Asymptotic Dynamics of Spiral and Scroll Waves

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Model Reduction Across Disciplines, Leicester, August 19–22, 2014
Conference dedicated to the 60th birthday of Alexander Gorban

1 A brief introduction

2 Theory

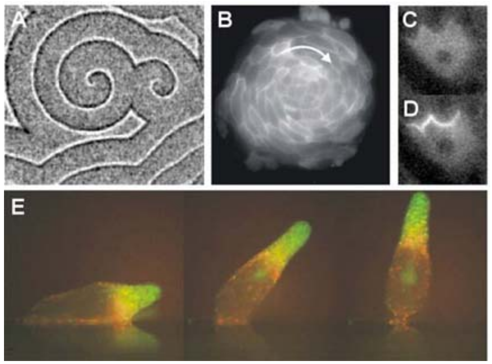
3 Examples: spirals

4 Examples: scrolls

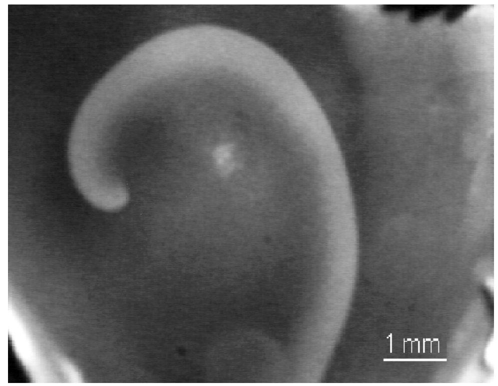
5 Examples: between 2D and 3D

6 Conclusions

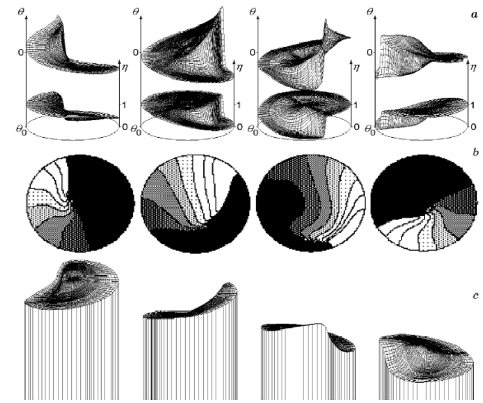
Various spiral waves in nature



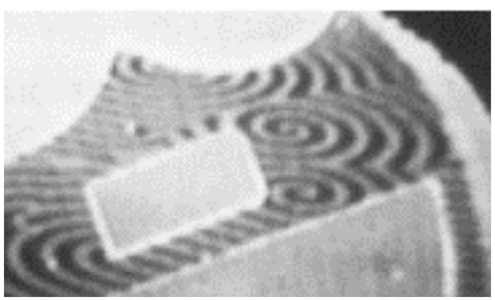
Dictyostelium discoideum (C. Weijer, Dundee)



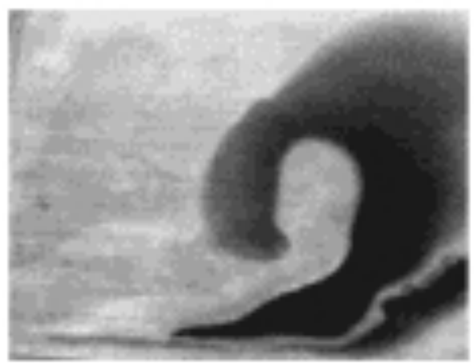
Retina (M.A.Dahlem, Magdeburg)



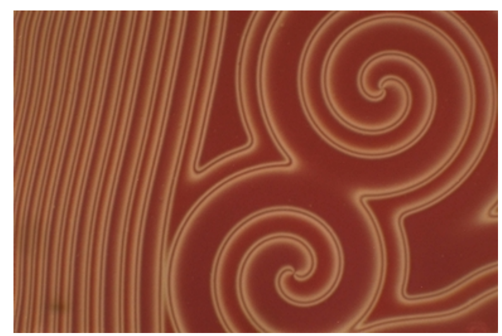
Combustion (A.Merzhanov, Chernogolovka)



Oxidation of CO on Pt (Y. Kevrekidis, Princeton)



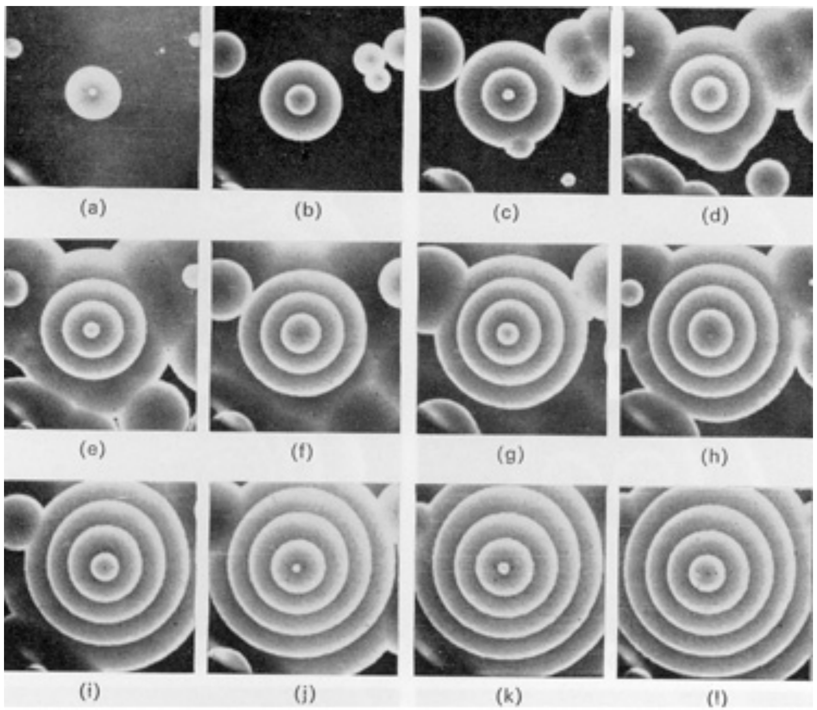
Rusting of steel (O. Steinbock, Florida)



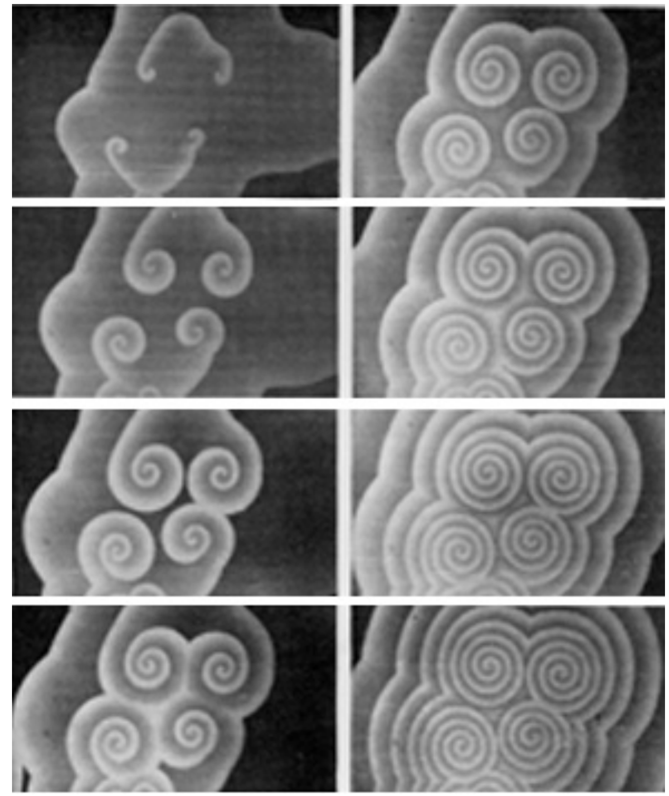
Liquid crystal (S. Residori, Nice)

Belousov-Zhabotinsky reaction

Non-stirred: concentric waves



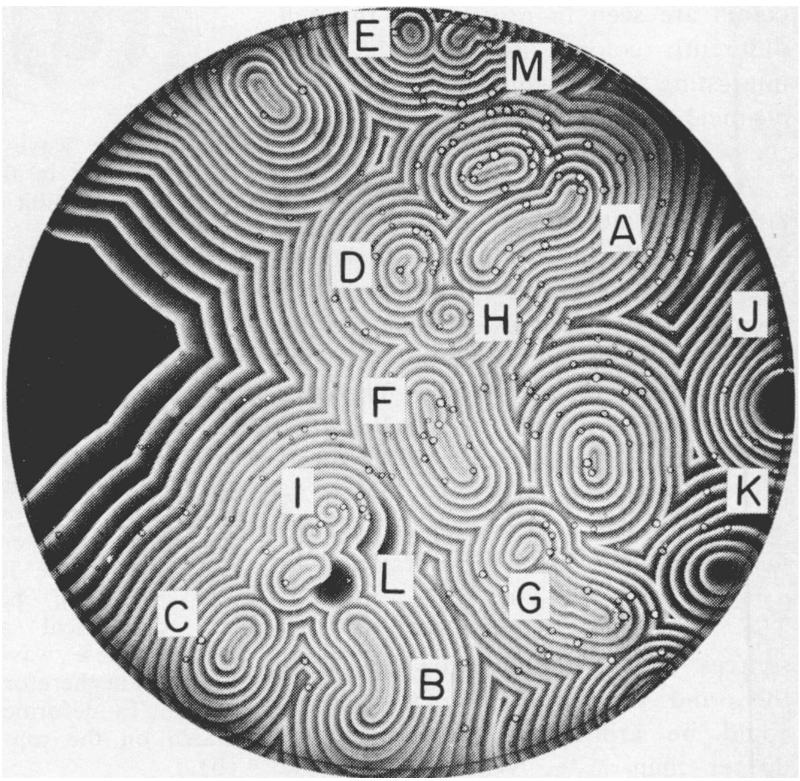
Stirred: spiral waves



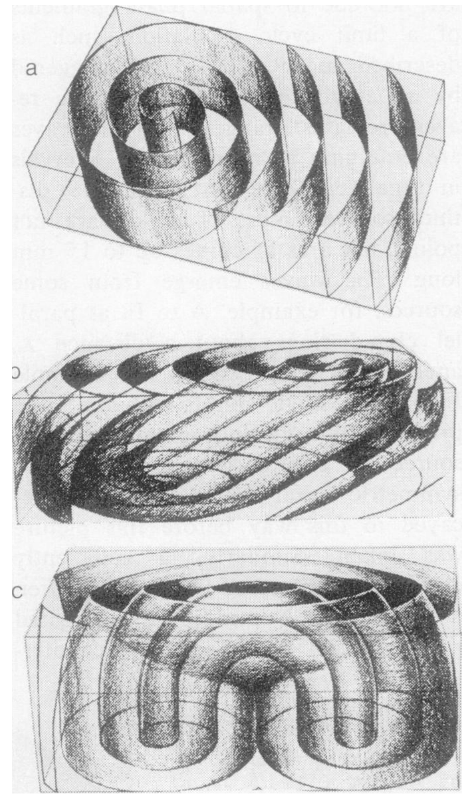
A. M. Zhabotinsky and A. N. Zaikin, "Spatial effects in a self-oscillating chemical system", in *Oscillatory processes in biological and chemical systems II*, Sel'kov E. E. Ed., Science Publ., Puschino (1971)

Scroll waves in BZ

Experiment

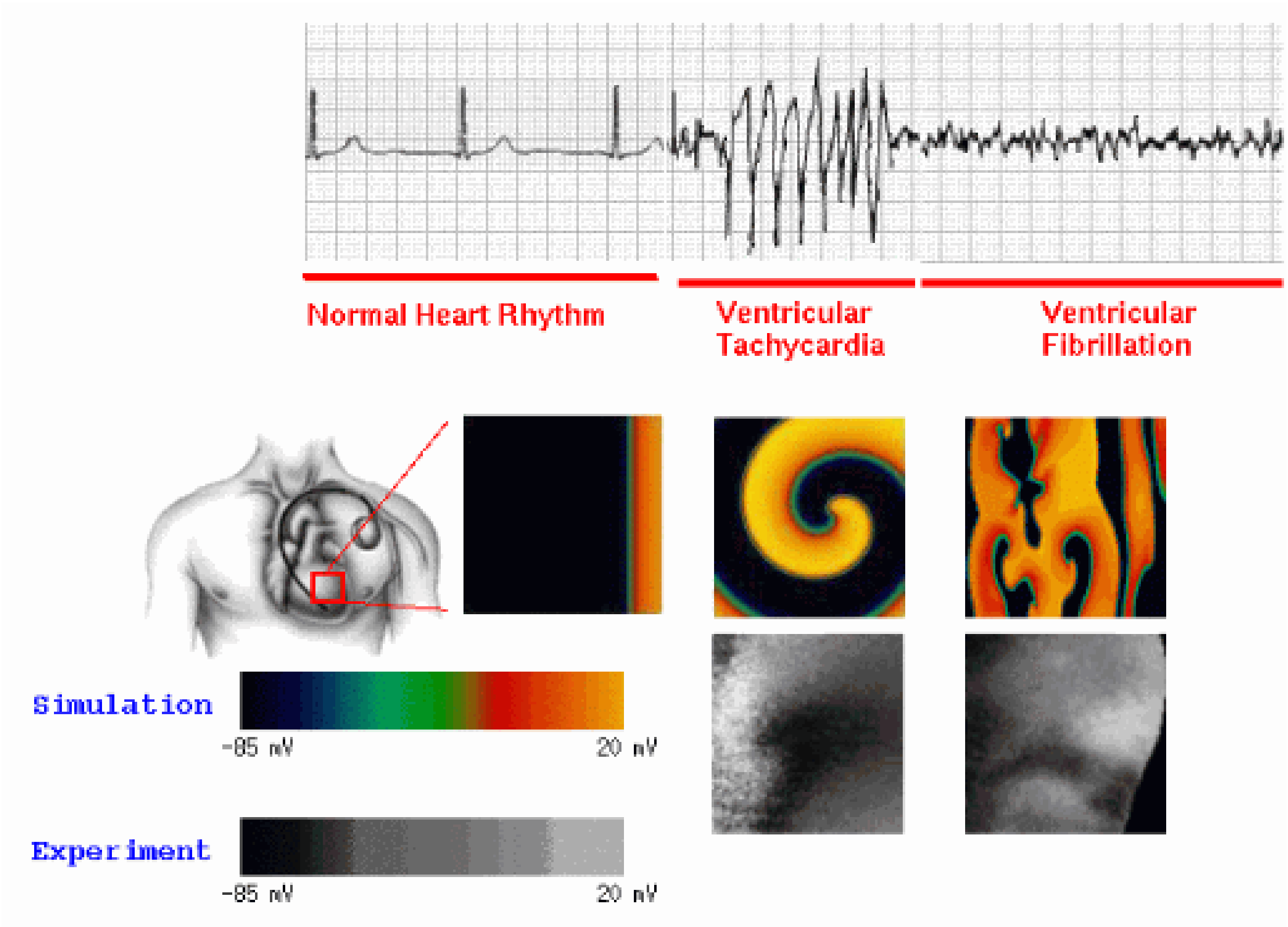


Explanation



A. T. Winfree, "Scroll-shaped waves of chemical activity in three dimensions", *Science* **181**:937-939 (1973)

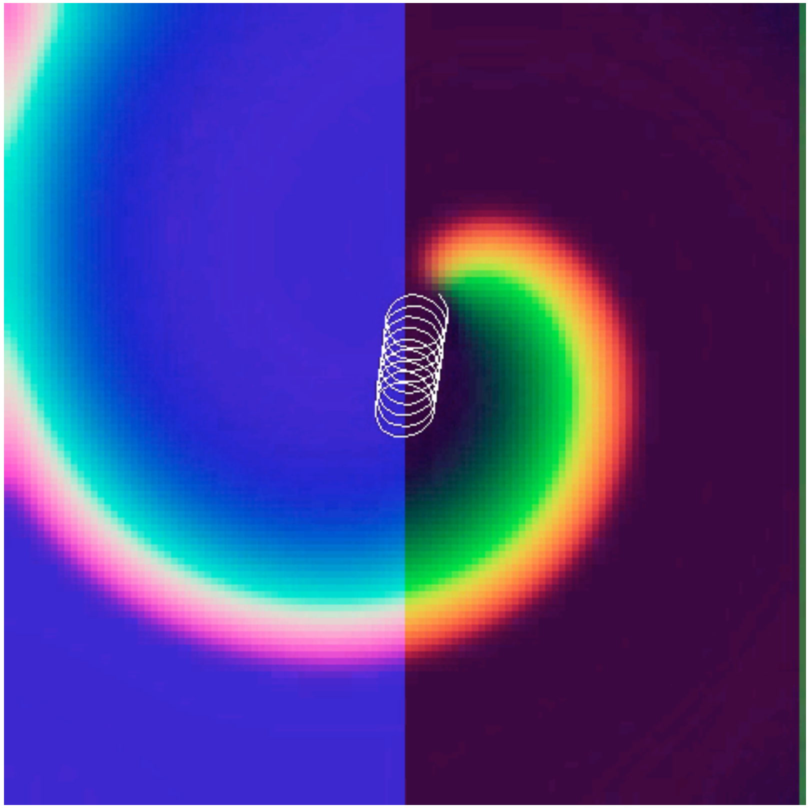
Spiral waves in heart



Picture from: `http://thevirtualheart.org` (F. Fenton)

