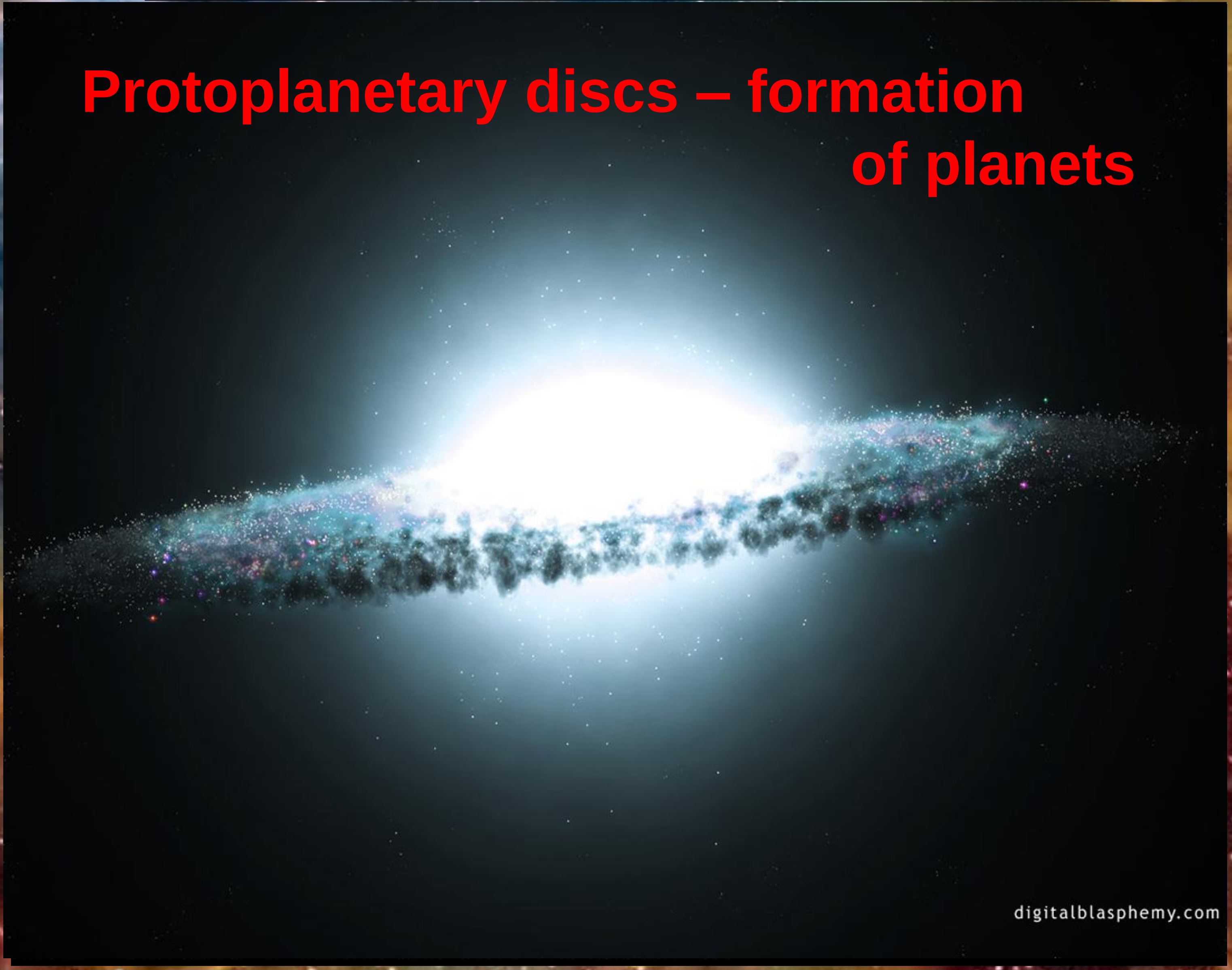
University of Leicester

Kinetic Theory of Ballistic Aggregation and Fragmentation. (Application to Planetary Rings)

