Supercomputing for Superproblem: a computational travel in pure mathematics

On the eve of the XXth century, in 1900, the great mathematician David Hilbert published a list of problems that should be the main challenge for the century. In his talk to the International Congress of Mathematicians he had presented ten problems and in the following publication he extended this list to the twenty-three problem. Many mathematicians sacrificed their life to these problems. Some of them were solved.

On the eve of the XXIth century (new Millennium), the Clay Mathematics Institute followed the Hilbert's way and had published in 2000 seven problems, the so-called "Millennium problems". One of these problems (Poincaré conjecture) is already solved by G. Perelman.

The only problem belongs to both 1900 and 2000 lists: this is the famous Riemann hypothesis ("the real part of any non-trivial zero of the Riemann zeta function is ½", 1859). This is a question about one function of complex variable, the Riemann zeta function. It is defined as a sum of the infinite series for a complex number *s*:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

This series converges for *s* with real part greater than 1 and can be analytically continuated to other values f *s*. It has zeros at the negative even integers, $s = -2, -4, -6, \ldots$, (the so-called trivial zeros). Where are the other zeros of $\zeta(s)$? If all these zeros have the real part ½ then the Riemann hypothesis is right.

The question sounds very simple and special. Indeed, why the location of zeros of a particular function may be so important that it is considered as one of the main problems in mathematics for two hundreds of years? It seems miraculous but the proof of the Riemann hypothesis will give us the Cornucopia with solutions of many famous problems in number theory and beyond.

Prime numbers are like genes of the integers as every integer has a product factorization in powers of prime numbers, which is something like the genotype of the integer. It is therefore fundamentally important to understand the distribution of prime numbers among the integers. Riemann showed that the number of primes less than a given natural number is represented by an explicit formula in terms of a sum over the zeros of the Riemann zeta function. The magnitude of the oscillations of primes around their expected position among all the integers is controlled by the real pats of the zeros of the zeta functions.

Many gifted mathematicians would be happy to prove or disprove the Riemann hypothesis. Yuri Matiyasevich is a famous mathematician because he had already solved the 10th Hilbert problem: Find an algorithm to determine whether a given polynomial equation with integer coefficients ("Diophantine equation") has an integer solution. Matiyasevich's theorem gives the negative answer: there is no such algorithm.

Now, Matiyasevich is working on the Riemann hypothesis. One of the ways to generate working hypotheses ("guesses") and disprove the wrong guesses he use is computing. We have to study the zeros of the Riemann zeta function, do we? Well, let us find as many zeros as we can with the highest accuracy we can, then generate hypotheses, then try again to disprove this guess and so on. The precision should be high enough (sometimes, he needed more than thousand significant figures).

This summer, Matiyasevich visited UK by invitation of the Isaac Newton Institute for Mathematical Sciences. He gave a talk at the University of Leicester and now published a report about his research of the Riemann zeta function.

The goal of this paper is to present numerical evidence for a new method for revealing all divisors of all natural numbers from the zeroes of the zeta function. This approach required supercomputing power.

There exist some examples of solution of famous pure mathematical problems using massive computations. The well-known example is the four color problem. Now, the four color map theorem states that, given any separation of a plane into contiguous regions (a map), no more than four colors are required to color the regions so that no two adjacent regions have the same color. First, it was proven that there exists a finite number of maps which can include a counterexample (1,936 maps) and then these maps were studied by computers.

Unfortunately, the Riemann hypothesis is not reduced now to a finite problem and, therefore, the computations can disprove but cannot prove it. Computations here provide the tools for guessing and disprove the guesses. Matiyasevich selected the following epigraph to his work published in Leicester:
"The physicist George Darwin used to say that
every once in a while one should do a completely
crazy experiment, like blowing the trumpet to
the tulips every morning for a month. Probably
nothing will happen, but if something did happen,
that would be a stupendous discovery."
[Ian Hacking, Representing and Intervening. Cambridge University Press, 1983. p.154]

Now the report about this supercomputing travel into the world of the Riemann hypothesis is available online together with the presentation of the Matiyasevich talk. New guesses are founded, some old guesses are rejected and no final answer yet.
http://www2.le.ac.uk/departments/mathematics/research/research-reports-2/research-reports-2012