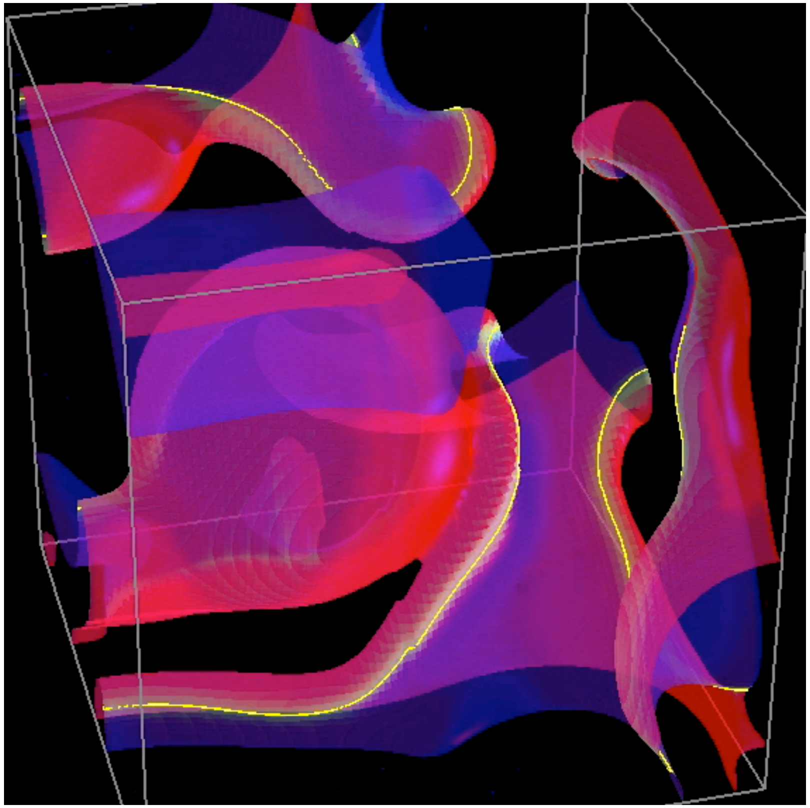
Spiral and scroll waves

Drift due to inhomogeneity



A.M. Pertsov, E.A. Ermakova, "Mechanism of the drift of a spiral wave in an inhomogeneous medium". *Biofizika*, **33**(2): 338-342, 1988.

Spontaneous evolution



V.N. Biktashev, A.V. Holden & H. Zhang, ``Tension of Organizing Filaments of Scroll Waves" *Phil. Trans. Roy. Soc. London, ser A* **347**: 611-630 (1994); V.N. Biktashev ``A Three-Dimensional Autowave Turbulence" Int. J. Bifurcation & Chaos, 8(4): 677-684, (1998)

The challenge and the intrigue

- In describing spiral and scroll waves, it is convenient to talk in terms of spiral core and scroll filaments, as particle-like and string-like objects.
- What corresponds to this convenience mathematically?
- The answer involves model reductions (2D→0D and 3D→1D), but also some unique feature of spiral waves.
- There will be also one more reduction (3D→2D) towards the end. . .

1 A brief introduction

2 Theory

3 Examples: spirals

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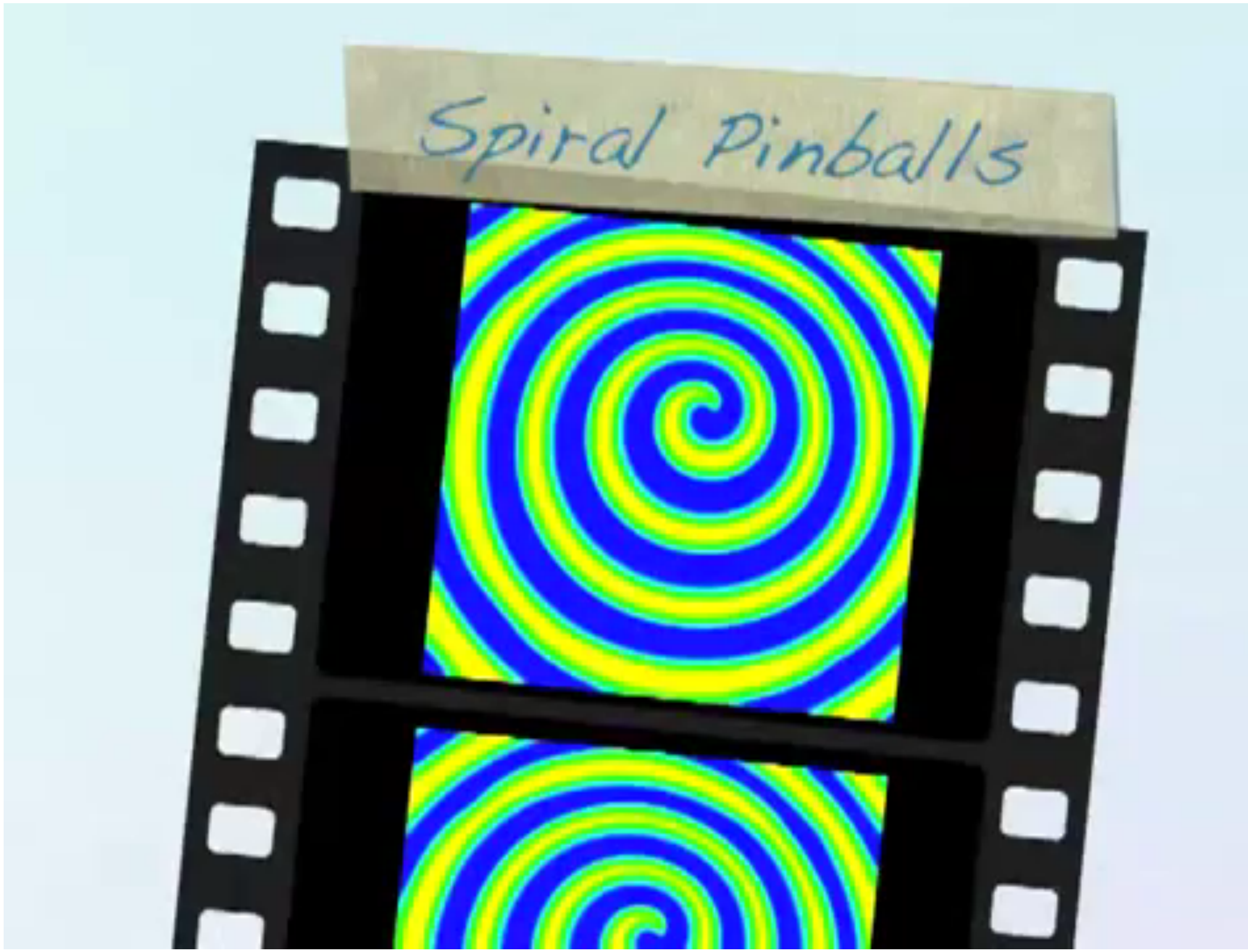
6 Conclusions

Drift of spirals:
a popular introduction by Dwight Barkley

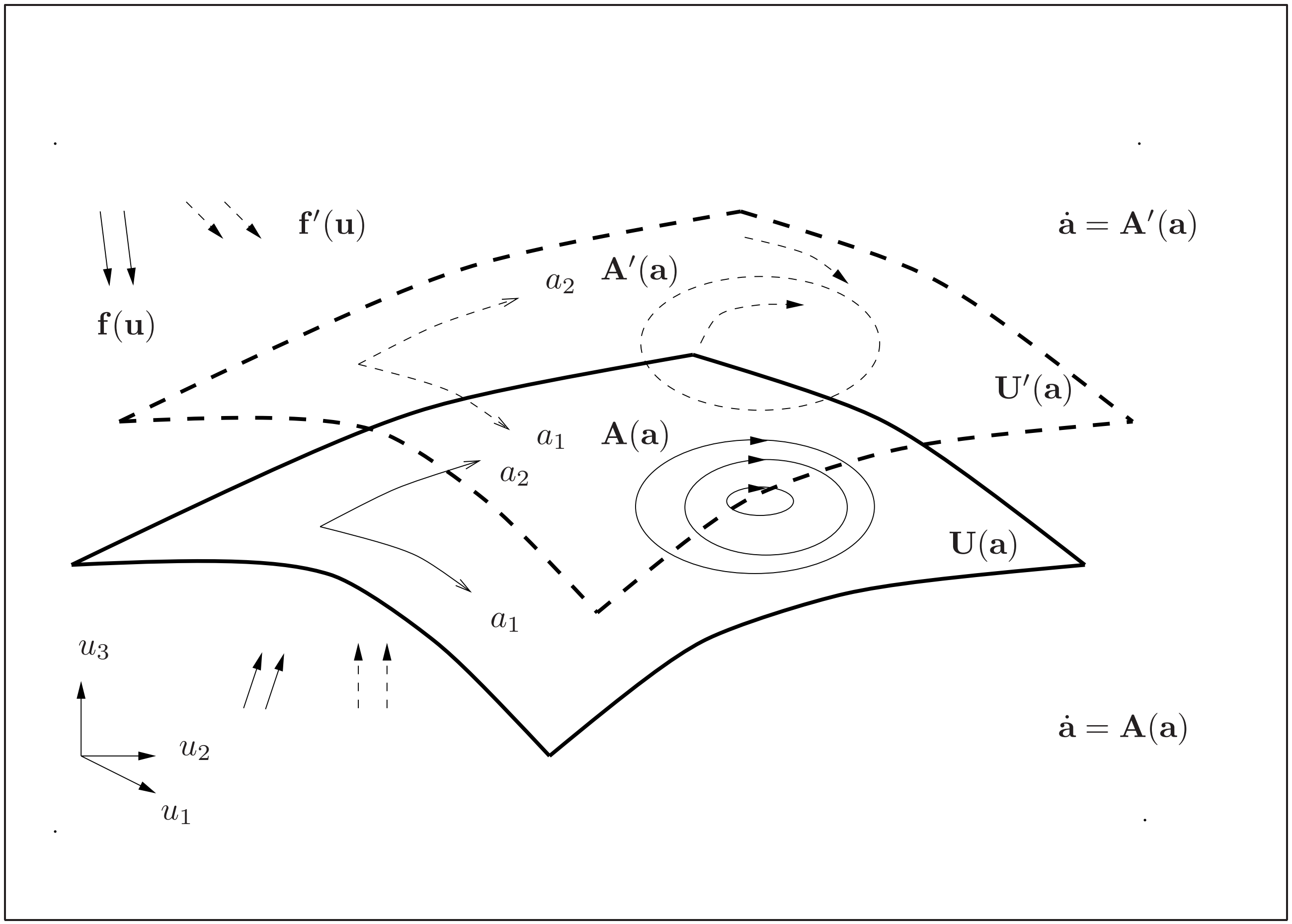
(with some help from Lady Gaga)

https://www.youtube.com/watch?feature=player_embedded&v=YGVvZVD_ddo

https://sites.google.com/site/barkleyvideos/



Spiral waves reduction (2D→0D): the idea



Spiral waves reduction (2D→0D): ansatz

- (Perturbed) reaction-diffusion system for ℓ components on the plane

$$\partial_t \mathbf{u} = \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2\mathbf{u} + \epsilon\mathbf{h},$$

$$\mathbf{u}(\vec{r},t), \mathbf{f}(\mathbf{u}), \mathbf{h}(\cdot) \in \mathbb{R}^\ell,\ \mathbf{D} \in \mathbb{R}^{\ell\times\ell},\ \ell \geq 2,\ \vec{r} \in \mathbb{R}^2.$$

- Steadily rotating spiral wave solutions ($\epsilon = 0$):

$$\mathbf{u}(\vec{r},t) = \mathbf{U}(\rho(\vec{r}-\vec{R}), \vartheta(\vec{r}-\vec{R}) + \omega t - \Phi).$$

($\vec{r} = (x,y)$, $\vec{R} = (X,Y) = \mathrm{const}$, $\Phi = const$, ω is an eigenvalue).

- For $\epsilon \neq 0$, the spiral drifts: solution remains approximately as above, but $\mathrm{d}\vec{R}/\mathrm{d}t = \mathcal{O}(\epsilon)$, $\mathrm{d}\Phi/\mathrm{d}t = \mathcal{O}(\epsilon)$.

