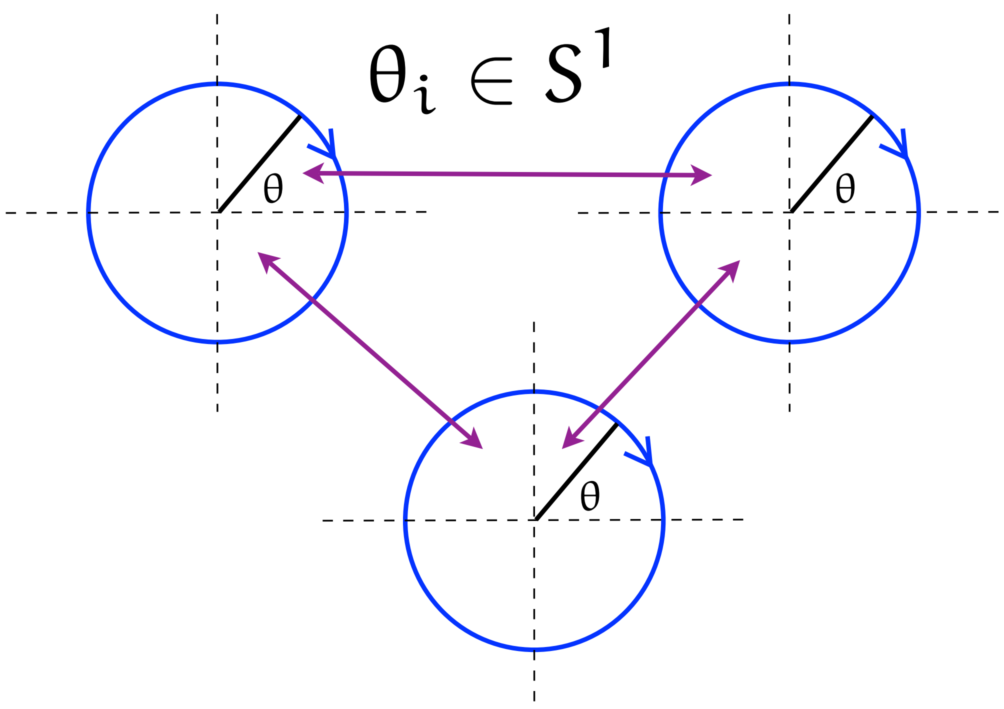
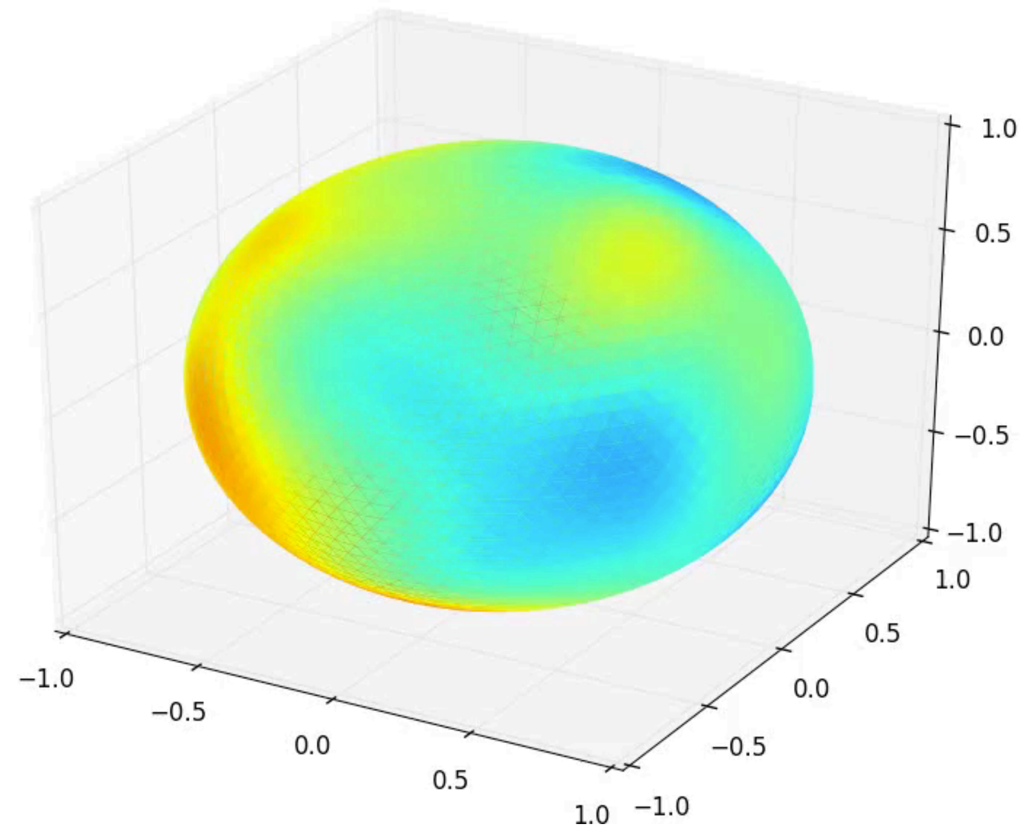
Model reduction in mathematical neuroscience



