Compressible Navier–Stokes limit of binary mixture of gas particles

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Yellow dust

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United Nations Convention to Combat Desertification (UNCCCD)
Collision dynamics

- We put $A$ and $B$ to denote two different species with particle mass $m^A$ and $m^B$.

- As an important parameter we introduce the mass ratio

  \[ \gamma = \frac{m^A}{m^B}. \]  

- The pre- and post-collision velocities satisfy the momentum conservation

  \[ \gamma u + u_1 = \gamma \tilde{u} + \tilde{u}_1 \]  

  and the energy conservation

  \[ \gamma |u|^2 + |u_1|^2 = \gamma |\tilde{u}|^2 + |\tilde{u}_1|^2. \]
The post-collision velocities are

\[
\begin{align*}
\tilde{v} &= v + \frac{2}{\gamma + 1} \langle v_1 - v, \omega \rangle \omega, \\
\tilde{v}_1 &= v_1 - \frac{2\gamma}{\gamma + 1} \langle v_1 - v, \omega \rangle \omega.
\end{align*}
\] (4)

The mass densities of A and B particles at \((t, x, v)\) (time \(t\), position \(x\) and velocity \(v\)) are denoted by \(f(t, x, v)\) and \(g(t, x, v)\) respectively.

We have two self collision kernels \(b^{AA}\) and \(b^{BB}\) and two cross collision kernels \(b^{BA}\) and \(b^{AB}\).

All of them are Grad’s hard cutoff potentials.
• For convenience, we define for a given function $f$,
\[
f_1 = f(v_1), \quad \hat{f} = f(\hat{v}), \quad \tilde{f} = f(\tilde{v}), \quad \tilde{f}_1 = f(\tilde{v}_1),
\]
\[
\bar{f} = f(\bar{v}), \quad \bar{f}_1 = f(\bar{v}_1)
\]
and collision kernels
\[
Q^{AA}(f, f) = \int_{\mathbb{R}^3} \int_{S^2} [\hat{f}_1 \hat{f} - f_1 f] b^{AA}(v_1 - v, \omega) \, d\omega \, dv_1;
\]
\[
Q^{BB}(g, g) = \int_{\mathbb{R}^3} \int_{S^2} [\bar{g}_1 \bar{g} - g_1 g] b^{BB}(v_1 - v, \omega) \, d\omega \, dv_1;
\]
\[
Q^{BA}(g, f) = \int_{\mathbb{R}^3} \int_{S^2} [\tilde{g}_1 \tilde{f} - g_1 f] b^{BA}(v_1 - v, \omega) \, d\omega \, dv_1;
\]
\[
Q^{AB}(f, g) = \int_{\mathbb{R}^3} \int_{S^2} [\bar{f}_1 \bar{g} - f_1 g] b^{AB}(v_1 - v, \omega) \, d\omega \, dv_1.
\]

• Moreover from mass conservation we have
\[
Q^{AB}(f, g) = Q^{BA}(g, f)
\]
for all admissible $f$ and $g$ and considering scattering cross section we have a constant $\mu$ such that
\[
b^{AB} = \mu b^{BA}.
\]
Boltzmann system

- The balance of mass is described by the Boltzmann system

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f &= \frac{1}{m_A} Q^{AA}(f, f) + \frac{1}{m_B} Q^{BA}(g, f), \\
\partial_t g + v \cdot \nabla_x g &= \frac{1}{m_B} Q^{BB}(g, g) + \frac{1}{m_A} Q^{AB}(f, g). 
\end{align*}
\] (7)

- For the mathematical model (7) we refer Chapter 7 and 8 in Chapman–Cowling 1934.

- With the perspective of a mathematical existence by DiPerna and Lion, I assume there is a weak $L^1$ renormalized solution of the Boltzmann system.
Objective

- We derive the compressible Navier–Stokes system formally when the Knudsen number $\epsilon = K_n^A$ of $A$ particle goes to zero and the mass ratio $\gamma$ is of the same order as the Knudsen number.
- Consequently, we assume the Knudsen number $K_n^B$ of $B$ particle equals to 1.
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We assume the collision kernels satisfy Grad’s hard potential assumption: There are positive constants $\alpha \in [0, 1]$ and $C$ such that

$$0 < b^{IJ}(z, \omega) \leq C(1 + |z|)^\alpha \quad \text{and} \quad \int_{S^2} b^{IJ}(z, \omega) \geq \frac{1}{C}(1 + |z|)^\alpha$$

for almost all $z \in \mathbb{R}^3$, and $I = A, B$ and $J = A, B$. 