Spiral waves reduction (2D→0D): equation of motion

- Drift velocity due to perturbation:

$$\dot{R} = \epsilon \int_{\phi-\pi}^{\phi+\pi} e^{-i\xi} \left\langle \mathbf{W}_1 \,,\, \tilde{\mathbf{h}}(\mathbf{U}; \rho, \theta, \xi) \right\rangle \frac{\mathrm{d}\xi}{2\pi} + \mathcal{O}\left(\epsilon^2\right),$$

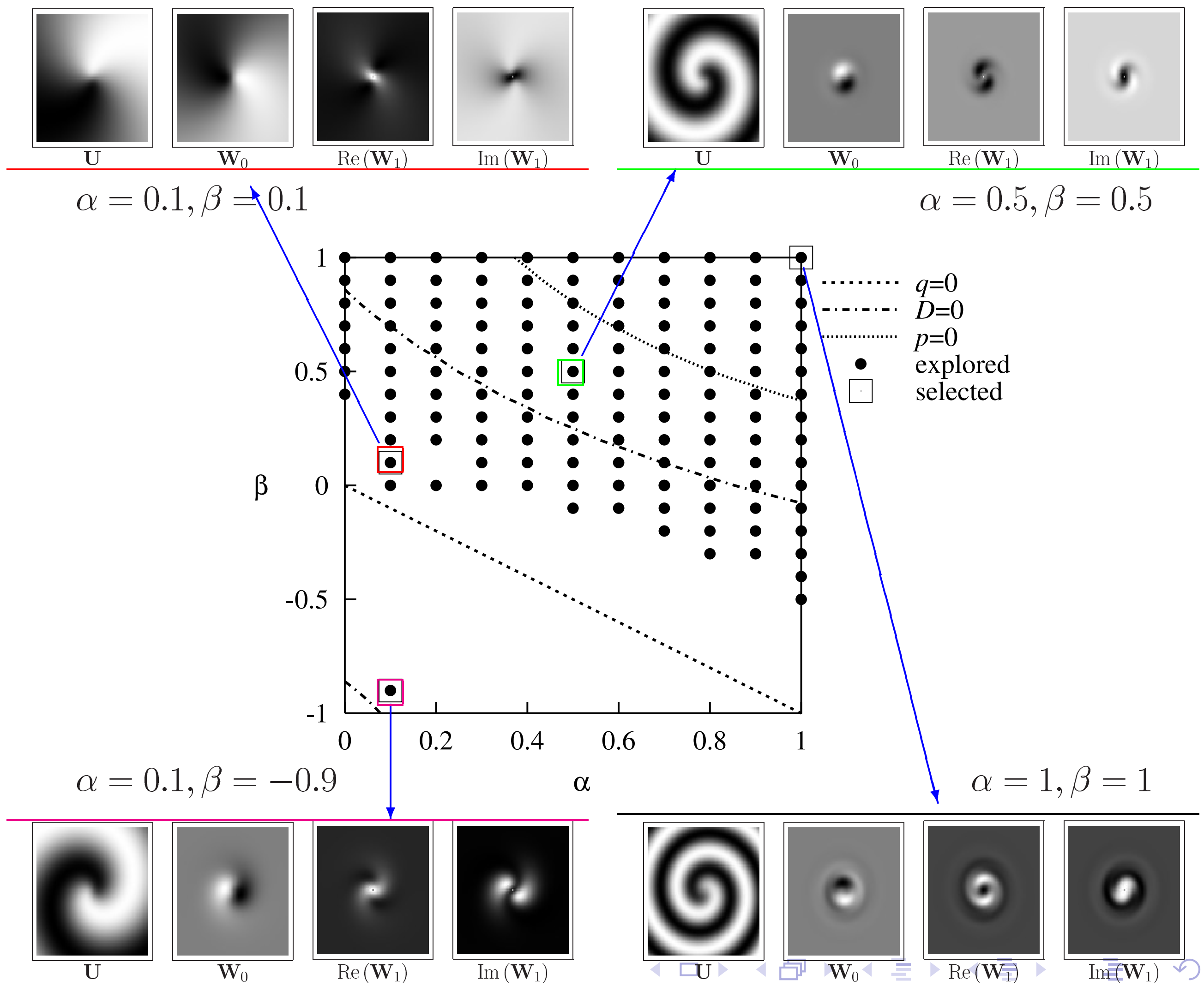
where (ρ, θ) are corotating polar coords, $\phi = \omega t - \Phi(t)$, and

$$\langle \mathbf{w} \,,\, \mathbf{v} \rangle = \int_{\mathbb{R}^2} \mathbf{w}^+(\vec{r})\, \mathbf{v}(\vec{r})\, \mathrm{d}^2\vec{r} = \oint \int_0^\infty \mathbf{w}^+(\rho, \theta)\, \mathbf{v}(\rho, \theta) \rho \,\mathrm{d}\rho \,\mathrm{d}\theta.$$

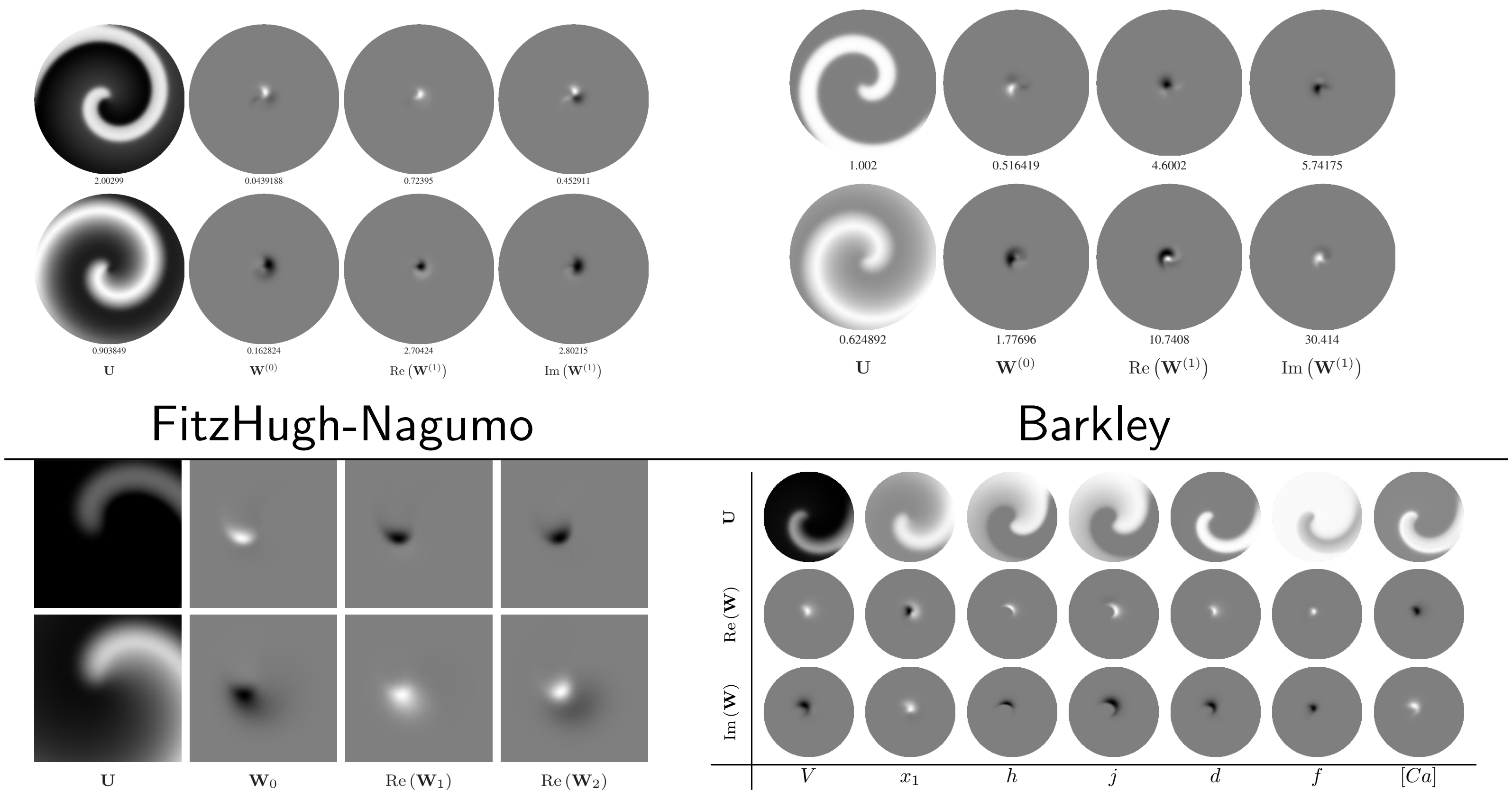
- (Translational) response function $\mathbf{W}_1(\rho, \theta) \in \mathbb{C}$: eigenfunction of the adjoint linearized operator, corresponding to eigenvalue $\mathrm{i}\omega$.
- Linear expressions, hence superposition principle.

Spiral wave reduction: why do the integrals converge

Complex Ginzburg-Landau Equation



Spiral wave reduction: why do the integrals converge



FitzHugh-Nagumo

Barkley

Oregonator
(BZ reaction)

Beeler-Reuter
(heart ventricles)

Spiral wave reduction: why do the integrals converge

SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 3, No. 1, pp. 1–68

Defects in Oscillatory Media: Toward a Classification*

Björn Sandstede† and Arnd Scheel‡

. . .

Corollary 4.6. *Assume that $u_{\mathrm{d}}(\xi,\tau)$ is a transverse source. The null space of the adjoint operator $\Phi_{\mathrm{d}}^{\mathrm{ad}} - 1$ on $L^2(\mathbb{R},\mathbb{C}^n)$ is at least two-dimensional and contains two linearly independent functions $\psi_{\mathrm{d}}^{c}(\xi,0)$ and $\psi_{\mathrm{d}}^{\omega}(\xi,0)$ that satisfy*

$$\int_{\mathbb{R}} \begin{pmatrix} \langle \psi_{\mathrm{d}}^{c}(\xi,0), \partial_\xi u_{\mathrm{d}}(\xi,0)\rangle & \langle \psi_{\mathrm{d}}^{c}(\xi,0), \partial_\tau u_{\mathrm{d}}(\xi,0)\rangle \\ \langle \psi_{\mathrm{d}}^{\omega}(\xi,0), \partial_\xi u_{\mathrm{d}}(\xi,0)\rangle & \langle \psi_{\mathrm{d}}^{\omega}(\xi,0), \partial_\tau u_{\mathrm{d}}(\xi,0)\rangle \end{pmatrix} \mathrm{d}\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, the corresponding solutions $\psi_{\mathrm{d}}^{c}(\xi,\tau)$ and $\psi_{\mathrm{d}}^{\omega}(\xi,\tau)$ of (4.8) decay exponentially with a uniform rate as $\xi \to \pm\infty$ for all τ.

B. Sandstede claims this result extends to 2D, i.e. spiral waves (private communication).

Scroll waves reduction (3D→1D): ansatz

- (Unperturbed) reaction-diffusion system for ℓ components in space,

$$\partial_t \mathbf{u} = \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2\mathbf{u} \cancel{+\epsilon\mathbf{h}},$$

$$\mathbf{u} = \mathbf{u}(\vec{r}, t); \vec{r} \in \mathbb{R}^3.$$

- Steadily rotating spirals in 2D:

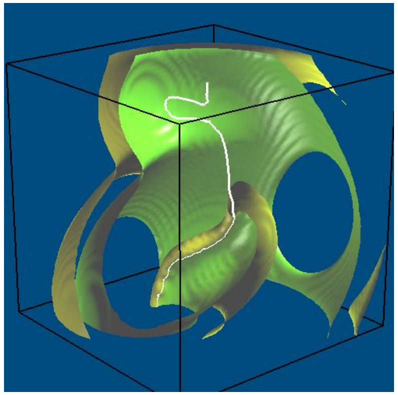
$$\mathbf{f}(\mathbf{U}) - \omega\mathbf{U}_\theta + \mathbf{D}\nabla^2\mathbf{U} = 0,$$

- Looking for a bended and twisted scroll in 3D:

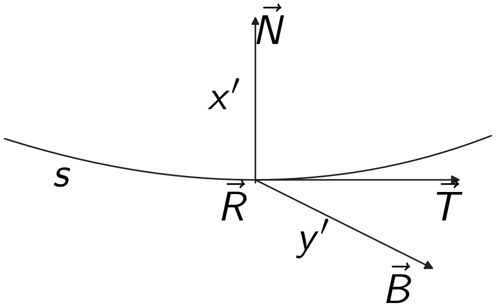
$$\mathbf{u}(\vec{R}+\vec{N}\rho'\cos\theta'+\vec{B}\rho'\sin\theta', t') =$$

$$\mathbf{U}(\rho', \theta' + (\omega t' - \Phi)) + \mathbf{v}, \quad |\mathbf{v}| \ll 1,$$

A twisted and bent scroll:



Frenet-Serret frame:



Scroll waves reduction (3D→1D): equation of motion

- Asymptotically, when filament is locally almost straight and twist is small,

$$
\begin{aligned}
(\dot{\vec{R}})_\perp = & \left(\gamma_1 + \gamma_2 \partial_\sigma \vec{R} \times\right) \partial_\sigma^2 \vec{R} && \text{(linear) "tension"} \\
& - \left(e_1 + e_2 \partial_\sigma \vec{R} \times\right) (\partial_\sigma^4 \vec{R})_\perp && \text{"rigidity"} \\
& + |\partial_\sigma^2 \vec{R}|^2 \left(b_1 + b_2 \partial_\sigma \vec{R} \times\right) \partial_\sigma^2 \vec{R} && \text{"nonlinear tension"}
\end{aligned}
$$

- Spirals are “building blocks” of scrolls, hence coefficients of the EoM are overlap integrals involving the same response functions.

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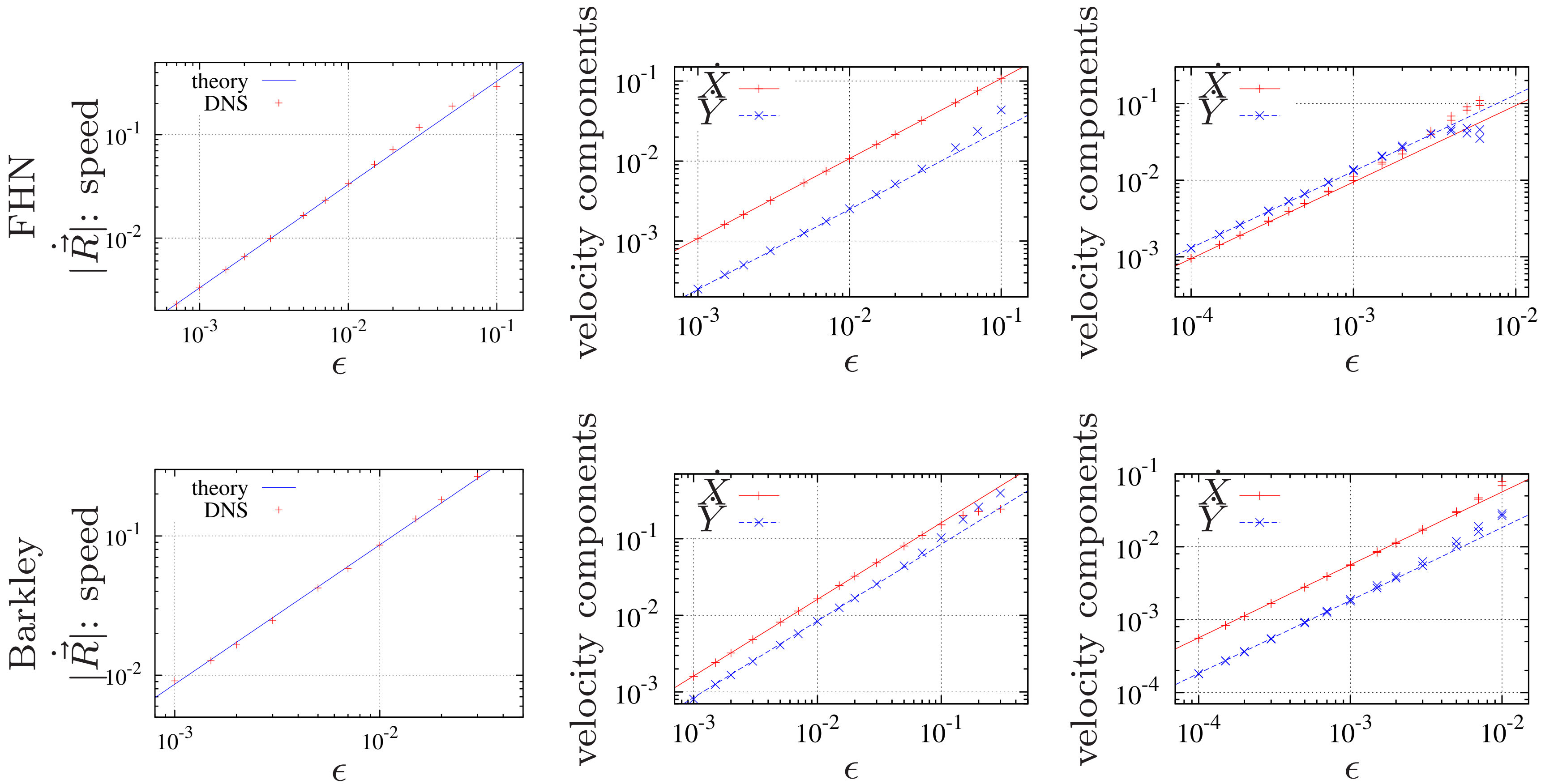
Simple drifts



