Quasi steady-state distributions for the Landau-Fokker-Planck equation with energy/particle sources

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• Weakly collisional plasma - nonlinear kinetic LFP equation

Influence of heating/acceleration on the temporal plasma relaxation on the base of space uniform isotropic collisional operator

• Motivation – additional heating:

  of fusion plasmas by RF waves (a quasi-linear diffusion operator), neutral beam injection (mono energetic distribution);

  the semiconductors behavior with an intrinsic or extrinsic conductivity under the action of particle fluxes or electromagnetic radiation, etc

• The analytical consideration goes with

  the high-accuracy numerical computing and comparison with the experimental results
OUTLINE

Boltzmann equation, Landau-Fokker-Planck equation

Quasi-steady state distribution functions in the presence of energy/particle sources - analytical asymptotic results

Numerical issues and simulation results:

  relaxation of the distribution function tails,
  the formation of the nonequilibrium distribution functions in the presence of weak sources
1 The Boltzmann equation

Spatially homogeneous \textbf{Boltzmann equation} for the distribution function \( f(v, t) \)

\[
\frac{\partial f}{\partial t} = \int d\mathbf{w} d\mu d\phi \ u \sigma(u, \cos \theta) \ [f(v')f(w') - f(v)f(w)] , \quad t \geq 0,
\]

\( \theta \) - scattering angle between relative velocity before \( \mathbf{u} = \mathbf{v} - \mathbf{w} \) and after collision \( \mathbf{u}' = \mathbf{v}' - \mathbf{w}' \)

\( \sigma(u, \mu) \) - differential cross section, \( \mu = \cos \theta \) and \( u = |\mathbf{u}| > 0. \)
2 Landau equation (1937)

For Coulomb potential (gas of charged particles)

\[ U(r) \sim \frac{\alpha}{r}, \quad \text{Rutherford cross-section} \quad \sigma(u, \theta) \sim \frac{1}{u^{-4} \sin^{-4} \left(\frac{\theta}{2}\right)} \]

scattering at small angles \( \theta \leq \theta_0 \ll 1 \), L.Landau derived

\[
\frac{\partial f}{\partial t} = \hat{I}_L[f, f] = \Gamma \frac{\partial}{\partial v_i} \left\{ \int d\mathbf{W} \frac{(u^2\delta_{ij} - u_iu_j)}{u^3} \left( \frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v)f(w) \right\}
\]

Divergence of the integral at the limits. As a result of some physical motivations

\[
\Gamma = \frac{2\pi e^4 L}{m^2}, \quad L = \ln \frac{d_{\text{max}}}{d_{\text{min}}}; \quad d_{\text{max}} = \sqrt{\frac{T}{n e^2}}, \quad d_{\text{min}} = \frac{e^2}{4\pi T}
\]

where \( L \gg 1 \) - Coulomb logarithm
Remark -

**Landau-type equation** $L \gg 1$, $\theta \leq \theta_0 \ll 1$

If we have power-like intermolecular potentials

$$U(r) = \frac{\alpha}{r^\beta}, \quad \beta \geq 1,$$

then

$$u\sigma = |u|^{\gamma}g_\gamma(\mu), \quad \gamma = \gamma(\beta) = 1 - \frac{4}{\beta}.$$

Maxwell molecules

$$\beta = 4, \quad \gamma = 0, \quad u\sigma \sim \text{const.}.$$

Coulomb potential - $\beta = 1$

”Dipole interaction” - $\beta = 2$
3 The Fokker-Planck collision operator (Rosenbluth, et al., 1957)

\[ I_{FP}[f, f] = -\frac{\partial}{\partial v_i} \left\{ f \frac{\partial h}{\partial v_i} + \frac{1}{2} \frac{\partial}{\partial v_j} \left( f \frac{\partial^2 g}{\partial v_i \partial v_j} \right) \right\}, \]

Rosenbluth potentials

\[ 4\pi h = \int d\mathbf{w} f(\mathbf{w}, t) | \mathbf{v} - \mathbf{w} |^{-1}, \quad 8\pi g = \int d\mathbf{w} f(\mathbf{w}, t) | \mathbf{v} - \mathbf{w} |, \]

\[ \Delta \Delta g(\mathbf{v}) = \Delta h(\mathbf{v}) = -f(\mathbf{v}). \]

FP form - friction and diffusion in velocity space - is very useful for regular (deterministic) numerical methods

Isotropic LFP equation, spherical coordinates, \( f(\mathbf{v}, t) \rightarrow \)
LFP collision integral does not change macroscopic characteristics:

- density \( n = \int d\mathbf{v} f(\mathbf{v}, t) = const \),

- energy \( \langle \mathcal{E} \rangle = \frac{m}{2} \int d\mathbf{v} v^2 f(\mathbf{v}, t) = \frac{3}{2} T = const. \)

Relaxation of the initial distribution function

\[ f(\mathbf{v}, 0) = f_0(\mathbf{v}) : \Gamma^{-1} \frac{\partial f}{\partial t} = I_{LFP}[f, f] \implies \]

Maxwell distribution

\[ f_{Maxw} = n \frac{m^{3/2}}{(2\pi T)^{3/2}} \exp \left( -\frac{mv^2}{2T} \right) \]
4 LFP-type equation with weak sources

\[ I[f, f] = \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ \frac{1}{v} \int_0^\infty dw \, Q(v, w) \left[ w f(w) \frac{\partial f}{\partial v} - v f(v) \frac{\partial f}{\partial w} \right] \right\}, \]

where the symmetrical kernel \( Q(v, w) \) is

\[ Q(v, w) = a(v, w)(v + w)^{\eta+4} + b(v, w)|v - w|^{\eta+4} \]

\[ \frac{(\eta + 2)(\eta + 4)(\eta + 6)}{(\eta + 2)(\eta + 4)(\eta + 6)}, \quad \eta = \frac{\beta - 4}{\beta} \]

with

\[ a(v, w) = [(\eta + 4)vw - (v^2 + w^2)], \quad b(v, w) = [(\eta + 4)vw + (v^2 + w^2)]. \]

For Coulomb potential \( \beta = 1, \quad U \sim r^{-1} : \)

\[ Q(v, w) = (2/3)w^3, \quad w \leq v; \quad Q(v, w) = (2/3)v^3, \quad w \geq v. \]
Kinetic equation with external sources of energy/particles

\[ \frac{\partial f}{\partial t} = I[f, f] + \varepsilon S(v), \quad f|_{t=0} = f^0(v), \quad 0 \leq v < \infty, \quad t \geq 0. \]

The particle density \( n \) and temperature \( T \)

\[ n = (f, 1) = 4\pi \int_0^\infty dv v^2 f(v, t), \quad T = \frac{m}{3n} (f, v^2) = \frac{4\pi m}{3n} \int_0^\infty dv v^4 f(v, t). \]

Weak sources: \( \varepsilon \to 0 \) then \( t \Rightarrow \tilde{t}/\varepsilon \) and \( \partial f/\partial \tilde{t} = \hat{I}/\varepsilon + S \)

Then \( \frac{dn}{d\tilde{t}} = (S, 1), \quad \frac{d}{d\tilde{t}} \left( \frac{3nT}{m} \right) = (S, v^2), \)

and \( f_{Maxw}(\tilde{t}) = n \left( \frac{m}{2\pi T} \right)^{3/2} \exp \left( -\frac{v^2}{2T} \right). \)

The intermediate regime \( \varepsilon \ll \tilde{t} \ll 1 \iff n \simeq n_0, \quad T \simeq T_0 \)
Electron acceleration / $S(v)$: $v \gg v_{th} = \sqrt{3T/m}$ - high velocity region / linear collision LFP integral

Normalization procedure: $(v_{th}, t_C, f, S)$ and $v \Rightarrow x = v^2$

Dimensionless linear Landau-type equation

$$\frac{\partial f}{\partial t} = \frac{N}{x^{1/2}} \frac{\partial}{\partial x} \left\{ x^\gamma \left[ 2\theta \frac{\partial f}{\partial x} + f(x) \right] \right\} + \varepsilon \cdot \frac{S(x)}{x^{1/2}}, \quad \gamma = \frac{2\beta - 1}{\beta}.$$ 

For $\beta = 1$ LFP equation:

$$\frac{\partial f}{\partial t} = \frac{N}{x^{1/2}} \frac{\partial}{\partial x} \left[ 2\theta \frac{\partial f}{\partial x} + f(x) \right] + \varepsilon \frac{S(x)}{x^{1/2}}.$$ 

Density $N = \int_0^\infty f x^{1/2} dx$, temperature $\theta = \frac{1}{3N} \int_0^\infty f x^{3/2} dx$; $N_0 = \theta_0 = 1$.