(8)
Dust storm

Figure: ©UNCCD News
• Let \((v^A)_0\) and \((v^B)_0\) be the reference velocities and \((f)_0\) and \((g)_0\) be the reference mass density.
• Let \(x_0\) be the reference length and \(t_0\) be the reference time.
• Then the reference numbers are
\[
((n^A)_0, (n^B)_0) = \left( \frac{(v^A)_0^3 (f)_0}{m^A}, \frac{(v^B)_0^3 (g)_0}{m^B} \right).
\]
• Knudsen numbers are
\[
(K^n_A, K^n_B) = \left( \frac{x_0}{(n^A)_0 \sigma^A}, \frac{x_0}{(n^B)_0 \sigma^B} \right).
\]
• Mach numbers are
\[
(M^A_a, M^B_a) = \left( \frac{x_0/t_0}{(v^A)_0}, \frac{x_0/t_0}{(v^B)_0} \right).
\]
• Mach numbers \((M_a^A, M_a^B) = (1, 1)\).
  Knudsen numbers \((K_n^A, K_n^B) = (\epsilon, 1)\).
• If we let the scattering cross section of all collisions are equal, we obtain a scaled Boltzmann system

\[
\begin{align*}
\partial_t f + \mathbf{v} \cdot \nabla_x f &= \varepsilon^{-1} Q^{AA}(f, f) + Q^{BA}(g, f), \\
\partial_t g + \mathbf{v} \cdot \nabla_x g &= Q^{BB}(g, g) + \varepsilon^{-1} Q^{AB}(f, g).
\end{align*}
\] (9)
For the symmetrized operator we consider

\[ L_{M(\rho, u, \theta)}(g) = \frac{-1}{M(\rho, u, \theta/\gamma)} (Q^{BA}(M(\rho, u, \theta/\gamma) g, M(\rho, u, \theta)) \]

\[ + Q^{AB}(M(\rho, u, \theta), M(\rho, u, \theta/\gamma) g) \).

From mass conservation, if \( Q^{AB}(M(\rho, u, \theta), M(\rho, u, \theta/\gamma) g) = 0 \), then

\[ Q^{BA}(M(\rho, u, \theta/\gamma) g, M(\rho, u, \theta)) = 0. \]

It follows that

\[ \tilde{g}_1 + \tilde{g} - g_1 - g = 0. \]

We define temperature of B particle

\[ \theta^B = \frac{\theta^A}{\gamma}. \]
We analyze symmetry properties. First, we denote that

\[ \langle f \rangle = \int_{\mathbb{R}^3} f(v) \, dv. \]

As is stated in the monograph by Cercignani, we have the following lemma for the self-collision.

**Lemma 1**

The following statements for self collisions are equivalent.

1. \( \phi = \alpha + \beta \cdot v + \frac{\theta}{2} |v|^2 \) for constants \( \alpha \) and \( \theta \) and a constant vector \( \beta \).
2. \( \langle \phi \, Q^{II}(f, f) \rangle = 0 \) for all nonnegative \( f \), where \( I = A \) or \( I = B \).
3. \( \hat{\phi}_1 + \hat{\phi} - \phi_1 - \phi = 0 \) for all vectors \( v, v_1 \in \mathbb{R}^3 \) and \( \hat{v}_1, \hat{v} \) satisfying conservation laws.
The collision operators satisfy the conservation properties of mass such as

\[ \langle Q^{IJ}(g, f) \rangle = 0, \]

for \( I, J = A, B \).