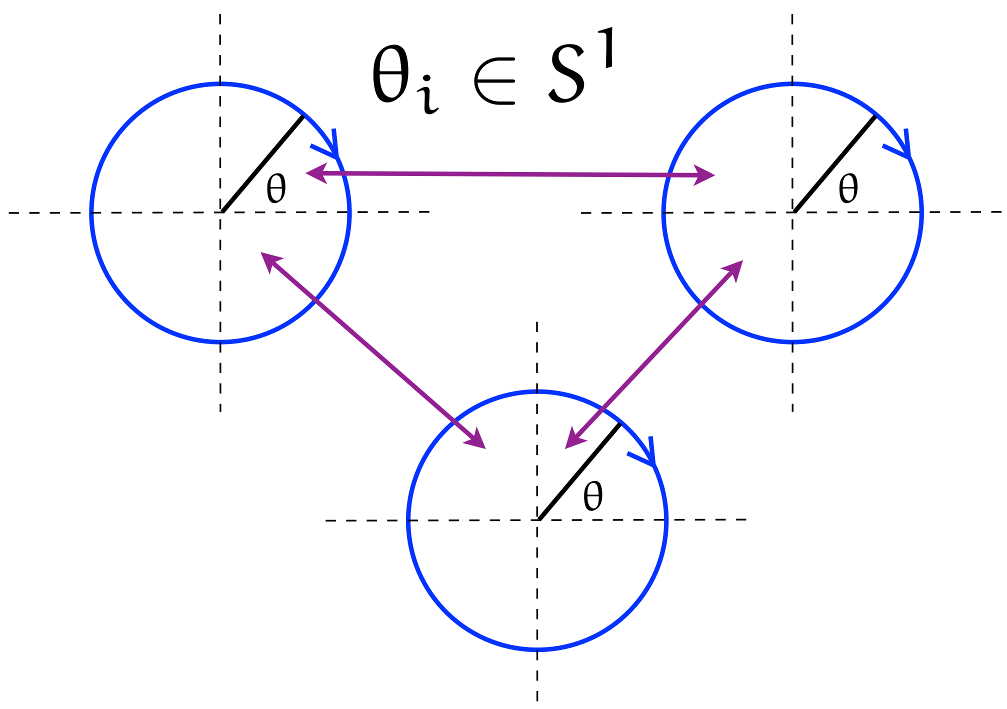


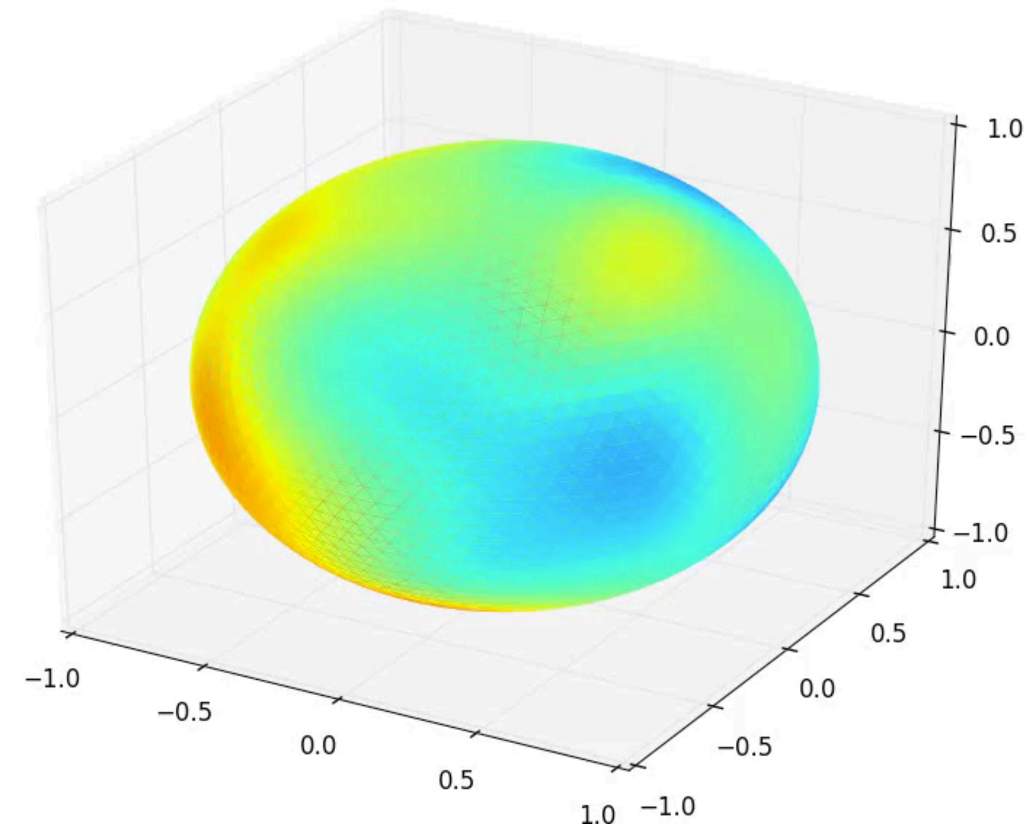
Stephen Coombes

Centre for Mathematical Medicine and Biology

The University of Nottingham

Model reduction in mathematical neuroscience





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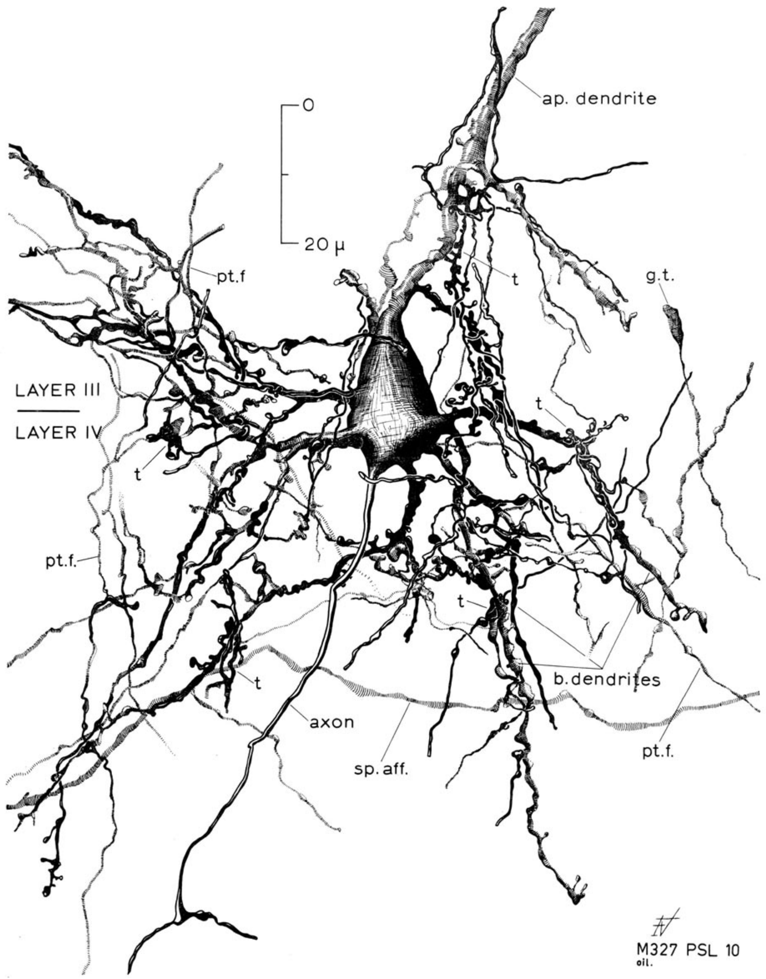
ap. dendrite
0
20 μ
pt.f
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LAYER III