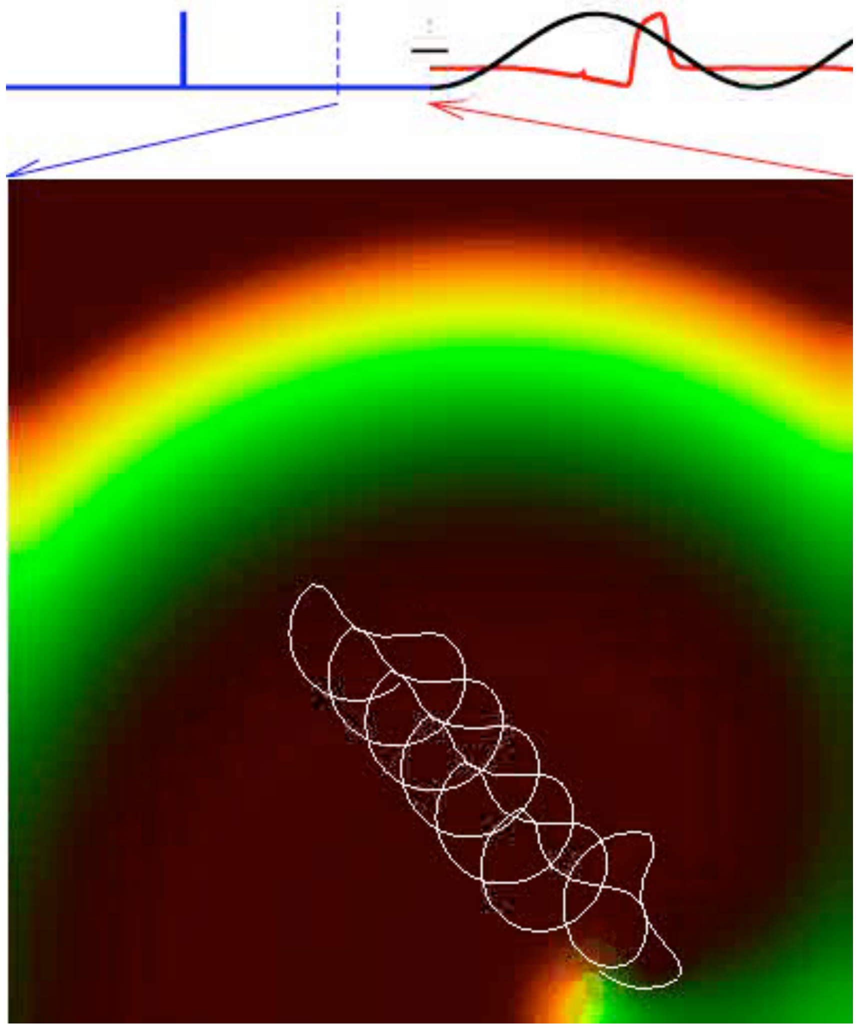
$\mathbf{h} \propto \cos(\omega t)$
resonant

$\mathbf{h} \propto \partial_x \mathbf{u}$
electrophoresis

$\mathbf{h} \propto (x - x_0)\partial f/\partial p$
linear parametric gradient

Resonant drift and Resonant repulsion

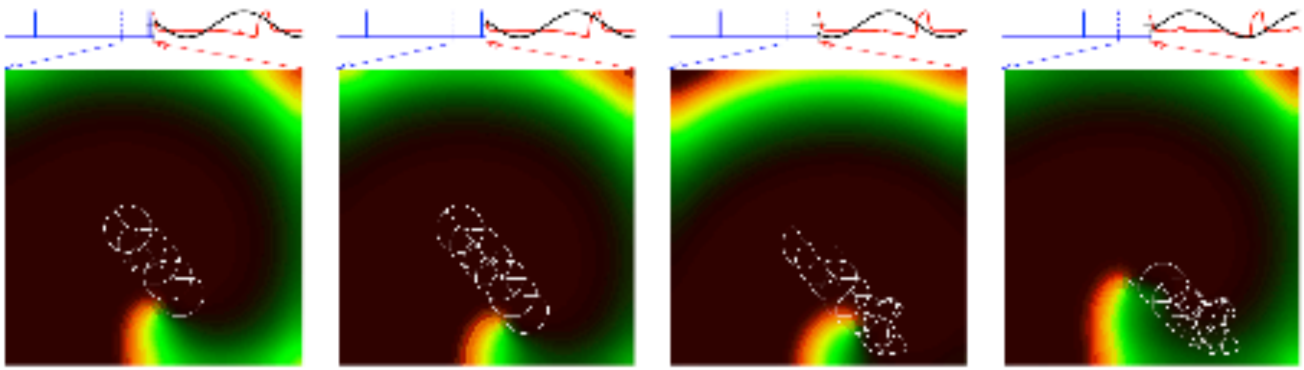
- Each stimulus shifts the spirals
- Stimulation period = spiral period => drift along straight line.
- However inhomogeneity
  (the boundary in this case) changes spiral period => direction of drift changes



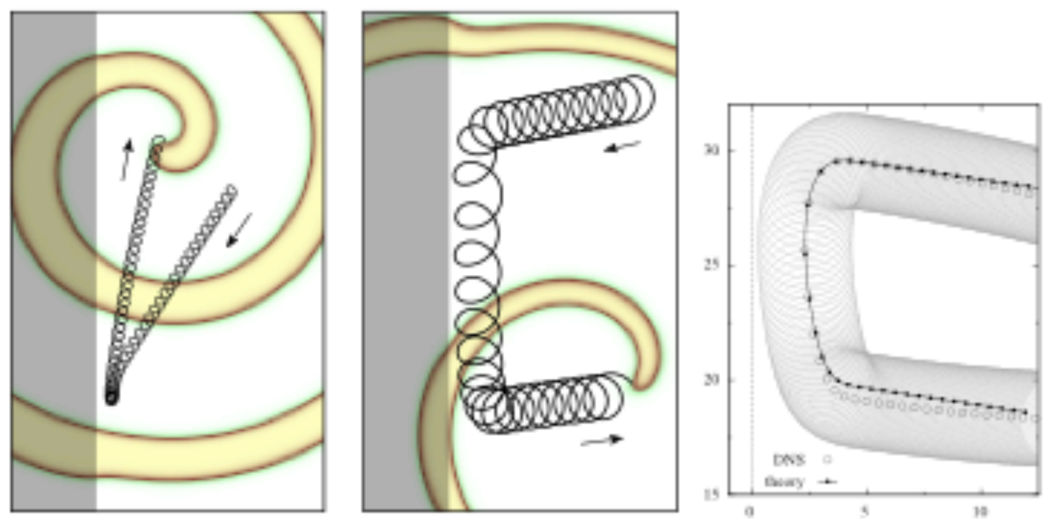
V.A. Davydov et al. “Drift and resonance of spiral waves in active media”, *Radiofzika* **31**(1988): 574-582; V.N. Biktashev, A.V. Holden ``Resonant Drift of an autowave vortex in a Bounded Medium" *Physics Letters A* **181**(3): 216-224, 1993

Resonant drift and “resonant repulsion”

Idea (Biktashev, Holden, 1993):

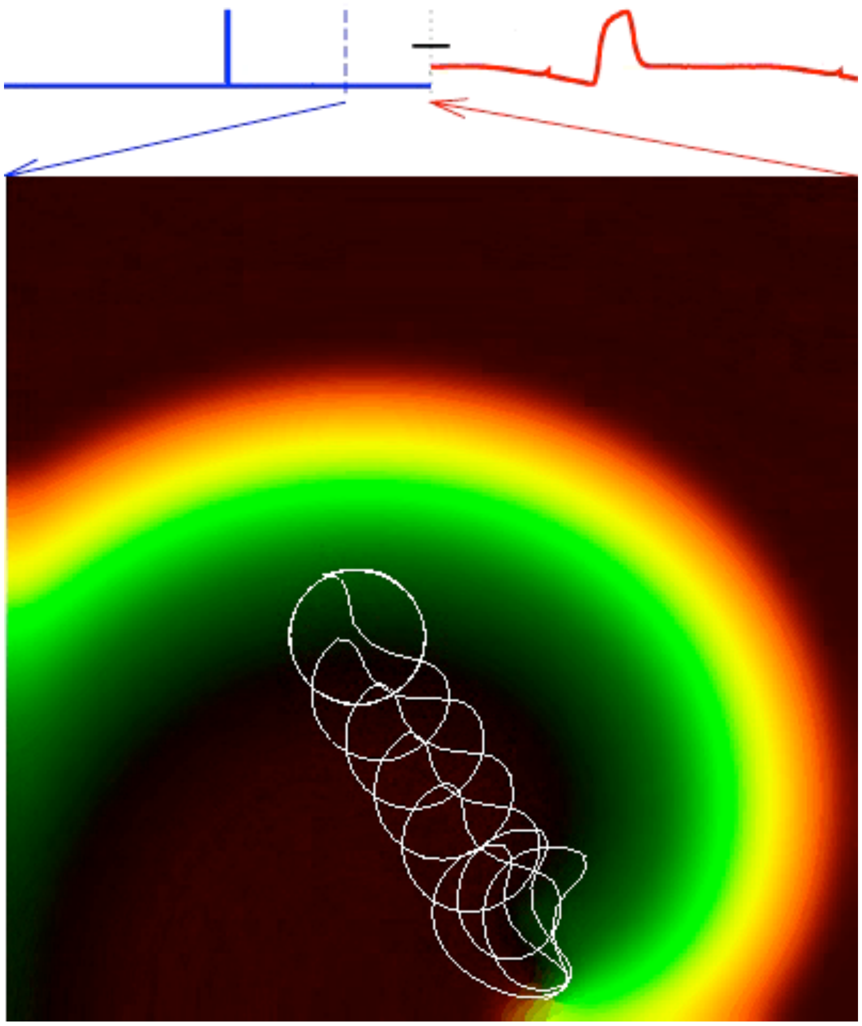


True asymptotic theory (Langham, Biktasheva, Barkley, 2014, under review):



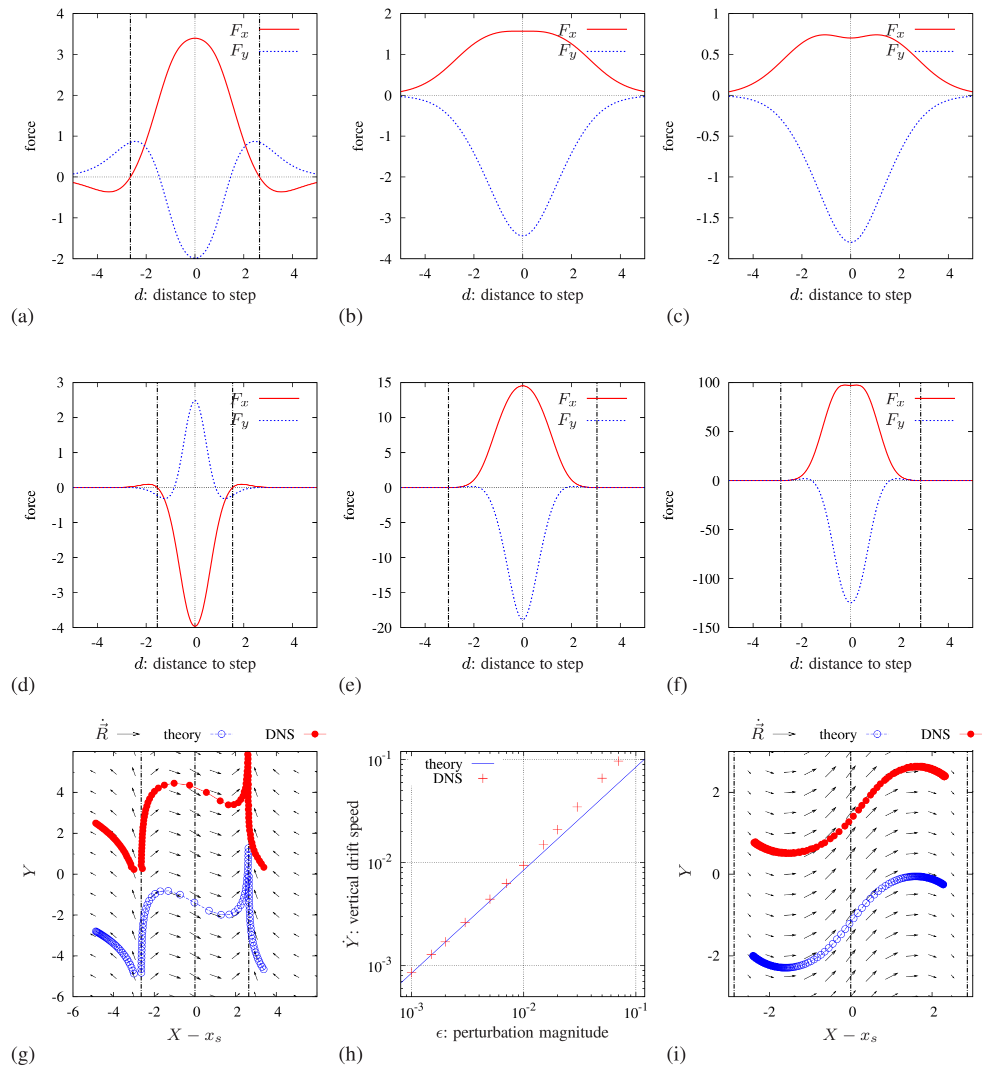
Feedback-controlled resonant drift

- Now stimulation period synchronized with spiral wave via a feedback loop.
- Drift proceeds notwithstanding obstacles => low-voltage defibrillation?



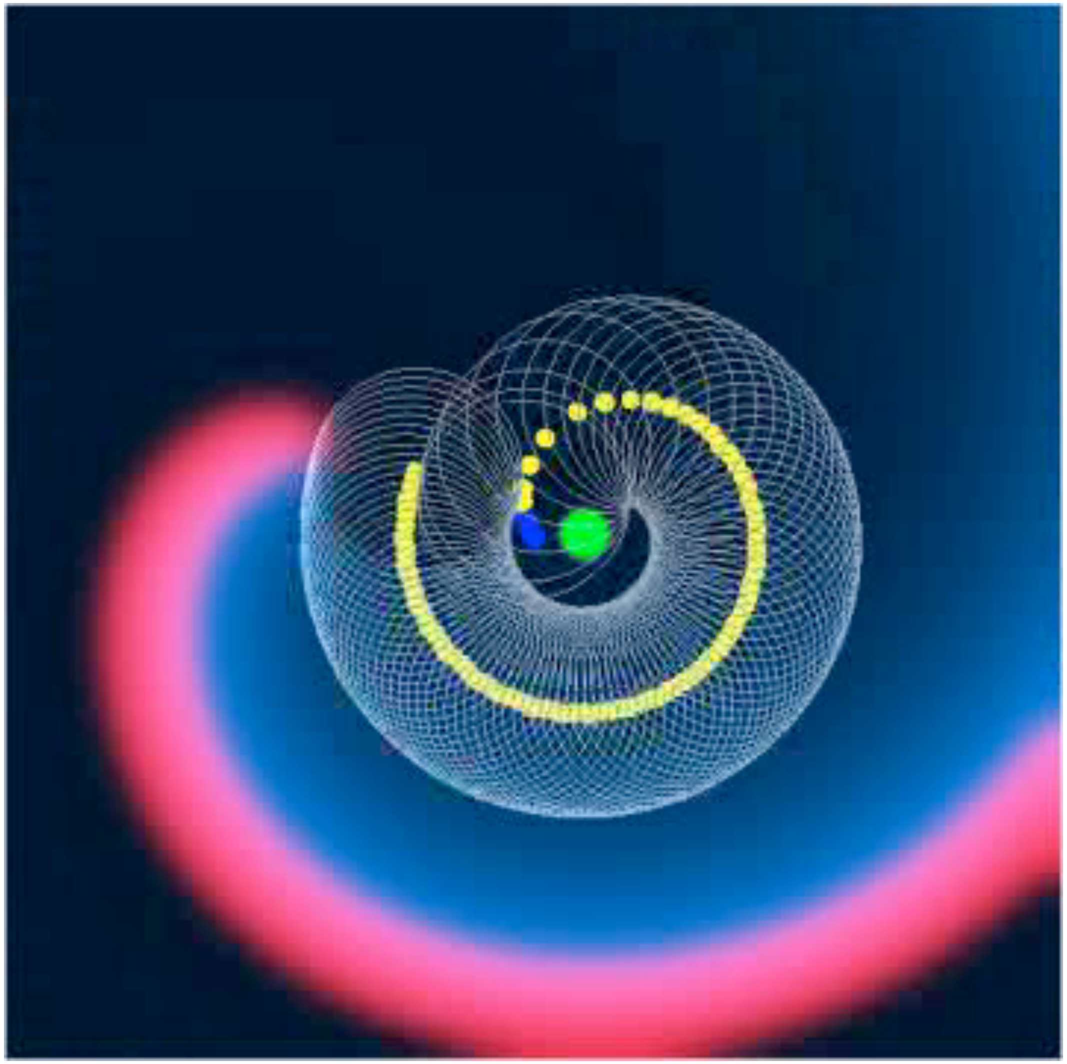
V.N. Biktashev & A.V. Holden ``Design Principles of a Low-Voltage Cardiac Defibrillator Based on the Effect of Feed-Back Resonant Drift" *J. Theor. Biol.* ***169(2): 101-113, 1994***

