Kinetic Theory of Ballistic Aggregation and Fragmentation. (Application to Planetary Rings)

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Conference dedicated to 60th birthday of Alexander Gorban
Aggregation of particles

Protoplanetary discs – formation of planets
Fragmentation of particles: grinding of large bodies to dust
Planetary Rings: Aggregation & Fragmentation
Saturn Rings

- Particle size: \( \approx 1 \text{ cm} - 10 \text{ m} \)
- Particle dimensions: \( \approx 10 - 100 \text{ m} \)
- Earth to Saturn distance: 384,000 kilometers (239,000 miles)
- Orbit speed: \( V_{\text{orbit}} = 20 \text{ km/s} \)
- Random speed: \( V_{\text{random}} = 0.1 - 0.01 \text{ cm/s} \)
Model

- Particles move ballistically between pair-wise collisions.
- Particles can aggregate upon collisions:
- Particles can break upon collisions:
Important features of particle collisions:

- Particles aggregate for small impact velocities
- Particles collide loosing kinetic energy for medium impact speeds
- Particles break for large impact velocities
Introduce two parameters, $E_{agg}$ and $E_{frag}$

$E_{12} = \frac{\mu_{12} \vec{V}_{12}^2}{2}$

$\mu_{12} = \frac{m_1 m_2}{m_1 + m_2}$

$V'_{12} = -\varepsilon V_{12}$

$\varepsilon \leq 1$

$\vec{V}_{12} = \vec{V}_1 - \vec{V}_2$

aggregation $E_{12} < E_{agg}$

fragmentation $E_{12} > E_{frag}$
**Boltzmann equation**

Distribution function: \( f(\vec{r}, \vec{V}, t) \)

\[
\frac{\partial}{\partial t} f(\vec{r}, \vec{V}, t) = I(f, f) \quad - \text{“collision integral”}
\]

\[
I = I_{\text{gain}}(f, f) - I_{\text{loss}}(f, f)
\]
2 The Source of Examples

2.1 The Boltzmann Equation
2.1.1 The Equation
2.1.2 The Basic Properties of the Boltzmann Equation
2.1.3 Linearized Collision Integral
2.2 Phenomenology and Quasi-Chemical Representation of the Boltzmann Equation
2.3 Kinetic Models
2.4 Methods of Reduced Description
2.4.1 The Hilbert Method
2.4.2 The Chapman–Enskog Method
2.4.3 The Grad Moment Method
2.4.4 Special Approximations
2.4.5 The Method of Invariant Manifold
2.4.6 Quasiequilibrium Approximations
E.g. for collisions with restitution only

\[ I_{\text{gain}}(f, f) = 4\pi R^2 \int \frac{\mathbf{v}_{12} \cdot \mathbf{e}}{\mathcal{E}} \left( \int d^2 \mathbf{r} \int d^3 \mathbf{r} \int \frac{1}{2} d^4 \mathbf{e} \right) f(\mathbf{r}, \mathbf{v}_{12}) \int f(\mathbf{r}, \mathbf{v}_{22}) d^4 t \]

\[ \mathbf{v}_{12} \cdot \mathbf{e} < 0, \quad \mathbf{v}_{12} \cdot \mathbf{E} < 0 \]

\[ \mathbf{v}_{12} \Delta t \]

\[ S = 4\pi R^2 \]

\[ \mathbf{V}_1, \mathbf{V}_2 \rightarrow \mathbf{V}_1, \mathbf{V}_2 \]

loss term:
Boltzmann Equation

\[ m_0 - \text{elementary mass} \]

\[ r_0 - \text{elementary radius} \]

\[ m_0, 2m_0, 3m_0, \ldots, km_0 \]

\[ f\left(m, \vec{V}, t\right) - \text{mass-velocity distribution function} \]

- Restitution collision
- Aggregative collision
- Disruptive collision
Generalized Boltzmann Equation

\[
\frac{\partial f_i(m_i, \vec{v}_i, t)}{\partial t} = I_{rest}(f, f) + I_{agg}(f, f) + I_{frag}(f, f) + I_{heat}(f, f)
\]

