From Lemma 1 in obvious fashion follows

LEMMA 2. Let $\underset{(k)}{\mathcal{F}}(x) \overset{\text{def}}{=} \underset{(k)}{a}_{ij}x^i x^j$, k = 1, . . . , n − 1; ij = 1, . . . , n + 1, be n − 1 real quadratic forms of n + 1 variables. Then there exist two real linearly independent vectors a and b such that each of the binary quadratic forms $\underset{(k)}{\Phi}(\lambda_1, \lambda_2) \overset{\text{def}}{=} \underset{(k)}{\mathcal{F}}(\lambda_1 a + \lambda_2 b)$, k = 1, . . . , n − 1, is either trivial or sign-definite, and all sign-definite forms $\underset{(k)}{\Phi}$ are proportional to each other.

The proofs of Theorems 2 and 3 now follow at once from the formulas expressing $K_\sigma(F)$ in terms of the coefficients of its first and second quadratic forms (the properties, used here, of the second quadratic forms are ensured by Lemmas 1 and 2).

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SETS OF REMOVABLE SINGULARITIES AND CONTINUOUS MAPS

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We examine sets of removable singularities of analytic functionals (negligible sets) in topological vector spaces (TVS). We prove that those of them any continuous image of which also is removable can be completely described in general topology terminology (compactness, minimal cardinality of everywhere dense subset). We examine only TVS over the complex number field C.

Let X be a TVS in which continuous linear functionals separate points, dim X > 1, and U be an open subset of X. The map $\varphi: U \to C$ is called an analytic functional if it is continuous and for any element $u \in U$ there exists a neighborhood $V \subset U$ in which φ is representable as the convergent series

$$\varphi(x) = \varphi(u) - \sum_{i=1}^{\infty} A_i(x-u),$$

where $A_i: X \to C$ is the restriction of the continuous i-linear functional $\tilde{A}_i: X^i \to C$ on a diagonal in X^i.

The following proposition is well known in Banach spaces [1]; the proof carries over from this case to the one presented below without change.

Proposition 1. Let H be a closed subset of X and for any $u \in U \cap H$ let there exist a straight line $L \subset X$ and a closed variety $Y \subset X$ of codimension 1, satisfying the following conditions:

1) $L \cap Y = \{u\}$ and the projection of $H \cap U$ onto Y, parallel to L, does not contain neighborhoods of u in Y;

2) a relatively compact domain $D \subset L \cap U$, bounded by a finite number of rectifiable curves, exists such that $u \in D$ and $\partial D \subset U \setminus H$.

Then for any functional φ analytic in $U \setminus H$ there exists a functional ψ analytic in U and coinciding with φ on $U \setminus H$.

A closed set $H \subset X$ is said to be negligible if for any open nonempty $U \subset X$ and for an arbitrary functional φ analytic in $U \setminus H$ there exists and is unique a functional ψ analytic in U and coinciding with φ on $U \setminus H$.

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Proposition 2. Let U be connected, q be a functional analytic in U and not equalling zero identically, and $S = \{x \in U | q(x) = 0\}$. In order that a closed set $H \subset S$ be negligible it is necessary and sufficient that it be nowhere dense in S.

Proof. The necessity is obvious. To prove the sufficiency we use the Weierstrass preparation theorem valid in the case of arbitrary TVS [1]. From it follows the existence, for any $s \in S$, of a decomposition $X = Y \oplus C$ such that for some neighborhood V of point s the projection of $S \cap V$ onto the first summand is a finite-to-one map and a local homeomorphism everywhere except for the preimage of a nowhere dense subset of Y. Hence follows the applicability of Proposition 1 in some neighborhood of any point $s \in H$, which proves what was required since negligibility is a local property.

Let T be a topological space.

Definition. We say that T is completely negligible relative to X if the set $[F(T)]$ is negligible for any continuous map $F: T \to X$. ($[A]$ denotes the closure of set A.)

THEOREM 1. For the complete negligibility of T relative to X it is necessary and sufficient that any continuous image of T into X be nowhere dense.

The necessity is obvious. We require a number of lemmas in order to prove the sufficiency.

LEMMA 1. Let Y be a closed linear subspace of space X, of codimension 1, and V be a neighborhood of zero in Y. Then a continuous map $F: V \to X$ exists such that $F(V)$ is a neighborhood of zero in X (not necessarily open).

Proof. We decompose X into the direct sum $X = Y \oplus C$; we decompose Y in the same way, but now over the real number field $R: Y = Z \oplus R$. Then $X = Z \oplus R \oplus C$ (over R). A continuous map $f: R \to R \oplus C$ exists such that $(\mathrm{id} \oplus f)(V)$ is a neighborhood of zero in X (f is a three-dimensional Peano curve).

In X we introduce an equivalence relation $\rho: x\rho y$ if $c \in C$, $|c| = 1$, exists such that $y = cx$. By $\bar{x}$ we denote the equivalence class containing x.

LEMMA 2. Let W be a neighborhood of zero in X. Then a continuous map $F: X/\rho \to X$ exists such that $F(W/\rho)$ is a neighborhood of zero in X.

