T}\right), \quad e = \frac{3\rho}{2m}K_B T$

- Local *H-theorem* :

$$\frac{dH_x(f)}{dt} = \int Q(f,f)\Big|_{f=f(x,v,t)} \ln f(x,v,t)\,d^3v \le 0$$

entropy production inequality: $\sigma = -k_B \frac{dH_x}{dt} \ge 0$

$f(x,v,t)$ solution of ***B.E.*** evolves, according to the vector field $J(f)$, over a "∞*-dimensional manifold*" F.

Notion of *Local Manifold*: given a finite-dimensional linear space of parameters *A*, a bounded domain *B* in *A*, and a family of functions $f(a,v,t)$ smoothly parameterized by $a \in B$ we consider all bounded and sufficiently smooth functions $a(\bullet):\Omega_x \to B$ and we define the *locally finite-dimensional* manifold as the set of functions

$$f(a(x),v,t)$$

A simple example give the local equilibria: *a* is the 5D vector of density, momentum density and temperature, *f* are Maxwellians.

Problem of reduced description for dissipative systems

$\Updownarrow$

Finding stable invariant manifolds in the space of distribution functions.

Locally finite-dimensional manifolds are the natural sources of approximations for the construction of invariant manifolds in the B.E. theory.

The notion of Invariant Manifold generalizes other historically previous methods to reduce description in dissipative systems:

- Hilbert method of normal solutions

- Chapman Enskog method

- Grad method.

Whatever the method, the technique is to seek for trial solutions of the B.E. through projections upon proper *locally-finite dimensional submanifolds*.
Still, usually, such manifolds are NOT invariant $\left(\Delta_f \neq 0\right)$.

Example: the locally five-dimensional manifold of local Maxwellians $\{f_{LM}(n,u,T)\}$

f_{LM} is the unique solution of the variational problem:

$$H(f)=\int f\ln f d^3v \to \min$$

for:

$$M_0(f)=\int f(x,v)d^3v=n$$

$$M_i(f)=\int f(x,v)v d^3v=nu_i,\ i=1,2,3$$

$$M_4(f)=\int f(x,v)v^2 d^3v=\frac{3nk_BT}{m}+nu^2$$

Hence, the LM manifold is the *quasiequilibrium* manifold.

Is it also an invariant manifold for the Boltzmann Equation?

Defect of invariance: $\Delta_y=(1-P_y)J(F(y))$

Thermodynamic projector: $P_{f_0(n,u,T)}(J)=\sum_{s=0}^{4}\frac{\partial f_0(n,u,T)}{\partial M_s}\int \psi_s J d^3v$

$$\Delta(f_{LM})=f_{LM}\left\{\left(\frac{m(v-u)^2}{2k_BT}-\frac{5}{2}\right)(v_i-u_i)\frac{\partial \ln T}{\partial x_i}+\frac{m}{k_BT}\left((v_i-u_i)(v_s-u_s)-\frac{1}{3}\delta_{is}(v-u)^2\right)\frac{\partial u_s}{\partial x_i}\right\}$$

• Chapman Enskog method: a sketch

Solutions to the B.E. are found from a singularly perturbed B.E.,

$$D_t f = \frac{1}{\varepsilon} Q(f, f)$$

where ε is a small parameter, and: $D_t f = \frac{\partial}{\partial t} f + \left(v, \frac{\partial}{\partial x} \right) f$

f is written as an expansion: $f_{CE} = \sum_{n=0}^{\infty} \varepsilon^n f_{CE}^{(n)}$

The procedure of evaluation of the functions is:

$$Q\left(f_{CE}^{(0)}, f_{CE}^{(0)}\right) = 0 \longrightarrow f_{CE}^{(0)} = f_{LM}$$

$$L f_{CE}^{(1)} = -Q\left(f_{CE}^{(0)}, f_{CE}^{(0)}\right) + \frac{\partial^{(0)}}{\partial t} f_{CE}^{(0)} + \left(v, \frac{\partial}{\partial x} \right) f_{CE}^{(0)}$$

The operator $\partial^{(0)} / \partial t$ is defined as:

$$\frac{\partial^{(0)}}{\partial t} \{\rho, \rho u, e\} = -\int \left\{ m, mv, \frac{mv^2}{2} \right\} \left(v, \frac{\partial}{\partial x} \right) f_{CE}^{(0)} d^3 v$$

EULER EQUATIONS

Further, $\partial^{(0)}/\partial t$ acts upon $f_{CE}^{(0)}$ according to the chain rule:

$$\frac{\partial^{(0)}}{\partial t} f_{CE}^{(0)} = \frac{\partial f_{CE}^{(0)}}{\partial \rho}\frac{\partial^{(0)}}{\partial t}\rho + \frac{\partial f_{CE}^{(0)}}{\partial(\rho u)}\frac{\partial^{(0)}}{\partial t}(\rho u) + \frac{\partial f_{CE}^{(0)}}{\partial e}\frac{\partial^{(0)}}{\partial t}e$$

According to the theory of linear integral equations, the function $f_{CE}^{(1)}$ is unique, once the following "*solubility condition*" *is* provided*:*

$$\int \{1, v, v^2\} f_{CE}^{(1)} d^3 v = 0$$

The first correction adds the terms:

$$\frac{\partial^{(1)}}{\partial t}\{\rho, \rho u, e\} = -\int \left\{m, mv, \frac{mv^2}{2}\right\}\left(v, \frac{\partial}{\partial x}\right) f_{CE}^{(1)} d^3 v$$

DISSIPATIVE HYDRODYNAMICS (expressions for the stress tensor and the heat flux)

The sequence provides higher order corrections, corresponding to different hydrodynamic models:

$f_{CE}^{(0)} \rightarrow$ Euler equations

$f_{CE}^{(0)} + \varepsilon f_{CE}^{(1)} \rightarrow$ Navier Stokes – Fourier equations

$f_{CE}^{(0)} + \varepsilon f_{CE}^{(1)} + \varepsilon^2 f_{CE}^{(2)} \rightarrow$ Burnett equations

$\ldots \rightarrow \ldots$

• Grad method: a sketch

Assumption behind: decomposition of motions:

• during fast evolution (time τ), a set of distinguished moments M' don't change significantly in comparison to the rest of moments M" (fast dynamics)

• towards the end of the fast evolution, the values of M" become determined by the values of M'.