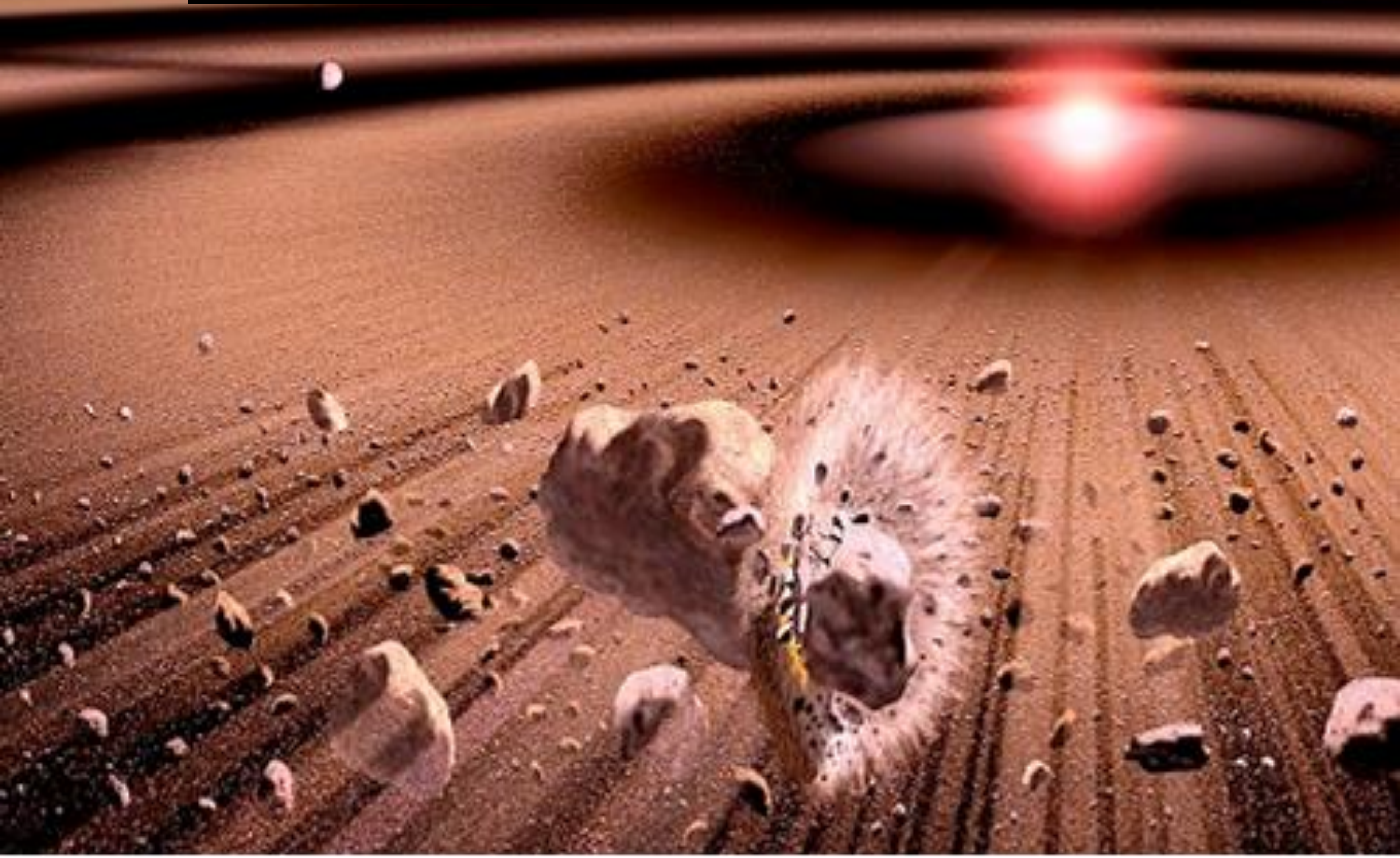
Nikolai Brilliantov

Conference dedicated to 60th birthday of Alexander Gorban

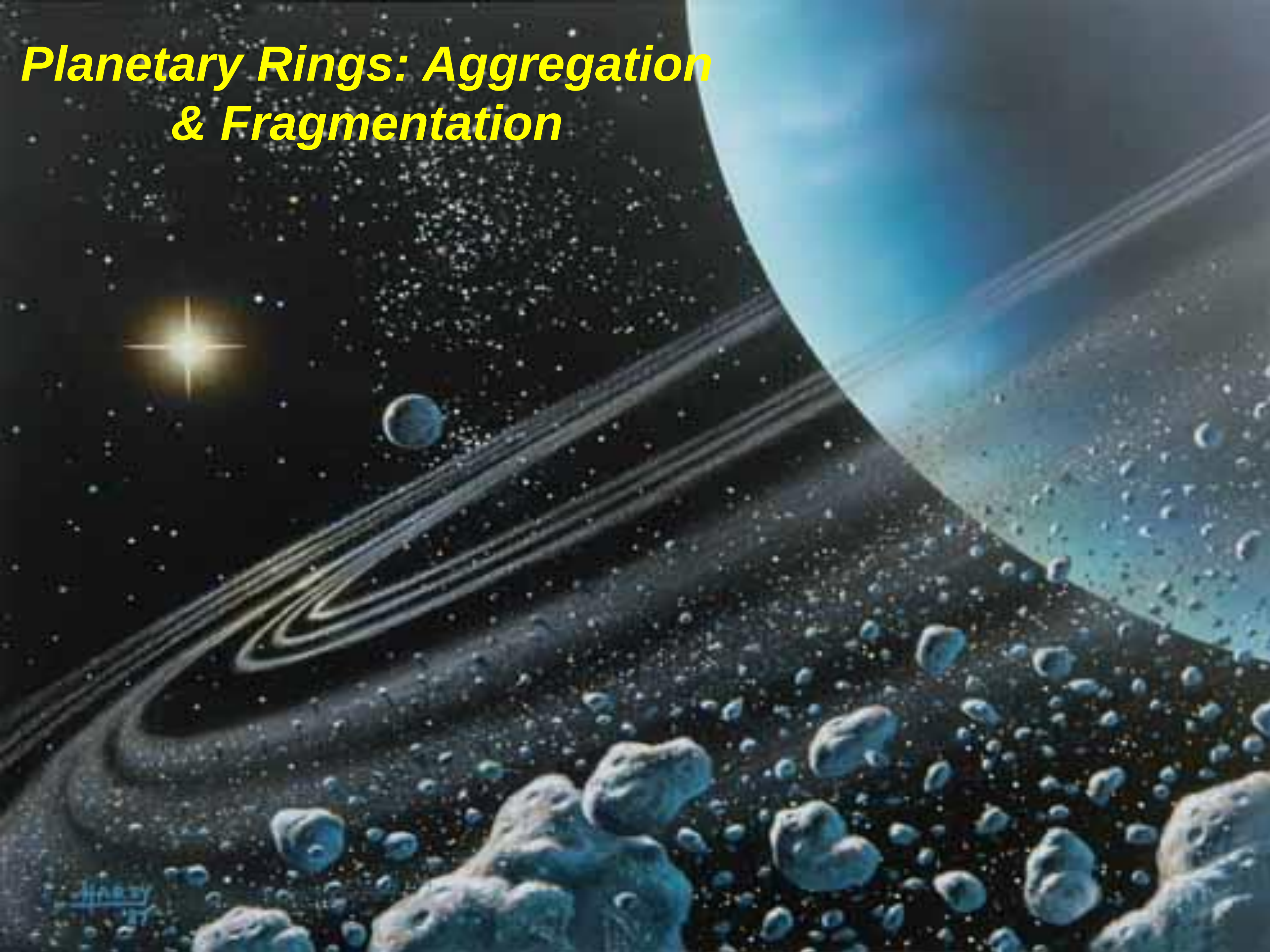
Aggregation of particles



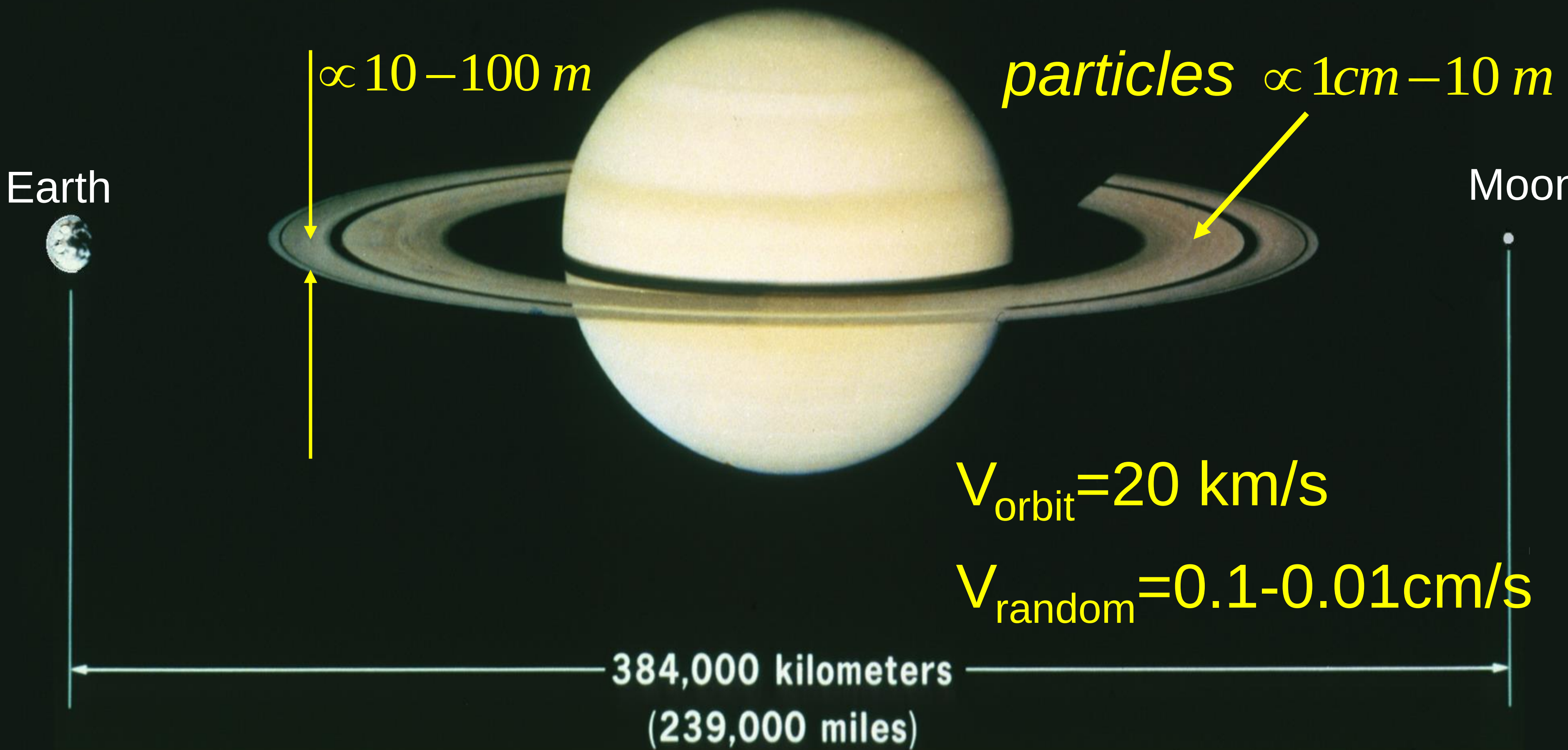
Fragmentation of particles:
grinding of large bodies to dust



Planetary Rings: Aggregation & Fragmentation

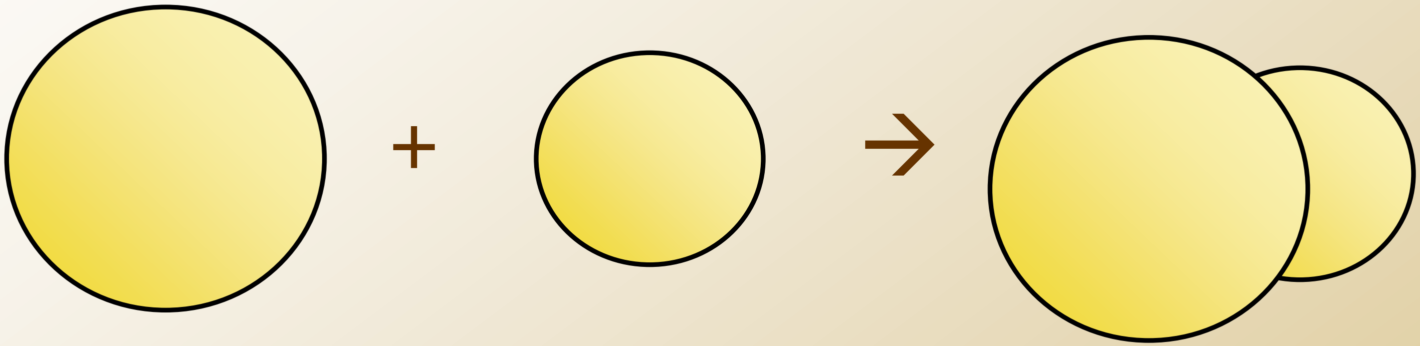


Saturn Rings

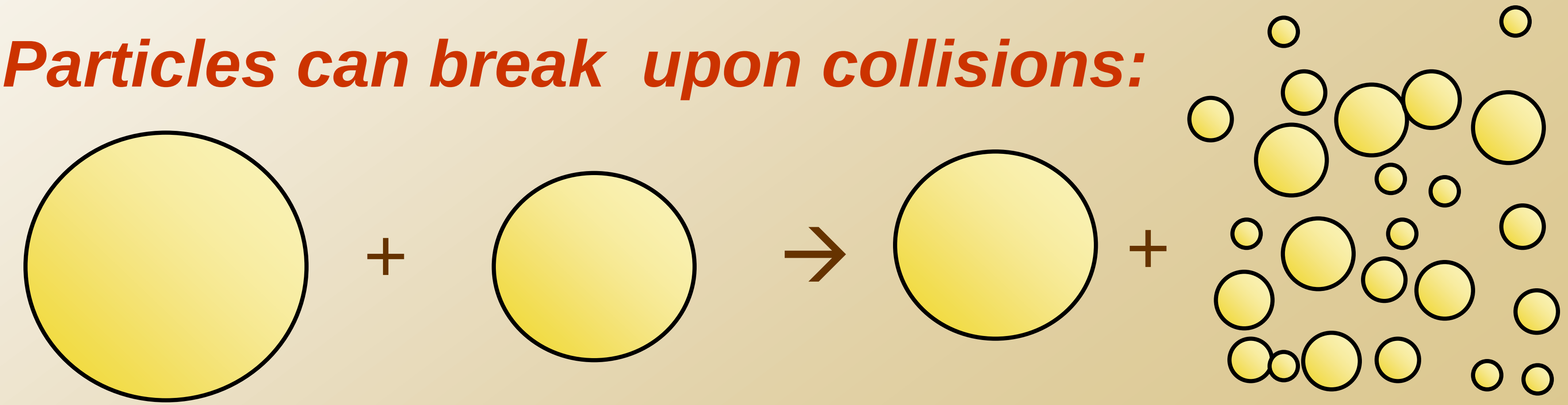


Model

- ***Particles move ballistically between pair-wise collisions***
- ***Particles can aggregate upon collisions:***