Interaction with a parametric step



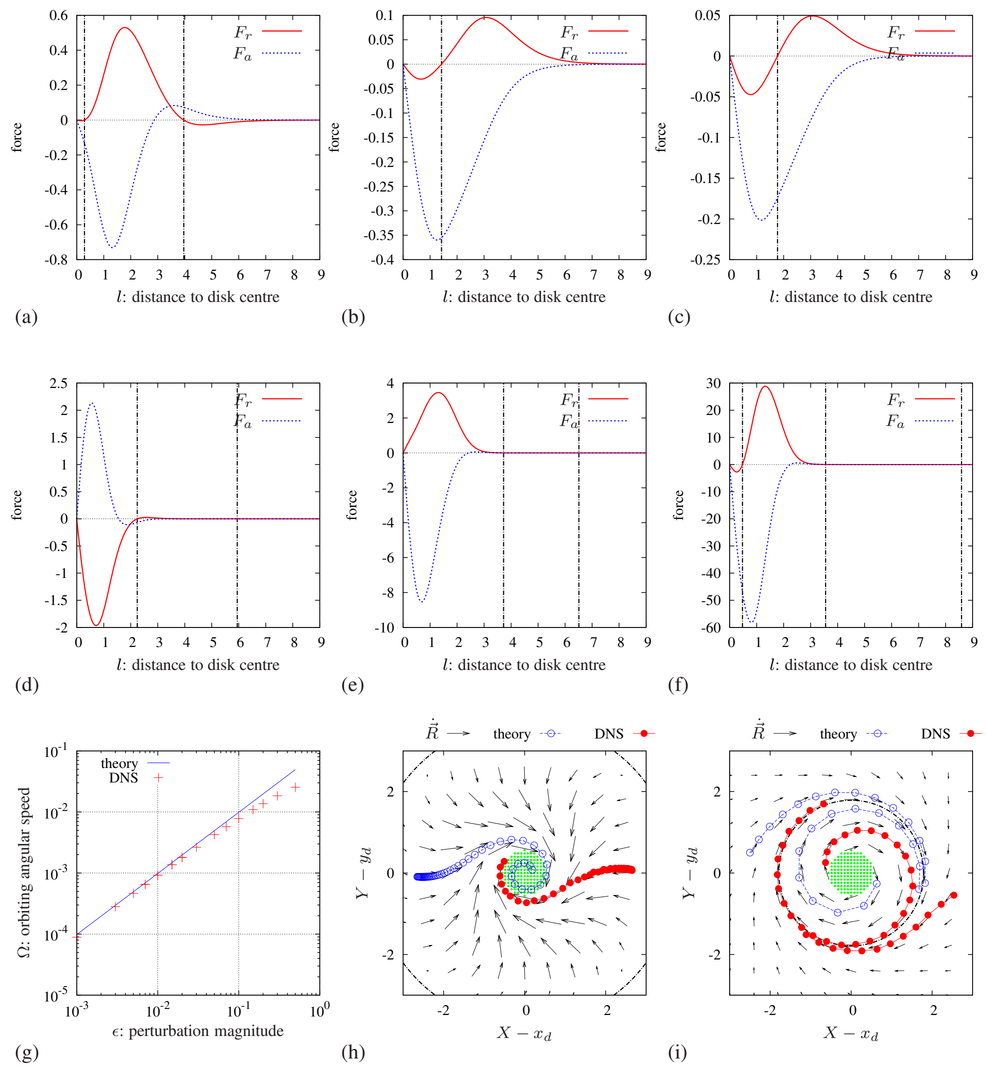
Orbital motion around a local inhomogeneity

- In this example, the inhomogeneity is repelling at short distance and attracting at long distance
- Therefore the spiral is kept at a stable distance
- This stable distance depends on the response functions (ie. medium parameters) not inhomogeneity strength!



V.N. Biktashev, D. Barkley and I.V. Biktasheva ``Orbital motion of spiral waves in excitable media'' *Phys. Rev. Lett.*, **104**(5): 058302, 2010

Orbital movement around local heterogeneity



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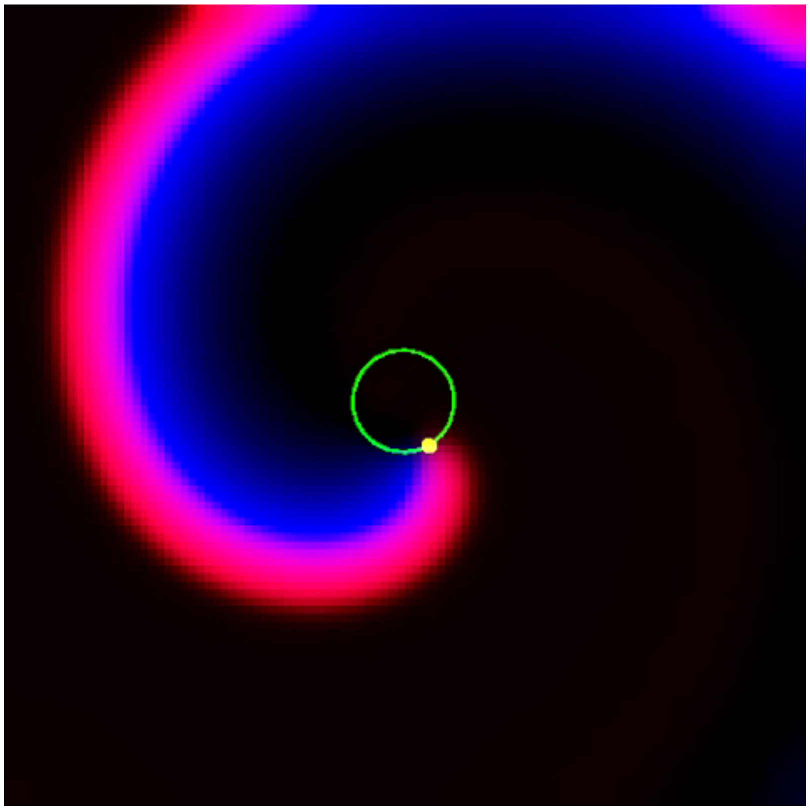
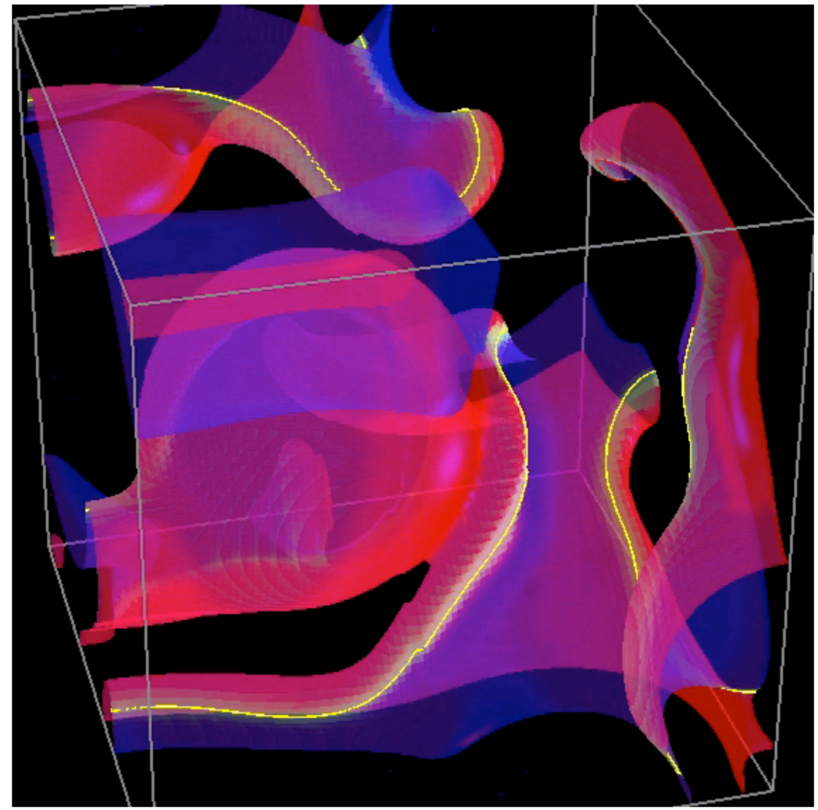
Scroll turbulence: a 3D phenomenon

2D: stationary spiral



3D: instability



(FitzHugh-Nagumo model)

V.N. Biktashev, A.V. Holden & H. Zhang, ``Tension of Organizing Filaments of Scroll Waves" *Phil. Trans. Roy. Soc. London, ser A* **347**: 611-630 (1994); V.N. Biktashev ``A Three-Dimensional Autowave Turbulence" Int. J. Bifurcation & Chaos, 8(4): 677-684, (1998)

Filament tension

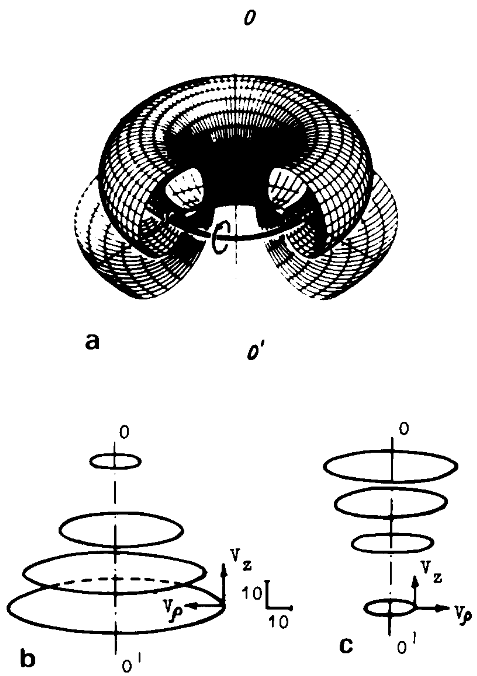


Fig. 1. Two regimes of the scroll ring drift. (a) Scroll ring. The direction of rotation is shown by the arrow. (b) The contraction regime. Evolution of a scroll filament in time intervals $\Delta T = 1000$. The bottom is the initial location of the ring. $g_f = 1.0$. (c) The extension regime. $\Delta T = 300$, $g_f = 0.775$.

A.V. Panfilov and A.N.Rudenko, *Physica* **28D**:215-218 (1987)

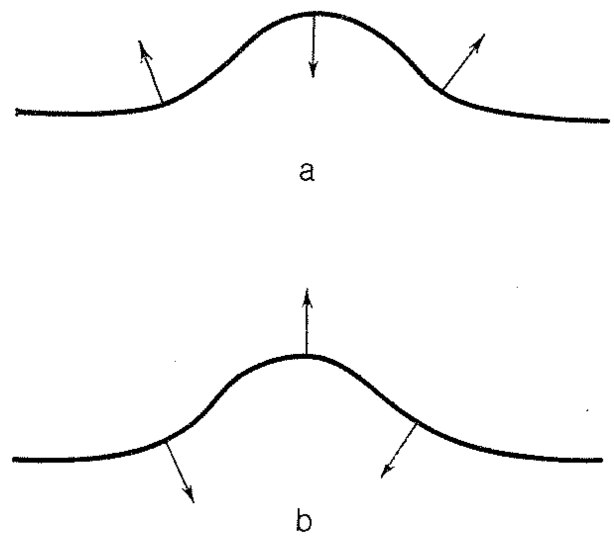
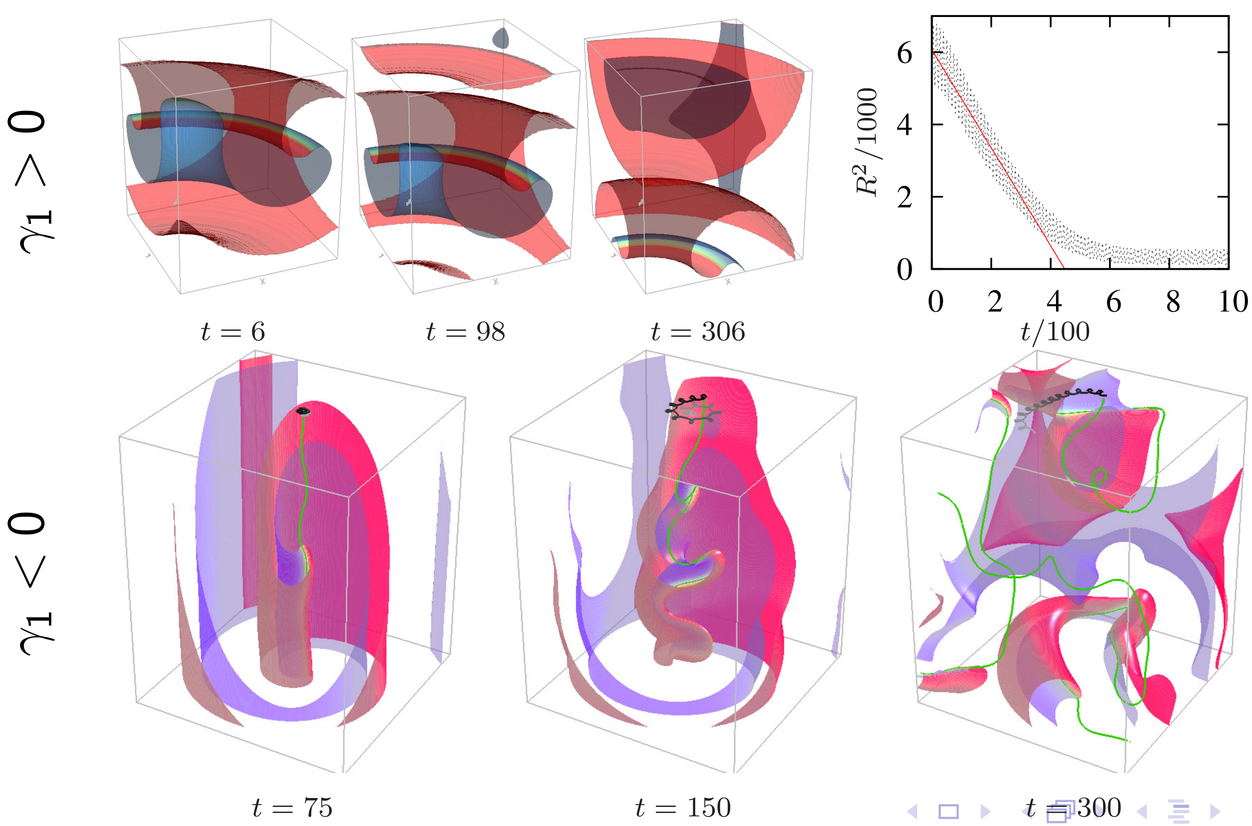


FIG. 3.

