Model reduction of nonlinear systems – use of multi-time scale models and model reduction techniques

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Concerned with large-scale non-linear circuits.

Their behaviour is described by:

\[ \dot{x}(t) = f(x(t)) + b(t) \]

\( x(t) \) are the states

\( b(t) \) are the inputs to the circuit
Concerned with envelope modulated signals
Such systems are suitable for **MULTI-TIME SCALE** analysis

We set \( x(t) = \hat{x}(t_1, \cdots t_p) \)

\( p \) is the number of different time scales

We rewrite the governing equation in multi-time scale format

\[
\frac{\partial \hat{x}}{\partial t_1} + \frac{\partial \hat{x}}{\partial t_2} + \cdots + \frac{\partial \hat{x}}{\partial t_p} = f(\hat{x}(t_1, t_2, \cdots t_p)) + \hat{b}(t_1, t_2, \cdots t_p)
\]
We can solve the multi-time scale partial differential equation using:

- Complete time-domain methods
- Mixed time-domain frequency-domain methods
- Mixed time-domain wavelet methods
We will consider **wavelet** approach

Why? –
• Suitable for highly nonlinear circuits
• Enables us to move from one level of accuracy to another in an incremental manner

To simplify matters, consider *two* time scales.

\[
\frac{\partial \hat{x}}{\partial t_1} + \frac{\partial \hat{x}}{\partial t_2} = f(\hat{x}(t_1, t_2)) + \hat{b}(t_1, t_2)
\]

\(x(t)\) is evaluated along the diagonal of the \(t_1, t_2\) plane

\[x(t) = \hat{x}(t, t)\]
$x(t)$ is represented by:

- a series of wavelets in one time scale
- coefficients which vary in the other time scale

$$\hat{x}_j(t_1, t_2) = \sum_{k=1}^{N} \overline{x}_k(t_1) \Psi_k(t_2)$$

If written for all collocation points in $t_2$

$$\hat{x}_{jN}(t_1) = E \overline{x}(t_1)$$

$E$ is a constant square matrix.
Its columns comprise the values of the $N$ wavelet functions, $\Psi_k(t_2)$, at the $N$ collocation points
The governing equation now becomes:

\[ E \frac{d\bar{x}}{dt} = -D \bar{x} + f_N(\bar{x}) + b_N \]

This is an ODE – but it concerns ONE time scale –

-so can be solved numerically in an efficient manner
At this point we employ MODEL REDUCTION techniques

We expand $\bar{x}(t_i)$ in a Taylor’s series $t_i^0$ is the initial time

$$\bar{x}(t_i) = \sum_{i=0}^{\infty} a_i (t_i - t_i^0)^i$$

The coefficients $a_i$ may be computed recursively
A Krylov space is formed from \( a_i \)

\[
K = \begin{bmatrix}
a_0 & a_1 & \cdots & a_q
\end{bmatrix}
\]

\( q \) is the order of the reduced system
We perform a QR decomposition of the Krylov space

\[ K = QR \]

\[ Q^T Q = I_q \]

Then Q is employed to perform a congruent transformation:

\[ \bar{x} = Q \hat{x} \]
\[ Q^T EQ \frac{d\hat{x}}{dt} = -Q^T DQ\hat{x} + Q^T f_N(Q\hat{x}) + Q^T b_N \]

or

\[ \hat{E} \frac{d\hat{x}}{dt} = -\hat{D}\hat{x} + Q^T f_N(Q\hat{x}) + \hat{b}_N \]

where

\[ \hat{E} = Q^T EQ \quad \hat{D} = Q^T DQ \quad \hat{b}_N = Q^T b_N \]

Reduced system is of size \( q \ll N \) – where \( N \) is the original size
For oscillators or systems subject to Frequency Modulated signals we use the Warped Multi-time scale model

\[
\omega(\tau) \frac{\partial \hat{x}}{\partial t_1} + \frac{\partial \hat{x}}{\partial t_2} = f(\hat{x}(t_1, t_2)) + b(t_2)
\]

It bends the path along which \( x(t) \) is evaluated to account for frequency variation

\[
x(t) = \hat{x}(\varphi(t), t) \\
\varphi(t) = \int_0^t \omega(t) dt
\]
Again we apply wavelet analysis

\[
\hat{x}_J(t_1, t_2) = \sum_{k=1}^{N} \bar{x}_k(t_2) \Psi_k(t_1)
\]
Warped Equation now becomes:

\[
\omega(\tau_2)D\bar{x} + E \frac{\partial \bar{x}}{\partial t_2} = f(E\bar{x}) + b_v(t_2)
\]

We apply the same reduction procedure

\[
\bar{x}(t_2) = \sum_{i=0}^{\infty} a_i (t_2 - t_2^0)^i
\]

\[
K = [a_0 \quad a_1 \quad \cdots \quad a_q] = QR
\]
Reduced equation is of size $q << N$

$$\omega(\tau_2) \hat{D}\hat{x} + \hat{E} \frac{\partial \hat{x}}{\partial t_2} = Q^{r} f(Q\hat{x}) + \hat{b}_n(t_2)$$
Example:

Input signal: \[ b(t) = \sin\left(\frac{2\pi}{T_1} t\right) \sin\left(\frac{2\pi}{T_2} t\right) \]

Nonlinear circuit
Output with full model

Output from reduced wavelet model
CONCLUSIONS

• Multi-timescale analysis suitable for nonlinear circuits with modulated signals

• Multi-timescale model can be reduced using Krylov space techniques
MORE RESEARCH WORK REQUIRED
FOR NONLINEAR MODEL REDUCTION

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