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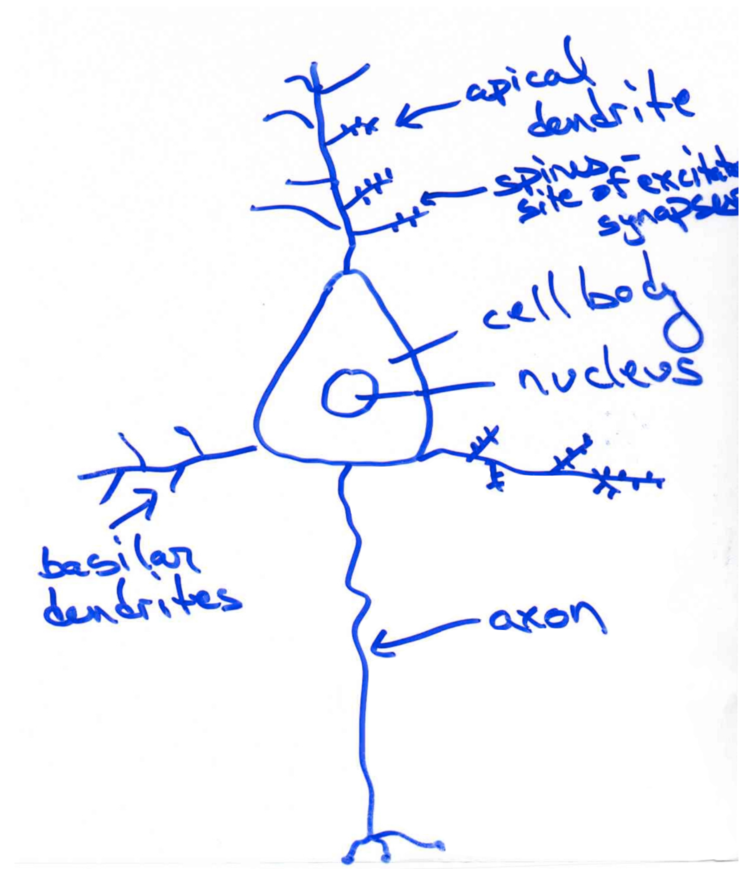
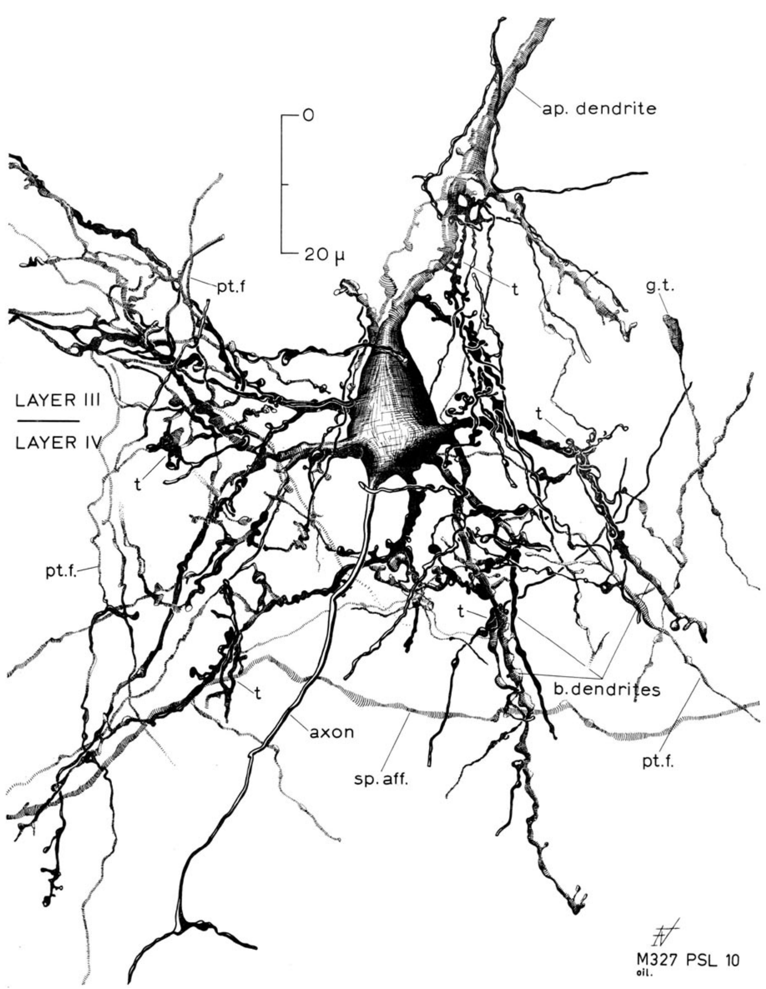
axon
sp.aff.
pt.f.
M327 PSL 10


apical dendrite
spines-
cell body
nucleus
basilar dendrites
axon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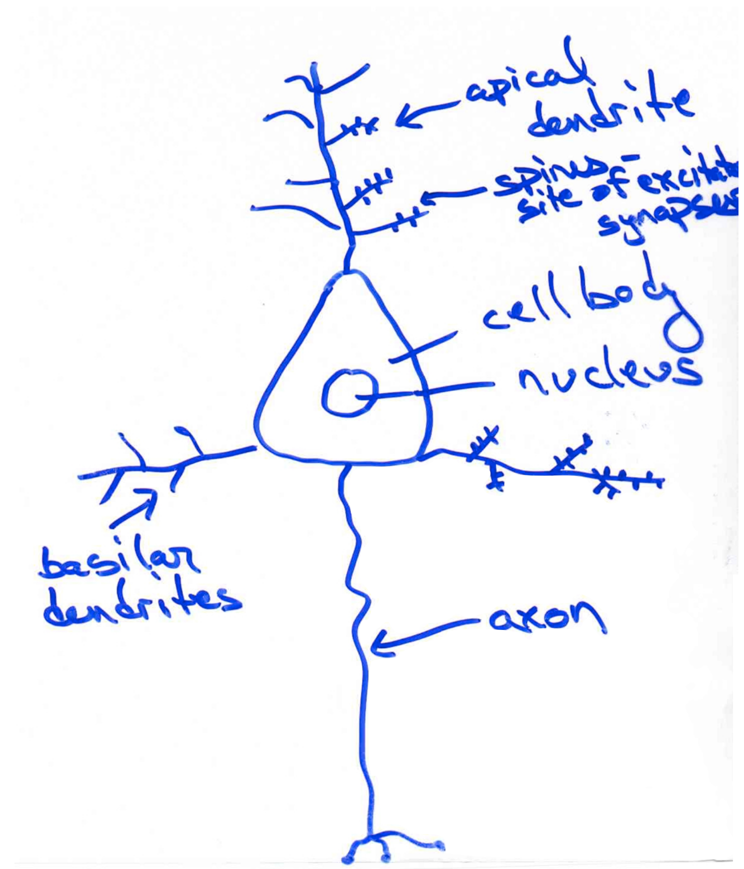
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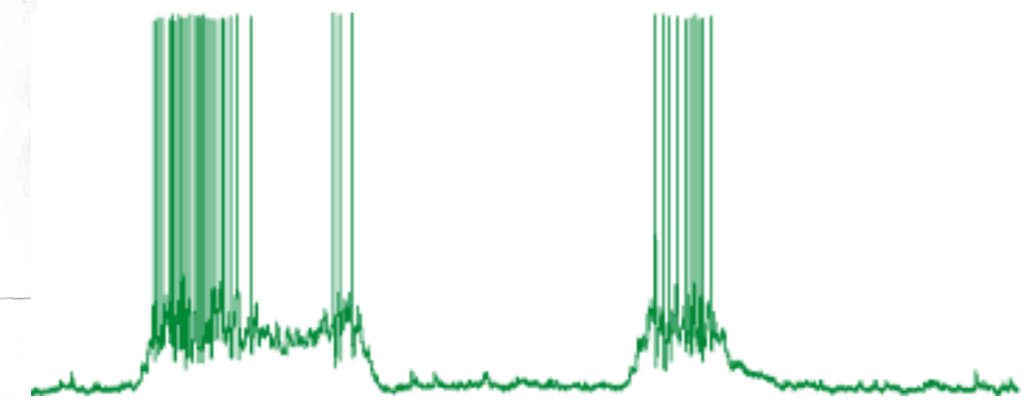

