Invariant Manifold-Based Model Reduction Method for a Class of Nonlinear Discrete-Time Dynamical Systems Using Functional Equations

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Problem formulation

- Given a nonlinear discrete-time dynamical system:

$$x(k+1) = f\left(x(k), w(k)\right)$$

driven by: $$w(k+1) = g\left(w(k)\right)$$

with: $x \in R^n, w \in R^m$

$k \in N^+$ Discrete time index

$f: R^m \times R^n \rightarrow R^n$ Analytic vector functions

$g: R^m \rightarrow R^m$

$\left(x^0, w^0\right) = (0, 0)$ Equilibrium point

develop an analytical model reduction method based on the computation of an invariant manifold that would lead to reduced-order system dynamics:

$$\begin{aligned} x(k) &= \pi(w(k)) \\ w(k+1) &= g(w(k)) \end{aligned}$$

The context

$$\begin{aligned} x(k+1) &= f(x(k), w(k)) \\ w(k+1) &= g(w(k)) \end{aligned}$$

- $w(k)$ - dynamics may represent:
  - Input/Disturbance model
  - The autonomous dynamics of an upstream process
  - Dynamics associated with a time-varying process parameter vector $w(k)$ that models phenomena such as catalyst deactivation, enzymatic activity degradation, heat transfer coefficient changes, etc.

- The class of problem under consideration naturally surfaces in:
  - System/Process condition monitoring (scheduling of possible replacement activities of the catalyst or enzyme)
  - Dynamic system analysis: A framework for the study of the dynamic response of driven systems/processes; skew product systems.

The proposed approach

- Under certain conditions, the proposed approach allows the **explicit** construction of an invariant manifold, upon which the restriction of the system dynamics leads to a reduced-order representation of the system dynamics in state space (model - reduction).

- Under the proposed problem formulation, the new method is based on a fairly general set of assumptions

- It is technically based on methods and results from nonlinear functional equations theory

The proposed approach

$$\begin{array}{rcl} x(k+1) & = & f(x(k), w(k)) \\ w(k+1) & = & g(w(k)) \end{array} \qquad (S)$$

- For the map $x = \pi(w), \pi \colon R^m \to R^n$ to be rendered invariant under the discrete-time dynamics (S), the following system of nonlinear functional equations (NFEs) must be satisfied:

$$\pi\left(g(w)\right) = f\left(\pi(w), w\right)$$

- A system of n nonlinear functional equations with n dependent and m independent variables.

Main results

- If:
  - The eigenvalues $\kappa_i; (i = 1, \ldots, m)$ of $A = \frac{\partial g}{\partial w}(0)$ all lie in the Poincaré domain (within the unit disc)
  - The eigenvalues $\lambda_i; (i = 1, \ldots, n)$ of $B = \frac{\partial f}{\partial x}(0, 0)$ are not

  related to the eigenvalues $\kappa_\mathbf{i}$ of $\mathbf{A}$ through any equation of the following form:

$$\prod_{i=1}^{m} \kappa_i^{\nu_i} = \lambda_j$$

  with $\nu_\mathbf{i}$ being non-negative integers such that $\sum_{i=1}^{m} \nu_i \geq 1$

- Then, there exists a neighborhood $U \subset R^m$ of $w^0 = 0$ and a unique and locally analytic map $x = \pi(w), \pi\colon U \to R^n$ such that:

$$M = \{(x, w) \in R^n \times U\colon x = \pi(w), \pi(0) = 0\}$$

  is an invariant manifold, with $x = \pi(w)$ being the solution to the associated system of NFEs.