• on the time of order $\theta >> \tau$, dynamics of the distribution function is determined by the dynamics of M' (slow evolution period).

$$f_G(M',v) = f_{LM}(\rho,u,e,v)\left[1+\sum_{(\alpha)}^{N} a_{(\alpha)}(M')H_{(\alpha)}(v-u)\right]$$

Upon inserting f_G into the B.E. and finding the moments of the resulting expression, we get the *Grad's moment equations.*

2. The meaning of stability : *H-theorem*

- If the gas is in an uniform state (f space independent), the H-function reads as:

$$H(f) = \int f(v) \ln f(v) d^3 v$$

then: $\dfrac{dH}{dt} \leq 0$. Thus, B.E. describes relaxation to the unique Global Maxwellian (whose parameters are fixed by initial conditions).

- Entropy density $S = -k_B H(f) + const.$ grows monotonically (Lyapunov functional) along the solutions (relation defining entropy also for beyond equilibrium states).

- H-theorem and Grad system:

$f_G = f_{LM}(1+\varphi)$ then expanding H-function in the neighborhood of f_{LM}

$$\Delta H(x,t) \equiv \Delta H[f_{LM}, \varphi] = H^0 + \int f_{LM} \ln f_{LM} \varphi(v) d^3 v + \frac{1}{2} \int f_{LM} \varphi^2(v) d^3 v$$

for 13 Moment Grad approximation: $\Delta H = H^0 + n \dfrac{\sigma_{ik}\sigma_{ik}}{4P^2} + n \dfrac{q_k q_k \rho}{5P^3}$

3. The concept of hyperbolicity for O.D.E.'s

Let E be a unitary space with (complex) scalar product $(\cdot,\cdot)$
We consider the *Cauchy* problem for a vector $u(t) \in E, t \geq 0$

$$\partial_t u = iBu + \frac{1}{\varepsilon} Au \quad , \ \varepsilon > 0 \qquad (3.1)$$

$$u\big|_{t=0} = u_0$$

under the following assumption about the operators A, B :

- both A and B are real and symmetric, i.e.:

$$\overline{A} = A, \overline{B} = B, (u_1, Au_2) = (Au_1, u_2), (u_1, Bu_2) = (Bu_1, u_2)$$

- A is negative semi-definite, i.e.:

$$(u, Au) \leq 0 \ , \forall u \in E$$

- the equation $Au = 0$ has precisely $1 \leq m \leq \dim E$ linearly independent solutions $u = e_\alpha, \alpha = 1, \ldots, m$, then:

$$N(A) = KerA = Span(e_1, \ldots, e_m)$$

It follows:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \frac{1}{\varepsilon}(Au, u) \ , \ \|u\|^2 = (u,u) \qquad (3.2)$$

We assume: $\dim E < \infty \longrightarrow E = N(A) \oplus R(A)$

4. Relation between hyperbolicity and stability (by Bobylev) for Burnett equations derived from B.E.:

Linearization of B.E. near the global Maxwellian: $f = f_{GM}(1+g)$

$$\partial_t g + v \cdot \partial_x g = \frac{1}{\varepsilon} L g$$

Fourier transform: $\hat{g}(k,v,t) = \int_{\Re^3} g(x,v,t) e^{-ik\cdot x} d^3x$

and obtain: $\partial_t \hat{g} + ik \cdot v \hat{g} = \frac{1}{\varepsilon} L \hat{g}$ (cfr. with (3.1)).

Then, application of C.E. method, leads to the general equation of hydrodynamics:

$$\partial_t x = [iB(\varepsilon) + \varepsilon A(\varepsilon)]x \;,\; \varepsilon > 0 \,,\; x \in N(A) \qquad (4.1)$$

A, B symmetric for Euler and N-S level $\Longrightarrow$

- $\|x(t)\| = \|x(0)\|$ for Euler
- $\|x(t)\| \le \|x(0)\|$ for N-S

B not symmetric for Burnett level $\Longrightarrow$ (3.2) does NOT hold!

Loss of symmetry of operator B is the reason of the instability of Burnett equations

• Bobylev instability (1982) :

Boltzmann equation, f (∞-dimensional manifold)

CHAPMAN ENSKOG

$$f \rightarrow f_{LM} + \varepsilon f_{CE}^{(1)} + \varepsilon^2 f_{CE}^{(2)} + \ldots$$

Hydrodynamics: $\partial_t \int \psi_i \left(f_{LM} + \varepsilon f_{CE}^{(1)} + \varepsilon^2 f_{CE}^{(2)} + \ldots \right) d^3 v = \int \psi_i J\left(f_{LM} + \varepsilon f_{CE}^{(1)} + \varepsilon^2 f_{CE}^{(2)} + \ldots \right) d^3 v$

Violation of the H-theorem at the Burnett level, was found by Bobylev for Maxwell molecules and then also studied by Uribe *et al.* for hard sphere molecules.