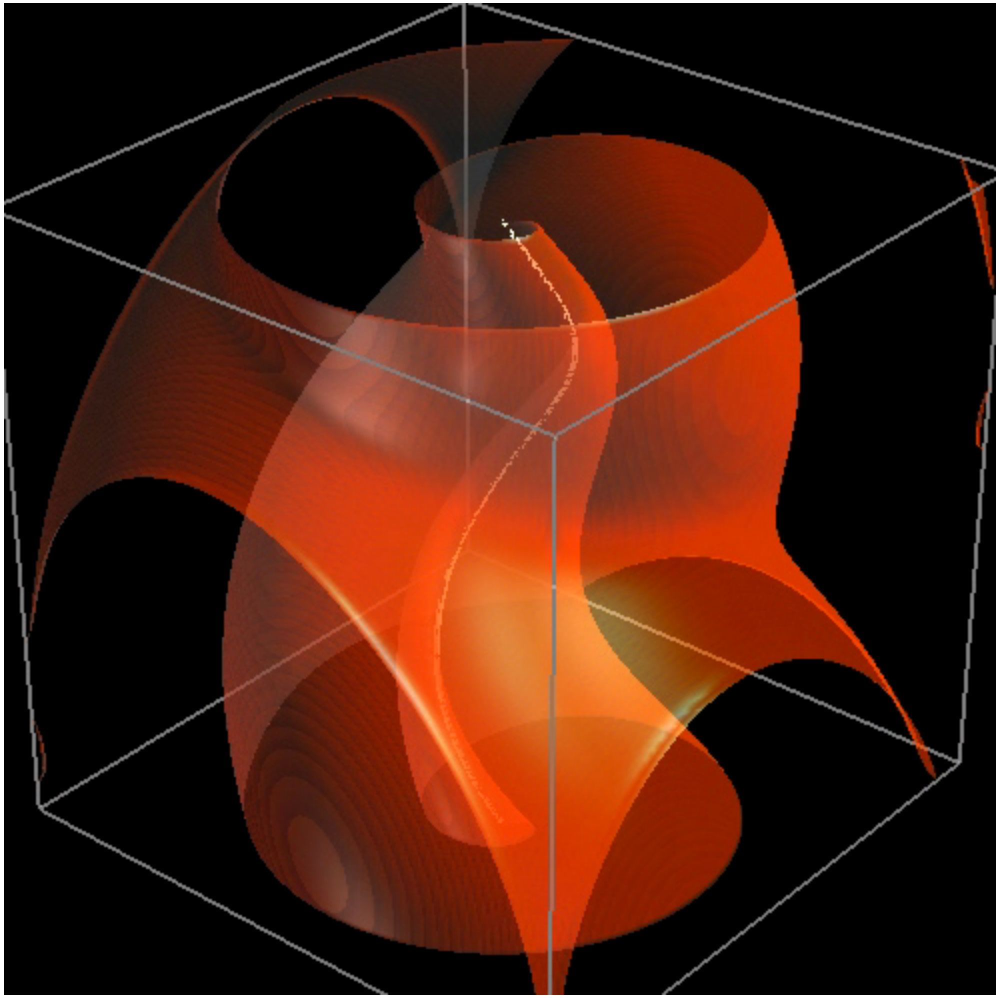
P.K. Brazhnik et al. *Sov. Phys. JETP* **64**:984-990 (1987)

Scroll wave turbulence

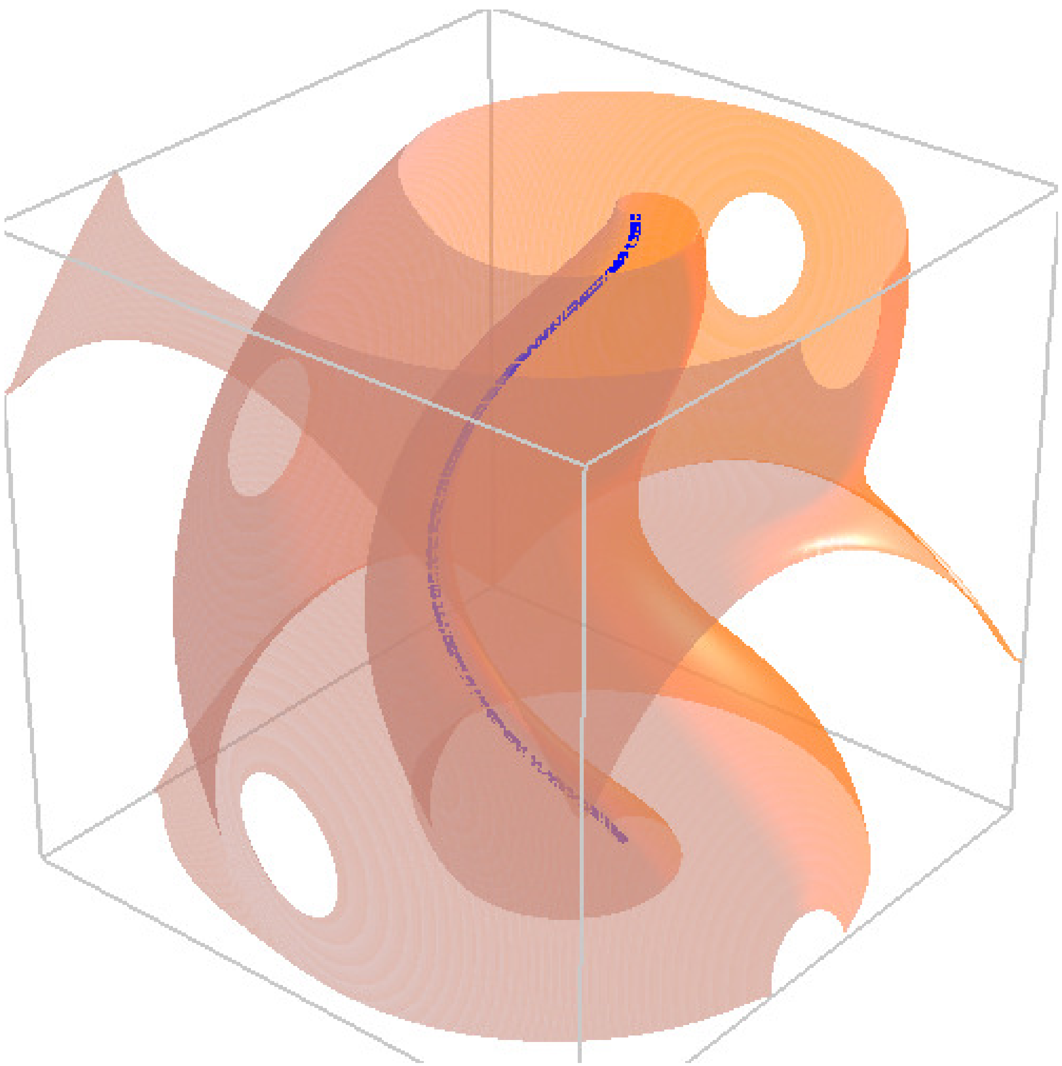
$$S(t) = \int \mathrm{d}s = \int \left|\partial_\sigma \vec{R}\right| \mathrm{d}\sigma \quad \Rightarrow \quad \frac{\partial S}{\partial t} = -\gamma_1 \int \left(\partial_\sigma^2 \vec{R}\right)^2 \mathrm{d}s + \mathcal{O}\left(\epsilon^2\right)$$

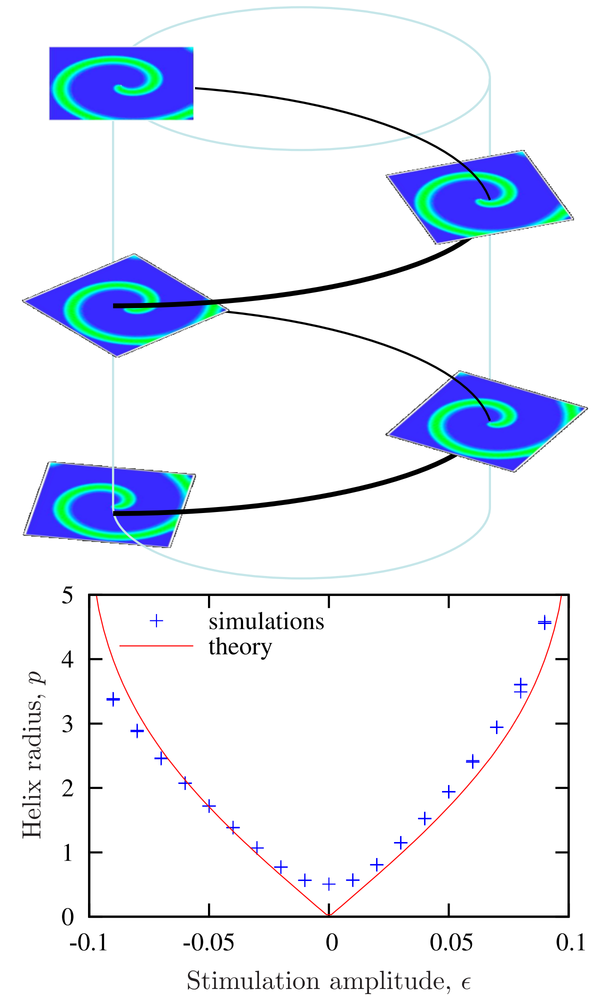


Precessing helical scroll: constant frequency near-resonant perturbation



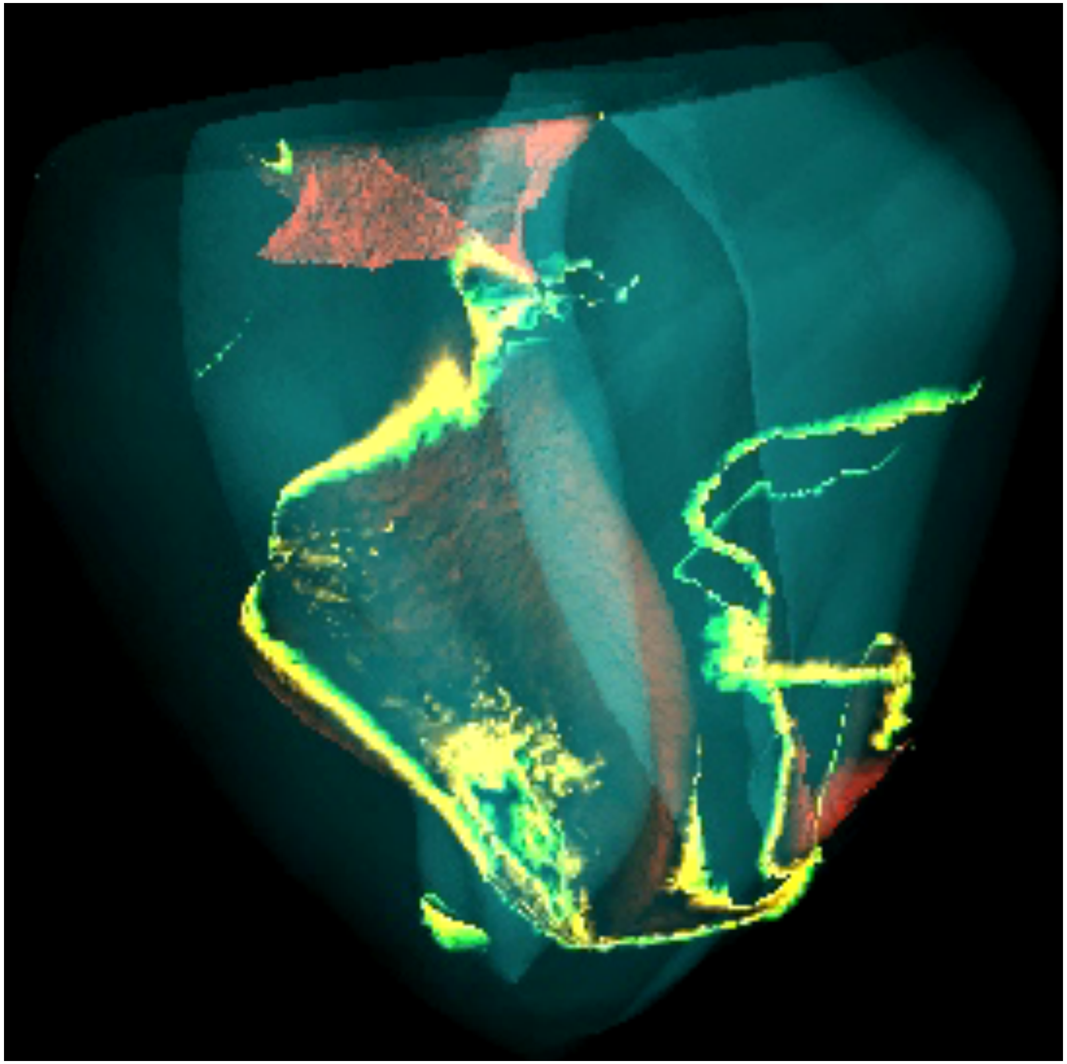
Helix produced by resonant stimulation in 3D





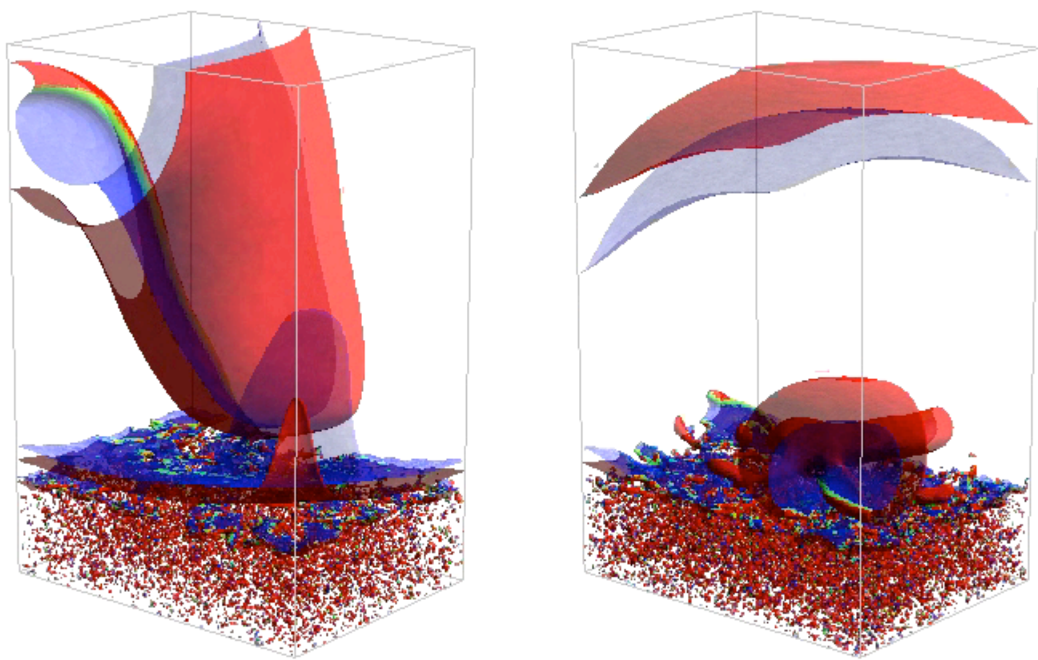
Helical scroll in rabbit heart geometry

- Here the “low-voltage defibrillation” failed.
- Possible reason: fiber orientation gradient => twist of a vortex => stationary helical twisted vortex by the mechanism described above.