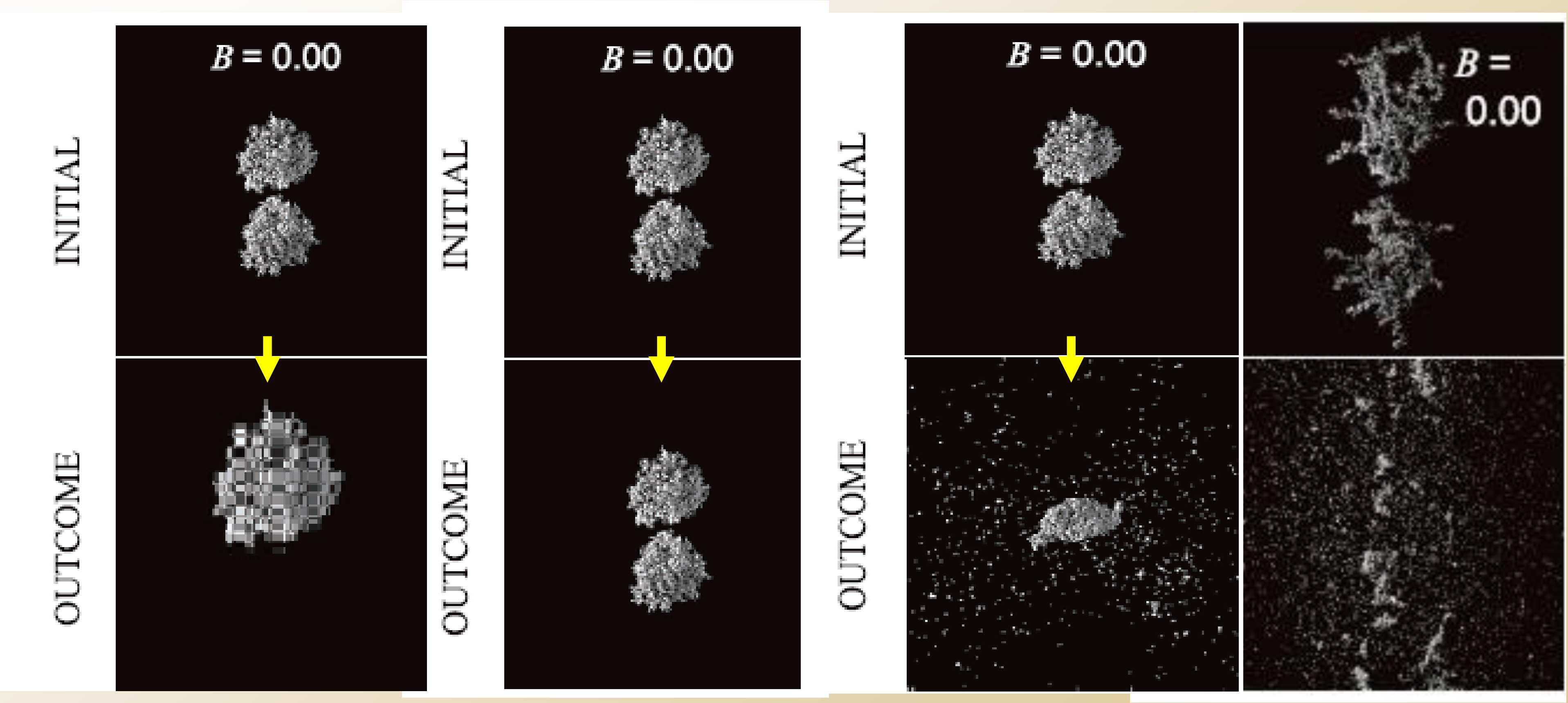


- ***Particles can break upon collisions:***



Important features of particle collisions:

- *Particles aggregate for small impact velocities*
- *Particles collide loosing kinetic energy for medium impact speeds*
- *Particles break for large impact velocities*



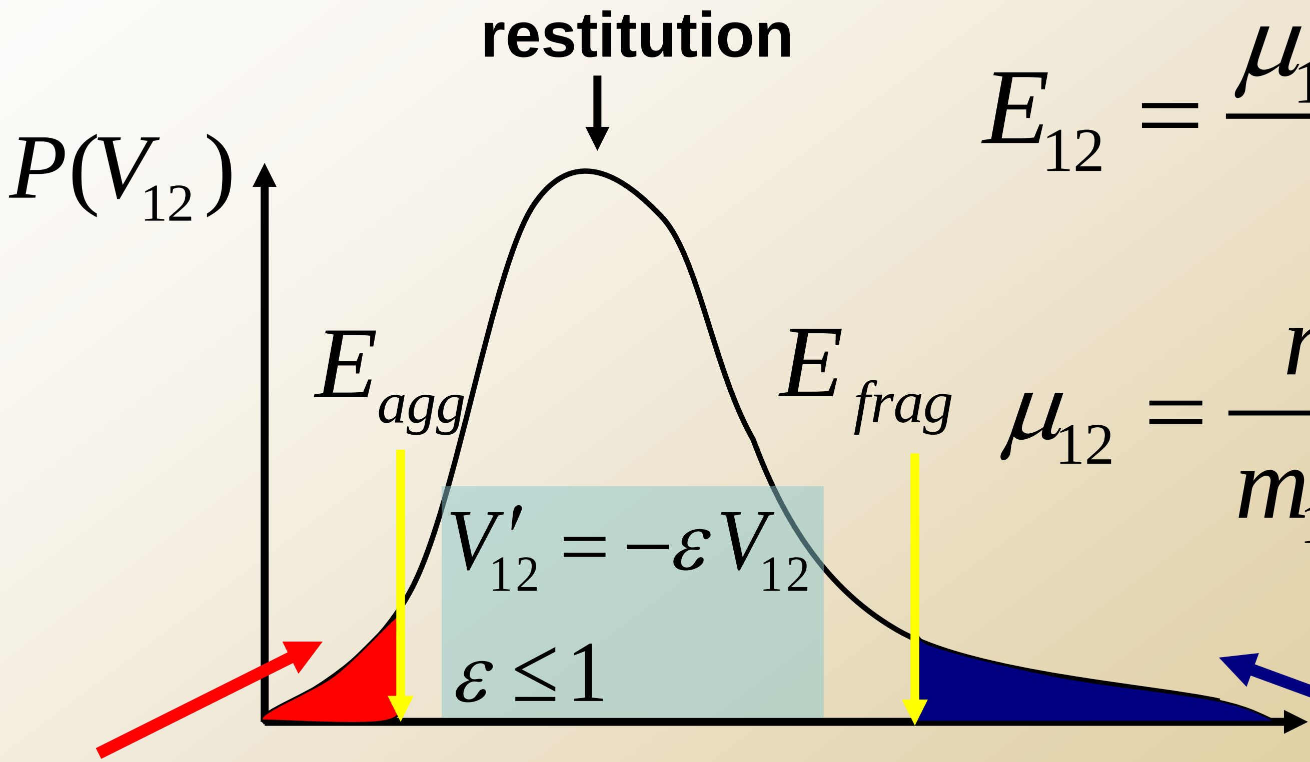
Small V_{imp} ***Medium*** V_{imp} ***Large*** V_{imp}

using pictures of K.Wada Astrophys. J., 2009

Introduce two parameters, E_{agg} and E_{frag}

$$E_{12} = \frac{\mu_{12}\vec{V}_{12}^{\,2}}{2}$$

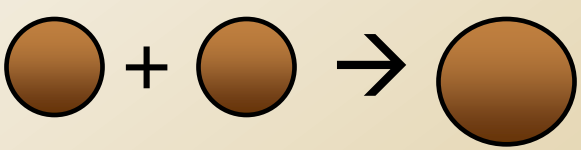
$$\mu_{12} = \frac{m_1 m_2}{m_1 + m_2}$$



$$\vec{V}_{12} = \vec{V}_1 - \vec{V}_2$$

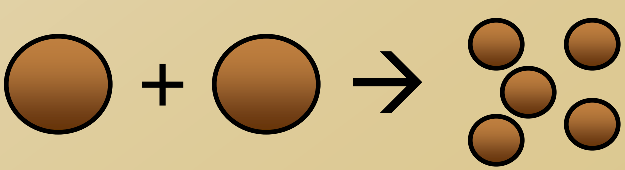
aggregation

$$E_{12} < E_{agg}$$



fragmentation

$$E_{12} > E_{frag}$$



Boltzmann equation

Distribution function: $f(\vec{r},\vec{V},t)$

$$\frac{\partial}{\partial t} f(\vec{r},\vec{V},t) = I(f,f)$$ - "collision integral"

$$I = I_{gain}(f,f) - I_{loss}(f,f)$$

2 The Source of Examples

2.1 The Boltzmann Equation
2.1.1 The Equation
2.1.2 The Basic Properties of the Boltzmann Equation
2.1.3 Linearized Collision Integral
2.2 Phenomenology and Quasi-Chemical Representation of the Boltzmann Equation
2.3 Kinetic Models
2.4 Methods of Reduced Description
2.4.1 The Hilbert Method
2.4.2 The Chapman–Enskog Method
2.4.3 The Grad Moment Method
2.4.4 Special Approximations
2.4.5 The Method of Invariant Manifold
2.4.6 Quasiequilibrium Approximations