• Bobylev regularization: HBEs (2006) :

General equation of hydrodynamics:

$$\partial_t x = \left[iB(\varepsilon) + \varepsilon A(\varepsilon)\right]x = \left[i\left(B_0 + \varepsilon^2 B_1 + \ldots\right) + \varepsilon\left(A_0 + \varepsilon^2 A_1 + \ldots\right)\right]x$$

change of coordinates: $z = Tx = \left(1 + \varepsilon^2 R\right)x \longrightarrow x = T^{-1}z = \left[1 - \varepsilon^2 R + O\left(\varepsilon^4\right)\right]z$

$$\partial_t z = T\left[iB(\varepsilon) + \varepsilon A(\varepsilon)\right]T^{-1}z = \left\{i\left[B_0 + \varepsilon^2\left(B_1 + RB_0 - B_0 R\right)\right] + \varepsilon A_0\right\}z + O\left(\varepsilon^3\right)$$

- A_0 already symmetric and negative semi-defined.
- $\widetilde{B} = B_1 + RB_0 - B_0 R$ real and symmetric

Hyperbolicity achieved through a “regularization” operator R (real and symmetric)

$\downarrow$

H-Theorem restored

- The idea of the work :

Boltzmann equation, f

GRAD METHOD $f \to f_{LM}(1+\varphi)$

CHAPMAN ENSKOG

Moment system:

$$\partial_t \int v_1^{i_1} v_2^{i_2} v_3^{i_3} f_{LM}(1+\varphi) d^3v = \int v_1^{i_1} v_2^{i_2} v_3^{i_3} J\left[f_{LM}(1+\varphi)\right] d^3v$$

CHAPMAN ENSKOG

$$\sigma \to \varepsilon\sigma^{(0)}\varepsilon^2\sigma^{(1)} + \ldots$$

$$q \to \varepsilon q^{(0)}\varepsilon^2 q^{(1)} + \ldots$$

Hydrodynamics

Burnett hydrodynamics, as derived from Grad system, displays the same instability seen through derivation from B.E. (consistency):

$$d_t \Delta H = d_t \int \left(H^{(0)} + n \frac{\sigma_{ik}\sigma_{ik}}{4P^2}\bigg|_{\sigma = \varepsilon\sigma^{(0)} + \varepsilon^2\sigma^{(1)}} + n \frac{q_k q_k \rho}{5P^3}\bigg|_{q = \varepsilon q^{(0)} + \varepsilon^2 q^{(1)}} \right) d^3x = ?$$

<u>Questions</u>: does the hyperbolic regularization apply on the 13 Moment Grad system and restore the H-Theorem at the Burnett level? Does it work also to higher order levels?

5. *Exact Hydrodynamics* from 13M Grad System :

1D13 Moment (linear) Grad system attains the following form (1D case):

$$
\begin{aligned}
&\partial_t \rho = -\partial_x u \\
&\partial_t u = -\partial_x \rho - \partial_x T - \partial_x \sigma \\
&\partial_t T = -\frac{2}{3}\partial_x u - \frac{2}{3}\partial_x q \\
&\partial_t \sigma = -\frac{4}{3}\partial_x u - \frac{8}{15}\partial_x q - \sigma \\
&\partial_t q = -\frac{5}{2}\partial_x T - \partial_x \sigma - \frac{2}{3} q
\end{aligned}
\tag{5.1}
$$

Turning into the Fourier space, we seek for solutions of the form $\zeta = \zeta_k \exp(\omega t + ikx)$ where ζ is a generic function and k is a real valued wavenumber proportional to the Knudsen number ε.

Application of the Chapman-Enskog method to the reduction of the system (1) results in the following series expansion of the nonhydrodynamic variables:

$$\sigma_k = \sum_{n=0}^{\infty} \sigma_k^{(n)}, q_k = \sum_{n=0}^{\infty} q_k^{(n)}$$

It can be proven that functions σ_k and q_k have the following structure, for all $n = 0,1,...$

$$\sigma_k^{(2n)} = a_n\left(-k^2\right)^n iku_k$$

$$\sigma_k^{(2n+1)} = b_n\left(-k^2\right)^{n+1}\rho_k + c_n\left(-k^2\right)^{n+1}T_k$$

$$q_k^{(2n)} = x_n\left(-k^2\right)^n ik\rho_k + y_n\left(-k^2\right)^n ikT_k$$

$$q_k^{(2n+1)} = z_n\left(-k^2\right)^{n+1}u_k$$

Hence we can express σ_k and q_k as:

$$\sigma_k = ikAu_k - k^2B\rho_k - k^2CT_k \qquad (5.2)$$

$$q_k = ikX\rho_k + ikYT_k - k^2Zu_k$$

where: $$A(k) = \sum_{n=0}^{\infty} a_n\left(-k^2\right)^n, B(k) = \sum_{n=0}^{\infty} b_n\left(-k^2\right)^n,... \qquad (5.3)$$

Substituting these expressions into the Fourier-transformed balance equations (5.1), we obtain the closed (reduced) system of hydrodynamic equations which is conveniently written in a vector form:

$$\partial_t x = Mx \qquad (5.4)$$

with: $x=\{\rho,u,T\},\ M=\begin{pmatrix} 0 & ik & 0 \\ ik\left(1-k^2B\right) & -k^2A & ik\left(1-k^2C\right) \\ -\frac{2}{3}k^2X & \frac{2}{3}ik\left(1-k^2Z\right) & -\frac{2}{3}k^2Y \end{pmatrix}$ (5.5)

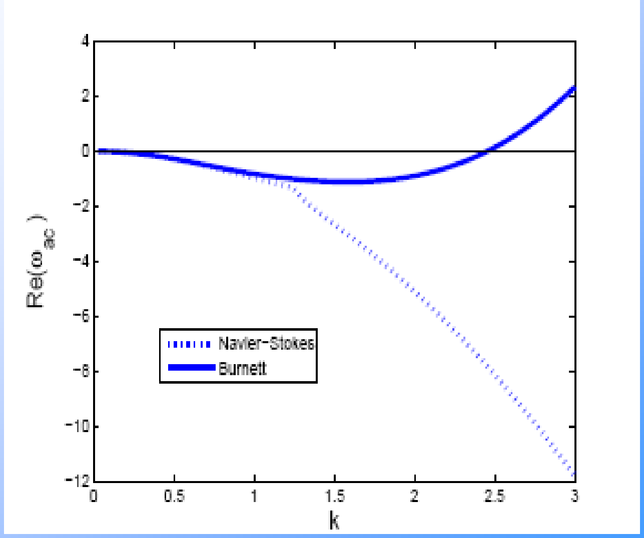
Finally we find the dispersion relation for the hydrodynamic modes $\omega(k)$ by solving the characteristic equation:

$$\det\left[M-\omega I\right]=0 \tag{5.6}$$

Chapman Enskog approximates the functions (5.3) with polynomials whose coefficients can be explicitly determined through a *nonlinear* recurrence procedure.
At the Burnett level:

$$\sigma_k=-\frac{4}{3}iku_k+\frac{4}{3}k^2\rho_k-\frac{2}{3}k^2T_k$$

$$q_k=-\frac{15}{4}ikT_k+\frac{7}{4}k^2u_k$$



The standard CE polynomial approximations lead to unstable hydrodynamic equations.
Dealing with a linear kinetic model it is still possible to search for *exact hydrodynamic solutions*, i.e., constitutive relations for the stress tensor and the heat flux, achieved by performing exact summation of the functions in (5.3).