S.W. Morgan, G. Plank, I.V. Biktasheva, and V.N. Biktashev, "Low energy defibrillation in whole ventricle model: a simulation study", in preparation

Moving boundary generating scrolls: filament tension role



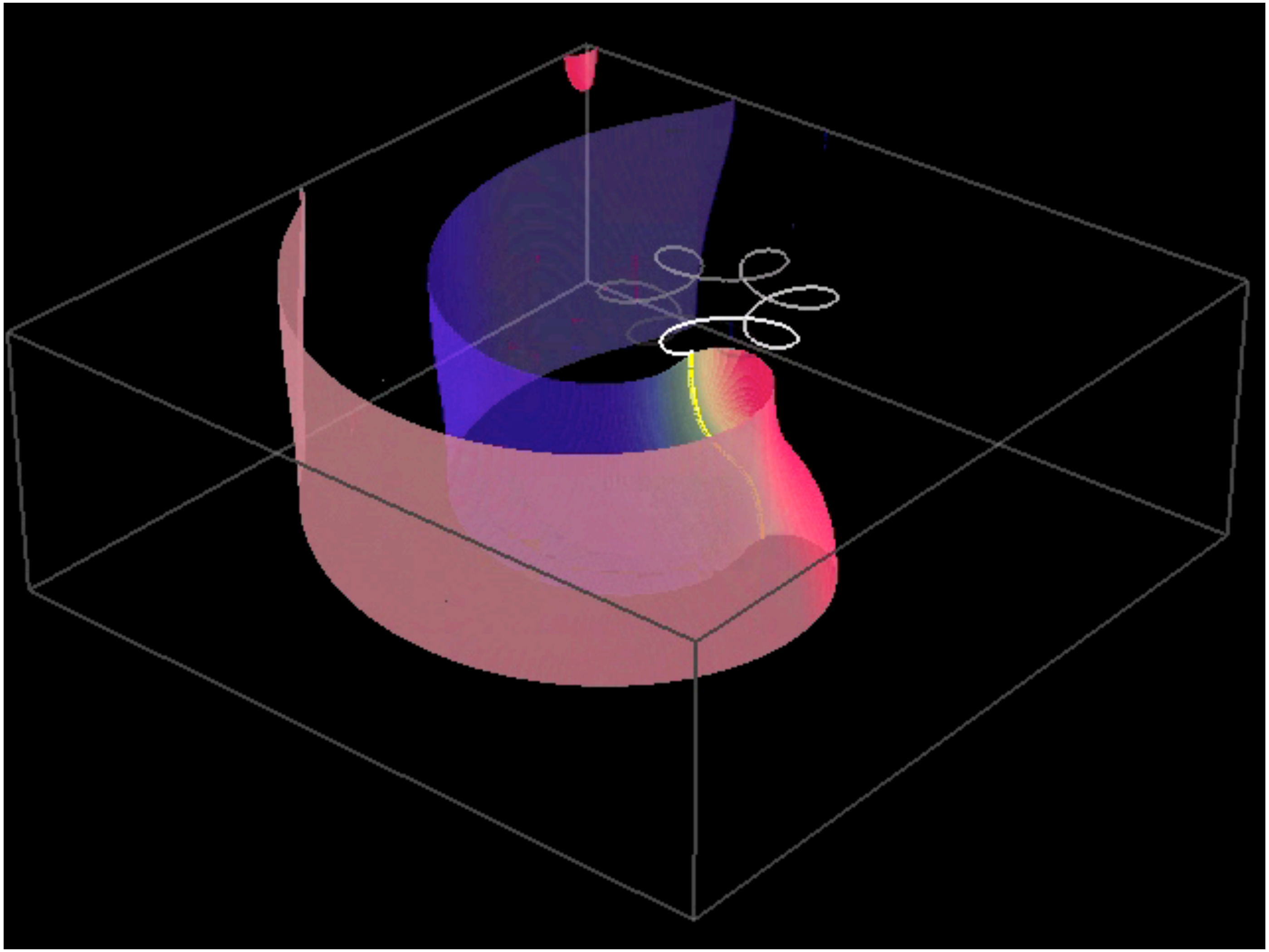
Low excitability, Negative filament tension

High excitability, Positive filament tension

V.N. Biktashev, I.V. Biktasheva and N.A. Sarvazyan, ``Evolution of spiral and scroll waves of excitation in a mathematical model of ischaemic border zone'' PLoS ONE, 6(9):e24388, 2011

Buckling of a negative tension filament in a thin layer: between 2D and 3D

Negative tension is tamed by filament “rigidity” and nonlinear effects



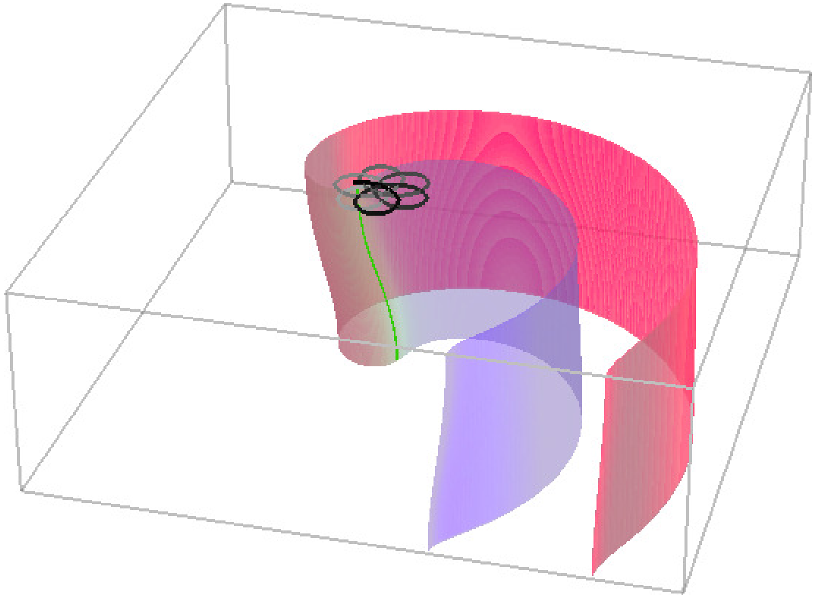
(Barkley model)

H.Dierckx, H.Verschelde, O.Selsil, V.N.Biktashev, Buckling of scroll waves, *PRL* **109**: 174102, 2012

Buckling of a scroll with negative filament tension

Rails, thermal expansion

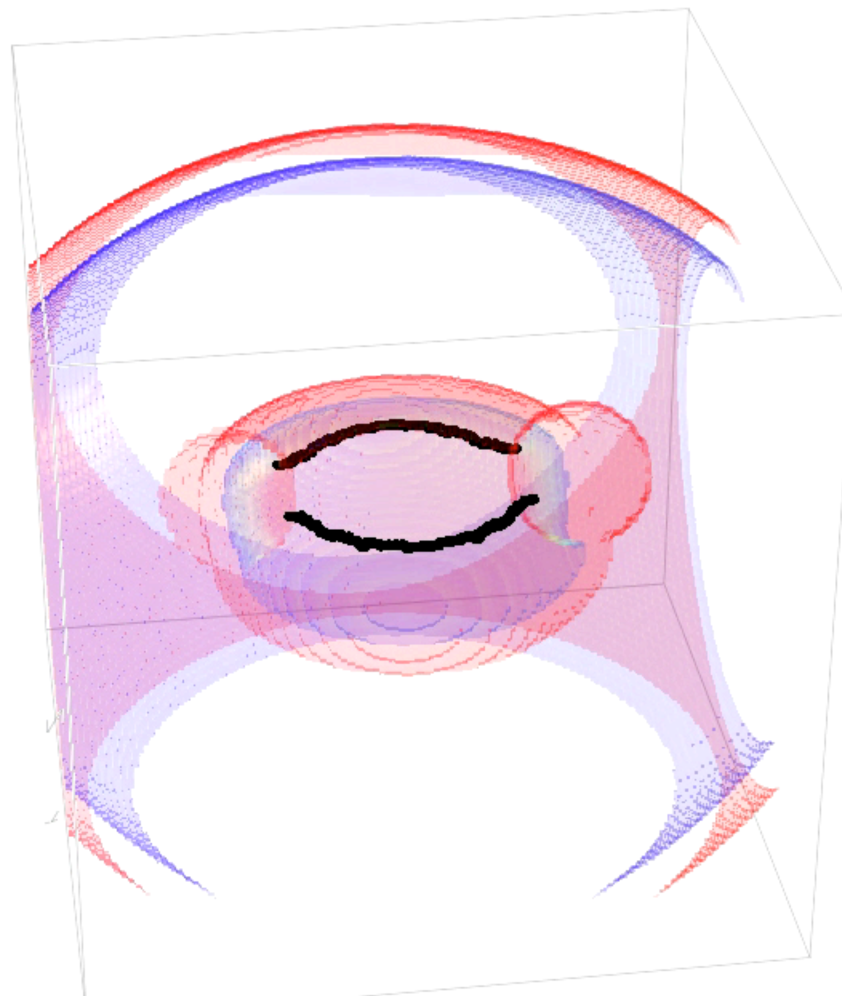
Scroll filament, negative tension



Stress/negative tension vs rigidity

Pinning of filament on two spherical beads

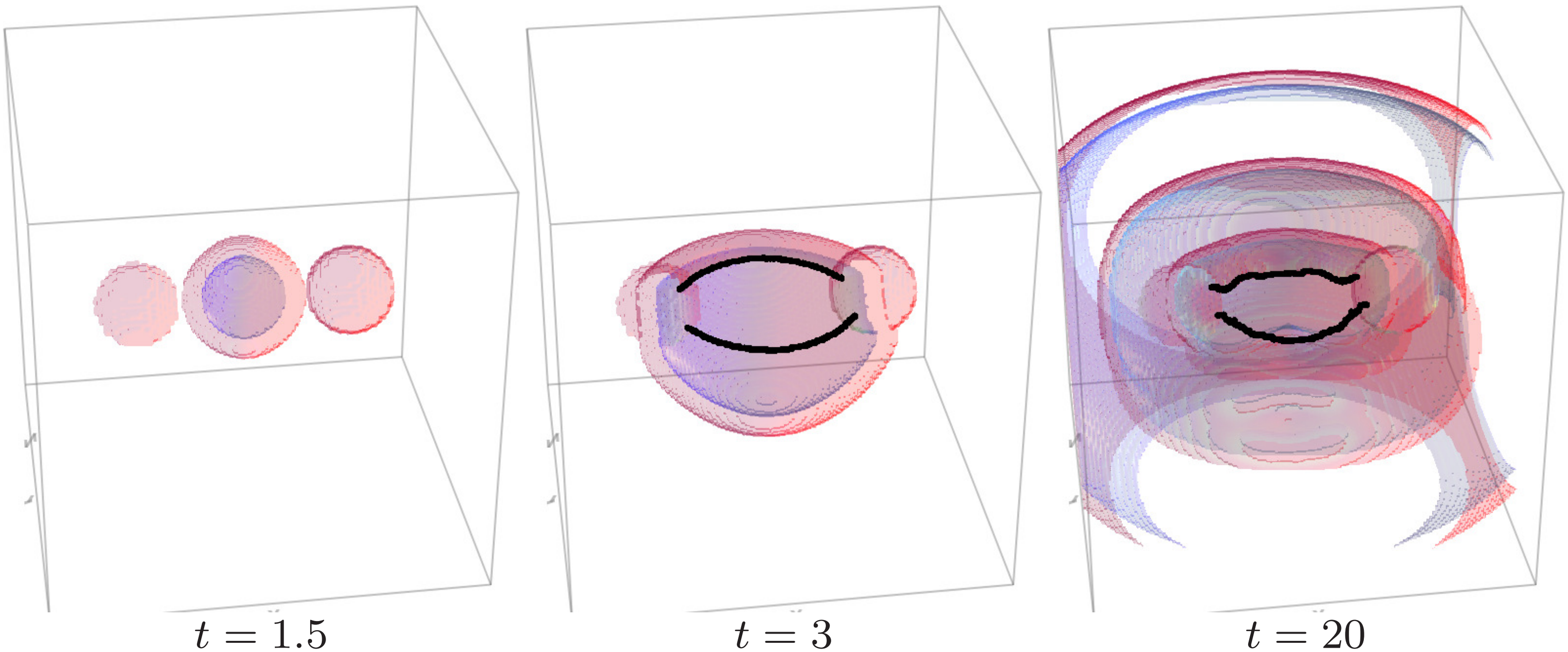
Established filament shape can be used to estimate the filament rigidity.



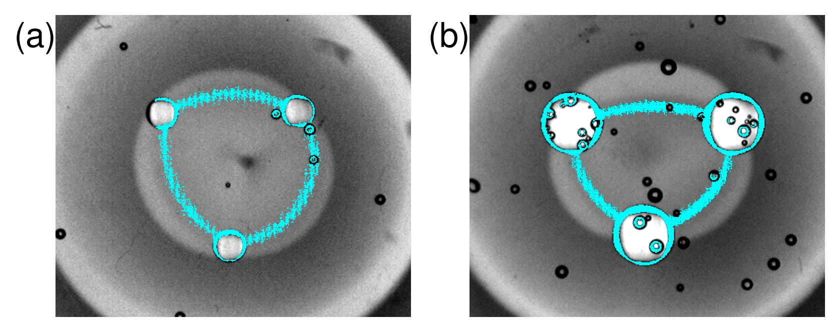
Oregonator model of BZ reaction.

E. Nakouzi, Z. A. Jiménez, V. N. Biktashev and O. Steinbock, ``Analysis of Anchor-Size Effects on Pinned Scroll Waves and Measurement of Filament Rigidity'' Phys. Rev. E, 89: 042901, 2014

Measuring rigidity of scroll filament in experiment



- Pinning of filament on spherical beads.
- Stat. shape: interaction of tension, rigidity and filaments' repulsion



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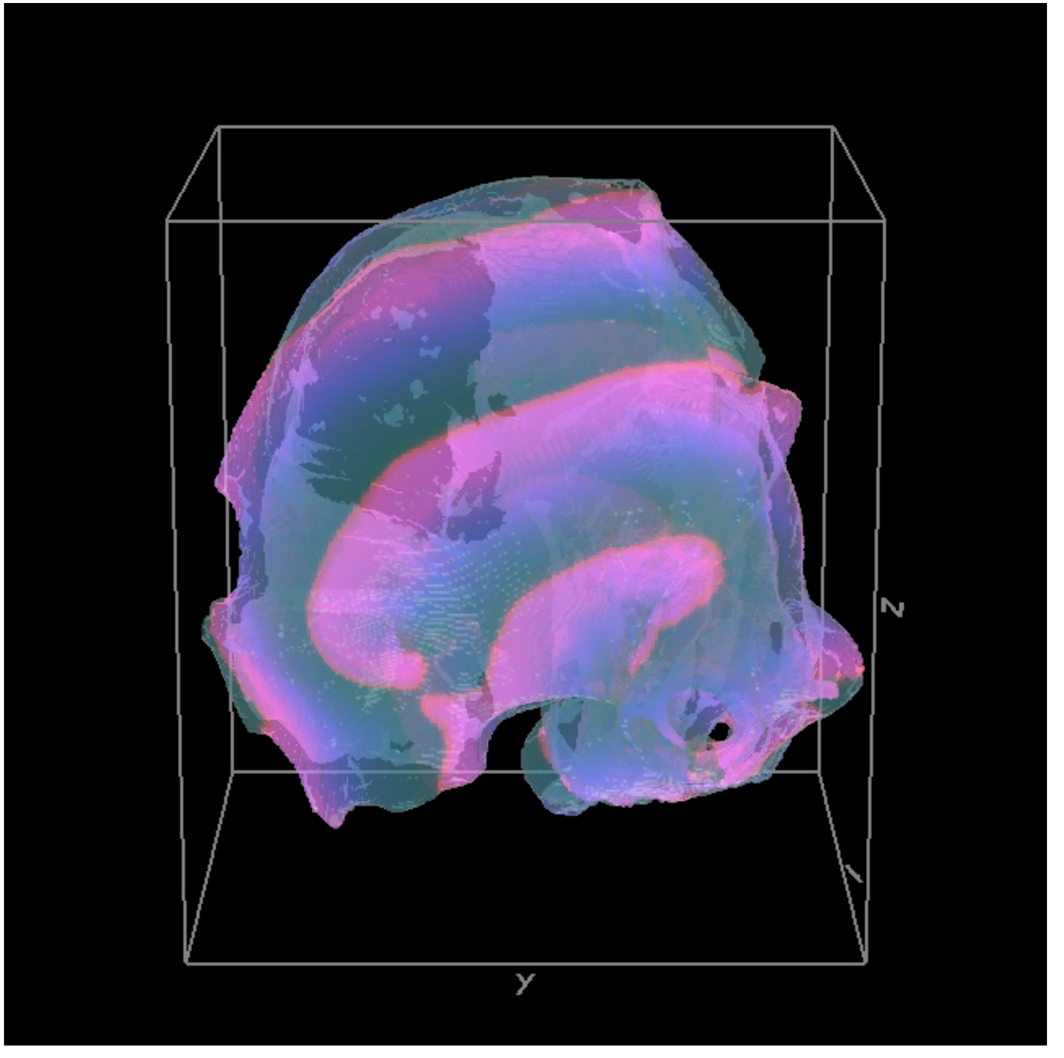
Re-entry in human atrium geometry

- Is it three-dimensional or two-dimensional?