E.g. for collisions with restitution only

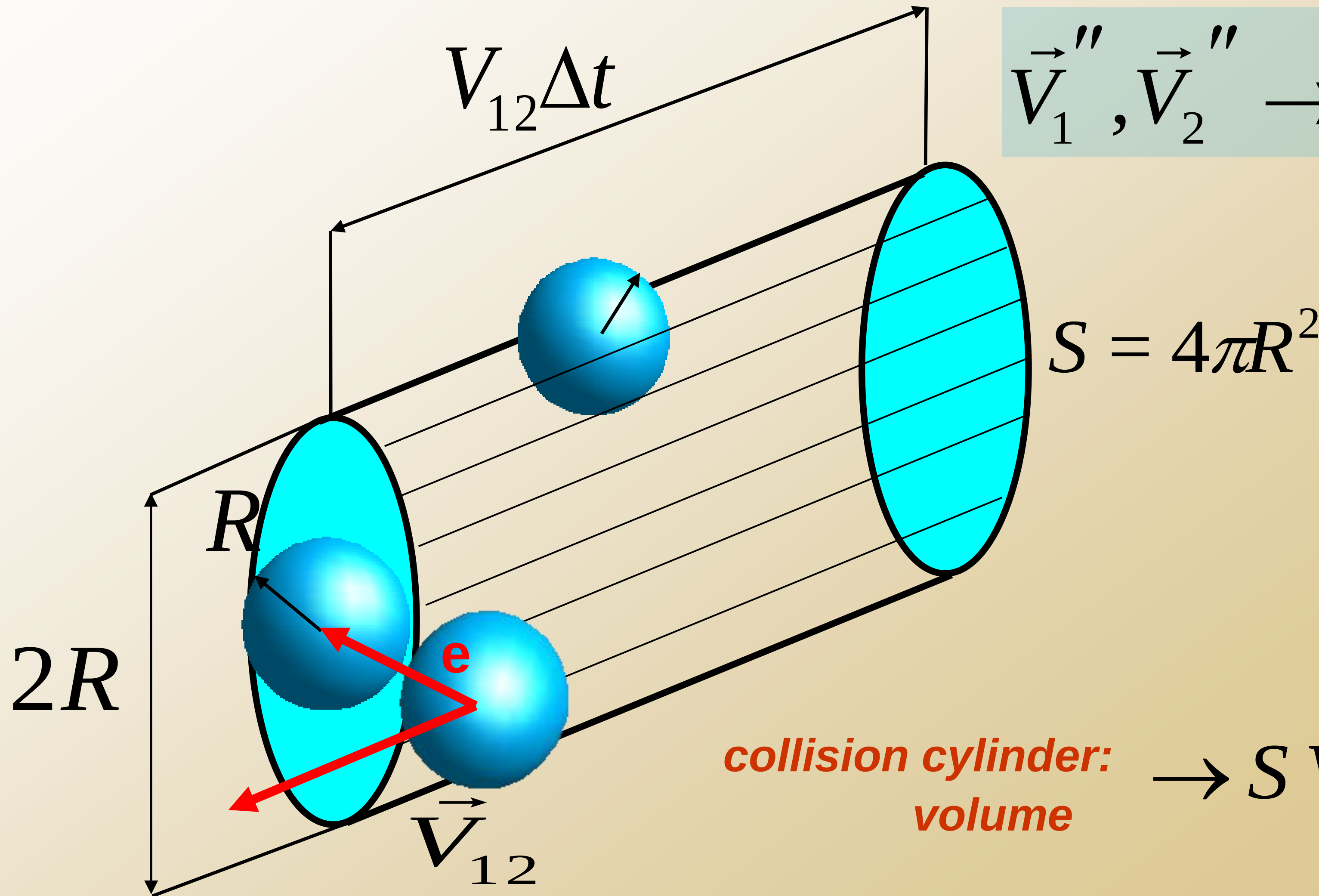
loss term:

$$I_{loss}(f,f) \equiv 4R^2 \int d\vec{V}_2 \int_{\vec{V}_{12}\cdot\vec{e}<0} d\vec{e}\, \left|\vec{V}_{12}\cdot\vec{e}\right| f(\vec{r},\vec{V}_1,t)\, f(\vec{r},\vec{V}_2,t)$$

gain term:

$$I_{gain}(f,f) \equiv 4R^2 \int d\vec{V}_2 \int_{\vec{V}_{12}\cdot\vec{e}<0} d\vec{e}\, \left|\vec{V}_{12}\cdot\vec{e}\right| \frac{1}{\varepsilon^2} f(\vec{r},\vec{V}_1'',t)\, f(\vec{r},\vec{V}_2'',t)$$

$$\vec{V}_1'', \vec{V}_2'' \rightarrow \vec{V}_1, \vec{V}_2$$

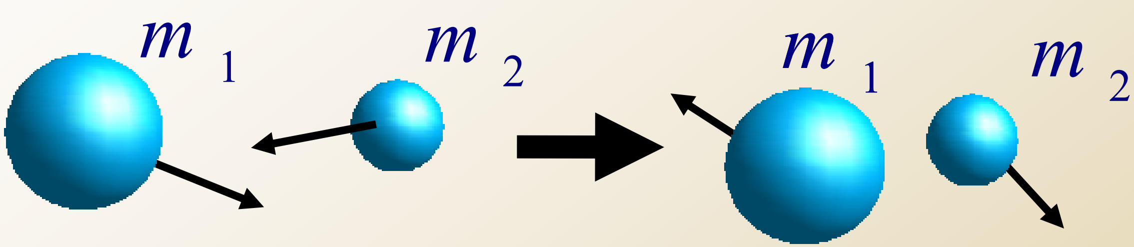


collision cylinder: volume $\rightarrow S\, V_{12} \Delta t$

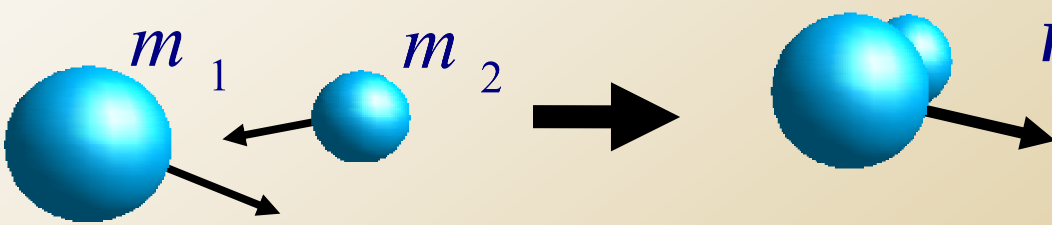
Boltzmann Equation

m_0 **-elementary mass** r_0 **-elementary radius**

$m_0 \quad 2m_0 \quad 3m_0 \quad ,\dots, \quad km_0$

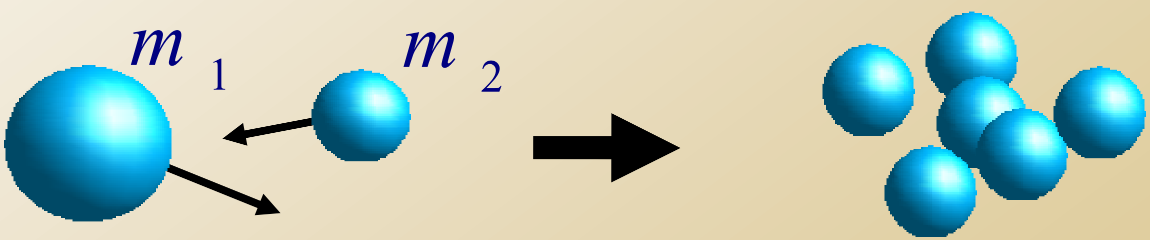


restitution collision



$$m = (m_1 + m_2)$$

aggregative collision



disruptive collision

$f(m, \vec{V}, t)$ - *mass-velocity distribution function*

Generalized Boltzmann Equation

restitution aggregation fragmentation

$$\frac{\partial f_i(m_i,\vec{v}_i,t)}{\partial t} = I_{rest}(f,f) + I_{agg}(f,f) + I_{frag}(f,f)$$

viscous heating

$$+ I_{heat}(f,f)$$

Keeps the temperature of ring particles constant

- ***Aggregation term***

$$I_k^{\mathrm{agg}} = \frac{1}{2}\sum_{i+j=k} \sigma_{ij}^2 \int d\vec{v}_i \int d\vec{v}_j \int d\vec{e}\,\Theta\left(-\vec{v}_{ij}\cdot\vec{e}\right)\left|\vec{v}_{ij}\cdot\vec{e}\right| \times$$

gain term

$$\times f_i\left(\vec{v}_i\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{\mathrm{agg}} - E_{ij}\right)\delta(m_k\vec{v}_k - m_i\vec{v}_i - m_j\vec{v}_j)$$

aggregation condition   momentum conservation

$$-\sum_j \sigma_{kj}^2 \int d\vec{v}_j \int d\vec{e}\,\Theta\left(-\vec{v}_{kj}\cdot\vec{e}\right)\left|\vec{v}_{kj}\cdot\vec{e}\right| \times$$

loss term

$$\times f_k\left(\vec{v}_k\right) f_j\left(\vec{v}_j\right)\Theta\left(E_{\mathrm{agg}} - E_{kj}\right).$$

- ***Fragmentation term has the similar structure, but requires a microscopic fragmentation model***

The heating term keeps the steady-state energy distribution for species. its particular form is not presently important.