Main results

- Assume that matrix $B = \frac{\partial f}{\partial x}(0,0)$ has stable eigenvalues.
- There exists a neighborhood V^0 of $(x^0, w^0) = (0,0)$ and real number $L \in (0,1)$ such that, if $(x(k=0), w(k=0)) \in V^0$ then:
$$\|x(k) - \pi((w(k))\| \leq (L)^k \|x(0) - \pi(w(0))\|$$
- **Remark 1:** All trajectories starting at a point sufficiently close to the origin (equilibrium point) are attracted to the invariant manifold M. Therefore, the reduced-order system dynamics is represented by:
$$\begin{aligned} x(k) &= \pi(w(k)) \\ w(k+1) &= g(w(k)) \end{aligned}$$
- **Remark 2:** It can be shown that under certain conditions, the proposed approach offers explicitly the system's stable manifold

Series solution method

- **Method:**
  - Expand $f(x,w), g(w)$ in multivariate Taylor series
  - Expand the unknown solution $\pi(w)$ in Taylor series
  - Match the Taylor coefficients of the same order on both sides of the associated system of NFEs

- **Recursions Formulas:**
  - Tensorial notation leads to a compact mathematical representation
  - Linear with respect to the Taylor coefficients of the unknown solution

- The series solution method can be easily implemented with the aid of a simple MAPLE code.

Case of the linear process

$$\begin{aligned} x(k+1) &= Bx(k) + Cw(k) \\ w(k+1) &= aw(k) \end{aligned} \qquad (S)$$

with $x \in R^n, w \in R$

- If $|a| < 1$ and $max_i|\lambda_i| \ll |a|$ (e.g. the catalyst deactivation dynamics is slower than the process dynamics due to poisoning (Fogler 1992)), then by solving recursion (S):

$$x(k) \underset{k\to\infty}{\approx} -(B - aI)^{-1}Ca^k w(0)$$

- Applying the proposed approach, the following functional equation needs to be solved: $\pi(aw) = B\pi(w) + Cw$
- All assumptions are satisfied and the solution is given by:

$$x = \pi(w) = -(B - aI)^{-1}Cw \Rightarrow x(k) \underset{k\to\infty}{\approx} -(B - aI)^{-1}Ca^k w(0)$$

  - The linear result is naturally reproduced

Case study

- Immobilized enzymatic continuous stirred tank bioreactor for the production of food grade linoleic acid from corn oil (Sehanputri and Hill 1999).

- Dynamic process model:

$$\begin{aligned}\frac{dS}{dt} &= \frac{k_1 E S}{1 - k_2 S} + \frac{v_0}{V}(S_0 - S) \\ \frac{dE}{dt} &= -k_d E\end{aligned}$$

- Discrete-time dynamic process model (Euler's method)

$$\begin{aligned}S(k+1) &= S(k) + \delta\left(\frac{k_1 E S}{1 - k_2 S} + \frac{v_0}{V}(S_0 - S(k))\right) &= f(S(k), E(k)) \\ E(k+1) &= E(k) - k_d E(k)\delta &= g(E(k))\end{aligned}$$

where δ is the time-discretization step

Case study

- Define deviation variables:

$$
\begin{aligned}
x &= S - S^0 \\
w &= E - E^0
\end{aligned}
$$

with respect to the reference equilibrium point:

$$(S^0, E^0) = (3.4, 0)$$

and denote:

$$
\begin{aligned}
\bar{f}(x, w) &= f(x + S^0, w + E^0) \\
\bar{g}(w) &= g(w + E^0)
\end{aligned}
$$

Case study

- Nonlinear functional equation:

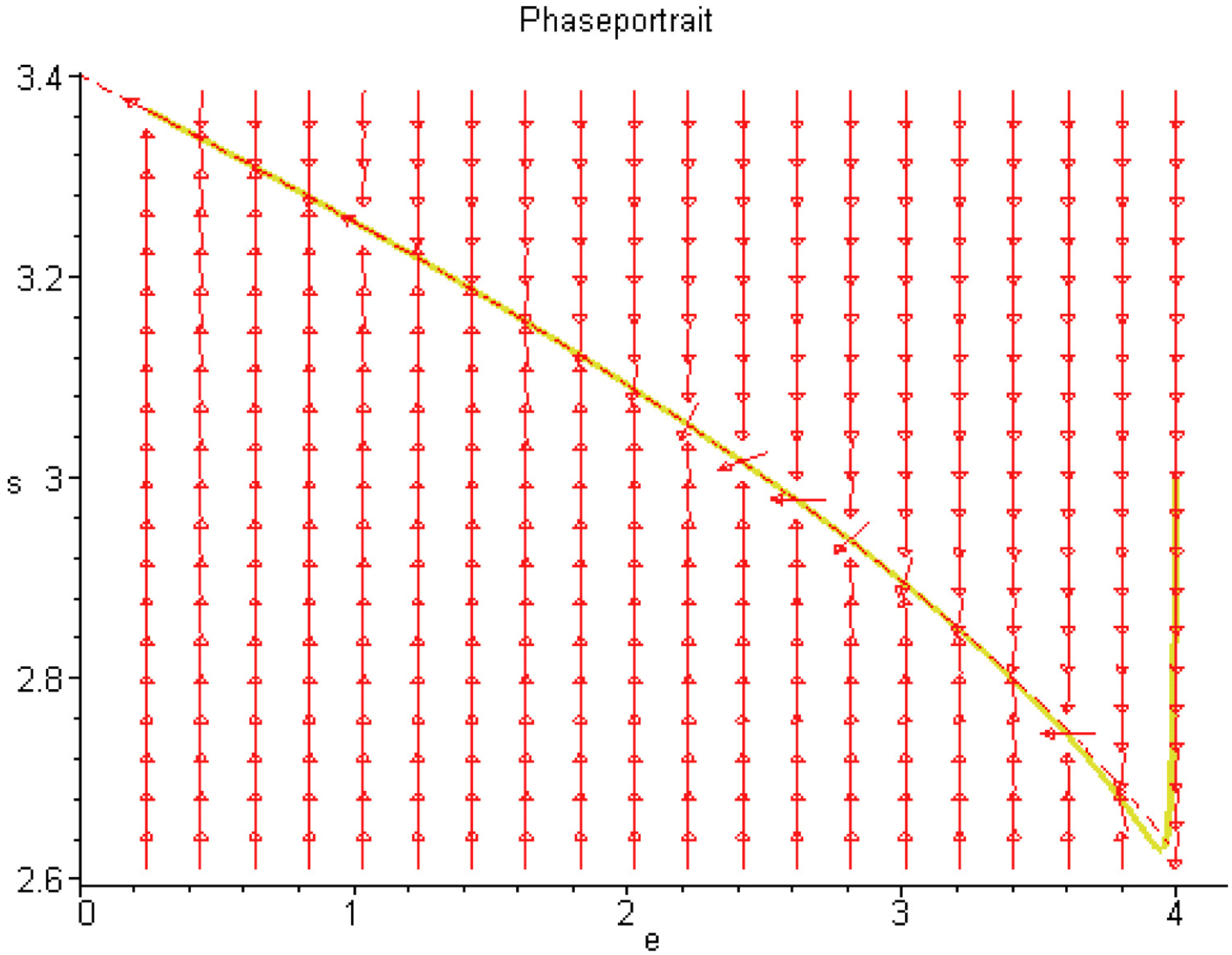
$$\pi(\bar{g}(w)) = \bar{f}(\pi(w), w)$$

- Up to N=10 series truncation orders were considered leading up to tenth-order Taylor polynomial approximations of the actual solution computed through a MAPLE code