The source is a finite function normalized to the condition

$$\int_0^\infty dx S(x) = 1.$$
Assumptions
Let \( t = \tau / \varepsilon \), substituting \( f \) with \( \varepsilon F(x)/N \), \( \varepsilon \to 0 \) - stationary equation

\[
\frac{\partial}{\partial x} \left( 2\theta \frac{\partial F}{\partial x} + F \right) + S(x) = 0, \quad (\tau - \text{the argument of } N_{\tau}, \theta_{\tau}).
\]

○ Source \( S(x - x_\varepsilon) \) is localized at the point \( x_\varepsilon = 2\theta_0 \left( \ln \frac{1}{\varepsilon} + y_1 \right) \), where \( y_1 \) - the number (positive or negative) independent of \( \varepsilon \).

○ The sewing condition: Maxwell distribution

\[
F_{Maxw}(x) = \sqrt{\frac{2}{\pi}} N \theta^{3/2} e^{\frac{-x}{2\theta}} = \varepsilon \sqrt{\frac{2}{\pi}} N \theta^{3/2} e^{\left\{ -\left[ \frac{x}{2\theta} - \ln \frac{1}{\varepsilon} \right] \right\}}
\]

\[
\equiv \varepsilon F_{Maxw}(0) e^{-y}, \quad y = \frac{x}{2\theta} - \ln \frac{1}{\varepsilon}, \quad F \to 0 \text{ as } y \to \infty.
\]

○ \( 1 << t << (\ln(1/\varepsilon))^{-1} \) \( \rightarrow N \simeq N_0, \quad \theta \simeq \theta_0 \)
\( \gamma \neq 0 \)

\[
F(x) = F_{Maxw}(0)e^{-\frac{x}{2\theta}} + \int_{-\infty}^{\infty} S(z)dz \int_{0}^{x} e^{-\frac{(x-y)}{2\theta}} y^{-\gamma} dy + \int_{0}^{x} S(z) \int_{0}^{z} y^{-\gamma} e^{-\frac{x-y}{2\theta}} dydz
\]

If \( S(x) = I_+ \delta(x - x+) \),

then for the Coulomb potential \( \beta = 1, \gamma = 0 \):

\[
F(x) = F_{Maxw}(0) e^{-\frac{x}{2\theta}} + I_+ \left\{ \eta[x_+ - x] + \eta[x - x_+] e^{-\frac{x-x_+}{2\theta}} \right\}
\]

The unit function is \( \eta[y] = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x < 0.
\end{cases} \)

Numerical simulation
Relaxation of the initial distribution function
Formation of the distribution tails $v \to \infty$, $t \to \infty$

Analytical asymptotic results. Let $f(v, t) = f_{Maxw}(v) G(v, t)$.

$0 \leq v \leq v_{th} = 1$, $t \approx 1$, $f(v, t) \approx f_{Maxw}$, $G \approx 1$.

For $v \gg 1$, $t \gg 1$, $G$ has a character of wave propagating towards high velocities:

$G(v, t) = \Phi \left\{ \frac{2}{5} v_f \left[ \frac{v - v_f(t)}{v_f(t)} \right]^{5/2} \right\}$, $\Phi$ — error function

the wave front $v_f(t) = (3t)^{1/3}$, and the front width $\Delta_f(t) = \sqrt{\frac{\pi}{2}}$. 
NUMERICAL SIMULATION RESULTS

Tail formation $\beta = 1, t \sim 100$
Tail formation $\beta = 2, t \sim 10$
Tail formation $\beta = 4, t \sim 2$
NUMERICAL SIMULATION RESULTS

\[ \varepsilon = 10^{-6}, \quad x_+ = 36, \quad S \sim \delta(x - x_+) \]
NUMERICAL SIMULATION RESULTS

STEADY-STATE FUNCTION: $\beta = 1, 2, 4$

$\varepsilon = 10^{-6}, \quad x_+ = 36, \quad S \sim \delta(x - x_+)$
NUMERICAL SIMULATION RESULTS

Experimental results - NBI

![Graph showing NBI heating and energy spectra](image)

**Fig. 4**

**NBI heating.** Slowing down energy spectra measured during H-NBI and D-NBI.

Solid line – theoretical prediction $\sim (E^{3/2} + E_{cr}^{3/2})^{-1}$ [7]

Dashed line – theoretical prediction with direct losses.
Conclusions

◦ For the particles interacting through the power-like potential in the interval between the source and the bulk in the momentum space, distribution tends to a steady-state nonequilibrium distribution which has a plateau or gradually decreasing form.

◦ Even the relative smallness of sources may imply a drastic deviation of the steady-state distribution function from the equilibrium

◦ The functional dependence of the steady-state nonequilibrium electron distribution is non sensitive to the extent to which the source are located in momentum space.

◦ Results of simulation are consistent with asymptotic and experimental results and can be useful in connection with the development of high-power particle and energy sources.
THANK YOU!