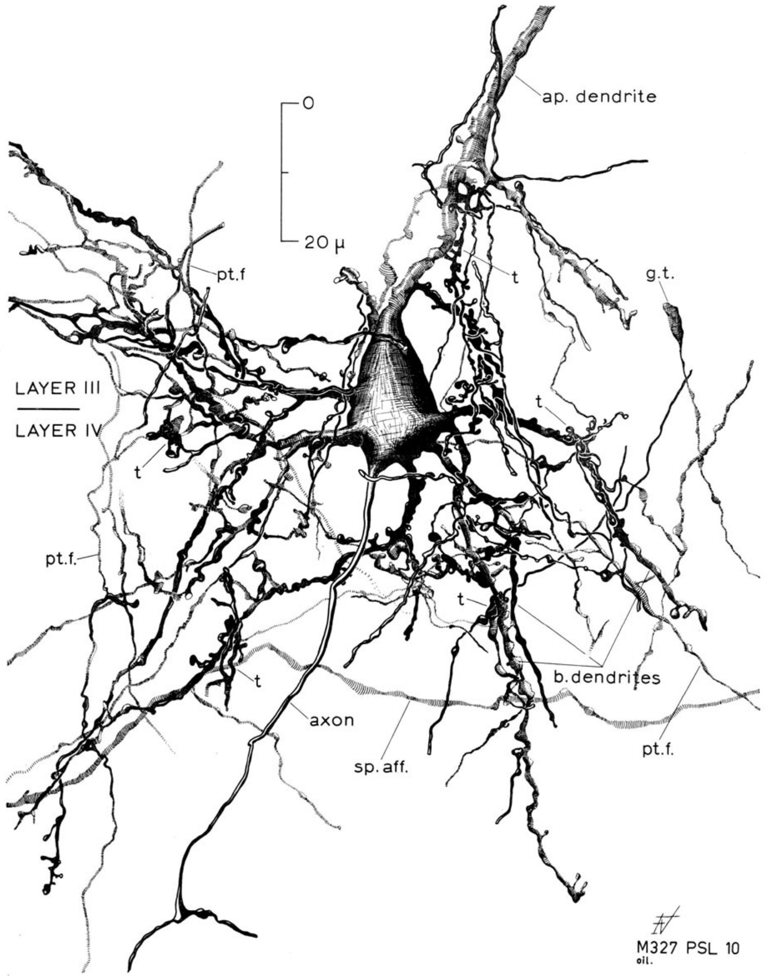
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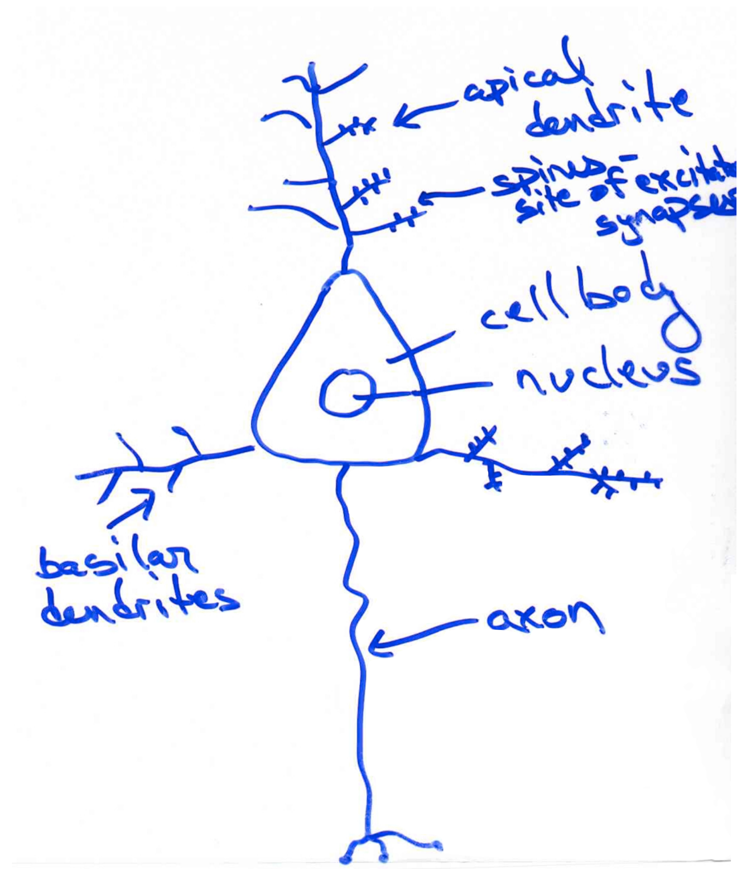
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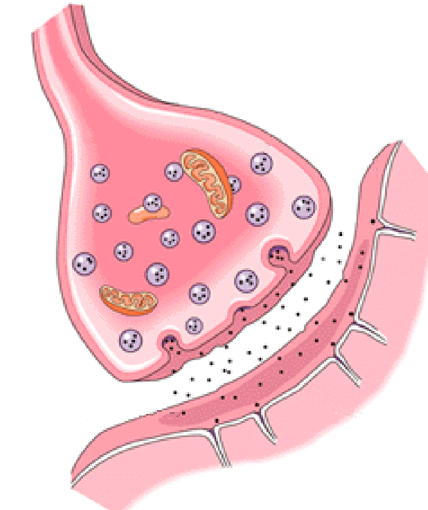