- **restitution**
- **aggregation**
- **fragmentation**

*Keeps the temperature of ring particles constant*
**Aggregation term**

\[
I_{k}^{\text{agg}} = \frac{1}{2} \sum_{i+j=k} \sigma_{ij}^{2} \int d\vec{v}_{i} \int d\vec{v}_{j} \int d\tilde{e} \Theta (-\vec{v}_{ij} \cdot \tilde{e}) |\vec{v}_{ij} \cdot \tilde{e}| \times
\]

\[
\times f_{i}(\vec{v}_{i}) f_{j}(\vec{v}_{j}) \Theta (E_{\text{agg}} - E_{ij}) \delta (m_{k} \vec{v}_{k} - m_{i} \vec{v}_{i} - m_{j} \vec{v}_{j})
\]

- **gain term**
- **aggregation condition**
- **momentum conservation**

\[
- \sum_{j} \sigma_{kj}^{2} \int d\vec{v}_{j} \int d\tilde{e} \Theta (-\vec{v}_{kj} \cdot \tilde{e}) |\vec{v}_{kj} \cdot \tilde{e}| \times
\]

\[
\times f_{k}(\vec{v}_{k}) f_{j}(\vec{v}_{j}) \Theta (E_{\text{agg}} - E_{kj})
\]

- **loss term**

**Fragmentation term has the similar structure, but requires a microscopic fragmentation model**

The heating term keeps the steady-state energy distribution for species. Its particular form is not presently important.
Generalized Boltzmann Equation

Assume:

- **Maxwellian:**
  \[ f_i(m_i, \vec{v}_i, t) = n_i \left( \frac{m_i}{2T(t)\pi} \right)^{3/2} \exp \left( -\frac{m_i\vec{v}_i^2}{2T(t)} \right) \]

Ask Alexander Why this is justified:

- **Complete fragmentation at collisions:**
  \[ n_i(t) = \int f_i(m_i, \vec{v}_i, t) d\vec{v}_i \]
  \[ T_i(t) = \int \frac{m_i\vec{v}_i^2}{2} f_i(m_i, \vec{v}_i, t) d\vec{v}_i = T(t) \]
**Kinetic equations for particles concentrations** \(n_i(t)\) **for space uniform systems**

\[
\frac{dn_k}{dt} = \frac{1}{2} \sum_{i+j=k} K_{ij} n_i n_j - n_k \sum_{j \geq 1} K_{kj} n_j - \lambda n_k \sum_{j \geq 1} K_{kj} n_j
\]

**aggregation**

\[
\frac{dn_1}{dt} = -n_1 \sum_{j \geq 1} K_{1j} n_j + \frac{\lambda}{2} \sum_{i,j \geq 2} (i + j) K_{ij} n_i n_j + \lambda n_1 \sum_{j \geq 2} jK_{1j} n_j
\]

**aggregation**

Relative ratio of aggregative and disruptive collisions

\[
K_{ij} = r_1^2 e^{-E_{\text{frag}}/T} \lambda^{-1} \sqrt{\frac{8\pi T}{m_1}} \sqrt{\frac{(i + j)}{ij}} \left(i^{1/3} + j^{1/3}\right)^2
\]

\[
\lambda = \left(1 + (1 + E_{\text{agg}}/T) e^{-E_{\text{agg}}/T}\right) e^{-E_{\text{frag}}/T}
\]
For planetary rings particles’ size ranges

from $\sim 10^{-3} \text{ m}$ to $\sim 1^0 \text{m}$

$10^9$ equations!

Analytics?
**Constant rate coefficients:**

\[
\frac{dn_1}{dt} = -n_1 N + \lambda(1 - n_1)N \\
\frac{dn_k}{dt} = \frac{1}{2} \sum_{i+j=k} n_i n_j - (1 + \lambda)n_k N;
\]

\[
\begin{cases}
\frac{dn_1}{dt} = -n_1 N + \lambda(1 - n_1)N \\
\frac{dN}{dt} = -N^2 + \lambda(1 - N)N
\end{cases}
\]

\[
t \rightarrow K_0 n_0 t \\
n_k \rightarrow n_0 n_k
\]

\[
K = K_0 \\
N(t) = \sum_{j \geq 1} n_j(t)
\]