(a variant of Courtemanche et al. 1998 human atrial kinetics model)

S.R.Kharche, I.V.Biktasheva, G.Seeman, H.Zhang, V.N. Biktashev, "*Mechanisms of spontaneous drift in the homogeneous human atrium*", in preparation, 2014



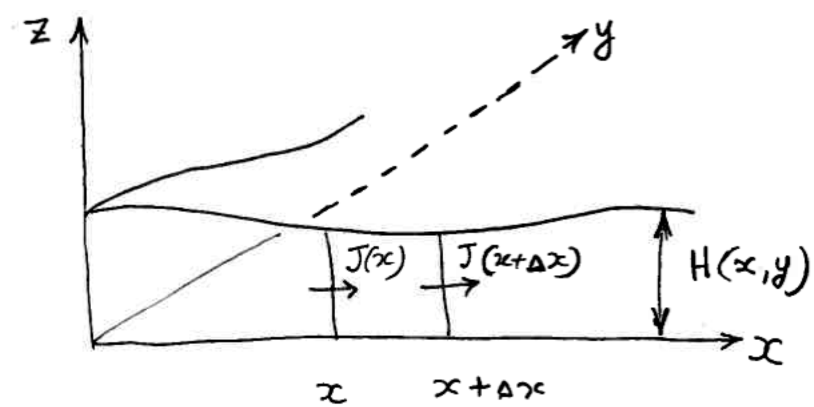
3D → 2D reduction for thin layers

$$\mathbf{v}_t = \mathbf{f}(\mathbf{v}) + \mathbf{D}\nabla^2\mathbf{v}, \qquad \mathbf{v} = \mathbf{v}(x, y, z, t)$$
$$(x, y) \in \mathbb{R}^2, \qquad 0 \le z \le H(x, y) = \mu\tilde{H}(x, y), \qquad \mu \ll 1.$$

with no-flux boundaries at $z = z_{\min}$ and $z = z_{\max}$. Then

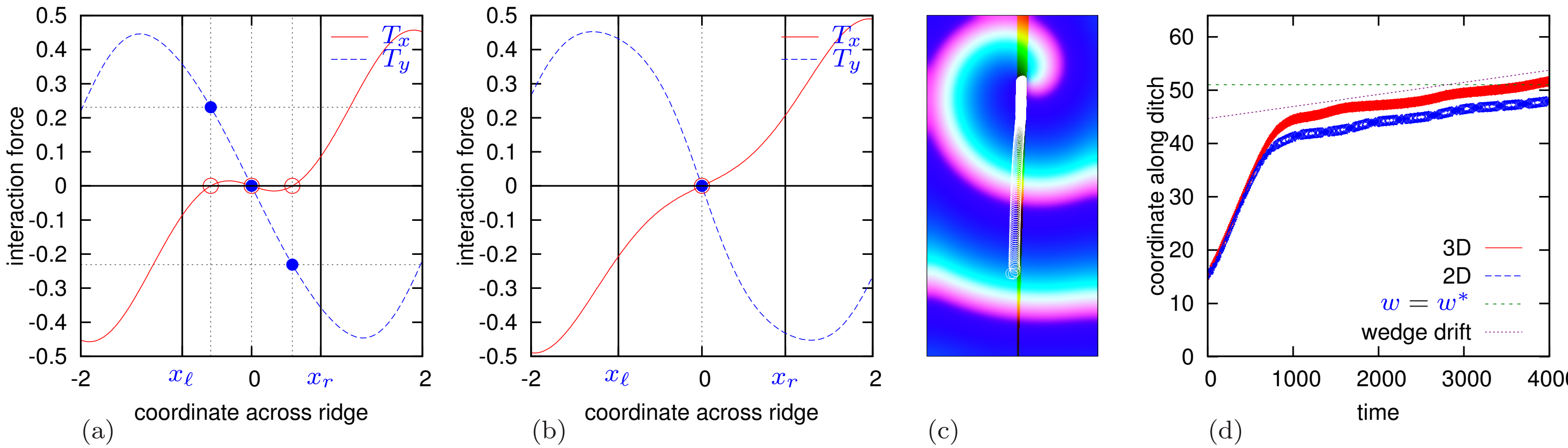
$$\mathbf{v}(x, y, z, t) = \mathbf{u}(x, y, t) + \mathcal{O}\left(\mu^2\right),$$

and



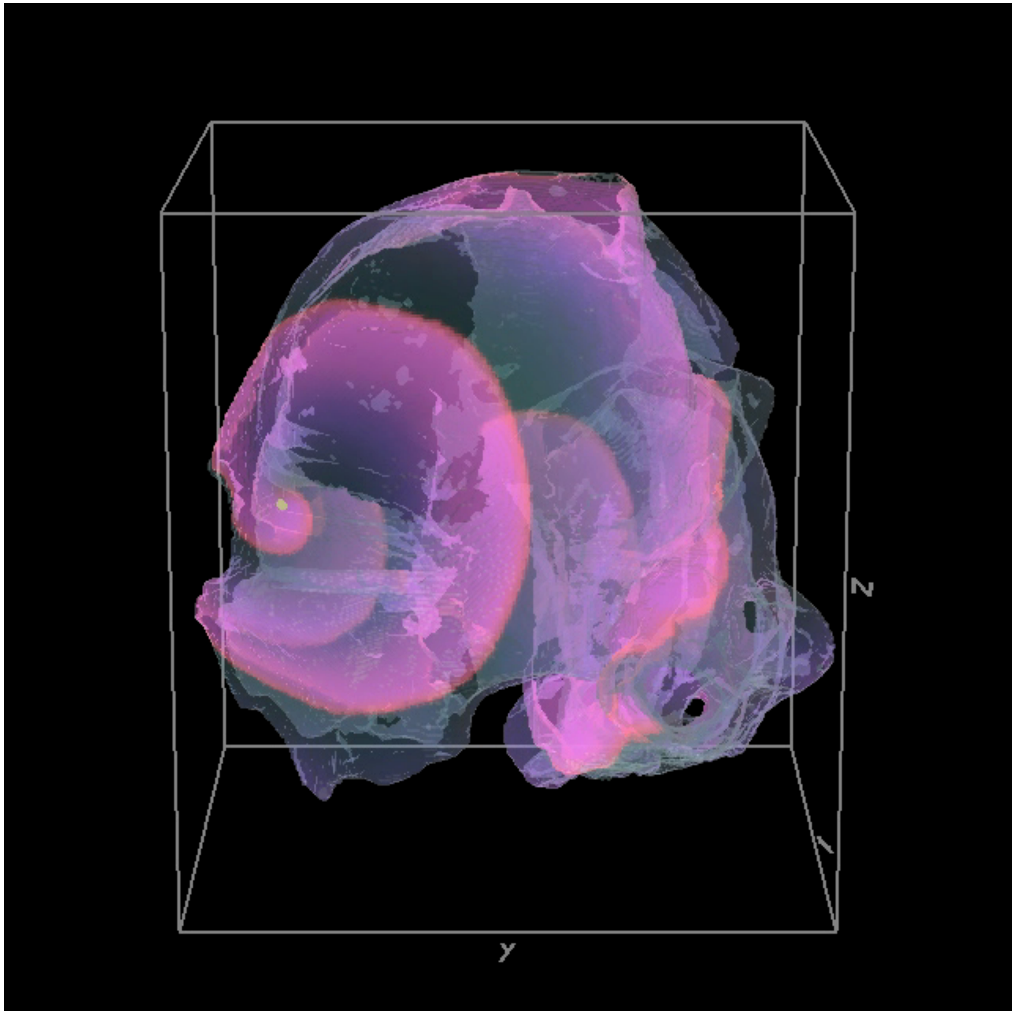
$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \mathbf{D}\frac{1}{H(x, y)}\nabla\cdot\left(H(x, y)\nabla\mathbf{u}\right) + \mathcal{O}\left(\mu^2\right)$$
$$\approx \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2\mathbf{u} + \mathbf{D}\left(\nabla(\ln H)\cdot\nabla\mathbf{u}\right)$$

Interaction of a scroll/spiral with a trough

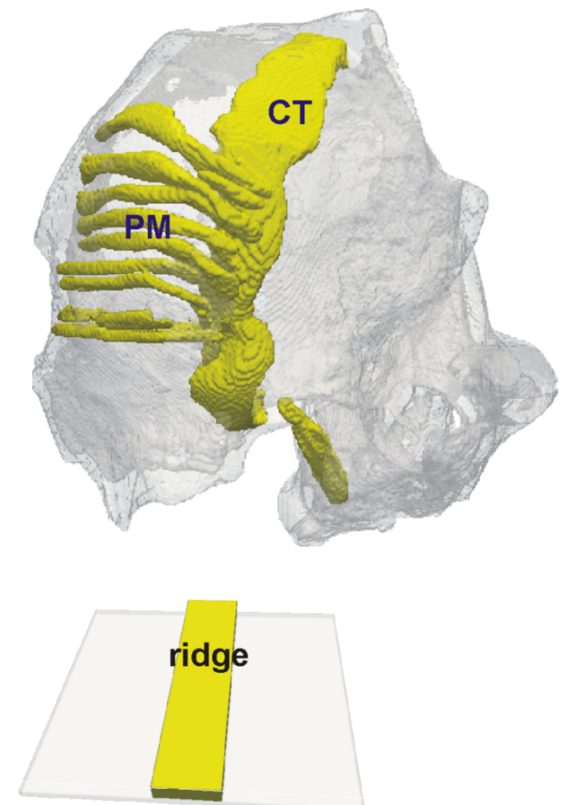


- Bifurcation: at some trough widths, there is “catching” solution, for some only “frozen” solution.
- If the trough width changes, there is also “wedging” force.

Anatomy induced drift in Human Atrium

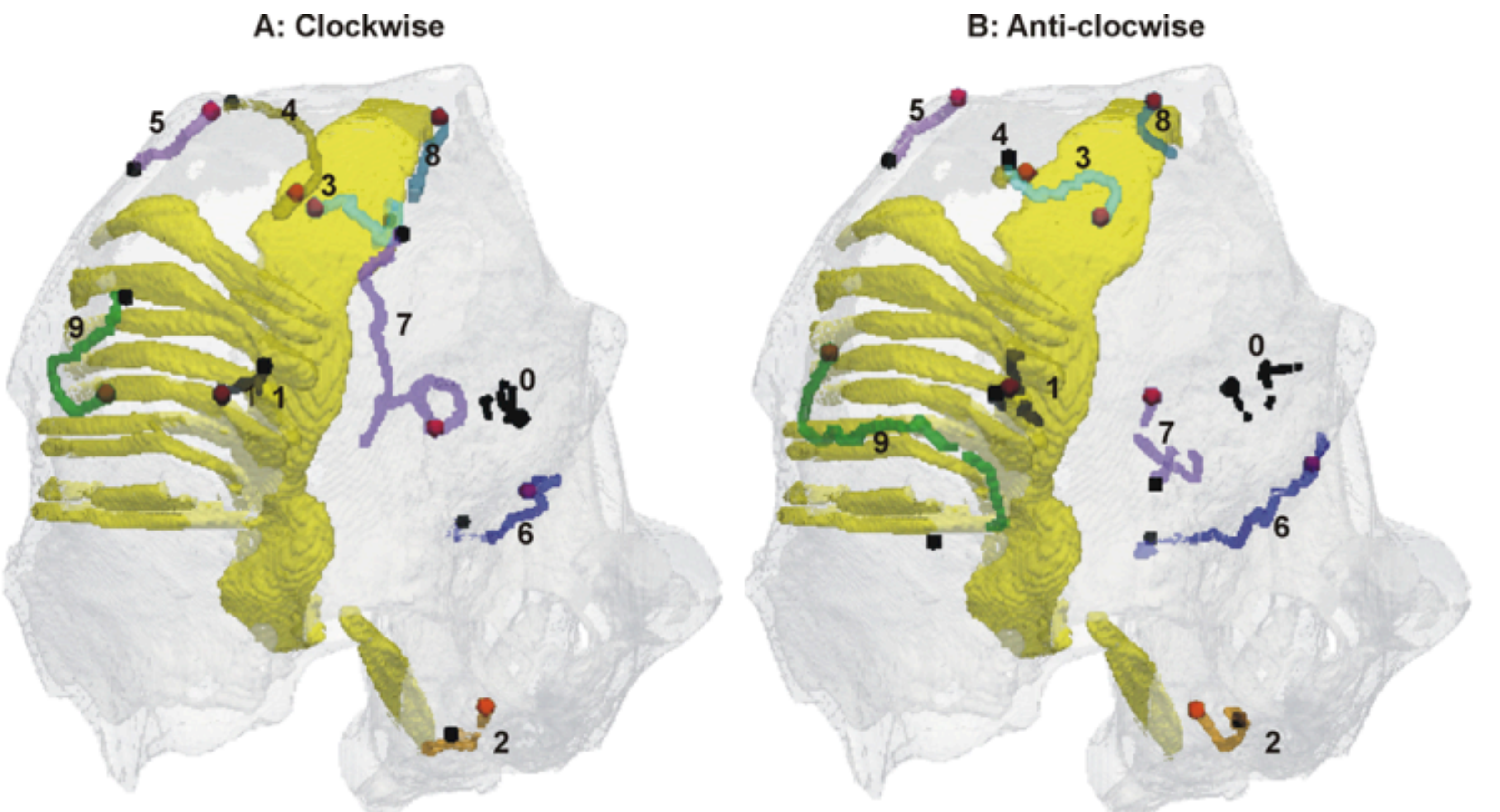


Epicardial View



- *Ridge* --- the CT and PM (attached to wall) ridge structures

Drift of spiral/scroll in human atrium geometry



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Conclusions

- Wave particle duality: spiral waves behave as particles and scroll waves as strings, with respect to small perturbations of generic nature. This is due to localization of the adjoints ("response functions"), which is a peculiar feature of this sort of dissipative patterns.
- Perturbation theory quantitatively agrees with direct simulations for sufficiently small perturbations.
- Perturbation theory can give useful qualitative insight even when perturbations are not small.
- Potential applications, particularly cardiology.

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GNU lincensed Software used

- Response functions: `dxspiral`*
- Direct numerical simulations: `BeatBox`*
- 3D visualization: `ezview`*, based on visualization code of Barkley and Dowle's EZSCROLL†

* http://empslocal.ex.ac.uk/people/staff/vnb262/

† http://homepages.warwick.ac.uk/~masax/

THE END