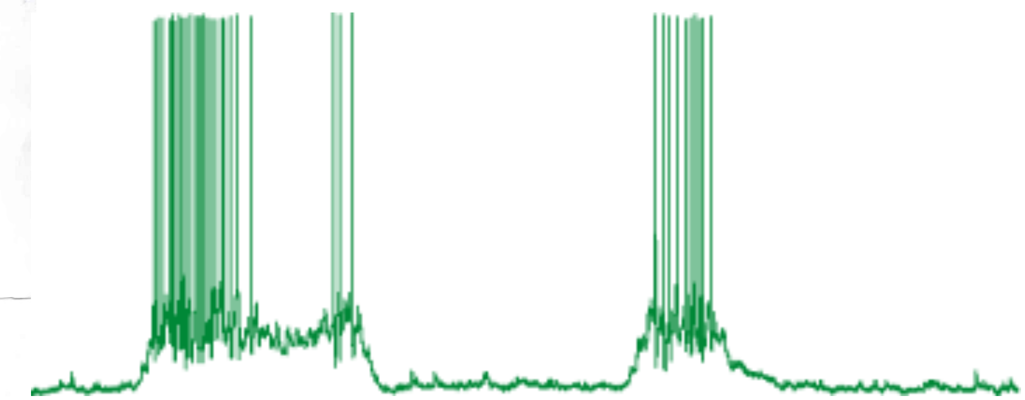

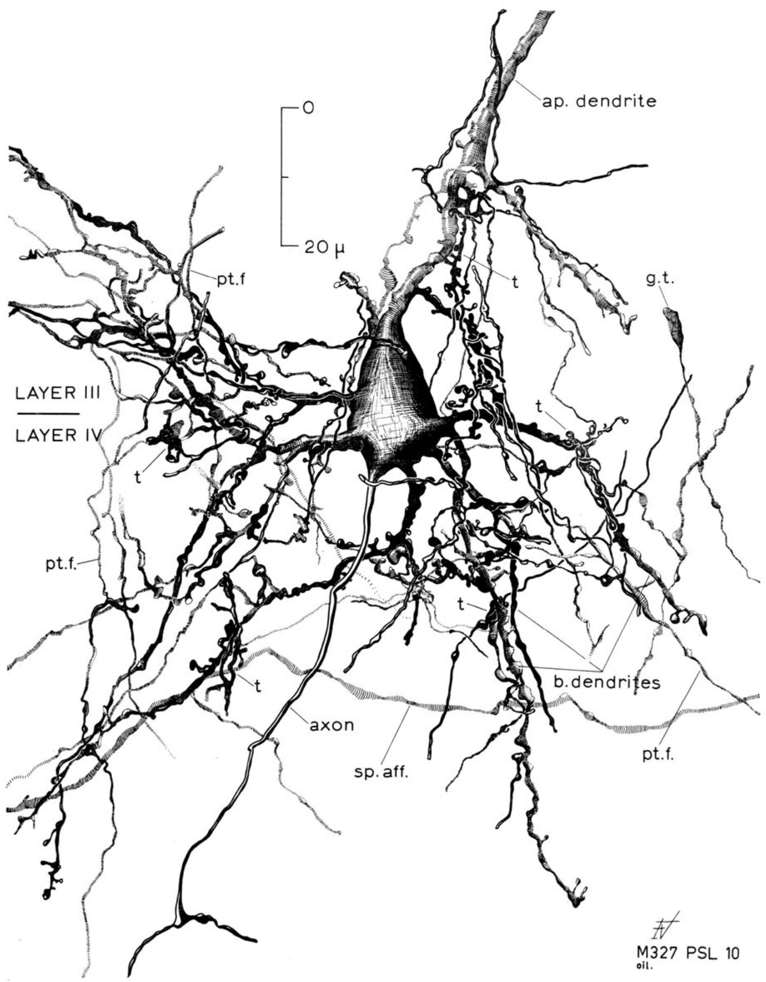
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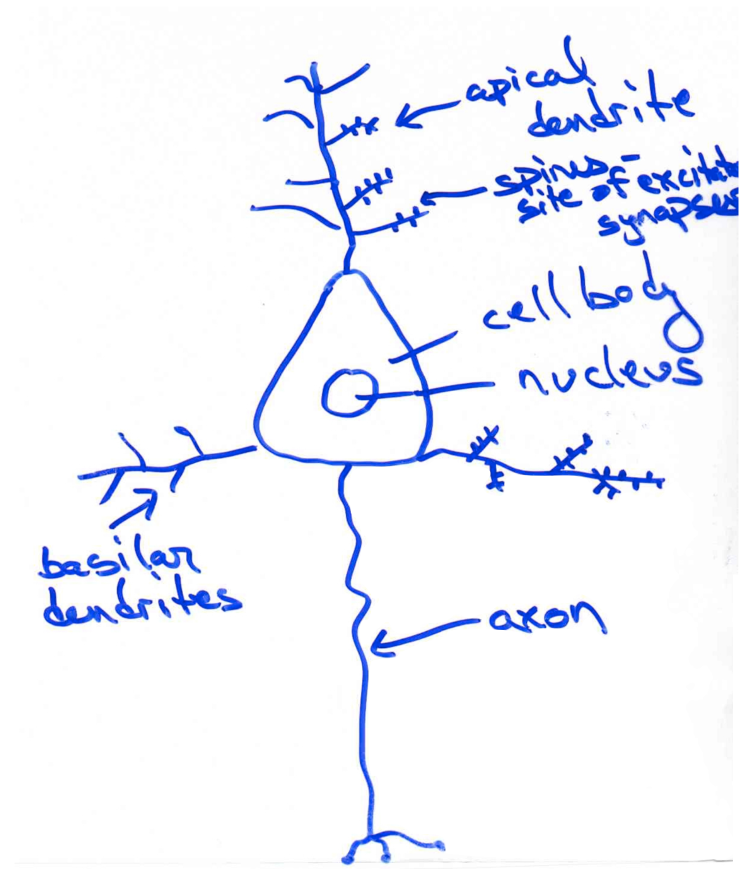
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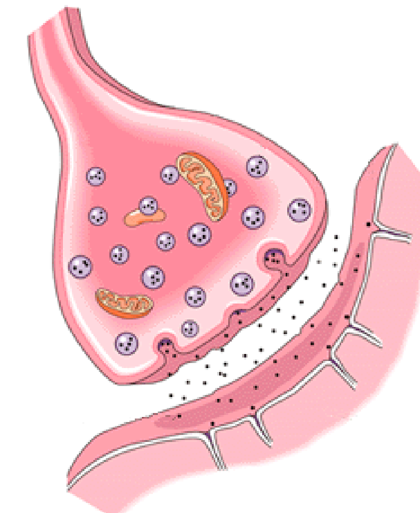