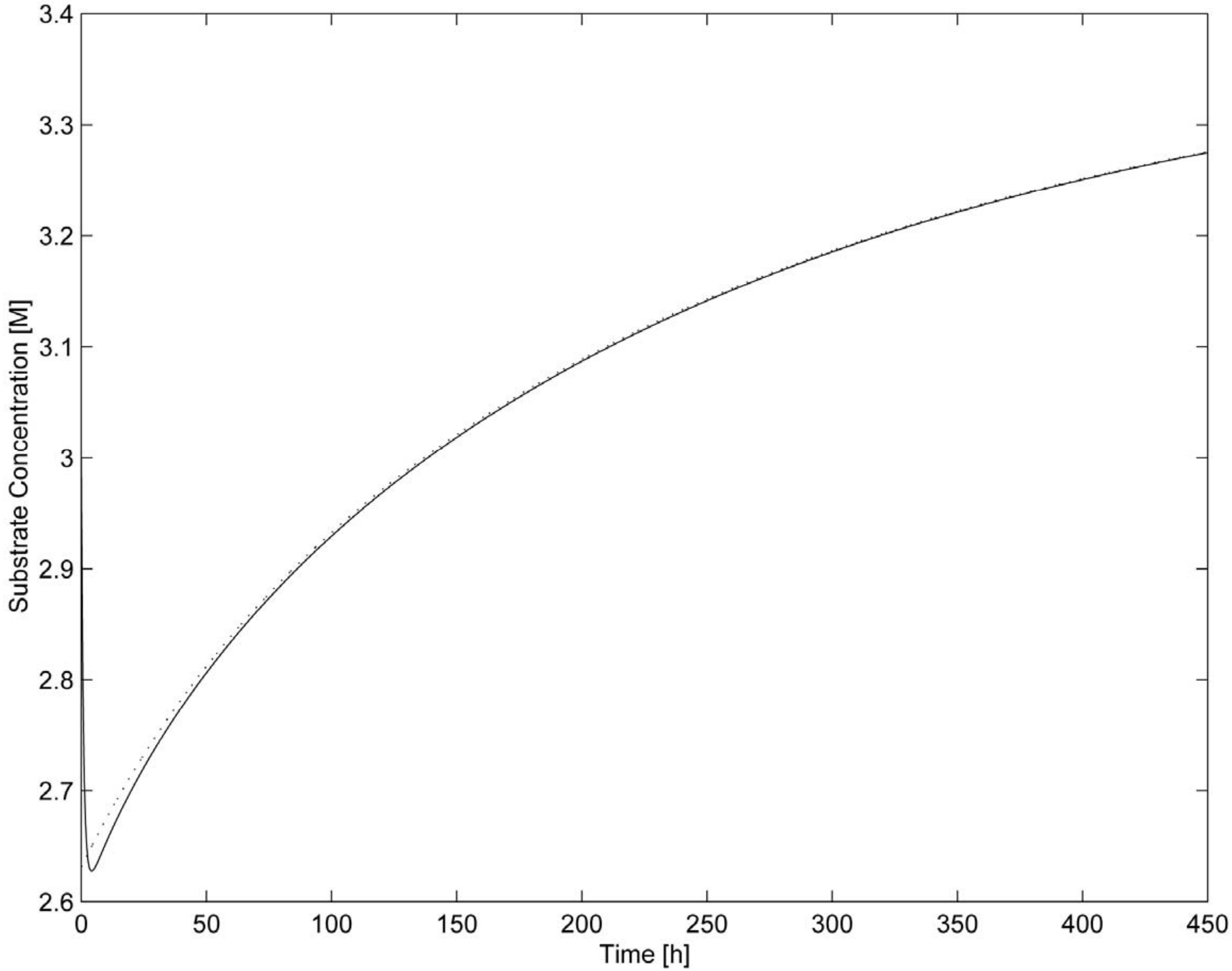
- Reduced-order system dynamics for a truncation order N:

$$\begin{aligned} x(k) &= \pi^{[N]}(w(k)) \\ w(k+1) &= \bar{g}(w(k)) \end{aligned}$$