Generalized Boltzmann Equation

Assume:

- ***Maxwellian:***

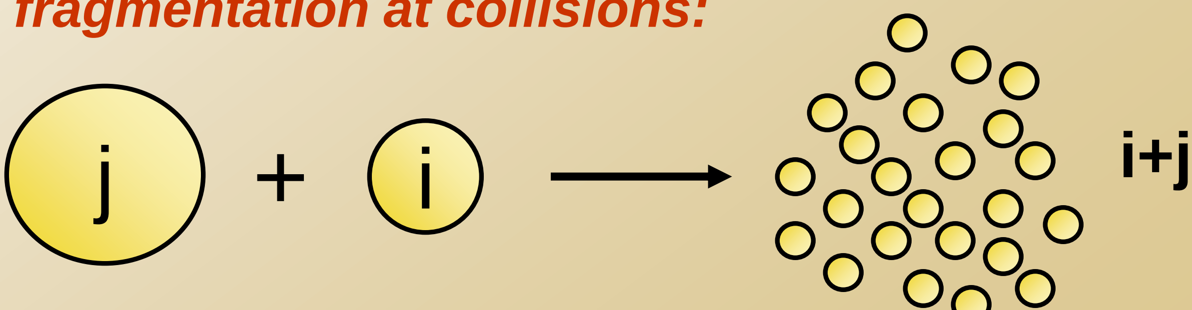
$$f_i(m_i,\vec{v}_i,t)=n_i\left(\frac{m_i}{2T(t)\pi}\right)^{3/2}\exp\left(-\frac{m_i v_i^2}{2T(t)}\right)$$

Ask Alexander Why this is justified:

$$n_i(t)=\int f_i(m_i,\vec{v}_i,t)\,d\vec{v}_i$$

$$T_i(t)=\int \frac{m_i v_i^2}{2} f_i(m_i,\vec{v}_i,t)\,d\vec{v}_i = T(t)$$

- ***Complete fragmentation at collisions:***



Kinetic equations for particles concentrations $n_i(t)$ for space uniform systems

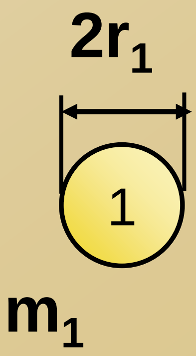
relative ratio of aggregative and disruptive collisions → λ

$$\frac{dn_k}{dt} = \underbrace{\frac{1}{2}\sum_{i+j=k} K_{ij} n_i n_j - n_k \sum_{j\ge 1} K_{kj} n_j}_{\text{aggregation}} - \underbrace{\lambda n_k \sum_{j\ge 1} K_{kj} n_j}_{\text{fragmentation}}$$

$$\frac{dn_1}{dt} = \underbrace{-n_1 \sum_{j\ge 1} K_{1j} n_j}_{\text{aggregation}} + \underbrace{\frac{\lambda}{2} \sum_{i,j\ge 2} (i+j) K_{ij} n_i n_j + \lambda n_1 \sum_{j\ge 2} j K_{1j} n_j}_{\text{fragmentation}}$$

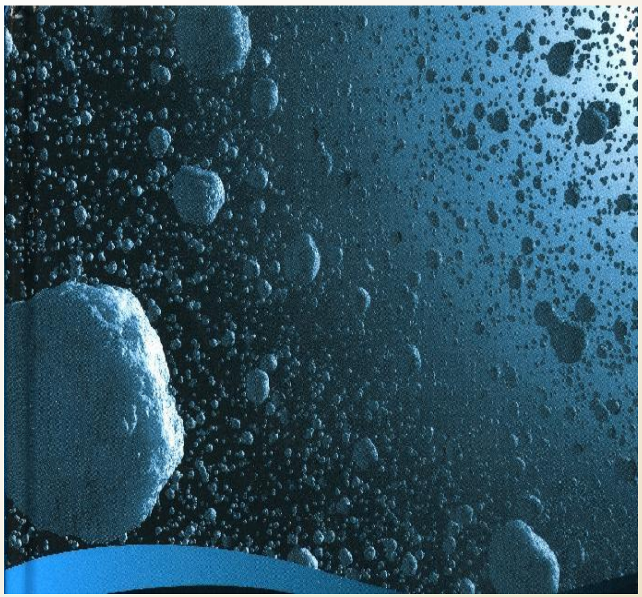
$$K_{ij} = r_1^{\,2} e^{-E_{\text{frag}}/T} \lambda^{-1} \sqrt{\frac{8\pi T}{m_1}} \sqrt{\frac{(i+j)}{ij}} \left(i^{1/3} + j^{1/3}\right)^2$$

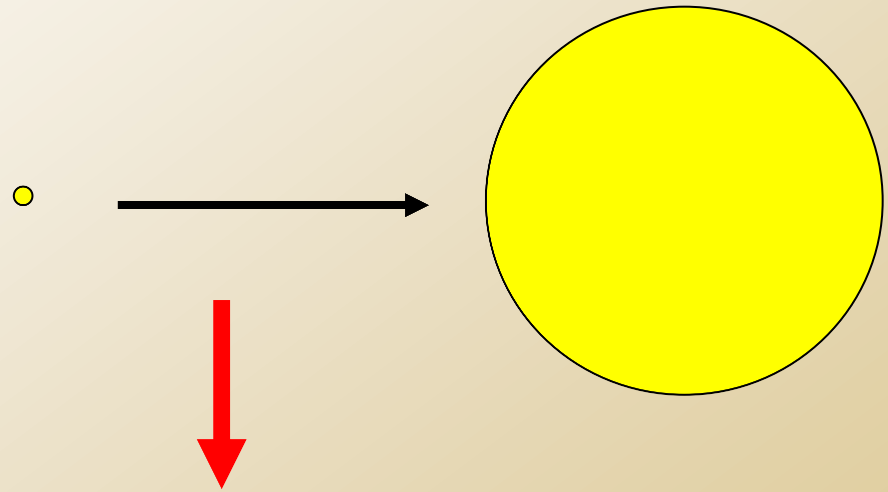
$$\lambda = \left(1 + (1 + E_{\text{agg}}/T) e^{-E_{\text{agg}}/T}\right) e^{-E_{\text{frag}}/T}$$



For planetary rings particles' size ranges

from ~ 10^{-3} m   to ~ 10^{0}m





10^{9} equations!

Analytics?

Constant rate coefficients: $K = K_0$

$$\frac{dn_1}{dt} = -n_1 N + \lambda(1-n_1)N$$

$$N(t) = \sum_{j\geq 1} n_j(t)$$

Total number of aggregates

$$\frac{dn_k}{dt} = \frac{1}{2}\sum_{i+j=k} n_i n_j - (1+\lambda) n_k N;$$

$$\begin{cases} \dfrac{dn_1}{dt} = -n_1 N + \lambda(1-n_1)N \\ \dfrac{dN}{dt} = -N^2 + \lambda(1-N)N \end{cases}$$

$$t \to K_0 n_0 t \qquad n_k \to n_0 n_k$$

Constant rate coefficients: $K = K_0$

For $n_k(t=0) = \delta_{k,1}$

$$N(t) = 2\lambda\left[1 + 2\lambda - e^{-\lambda t}\right]^{-1}$$

$$n_1(t) = \frac{\lambda}{1+\lambda} + \frac{1}{1+\lambda}\left[\frac{(1+2\lambda)e^{\lambda t} - 1}{2\lambda}\right]^{-\frac{2(1+\lambda)}{1+2\lambda}}$$

$$\tau_{rel}^{-1} = K_0 n_0 \lambda = 10^3 - 10^5 \text{ years}$$

The system relaxes to the steady state distribution of particles sizes!