exact summation of Chapman Enskog $\Leftrightarrow$ **Dynamic Invariance principle**

Dynamic Invariance Principle (DIP):

$$\partial_t^{micro}\sigma_k = -\frac{4}{3}iku_k - \frac{8}{15}ikq_k\left(X,Y,Z,k\right) - \sigma_k\left(A,B,C,k\right) \qquad \partial_t^{macro}\sigma_k = \frac{\partial\sigma_k}{\partial\rho_k}\partial_t\rho_k + \frac{\partial\sigma_k}{\partial u_k}\partial_t u + \frac{\partial\sigma_k}{\partial T_k}\partial_t T_k$$

$$\partial_t^{micro}q_k = -\frac{5}{2}ikT_k - ik\sigma_k\left(A,B,C,k\right) - \frac{2}{3}q_k\left(X,Y,Z,k\right) \qquad \partial_t^{macro}q_k = \frac{\partial q_k}{\partial\rho_k}\partial_t q_k + \frac{\partial q_k}{\partial u_k}\partial_t u_k + \frac{\partial q_k}{\partial T_k}\partial_t T_k$$

The DIP states that the two time derivatives coincide, since the set $\left\{\sigma_k, q_k\right\}$ has to solve both the full Grad system and the reduced system.

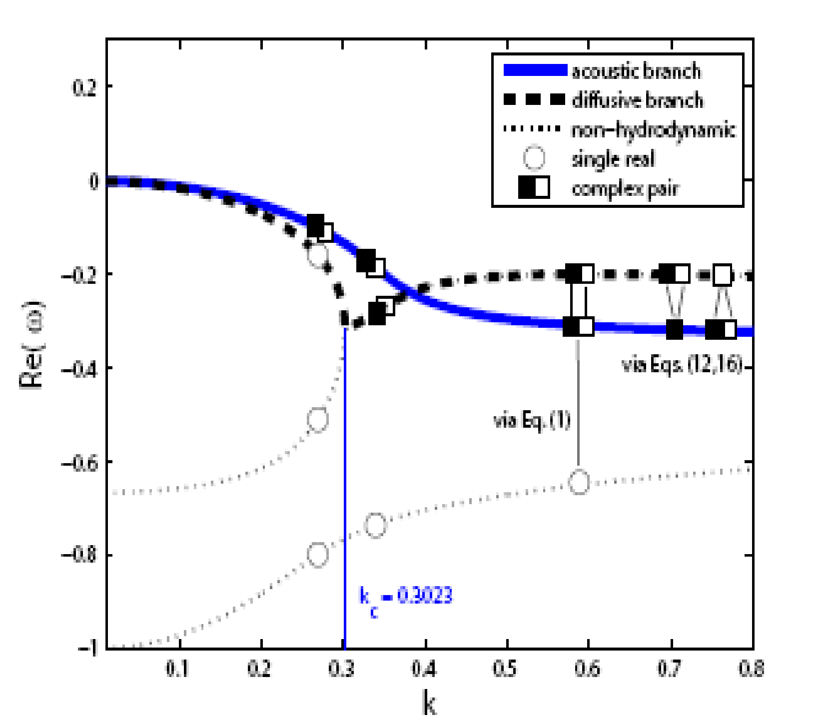
DIP implies a closed set of equations, here referred as invariance equations (IE), relating the six functions $A, B, \ldots, Z$

$$
\begin{aligned}
-\frac{4}{3} - A - k^2\left(A^2 + B - \frac{8Z}{15} + \frac{2C}{3}\right) + \frac{2}{3}k^4 CZ &= 0, \\
\frac{8}{15}X + B - A + k^2 AB + \frac{2}{3}k^2 CX &= 0, \\
\frac{8}{15}Y + C - A + k^2 AC + \frac{2}{3}k^2 CY &= 0, \\
A + \frac{2}{3}Z + k^2 ZA - X - \frac{2}{3}Y + \frac{2}{3}k^2 YZ &= 0, \\
k^2 B - \frac{2}{3}X - k^2 Z + k^4 ZB - \frac{2}{3}YX &= 0, \\
-\frac{5}{2} - \frac{2}{3}Y + k^2(C - Z) + k^4 ZC - \frac{2}{3}k^2 Y^2 &= 0.
\end{aligned}
\qquad (5.7)
$$

The dispersion relation was found by simultaneously solving numerically the IE (5.17) and the characteristic equation (5.6).

The resulting hydrodynamic spectrum consists of two modes:

- $\omega_{ac}(k)$ acoustic mode, represented by two complex-conjugated roots of (5.6)
- $\omega_{diff}(k)$ real-valued diffusive heat mode.