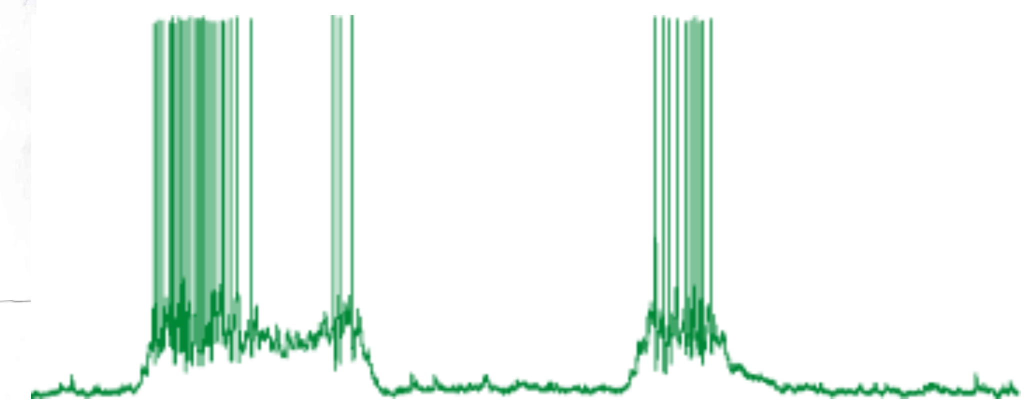
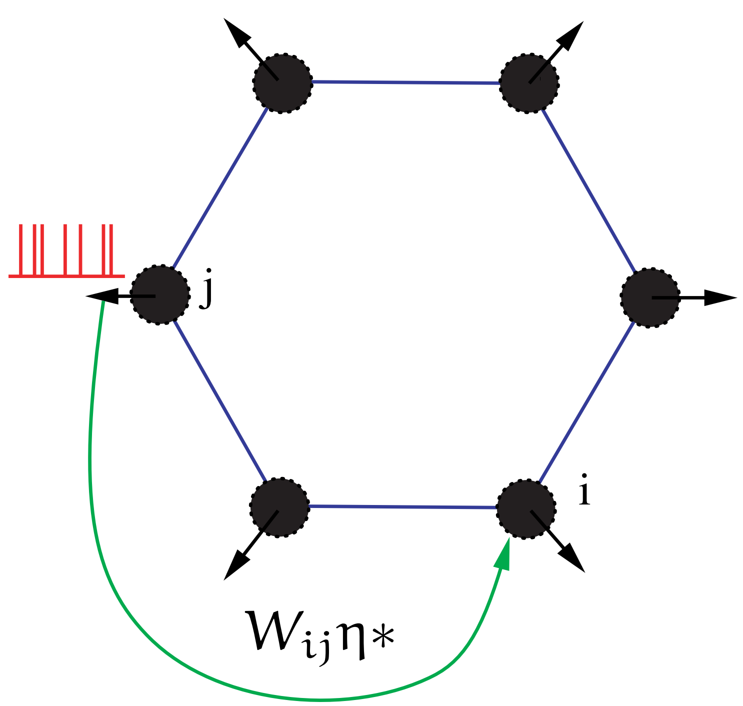
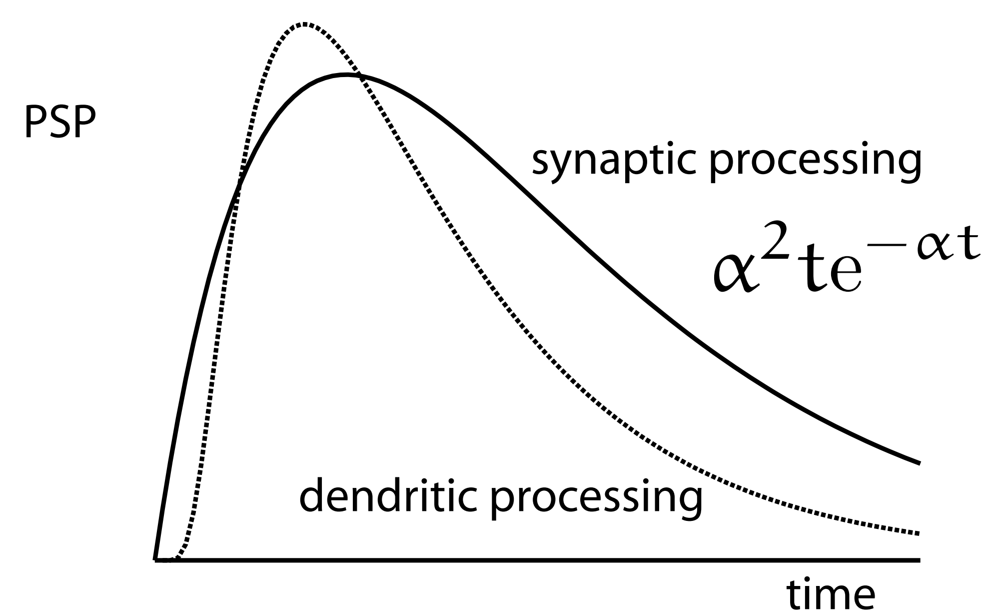
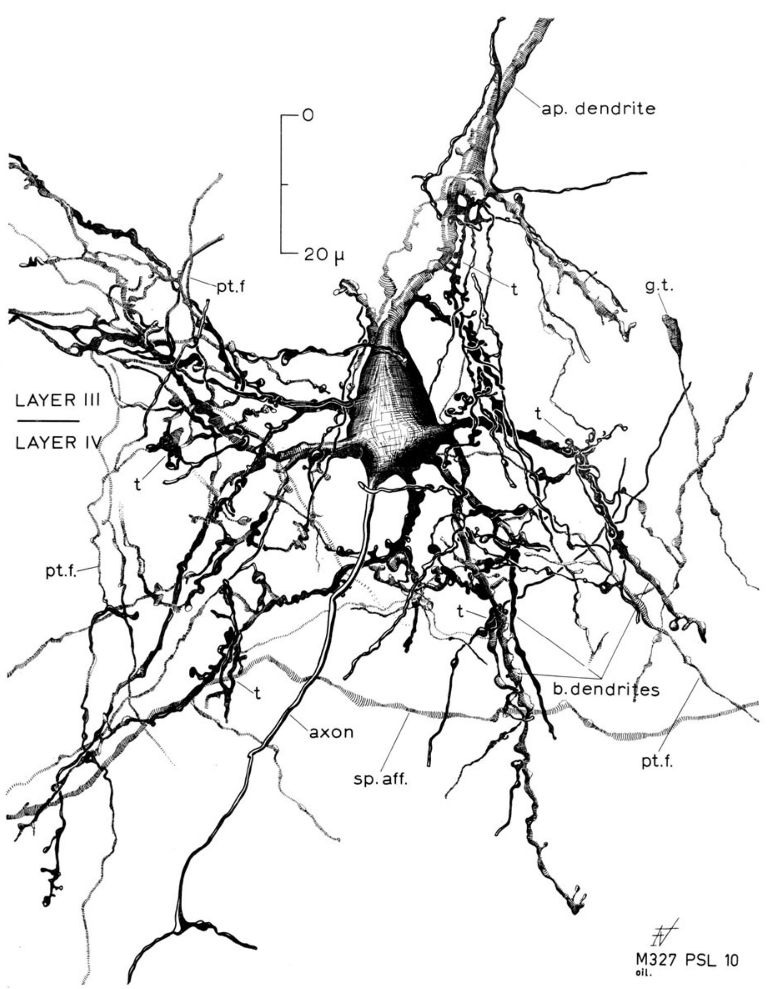
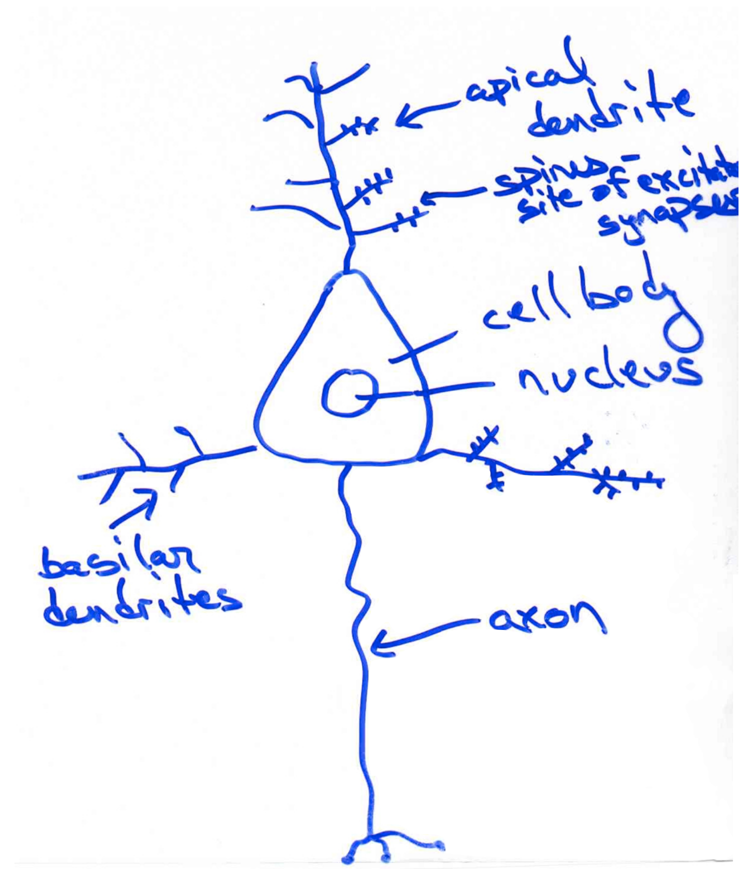