As far as the self-collision concerned, we have a conservation of momentum and energy such that for \( I = A, B \)

\[ \langle vQ^{II}(f, f) \rangle = 0, \quad \langle |v|^2 Q^{II}(f, f) \rangle = 0. \]

However, the momentum and energy conservations of the cross collision appear in different ways which are not necessary here.
Lemma 2

Suppose that \( Q^{AA}(f, f) = 0 \). Then there is a constant vector \((\rho^A, u^A, \theta^A)\) in \( \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \) such that

\[
f(v) = M_{(\rho^A, u^A, \theta^A)}(v) = \frac{\rho^A}{[2\pi \theta^A]^{3/2}} \exp \left( - \frac{|v - u^A|^2}{2\theta^A} \right).
\]
Linearised cross-collision operator I

Considering the structure of $L_{M(\rho^B, u^B, \theta^B)}$, we have

$$L_{M(\rho^B, u^B, \theta^B)}(g) = \int_{S^2} \int_{\mathbb{R}^3} (g_1 + g - \bar{g}_1 - \bar{g}) M(\rho^A, u^A, \theta^A)(v_1) b^{AB}(v_1 - v, \omega) dv_1 d\omega.$$ 

**Theorem 1**

$L_{M(\rho^B, u^B, \theta^B)}$ is a Fredholm operator in $L^2(M(\rho^B, u^B, \theta^B) dv)$ and $\text{Ker}(L_{M(\rho^B, u^B, \theta^B)}) = \text{span}\{1, v, |v|^2\}$, where $M(\rho^B, u^B, \theta^B) dv$ is the $M(\rho^B, u^B, \theta^B)$ weighted Lebesgue measure.
Now we consider a cross Boltzmann linear operator

\[
K_{M(\rho, u, \theta/\gamma)}(g) = \frac{-1}{M(\rho, u, \theta/\gamma)} Q^{AB}(M(\rho, u, \theta), M(\rho, u, \theta/\gamma) g)
\]

\[
= \int_{S^2} \int_{\mathbb{R}^3} (g - \bar{g}) M(\rho, u, \theta)(v_1) b^{AB}(v_1 - v, \omega) dv_1 d\omega.
\]

In the next theorem, we prove that the linear operator \( K_{M(\rho_B, u_B, \theta_B)} \) for cross collision is also a Fredholm operator and its kernel is one-dimensional.

**Theorem 2**

\( K_{M(\rho_B, u_B, \theta_B)} \) is a self-adjoint Fredholm operator in \( L^2(M(\rho_B, u_B, \theta_B) \, dv) \) and \( \text{Ker}(K_{M(\rho_B, u_B, \theta_B)}) = \text{span}\{1\} \), where \( M(\rho_B, u_B, \theta_B) \, dv \) is \( M(\rho_B, u_B, \theta_B) \) weighted Lebesgue measure.
Maxwellian for cross-collision

Corollary

Suppose that \( Q^{AB}(M(\rho^A, u^A, \theta^A), g) = 0 \) and \( g \) is nonnegative and rapidly decreasing. Then \( g \) is a Maxwellian such that for \( \theta^B = \theta^A / \gamma, u^B = u^A \)

\[
g(v) = M(\rho^B, u^B, \theta^B)(v) = \frac{c \rho^B}{[2\pi \theta^B]^{3/2}} \exp \left( -\frac{|v - u^B|^2}{2\theta^B} \right).
\]
3 Compressible limit
   Enskog–Chapman expansion
   Compressible Navier–Stokes limit
Enskog–Chapman expansion

We are looking for a solution to (9) by Enskog-Chapman expansion

\[(f, g) = \left( \sum_{n=0}^{\infty} \epsilon^n f_n, \sum_{n=0}^{\infty} \epsilon^n g_n \right).\]

We denote \(U^A, U^B\) and assume that \(f_n\) and \(g_n\) satisfy

\[
\left( \int_{\mathbb{R}^3} (1, v, |v|^2)f_0 dv, \int_{\mathbb{R}^3} (1, v, |v|^2)g_0 dv \right) = \left( U^A, U^B \right)
\]

\[
\left( \int_{\mathbb{R}^3} (1, v, |v|^2)f_n dv, \int_{\mathbb{R}^3} (1, v, |v|^2)g_n dv \right) = 0, \quad n \geq 1.
\]

Furthermore the hydrodynamic variable \((U^A, U^B)\) satisfies the hydrodynamic equation

\[
\partial_t (U^A, U^B) + \sum_{n=0}^{\infty} \epsilon^n \nabla \cdot \Phi_n (U^A, U^B) = 0.
\]
We list the sequence of equations up to the order 1 of $\epsilon$.