Among the many sets of solutions to the system (5.7), the relevant ones are continuous functions with the asymptotics: $\lim_{k \to 0} \omega_{hydr} = 0$

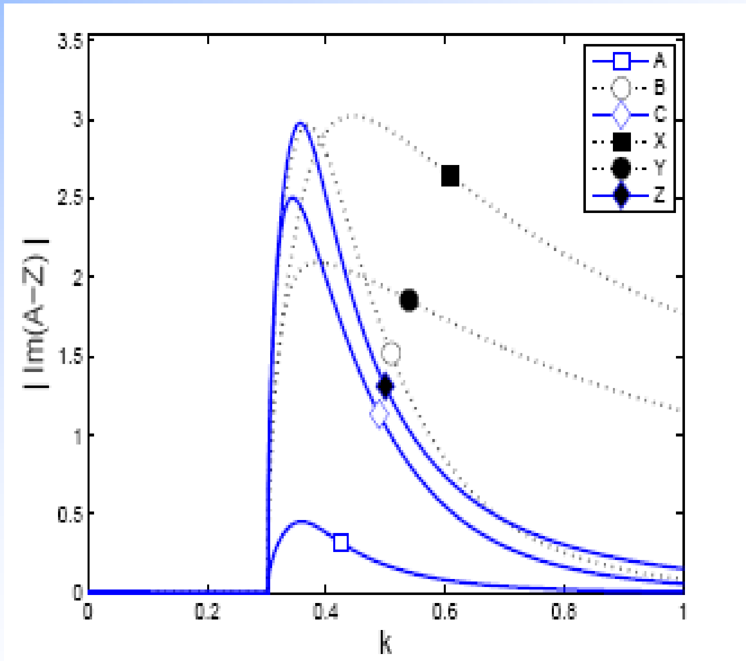
Remarkably, we find that the solution with this asymptotics is *unique*, and represented by a pair of complex conjugated sets $\{S, S^*\}$.

Notice that:

- **exact hydrodynamics is stable (i.e.: H-theorem exists)**.
  Hence the failure of the CE method does NOT lie in the method itself, but in the truncation to lower order levels.

- **a critical point occurs**, for $k = k_c$.
  For $k > k_c$, the CE method does not recognize any longer the resulting diffusive branch as an extension of a hydrodynamic branch.

Also, the set of solutions $\{S, S^*\}$, real valued for $k \le k_c$, continues upon a complex manifold.



The first important message: there is no closed set of hydrodynamic equations after k_c , even though the acoustic mode extends smoothly beyond k_c.

Is exact (stable) hydrodynamics also hyperbolic?

$$\partial_t x = [\Re(M) - i\Im(M)]x \qquad (5.8)$$

$$\Re(M) = \sum_{n=0}^{\infty} (-1)^n R^{(n)} \varepsilon^{2n+1} = \varepsilon R^{(0)} - \varepsilon^3 R^{(1)} + O(\varepsilon^5)$$

$$\Im(M) = \sum_{n=0}^{\infty} (-1)^n R^{(n)} \varepsilon^{2n} = I^{(0)} - \varepsilon^2 I^{(1)} + \varepsilon^4 I^{(2)} O(\varepsilon^6)$$

We find that the operators $\Re(M)$ and $\Im(M)$ involve the following real-valued operators:

$$I^{(n)} = k^{2n+1}\begin{pmatrix} 0 & \delta_{n,0} & 0 \\ b_{n-1} & 0 & c_{n-1} \\ 0 & \frac{2}{3}z_{n-1} & 0 \end{pmatrix} \qquad R^{(n)} = k^{2n+2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_n & 0 \\ \frac{2}{3}x_n & 0 & \frac{2}{3}y_n \end{pmatrix}$$

Equations of hydrodynamics (5.8) are **hyperbolic** and **stable** provided that we can find a transformation of hydrodynamic fields, such that:

- $\Re(M')$ and $\Im(M')$ are both real and symmetric
- $\Re(M')$ has negative semidefinite eigenvalues.

Therefore, we seek a transformation Ω such that $M' = \Omega M \Omega^{-1}$ is symmetric and $\Re(M')$ is semi-negative.

$$\Omega = \frac{1}{\Omega_{uu}}\begin{pmatrix} \frac{1}{\Omega_{\rho\rho}} & 0 & \frac{1}{\Omega_{\rho T}} \\ 0 & \Omega_{uu} & 0 \\ 0 & 0 & \Omega_{TT} \end{pmatrix} \qquad (5.9)$$

$$\Omega_{\rho\rho} = \frac{\Omega_{uu}^2}{\sqrt{3X + 2Y[[Z]]}}$$

$$\Omega_{uu} = \sqrt{X[[3B - 2Z[[C]] - 2C]] + 2Y[[B]][[Z]]}$$

$$\Omega_{\rho T} = \frac{3[[C]]X}{\sqrt{3X + 2Y[[Z]]}}$$

$$\Omega_{TT} = \sqrt{3[[C]](Y[[B]] - [[C]]X)}$$

with the symbolic notation:

$$[[\bullet]] = 1 - (k\varepsilon)^2 \bullet$$

With Ω provided by (5.9), we have that the resulting hydrodynamics is hyperbolic. We, next, calculate the eigenvalues of $\Re(M')$ (containing transport coefficients):

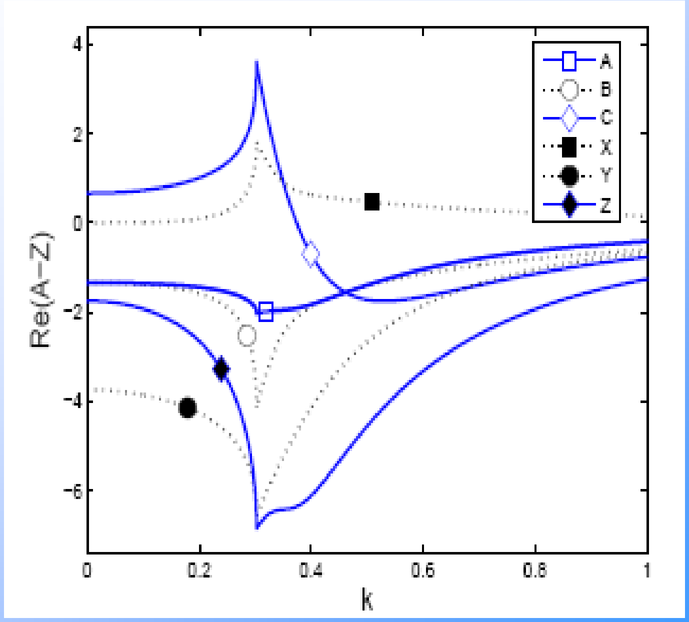
$$\lambda_1 = 0, \lambda_2 = k^2 \varepsilon A, \lambda_3 = \frac{2}{3} k^2 \varepsilon Y \qquad (5.10)$$