Constant rate coefficients: $K = K_0$

Steady-state:

$$n_1 = \frac{\lambda}{1+\lambda} \qquad N = \frac{2\lambda}{1+2\lambda}$$

$$0 = \frac{1}{2}\sum_{i+j=k} n_i n_j - (1+\lambda) n_k N; \quad k = 2,3,\ldots$$

Introduce Generating function: $G(z) = \sum_{k\geq 1} n_k z^k$

$$G(z)^2 - 2(1+\lambda)NG(z) + 2(1+\lambda)Nn_1 z = 0$$

Solving the quadratic equation we obtain:

$$G(z) = (1+\lambda)N\left[1-\sqrt{1-\frac{2n_1}{(1+\lambda)N}z}\right]$$

Expanding $G(z)$ ***we obtain exact steady-state***

$$n_k = \frac{N}{\sqrt{4\pi}}(1+\lambda)\left[\frac{2n_1}{(1+\lambda)N}\right]^k \frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma\left(k+1\right)}$$

For $k >> 1$

$$n_k = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 k} k^{-3/2}$$

General case

The rate kernel is uniform:

$$K_{ij} \propto \sqrt{\frac{(i+j)}{ij}}\left(i^{1/3}+j^{1/3}\right)^2$$

$$K_{ai\,aj} = a^{2\mu} K_{ij} \quad \text{with} \quad \mu = \frac{1}{12}$$

This suggests the approximation:

$$K_{ij} \approx K_0 \left(ij\right)^{\mu}$$

$$K_0 = r_1^{\,2} e^{-E_{\text{frag}}/T} \lambda^{-1} \sqrt{\frac{8\pi T}{m_1}}$$

for μ=0 aggregation and fragmentation does not depend on particles size (constant kernel)

Steady-state equation for $K_{ij} = K_0 (ij)^{\mu}$

$$0 = \frac{1}{2} \sum_{i+j=k} K_0 (ij)^{\mu} n_i n_j - (1+\lambda) \sum_{j \geq 1} K_0 (kj)^{\mu} n_k n_j$$

Introduce

$$l_k = k^{\mu} n_k \qquad L = \sum_{k \geq 1} l_k$$

$$0 = \frac{1}{2} \sum_{i+j=k} l_i l_j - (1+\lambda) l_k L; \qquad n_k \to l_k \qquad N \to L$$

$$F(z)^2 - (1+\lambda) M F(z) + (1+\lambda) M n_1 z = 0$$

$$M = F(1)$$

Solving quadratic equation and expanding F(z) we obtain:

$$n_k = \frac{(1+\lambda)M}{4\sqrt{\pi}}\left(1-\left(\frac{\lambda}{1+\lambda}\right)^2\right)^k \frac{\Gamma(k-1/2)}{k^{\mu}\Gamma(k+1)}$$

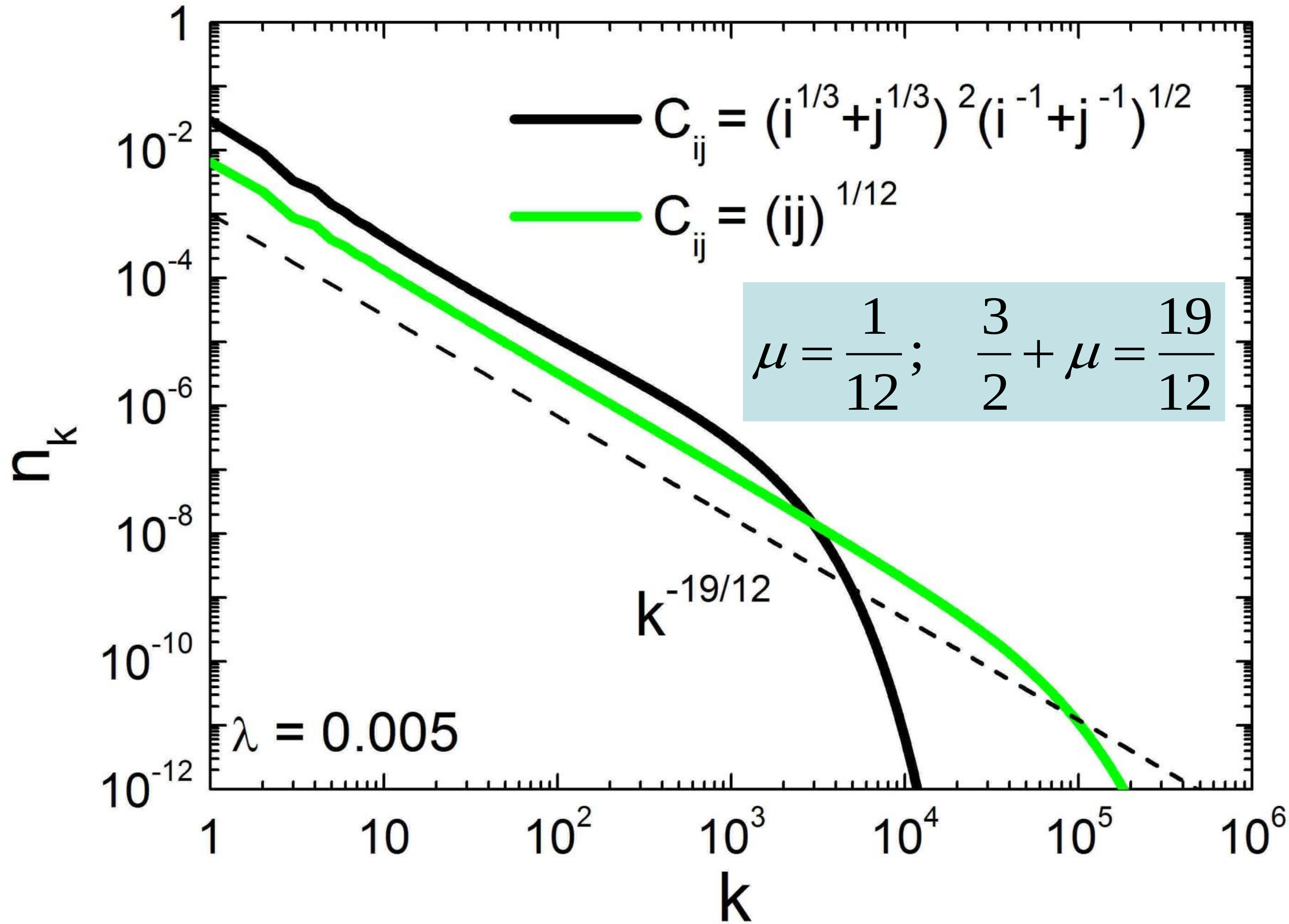
For large k>>1:

$$\tfrac{3}{2}+\mu=\tfrac{19}{12}=1.58$$

$$n_k \approx \left[\frac{\lambda^{5/6}}{2^{5/6}\Gamma\left(\frac{5}{12}\right)}\right]\frac{1}{k^{3/2+\mu}}\, e^{-\lambda^2 k}$$

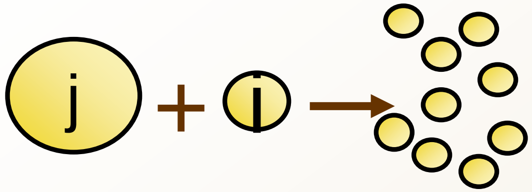
power-law distribution **exponential cutoff**

Numerical Solution of the rate equations.