Case study

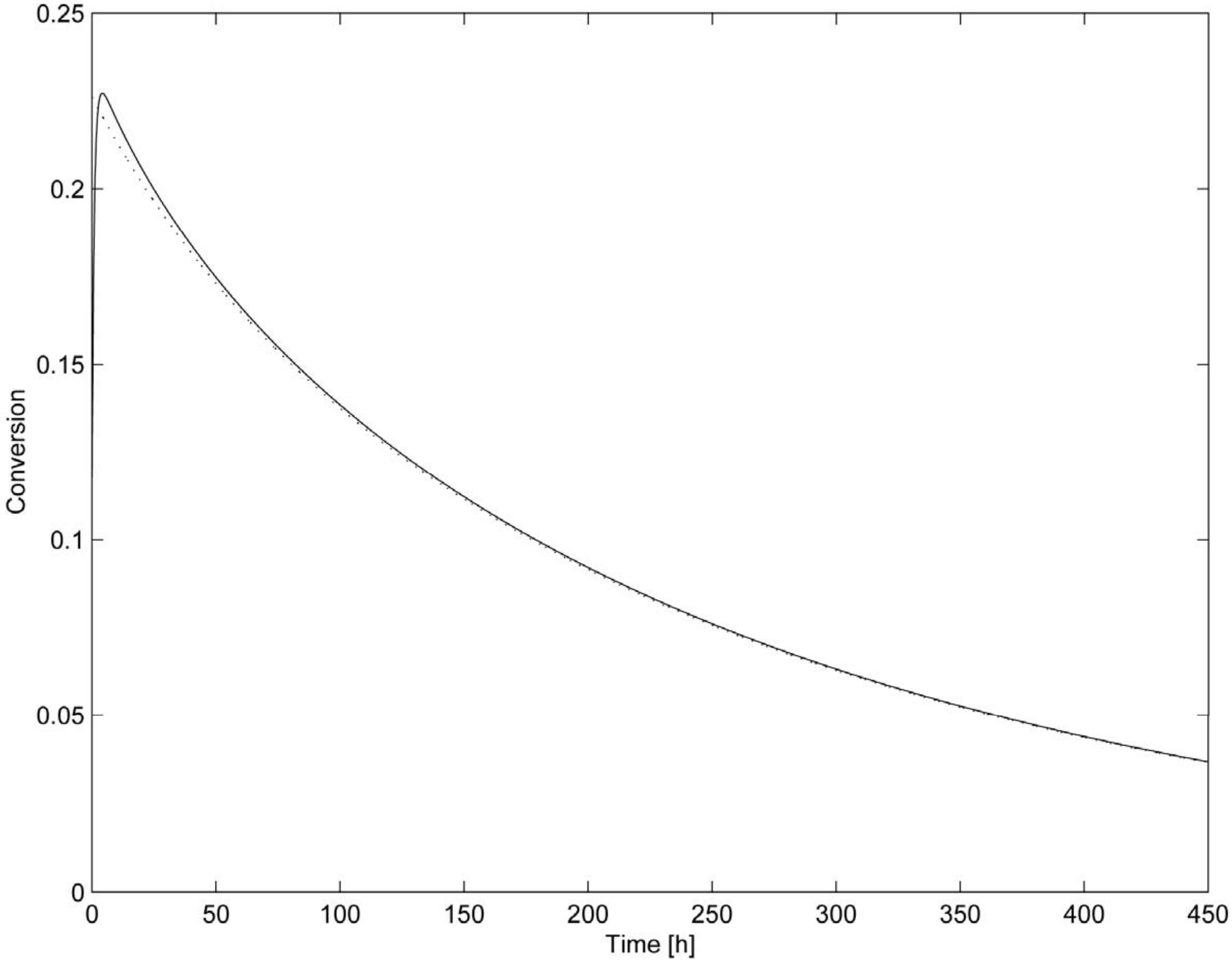


Case study



3.4
3.3
3.2
3.1
3
2.9
2.8
2.7
2.6
Substrate Concentration [M]
0
50
100
150
200
250
300
350
400
450
Time [h]

Case study


0.25
0.2
0.15
0.1
0.05
0
Conversion
0
50
100
150
200
250
300
350
400
450
Time [h]

Conclusions

- A new model-reduction method for a class of nonlinear discrete-time dynamical systems realized through the explicit construction of an invariant manifold is proposed

- On a theoretical level the proposed approach is:
  - Conceptually founded on the notion of invariance
  - Technically based on methods and results from nonlinear functional equations theory

- On a practical level, the proposed method can be easily implemented with the aid of a symbolic software package such as MAPLE.