Hence, the *T-transformed* hydrodynamics reads as:

$$\partial_t z = \Omega M \Omega^{-1} z = M' z$$

where: $(M')^T = M' \longrightarrow$ **HYPERBOLICITY**

$$\left\{ \begin{array}{l} Tr(M') \leq 0 \\ Det(M') \geq 0 \end{array} \right. \longrightarrow \textbf{DISSIPATIVITY}$$



•Hyperbolic dissipative hydrodynamics from 3D13M Grad system

$$\partial_t \rho_k = -ik \cdot u_k$$

$$\partial_t u_k = -ik(\rho_k + T_k) - ik \cdot \sigma_k$$

$$\partial_t T_k = -\frac{2}{3} ik \cdot (u_k + q_k)$$

$$\partial_t \sigma_k = -2i\overline{ku_k} - \frac{4}{5} i\overline{kq_k} - \sigma_k$$

$$\partial_t q_k = -\frac{5}{2} ikT_k - ik \cdot \sigma_k - \frac{2}{3} q_k$$

We decompose vectors and tensors into longitudinal and transversal parts:

$$e_k = \frac{k}{|k|}$$

$$u_k = u_k^p e_k + u_k^{\perp}; q_k = q_k^p e_k + q_k^{\perp}; \sigma_k = \sigma_k^p \overline{e_k e_k} + 2\sigma_k^{\perp}$$

to obtain two closed sets of equations which can be solved separately

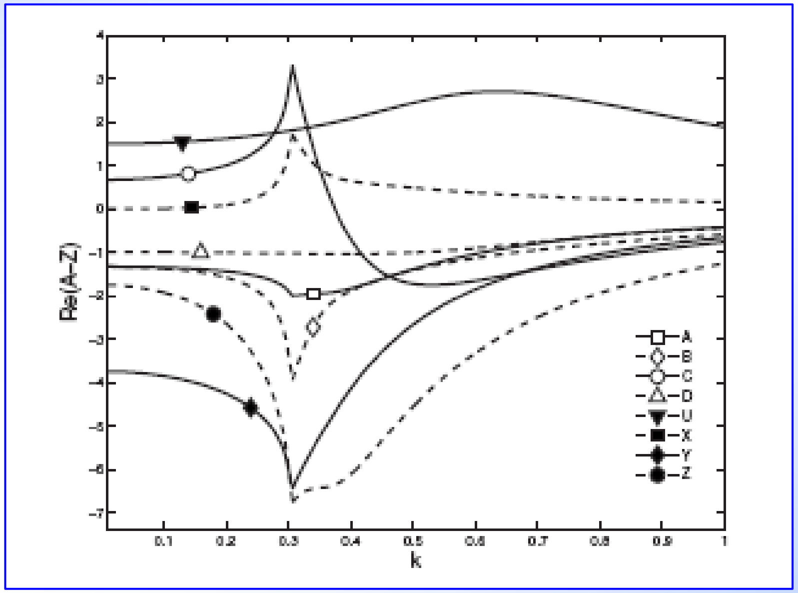
$$\sigma_k^p = ikAu_k^p - k^2 B\rho_k - k^2 CT_k \qquad \sigma_k^\perp = ikD\overline{eu_k^\perp}$$

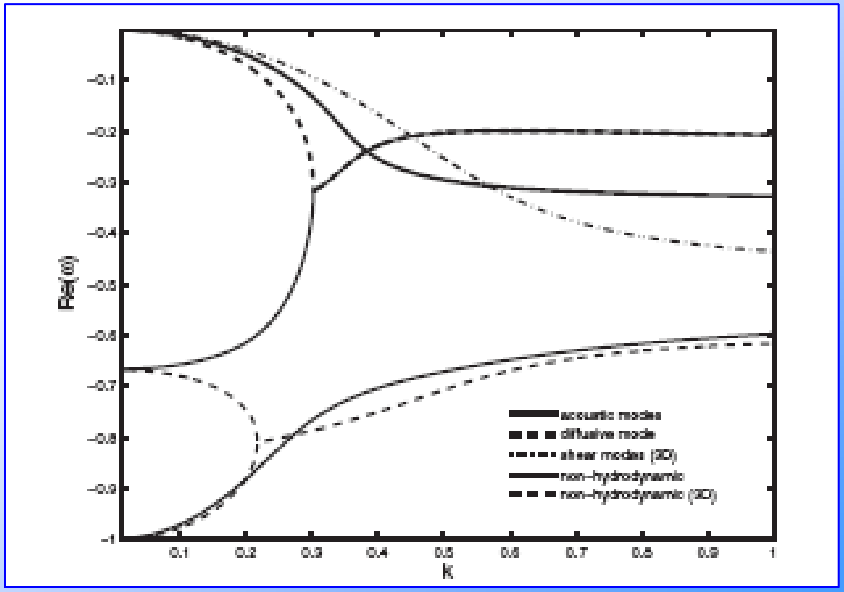
$$q_k^p = ikX\rho_k + ikYT_k - k^2 Zu_k^p \qquad q_k^p = -k^2 Uu_k^\perp$$

$$\Downarrow$$

3D exact hydrodynamics: $\boxed{\partial_t x_k = M_k x_k}$

with $x_k = \left(\rho_k, u_k^p, T_k, u_k^\perp\right) \qquad M_k = \begin{pmatrix} M_k^p & 0 \\ 0 & M_k^\perp \end{pmatrix}$