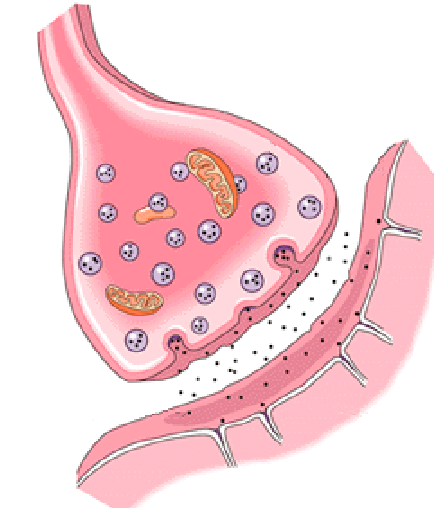


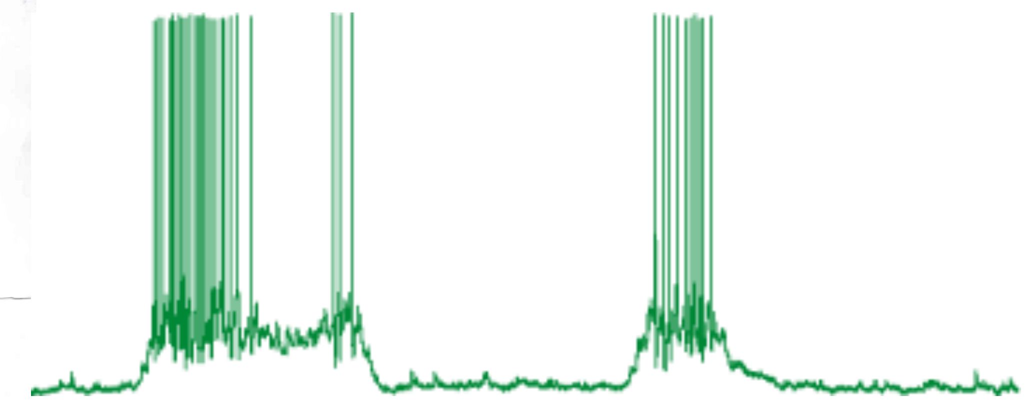

j
i
$W_{ij}\eta*$


PSP
synaptic processing
$\alpha^2 te^{-\alpha t}$
dendritic processing
time

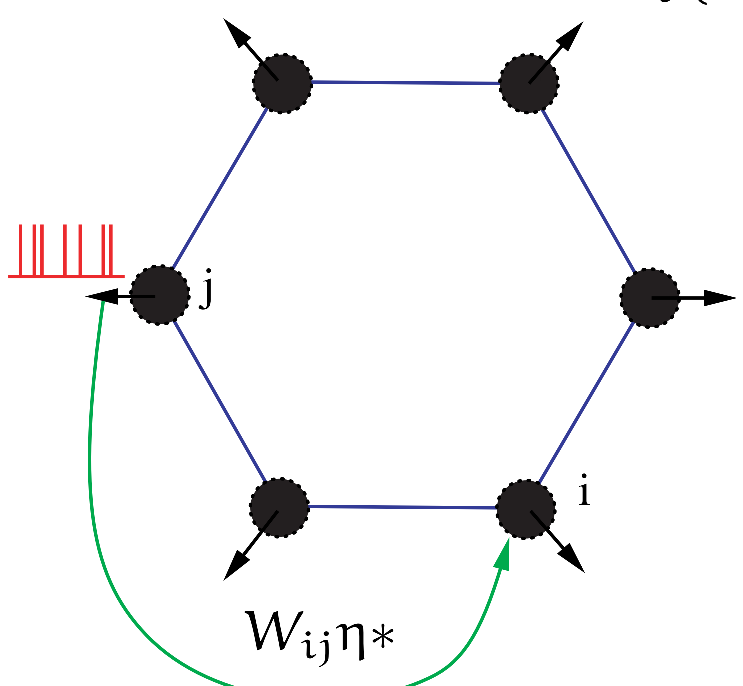