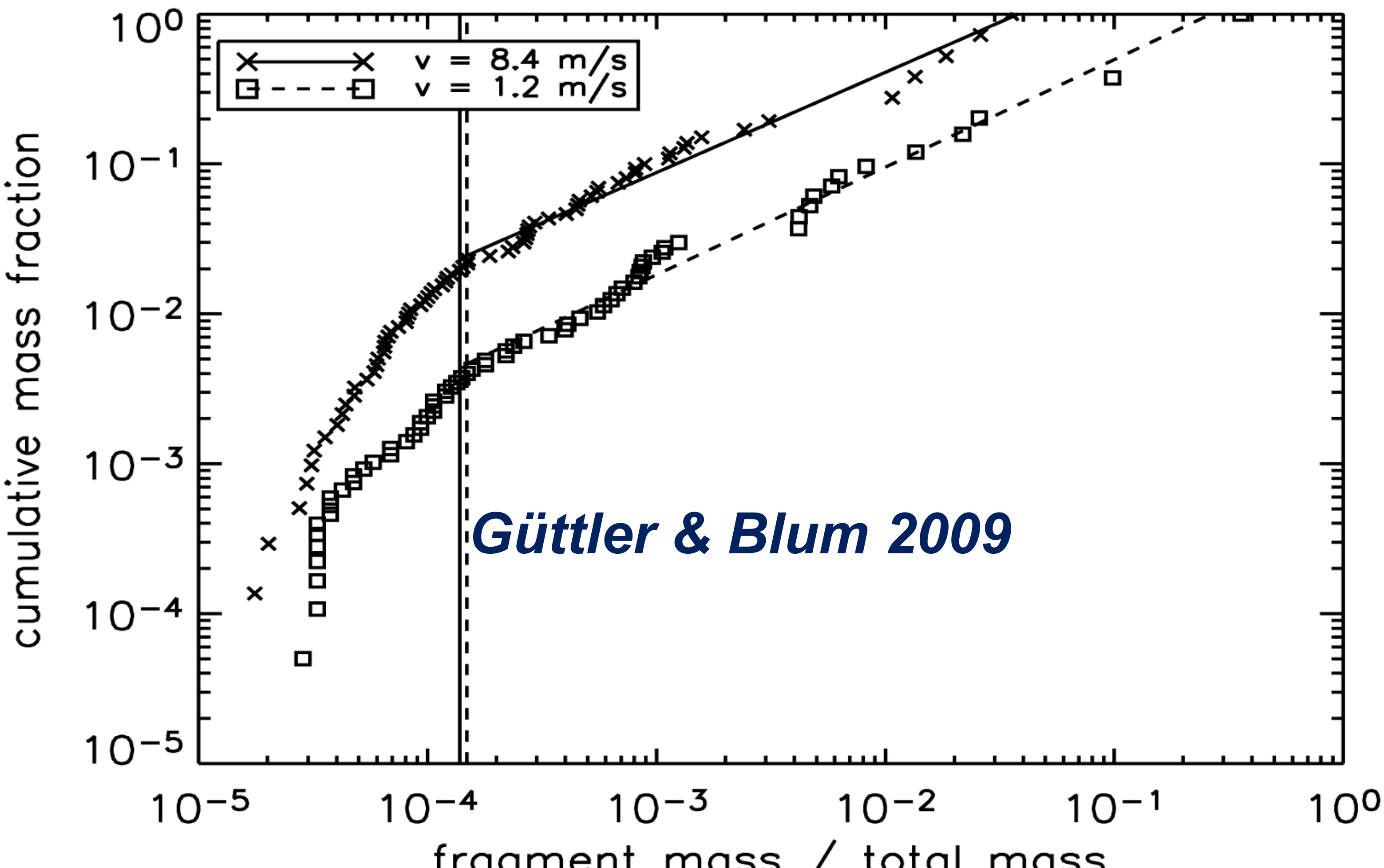


$$\mu = \frac{1}{12}; \quad \frac{3}{2} + \mu = \frac{19}{12}$$

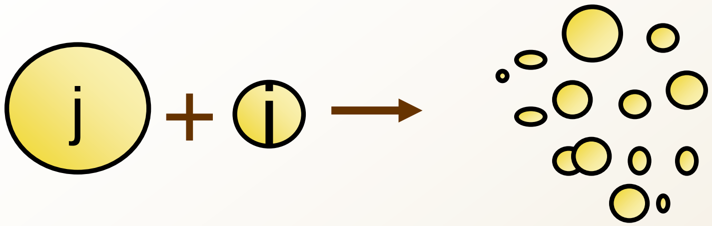
Fragmentation models



$P(m_k)$ - ***distribution of fragments' mass***



breakage of particles with a power-law distribution of debris



$$P(m_k) \propto k^{-\alpha}$$

$$\frac{dn_1}{dt} = -n_1 \underbrace{\sum_{j\geq 1} K_{1j} n_j}_{\text{aggregation}} + \underbrace{\frac{\lambda}{2} \sum_{i,j\geq 2} (i+j) K_{ij} n_i n_j + \lambda n_1 \sum_{j\geq 2} j K_{1j} n_j}_{\text{fragmentation}}$$

$$\frac{dn_k}{dt} = \frac{1}{2} \sum_{i+j=k} K_{ij} n_i n_j - (1+\lambda) \sum_{i=1}^{\infty} K_{ki} n_i n_k$$

old terms

$$+\lambda \sum_{i=1}^{k} n_i \sum_{j=k+1}^{\infty} K_{ij} n_j P_k(j) + \frac{\lambda}{2} \sum_{i,j\geq k+1}^{\infty} K_{ij} n_i n_j \left[P_k(i) + P_k(j) \right]$$

new terms

$$P_k(i) \propto i\, k^{-\alpha}$$

Assume distribution $n_k \propto k^{-\gamma} e^{-ak}$

Then:

Old terms scale as

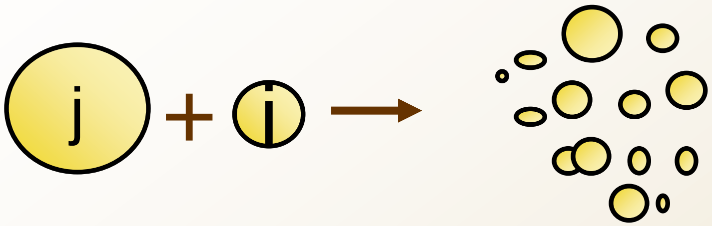
$$\propto k^{\mu-\gamma} \qquad \text{for} \qquad k >> 1; \quad ak < 1$$

New terms scale as

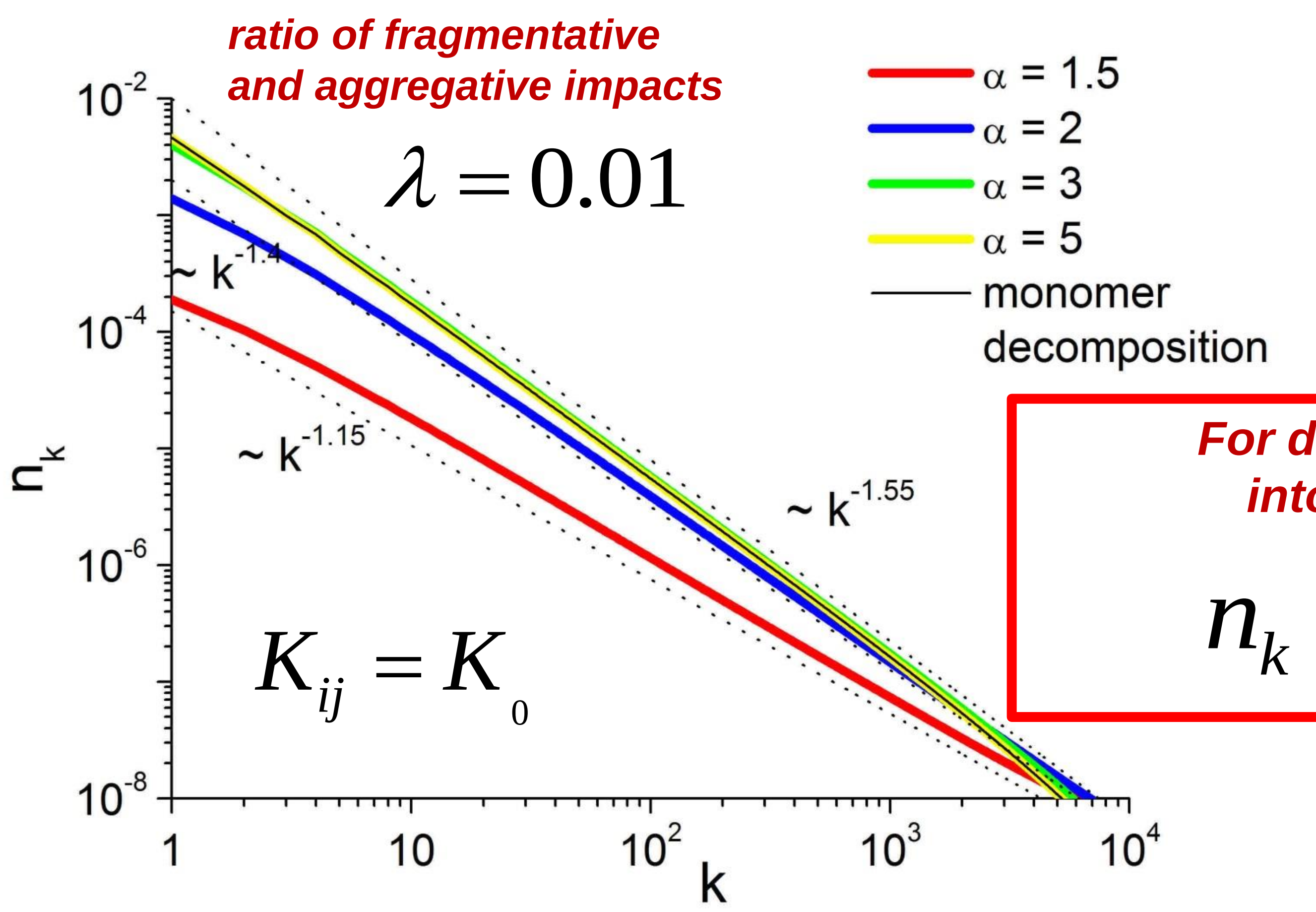
$$\propto k^{-\alpha} \ \text{or} \ \ k^{\mu-\gamma+1-\alpha} \qquad k >> 1; \quad ak < 1$$

For steep size distributions of debris size the resulting steady-state distribution is universal and coincides with this for the complete fragmentation into monomers

breakage of particles with a power-law distribution for debris



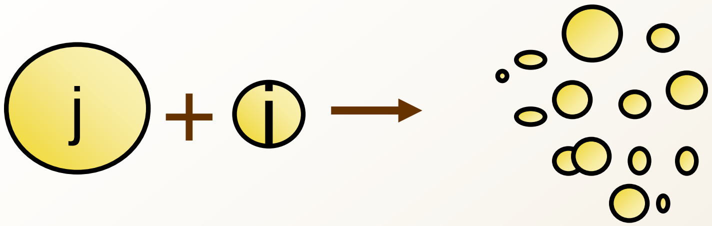
$$P(m_k) \propto k^{-\alpha}$$



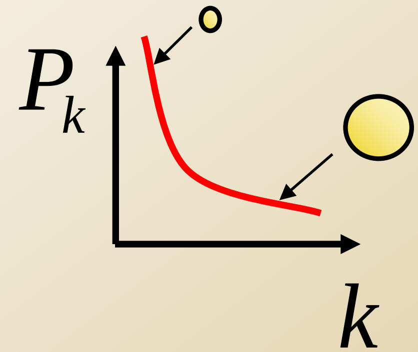
For decomposition into monomers

$$n_k \propto k^{-1.50}$$

General case: breakage of particles with a power-law distribution for debris



$$P(m_k) \propto k^{-\alpha}$$



Universality of steady-state size distribution in aggregation-fragmentation processes:

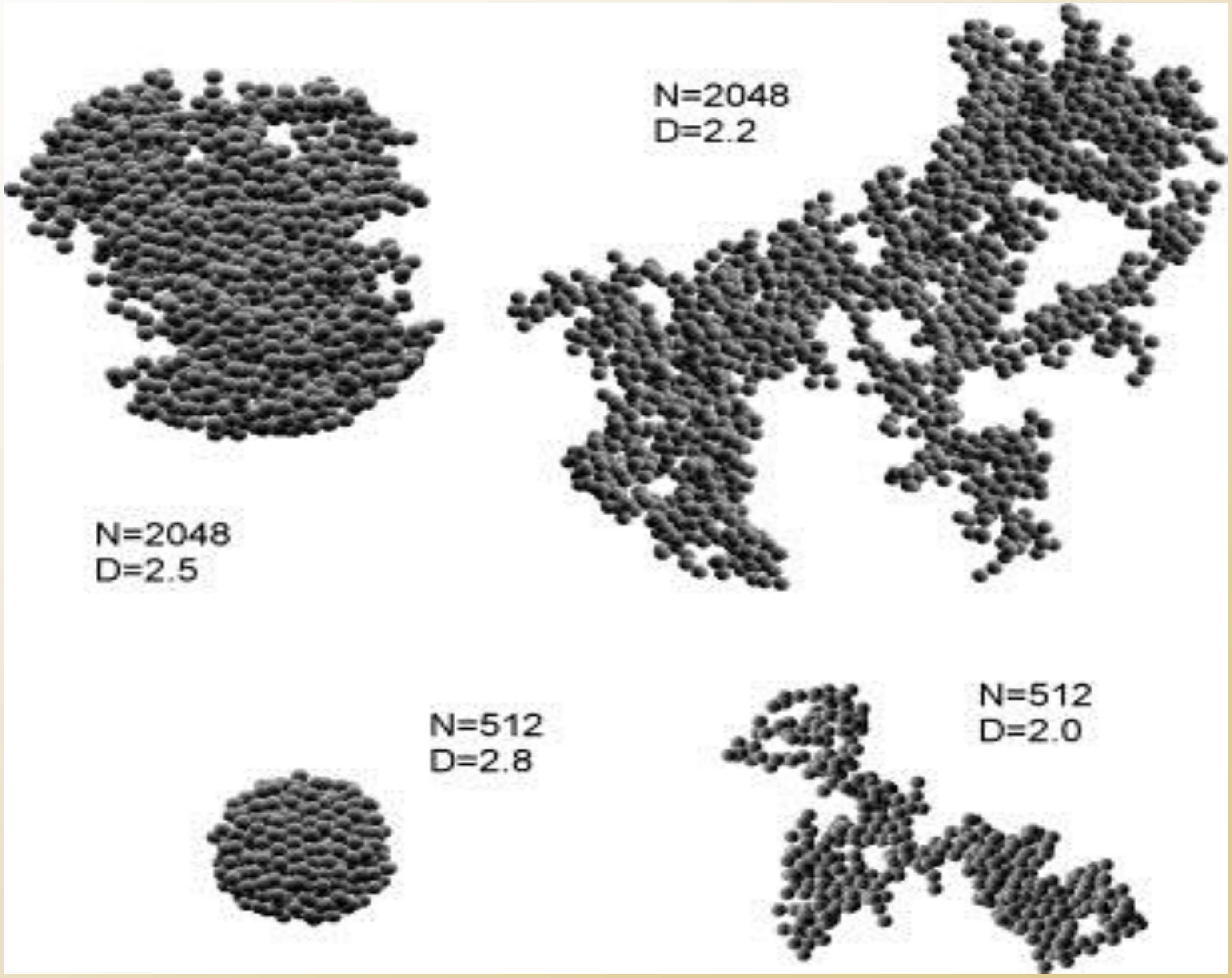
All steady–state aggregate size distributions have for k>>1 the same form

$$n_k \propto k^{-3/2-\mu} e^{-\lambda^2 k}$$

where 2μ *is the homogeneity degree of the kinetic coefficients, if the size distribution of debris is steep enough; it coincides with the form for the case of complete decomposition into monomers. For the case of power-law debris size distribution, the condition seemingly reads,*

$$\alpha \geq 3$$

Ring particles are "ephemeral dynamic bodies. They are very loose and weak with a low average coordination number. Hence a steep distribution of debris at a collisional decomposition is very plausible.



The radii distribution of particles in Planetary Rings:

$$k \propto R^3 \qquad n_k dk = f(R)dR$$

we obtain for the radii distribution function:

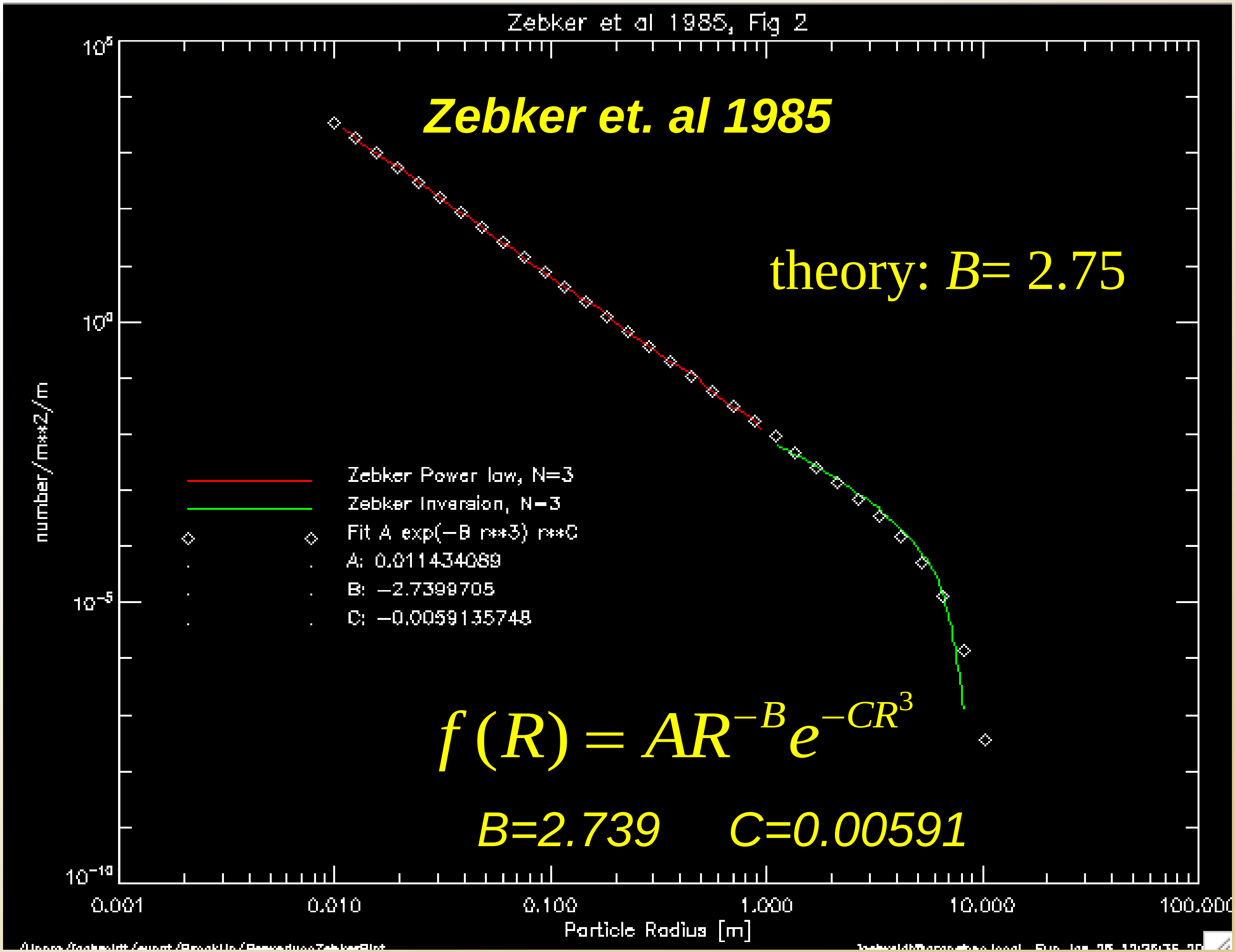
$$f(R) \propto R^{-(5/2+3\mu)} \exp\left(-R^3 / R_c^3\right)$$

$$\frac{5}{2} + 3\mu = \frac{11}{4} = 2.75$$

$$R_c = r_1 \lambda^{-2/3}$$

N. Brilliantov, et all. PNAS, 2014 submitted.

Comparison with the observational data for Planetary Rings



Conclusion

- ***Kinetic theory of ballistic aggregation and fragmentation is developed***
- ***Analytical result for the steady-state size distribution function is obtained***
- ***Theoretical results agree very well with the observation data for Saturn Rings***

N.Brilliantov, J.Schmidt, F.Spahn, *Planetary and Space Science*, 73 (2009) 327

J.Schmidt, N.Brilliantov, F. Spahn, S.Kempf, *Nature* 451 (2008) 685

F.Postberg, S.Kempf, J.Schmidt, N.Brilliantov, A.Beinsen, B.Abel, U.Buck, R. Srama, *Nature*, 459 (2009) 1098

N.Brilliantov, A.Bodrova, P.Krapivsky, *J. Stat. Mech.* (2009) 06/P06011

Still to be done:

- ***To take into account different granular temperatures T_k***
- ***To compute aggregation and fragmentaion energies E_{agg} and E_{frag}***
- ***To estimate the cutoff radius R_c and to compare with the observation data***
- ***Many other interesting issues….***

N.Brilliantov, J.Schmidt, F.Spahn, *Planetary and Space Science*, 73 (2009) 327

J.Schmidt, N.Brilliantov, F. Spahn, S.Kempf, *Nature* 451 (2008) 685

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