Total number of aggregates
**Constant rate coefficients:** \( K = K_0 \)

For \( n_k(t = 0) = \delta_{k,1} \)

\[
N(t) = 2\lambda \left[ 1 + 2\lambda - e^{-\lambda t} \right]^{-1}
\]

\[
n_1(t) = \frac{\lambda}{1 + \lambda} + \frac{1}{1 + \lambda} \left[ \frac{(1 + 2\lambda)e^{\lambda t} - 1}{2\lambda} \right]^{\frac{2(1+\lambda)}{1+2\lambda}}
\]

\[
\tau_{rel}^{-1} = K_0 n_0 \lambda = 10^3 - 10^5 \text{ years}
\]

The system relaxes to the steady state distribution of particles sizes!
**Constant rate coefficients:**

\[ K = K_0 \]

**Steady-state:**

\[ n_1 = \frac{\lambda}{1 + \lambda} \quad N = \frac{2\lambda}{1 + 2\lambda} \]

\[ 0 = \frac{1}{2} \sum_{i+j=k} n_i n_j - (1 + \lambda) n_k N; \quad k = 2, 3, \ldots \]

**Introduce Generating function:**

\[ G(z) = \sum_{k \geq 1} n_k z^k \]

\[ G(z)^2 - 2(1 + \lambda) NG(z) + 2(1 + \lambda) Nn_1 z = 0 \]
Solving the quadratic equation we obtain:

\[ G(z) = (1 + \lambda)N \left[ 1 - \sqrt{1 - \frac{2n_1}{(1 + \lambda)N}} z \right] \]

Expanding \( G(z) \) we obtain exact steady-state

\[ n_k = \frac{N}{\sqrt{4\pi}} (1 + \lambda) \left[ \frac{2n_1}{(1 + \lambda)N} \right]^k \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k + 1)} \]

For \( k \gg 1 \)

\[ n_k = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2k} k^{-3/2} \]
General case

The rate kernel is uniform:

\[ K_{ij} \propto \sqrt{\frac{(i+j)}{ij}} \left( i^{1/3} + j^{1/3} \right)^2 \]

\[ K_{ai_aj} = a^{2\mu} K_{ij} \quad \text{with} \quad \mu = \frac{1}{12} \]

This suggests the approximation:

\[ K_{ij} \approx K_0 \left( \frac{i}{j} \right)^\mu \]

\[ K_0 = r_1^2 e^{-E_{\text{frag}}/T} \lambda^{-1} \sqrt{\frac{8\pi T}{m_1}} \]

for \( \mu=0 \) aggregation and fragmentation does not depend on particles size (constant kernel)
Steady-state equation for \( K_{ij} = K_0(ij)^\mu \)

\[
0 = \frac{1}{2} \sum_{i+j=k} K_0(ij)^\mu n_i n_j - (1 + \lambda) \sum_{j \geq 1} K_0(kj)^\mu n_k n_j
\]

**Introduce**

\[
l_k = k^\mu n_k
\]

\[
L = \sum_{k \geq 1} l_k
\]

\[
0 = \frac{1}{2} \sum_{i+j=k} l_i l_j - (1 + \lambda) l_k L;
\]

\[
F(z)^2 - (1 + \lambda)MF(z) + (1 + \lambda)Mn_1 z = 0
\]

\[M = F(1)\]
Solving quadratic equation and expanding $F(z)$ we obtain:

\[ n_k = \frac{(1 + \lambda) M}{4\sqrt{\pi}} \left( 1 - \left( \frac{\lambda}{1+\lambda} \right)^2 \right)^k \frac{\Gamma(k - 1/2)}{k^\mu \Gamma(k + 1)} \]

\[
\frac{3}{2} + \mu = \frac{19}{12} = 1.58
\]

For large $k >> 1$:

\[ n_k \approx \left[ \frac{\lambda^{5/6}}{2^{5/6} \Gamma\left(\frac{5}{12}\right)} \right] \frac{1}{k^{3/2+\mu}} e^{-\lambda^2 k} \]

- power-law distribution
- exponential cutoff
Numerical Solution of the rate equations.