Proof. We decompose X into the direct sum $X = Y \oplus C$. Representing $C = R \oplus R$, we obtain (over R) $X = Y \oplus R \oplus R$. In each equivalence class with respect to ρ, not lying in Y, there is a unique representation of the form $y \oplus a \oplus 0$ $(a > 0)$. We define a map $\Phi: X \to Y \times R_+$ (R_+ is the set of nonnegative real numbers):

$$\text{if}\quad x \notin Y,\ \text{then}\ \Phi(x) = (ay, a),\ \text{where}\ \{y \oplus a \oplus 0\} = \bar{x} \cap (Y \oplus R_+ \oplus 0);$$
$$\text{if}\quad x \in Y,\ \text{then}\ \Phi(x) = (0, 0).$$

Then we should construct the map $\Psi: Y \times R_+ \to X$ in the same way as in the preceding lemma (using a Peano curve) and note that $\Phi(x_1) = \Phi(x_2)$ if $x_1 \rho x_2$.

LEMMA 3. Let H be a closed subset of X and for each $h \in H$ let there exist a neighborhood $K \subset X$ such that $K \cap H$ cannot be mapped onto a subset of X possessing a nonempty interior. Then H is a negligible set.

Proof. From Lemma 2 it follows that for any open $U \subset K$ and for an arbitrary $u \in U$ there exists a disk with center at u, whose boundary circle lies in $U \setminus H$. By Lemma 1 the projection of $H \cap K$ onto any closed linear subspace $Y \subset K$, of codimension 1, does not contain a nonempty open subset of Y, which proves the applicability of Proposition 1.

To complete the proof of Theorem 1 we note that for an arbitrary continuous map $F: T \to X$ the set $[F(T)]$ satisfies the hypotheses of Lemma 3 with $K = X$ if any continuous image of T into X is nowhere dense.

COROLLARY 1. If T is compact and X is infinite-dimensional, then T is completely negligible relative to X. (Since a compact subset of an infinite-dimensional Hausdorff TVS is nowhere dense in it; see [2], for instance.)

COROLLARY 2. If $s(T) < s(X)$, then T is completely negligible relative to X [$s(T)$ is the lower bound for the cardinality of an everywhere dense subset of T].

Below we take it that T is metrizable and X is infinite-dimensional.

LEMMA 4. For the complete negligibility of T relative to X it is necessary and sufficient that any closed discrete subspace of set T have a cardinality less than $s(X)$.

Proof. To show the necessity we assume the contrary: let there exist a closed discrete subspace $E \subset T$ of cardinality $s(X)$. Because T is metrizable there exists a family $\{V_e | e \in E\}$, $e \in V_e$, of nonintersecting open subsets of T. For each $e \in E$ we can define a function f_e, continuous in T, with support in V_e, equal to 1 at point e. Let $\Psi: E \to X$ be a map of E onto some everywhere dense subset of X. We continue it up to a continuous map $F: T \to X$ by setting $F(t) = f_e(t)\Psi(e)$ if $t \in V_e$ for some $e \in E$ and $F(t) = 0$ otherwise. To prove the necessity it remains to note that $[F(t)] = X$.

To prove the sufficiency in the case of a separable X we can refer to Corollary 1.

If, however, T is not separable,* then a closed discrete subspace of cardinality $s(T)$ exists in it. (This can be shown by examining an ε-net on T, in an arbitrarily taken metric, as $\varepsilon \to 0$.) Therefore, if a closed discrete subspace of cardinality $s(X) > \aleph_0$ does not exist in T, then $s(T) < s(X)$ and T is completely negligible relative to X (Corollary 2).

THEOREM 2. Let T be a metric space.

A. T is completely negligible relative to C^n ($n \geq 2$) if and only if T is compact and denumerable.

B. T is completely negligible relative to a separable infinite-dimensional TVS X if and only if T is compact.

C. T is completely negligible relative to a nonseparable TVS X if and only if $s(T) < s(X)$.

Proof. Assertions B and C follow from Lemma 4; the sufficiency in A follows from Theorem 1, the necessity of compactness follows from the proof of Lemma 4, and the necessity that T be no more than denumerable follows from the next lemma and from Theorem 1.

LEMMA 5. In order that a continuous map of a compact metrizable space T onto the unit segment $I = [0, 1]$ not exist, it is necessary that T be no more than denumerable.

Proof. Let us assume that a continuous mapping of T onto I does not exist. Then T is fully unconnected. (To prove this we can use, for example, the Urysohn function [3].) If T is nondenumerable, then by virtue of the metrizability, in it exists an infinite set of points, any neighborhood of each of which is nondenumerable. In this case a continuous surjective map $F: T \to I$ is constructed thus: I is divided in two into I_1 and I_2 and T is partitioned into nondenumerable open-closed sets T_1 and T_2; we assume $F(T_1) = I_1$ and $F(T_2) = I_2$; further, I_1 and I_2 are divided in two and T_1 and T_2 are partitioned into nondenumerable open-closed sets, etc. The fact that T is compact and fully unconnected guarantees that such a definition is well posed.

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*Translator's Note: There seems to be a discrepancy here: the author mentioned a "separable X" in the preceding sentence. I believe that it should be "X is not separable" here. This seems to be borne out by the statement and proof of Theorem 2 below.