1. 1st order

$0 = Q^{AA}(f_0, f_0), \quad Q^{AB}(f_0, g_0) = 0.$

2. 0th order

\[
\begin{align*}
(\partial_t + v \cdot \nabla)f_0 & = Q^{AA}(f_1, f_0) + Q^{AA}(f_0, f_1) + Q^{BA}(g_0, f_0), \\
(\partial_t + v \cdot \nabla)g_0 & = Q^{BB}(g_0, g_0) + Q^{AB}(f_1, g_0) + Q^{AB}(f_0, g_1).
\end{align*}
\]

3. 1st order

\[
\begin{align*}
(\partial_t + v \cdot \nabla)f_1 & = Q^{AA}(f_2, f_0) + Q^{AA}(f_0, f_2) + Q^{AA}(f_1, f_1) \\
& \quad + Q^{BA}(f_1, g_0) + Q^{BA}(f_0, g_1), \\
(\partial_t + v \cdot \nabla)g_1 & = Q^{AB}(f_2, g_0) + Q^{AB}(f_0, g_2) + Q^{AB}(f_1, g_1) \\
& \quad + Q^{BB}(g_1, g_0) + Q^{BB}(g_0, g_1).
\end{align*}
\]
The formal conservation of the first order for a particle is the compressible Navier-Stokes equations:

\[
\begin{align*}
\partial_t \rho^A + \nabla \cdot (\rho^A u^A) &= 0, \\
\partial_t (\rho^A u^A) + \nabla \cdot (\rho^A u^A \otimes u^A) + \nabla (\rho^A \theta^A) &= \epsilon \nabla \cdot (\mu^A D(u^A)), \\
\partial_t \left( \rho^A \left( \frac{1}{2} |u^A|^2 + \frac{3}{2} \theta^A \right) \right) + \nabla \cdot \left( \rho^A \left( \frac{1}{2} |u^A|^2 + \frac{5}{2} \theta^A \right) u^A \right) &= \epsilon \nabla \cdot (\kappa^A \nabla \theta^A) + \epsilon \nabla (\mu^A D(u^A) \cdot u^A),
\end{align*}
\]

where

\[
\begin{align*}
\mu^A &= \frac{2\theta^A}{15} \int_0^{\infty} \alpha^A(\theta^A, r) r^6 e^{-r^2/2} \frac{dr}{2\pi}, \\
\kappa^A &= \frac{\theta^A}{6} \int_0^{\infty} \beta^A(\theta^A, r) r^4 (r^2 - 5)^2 e^{-r^2/2} \frac{dr}{2\pi}.
\end{align*}
\]
Let
\[ D(u) = \frac{1}{2}(\nabla u + \nabla u^T). \]

**Theorem 4**

The formal conservation of the first order for B particle is the compressible Navier-Stokes equations:

\[
\begin{align*}
\partial_t \rho^B &+ \nabla \cdot (\rho^B u^B) = 0, \\
\partial_t (\rho^B u^B) &+ \nabla \cdot (\rho^B u^B \otimes u^B) + \nabla (\rho^B \theta^B) \\
&= \epsilon (\nabla \cdot (\mu^{BB} D(u^B)) + \nabla \cdot (\mu^{AB} D(u^A))), \\
\partial_t \left( \rho^B \left( \frac{1}{2} |u^B|^2 + \frac{3}{2} \theta^B \right) \right) &+ \nabla \cdot \left( \rho^B \left( \frac{1}{2} |u^B|^2 + \frac{5}{2} \theta^B \right) u^B \right) \\
&= \epsilon (\nabla \cdot (\kappa^{BB} \nabla \theta^B) + \nabla \cdot (\kappa^{AB} \nabla \theta^A)) \\
&+ \epsilon (\nabla (\mu^{BB} D(u^B) \cdot u^B) + \nabla (\mu^{AB} D(u^A) \cdot u^A)),
\end{align*}
\]

where \( \mu^{IJ}, \kappa^{IJ} \) for \( I = A, B \) and \( II = A, B \) can be evaluated explicitly.
Thank you!