\[ C_{ij} = (i^{1/3} + j^{1/3})^2 (i^{-1} + j^{-1})^{1/2} \]

\[ C_{ij} = (ij)^{1/12} \]

\[ \mu = \frac{1}{12}; \quad \frac{3}{2} + \mu = \frac{19}{12} \]

\[ \lambda = 0.005 \]

\[ n_k \]

\[ k^{-19/12} \]
Fragmentation models

\[ P(m_k) \] - distribution of fragments’ mass

Güttler & Blum 2009
breakage of particles with a power-law distribution of debris

\[ P(m_k) \propto k^{-\alpha} \]

\[
\frac{dn_1}{dt} = -n_1 \sum_{j \geq 1} K_{1j} n_j + \frac{\lambda}{2} \sum_{i, j \geq 2} (i + j) K_{ij} n_i n_j + \lambda n_1 \sum_{j \geq 2} j K_{1j} n_j
\]

aggregation

\[
\frac{dn_k}{dt} = \frac{1}{2} \sum_{i + j = k} K_{ij} n_i n_j - (1 + \lambda) \sum_{i = 1}^{\infty} K_{ki} n_i n_k
\]

fragmentation

new terms

\[
+ \lambda \sum_{i = 1}^{k} n_i \sum_{j = k+1}^{\infty} K_{ij} n_j P_k(j) + \frac{\lambda}{2} \sum_{i, j \geq k+1} K_{ij} n_i n_j [P_k(i) + P_k(j)]
\]

old terms

\[ P_k(i) \propto i^{k-\alpha} \]
Assume distribution \( n_k \propto k^{-\gamma} e^{-ak} \)

Then:

Old terms scale as

\( \propto k^{\mu-\gamma} \) for \( k \gg 1; \ ak < 1 \)

New terms scale as

\( \propto k^{-\alpha} \) or \( k^{\mu-\gamma+1-\alpha} \) \( k \gg 1; \ ak < 1 \)

For steep size distributions of debris size the resulting steady-state distribution is universal and coincides with this for the complete fragmentation into monomers.
breakage of particles with a power-law distribution for debris

\[ P(m_k) \propto k^{-\alpha} \]

ratio of fragmentative and aggregative impacts

\[ \lambda = 0.01 \]

For decomposition into monomers

\[ n_k \propto k^{-1.50} \]
General case: breakage of particles with a power-law distribution for debris

**Universality of steady-state size distribution in aggregation-fragmentation processes:**

All steady–state aggregate size distributions have for $k >> 1$ the same form

$$n_k \propto k^{-3/2-\mu} e^{-\lambda^2 k}$$

where $2\mu$ is the homogeneity degree of the kinetic coefficients, if the size distribution of debris is steep enough; it coincides with the form for the case of complete decomposition into monomers. For the case of power-law debris size distribution, the condition seemingly reads,

$$\alpha \geq 3$$
Ring particles are "ephemeral dynamic bodies. They are very loose and weak with a low average coordination number. Hence a steep distribution of debris at a collisional decomposition is very plausible."
The radii distribution of particles in Planetary Rings:

\[ k \propto R^3 \quad n_k \, dk = f(R) \, dR \]

we obtain for the radii distribution function:

\[ f(R) \propto R^{-\left(\frac{5}{2} + 3\mu\right)} \exp\left(-\frac{R^3}{R_c^3}\right) \]

\[ \frac{5}{2} + 3\mu = \frac{11}{4} = 2.75 \]

\[ R_c = r_1 \lambda^{-2/3} \]

Comparison with the observational data for Planetary Rings

Zebker et. al 1985

theory: $B = 2.75$

$$f(R) = AR^{-B} e^{-CR^3}$$

$B = 2.739, C = 0.00591$
Conclusion

- **Kinetic theory of ballistic aggregation and fragmentation is developed**

- **Analytical result for the steady-state size distribution function is obtained**

- **Theoretical results agree very well with the observation data for Saturn Rings**


Still to be done:

- **To take into account different granular temperatures** $T_k$
- **To compute aggregation and fragmentation energies** $E_{\text{agg}}$ and $E_{\text{frag}}$
- **To estimate the cutoff radius** $R_c$ and to compare with the observation data
- **Many other interesting issues….**


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