Due to the block diagonal structure of M_k and to the fact that the hyperbolic feature of M_k^p was already analyzed in 1D13M, a transformation exists and it is *unique* also in the 3D case:

$$\Omega_k = \begin{pmatrix} \Omega_k^p & 0 \\ 0 & \Omega_k^\perp \end{pmatrix}$$

where $\Omega_k^\perp$ is found to be manifestly simple: $\Omega_k^\perp = I$

Further, due to the occurrence of negative eigenvalues, ($\lambda_1 = 0, \lambda_2 = k^2 A, \lambda_3 = \frac{2}{3} k^2 Y, \lambda_{4,5} = k^2 D$) an H-Theorem holds:

Defining an “H functional” : $H(t) = \frac{1}{2} \int \left[\rho'^2(r,t) + u'^2(r,t) + T'^2(r,t) \right] d^3 r$

and turning onto the Fourier space, this leads to: $\partial_t H(t) = \sum_{s=1}^{5} \int \lambda_s \left| x'_{k,s} \right|^2 d_k^3 \le 0$

Thus, the H-theorem is proved for linearized exact hydrodynamics from 3D13M

(for $k \le k_c$)

6. The message of hyperbolicity

- Bobylev, dealing with Burnett hydrodynamics from B.E., pointed out the connection between hyperbolicity and stability.
  We found out that the same hyperbolic regularization works for Burnett hydrodynamics as derived from Grad system.

- Beyond Burnett level (up to an arbitrary order n, also in limit $n \to \infty$, of "exact hydrodynamics"), we need to take into account also dissipative properties of the "Ω- *transformed*" hydrodynamics. Namely:

$$\left\{\begin{array}{l} hyperbolicity \\ dissipativity \end{array}\right. \xrightarrow{\textit{sufficient condition}} \begin{array}{c} \textit{stability} \\ \textit{(existence of H-theorem)} \end{array}$$

- Exact (stable) hydrodynamics, from 13M Grad system, is hyperbolic.

- We found the occurrence of a critical point on the wavelength domain: there is no closed exact hydrodynamic description beyond k_c.

7. The goal: exact hydrodynamics from Boltzmann eq.

Boltzmann equation, f

GRAD METHOD $f \to f_{LM}\left(1+\varphi\right)$

CHAPMAN ENSKOG

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_{CE}^{(n)}$$

Moment system:

$$\partial_t \int v_1^{i_1} v_2^{i_2} v_3^{i_3} f_{LM}\left(1+\varphi\right) d^3 v = \int v_1^{i_1} v_2^{i_2} v_3^{i_3} J\left[f_{LM}\left(1+\varphi\right)\right] d^3 v$$

CHAPMAN ENSKOG

$$\sigma = \sum_{n=1}^{\infty} \varepsilon^n \sigma^{(n-1)}$$

$$q = \sum_{n=1}^{\infty} \varepsilon^n q^{(n-1)}$$

Hydrodynamics

Exact Hydrodynamics from B.E. =??

Exact Hydrodynamics from Grad method = hyperbolic+dissipative

$$\partial_t \int \psi_i \sum_{n=0}^{\infty} \varepsilon^n f_{CE}^{(n)} d^3 v = \int \psi_i J\left({}_i\sum_{n=0}^{\infty} \varepsilon^n f_{CE}^{(n)} \right) d^3 v$$

Question: is it possible to infer anything about the possible hyperbolicity of exact hydrodynamics derived from B.E., once we know that exact hydrodynamics derived from Grad approximation is hyperbolic?

More generally: is it possible to derive properties for hydrodynamics deriving from f (living upon the *unbounded manifold* associated to B.E.), once we know about hydrodynamics deriving from the projection of f over a *locally-finite dimensional* (sub)manifold?

• A sketch of the solution:

We seek for ***normal solutions*** (Hilbert, 1911), that is, solutions to the Boltzmann equation which depend on space and time only through five hydrodynamic fields:

$$f = f\left(v, n(r,t), u(r,t), T(r,t)\right)$$

then we linearize around the global equilibrium solution ($\rho = \rho_0; u = 0; T = T_0$), to obtain - after proper rescaling to adimensional variables:

$$f = f^{GM}[1+\varphi] = f^{GM}\left[1+\left(A^0+\delta A\right)n_k + \left(B^0+\delta B\right)u_k + \left(C^0+\delta C\right)T_k\right]$$

FT of the linearized Boltzmann equation reads as:

$$\partial_t f + c_j i k_j f = L(f)$$

where we denote: $L(f) = -\frac{1}{\tau}\left(f - f^{GM}\right)$ *BGK* collision operator

$f^{GM} = n_0 \frac{1}{\pi^{\frac{3}{2}} v_t^3} e^{-c^2}$ global Maxwellian

$v_t = \sqrt{2 k_B T_0}$ thermal velocity

τ relaxation time

This leads to equations of hydrodynamics in their general form:

$$\partial_t n_k = -ik \cdot u_k$$
$$\partial_t u_k = -ik\left(n_k + T_k\right) - ik \cdot \left[< \delta A c_i c_j > n_k + < \delta B c_i c_j > u_k + < \delta C c_i c_j > T_k \right]$$
$$\partial_t T_k = -\frac{2}{3} ik \cdot u_k - \frac{2}{3} ik \cdot \left[< \delta A c^2 c_i > n_k + < \delta B c^2 c_i > u_k + < \delta C c^2 c_i > T_k \right]$$

where: $< \xi(c) > = \int f^{GM} \xi(c) dc$

Invariance Equations: $\boxed{\Delta(\varphi) = (1 - P) J(\varphi) = 0}$

$$\left(\partial_t \varphi\right)^{MICRO} = J(\varphi) = -ik \cdot c\varphi - \varphi$$

$$\Updownarrow$$

$$\left(\partial_t \varphi\right)^{MACRO} = PJ(\varphi) = \frac{\partial \varphi}{\partial n_k} \partial_t n_k + \frac{\partial \varphi}{\partial u_k} \partial_t u_k + \frac{\partial \varphi}{\partial T_k} \partial_t T_k$$

we end up with a set of integral (nonlinear) equations:

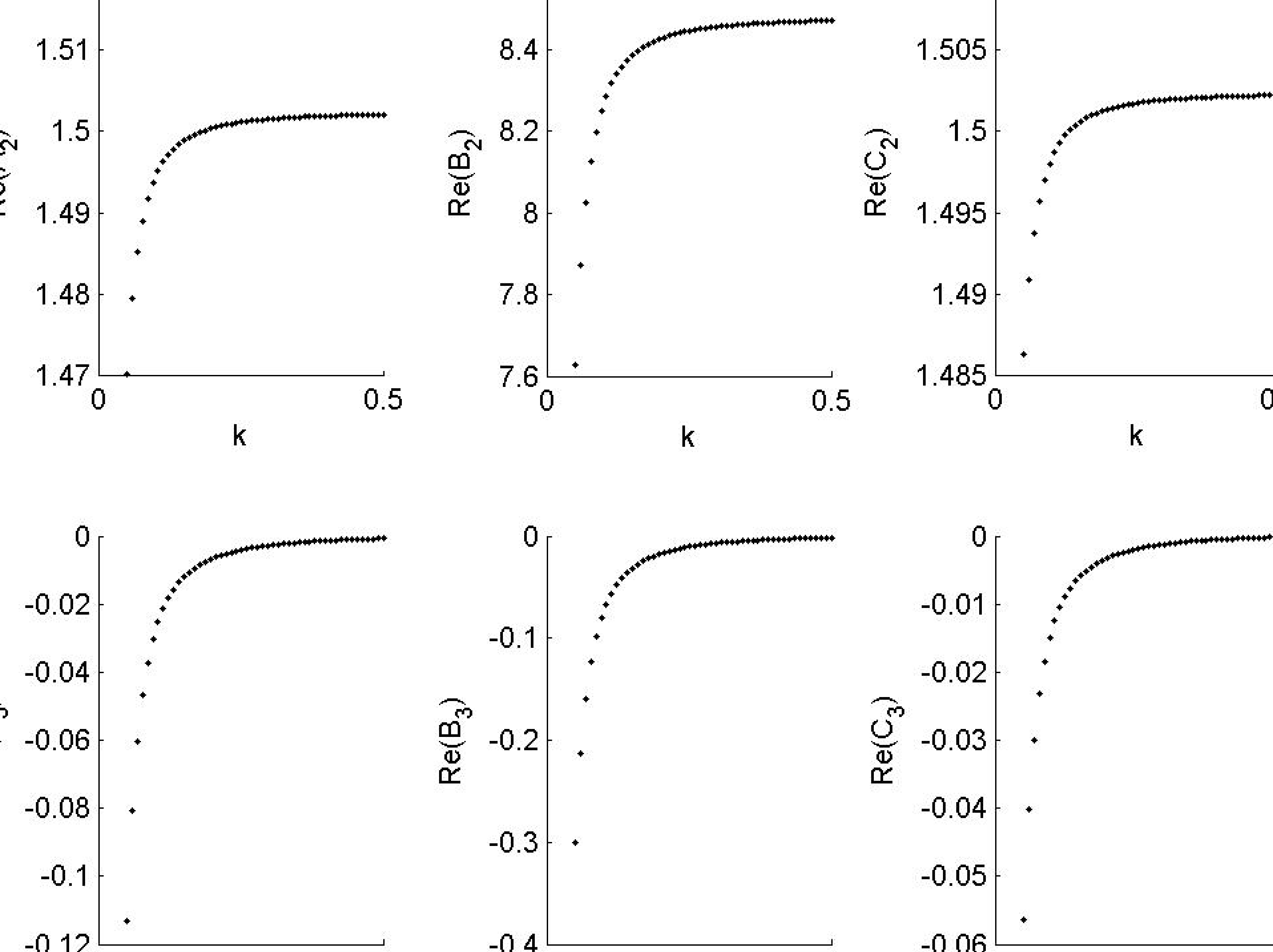
$$\begin{pmatrix} \left(1+ik_jc_j\right) & -ik_j\left(1+<\delta A c_i c_j>\right) & -ik_j<\delta A c^2 c_j> \\ -ik_j & -ik_j<\delta B c_i c_j>+ik_jc_j+1 & -ik_j\left(1+<\delta B c^2 c_j>\right) \\ 0 & -ik_j\left(1+<\delta C c_i c_j>\right) & -ik_j<\delta C c^2 c_j>+ik_jc_j+1 \end{pmatrix} \begin{pmatrix} A^0+\delta A \\ B^0+\delta B \\ C^0+\delta C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

solutions to this system provides the expressions
for the non-hydrodynamic variables:

$$\sigma_{ij,k} =<\delta A c_i c_j> n_k + <\delta B c_i c_j> u_k + <\delta C c_i c_j> T_k$$

$$q_{i,k} =<\delta A c^2 c_i> n_k + <\delta B c^2 c_i> u_k + <\delta C c^2 c_i> T_k$$

Observation: no critical point occurs along the spectrum of the hydrodynamic modes



1.51
1.5
1.49
1.48
1.47
0
0.5
k
8.4
8.2
8
7.8
7.6
Re(B$_2$)
1.505
1.5
1.495
1.49
1.485
Re(C$_2$)
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.1
-0.2
-0.3
-0.4
Re(B$_3$)
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
Re(C$_3$)

8. What's next?

• **The idea**: the occurrence of a critical point in the hydrodynamic modes is not a general feature of linear hydrodynamics, rather it is just a consequence of its derivation via a projection of the distribution function over some submanifold (like with 13M Grad).

$$\begin{tikzcd}
f\left[A_i(v,x)\right] \arrow[rr, "\textit{Projection}"] \arrow[d, "\text{Invariance Equations}"'] & & A_i(v,x) = \sum H^{(n)}(v)\cdot \tilde{A}_i^{(n)}(x) \arrow[d, "\text{Invariance Equations}"] \\
A_i(v,x) \arrow[rr, "\textit{Projection}"'] & & \tilde{A}_i^{(n)}(x)
\end{tikzcd}$$

• Do the two operations commute? $\left[IE, \Pi\right] = 0\,??$