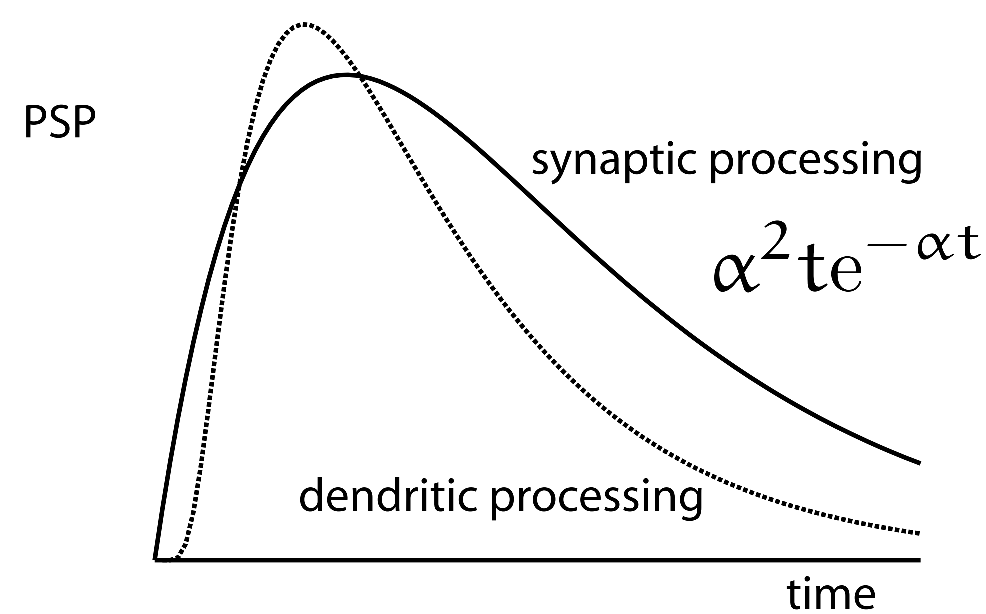


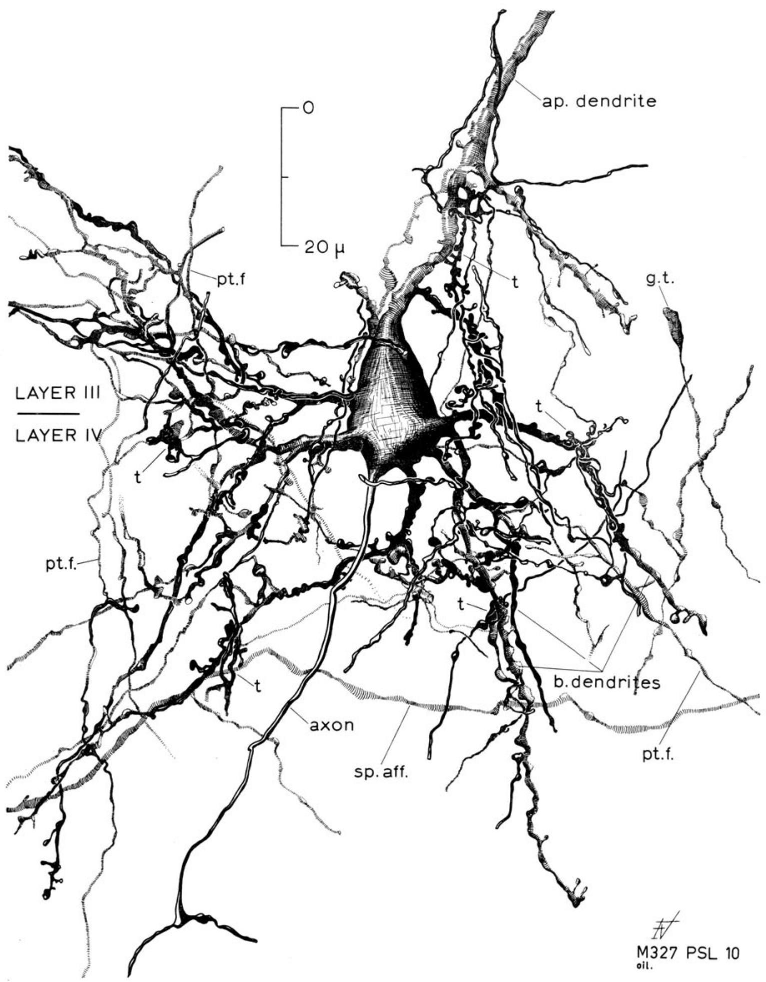


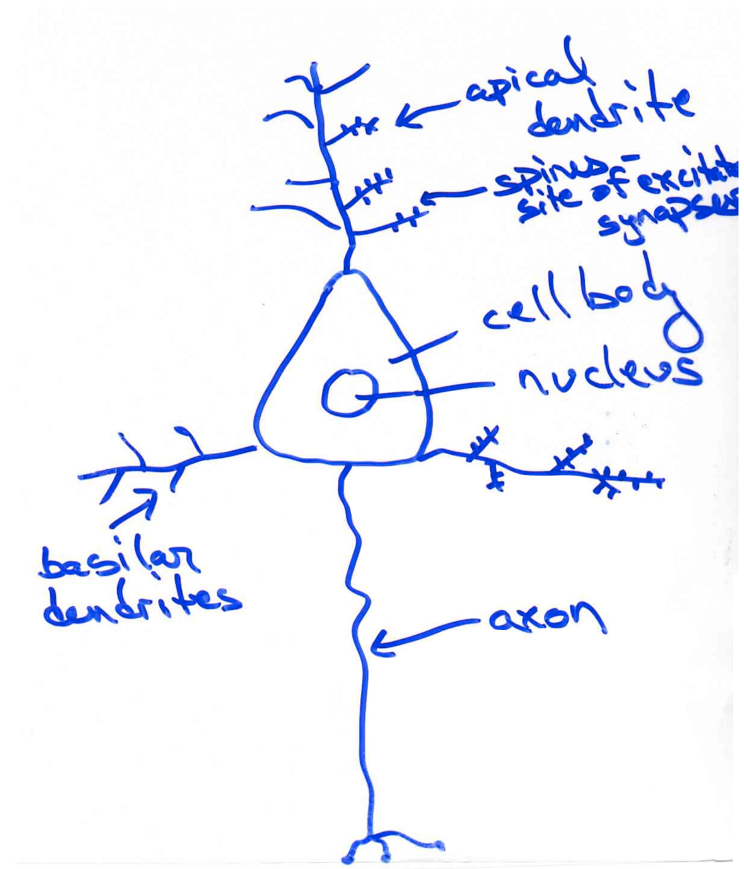


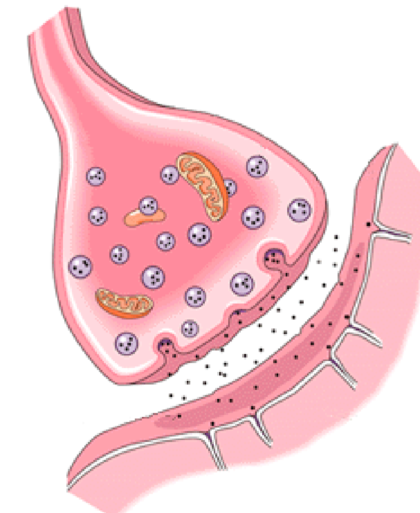
$$s_i(t) = \sum_{j=1}^{N} W_{ij} \sum_{m \in \mathbb{Z}} \eta(t - T_j^m)$$

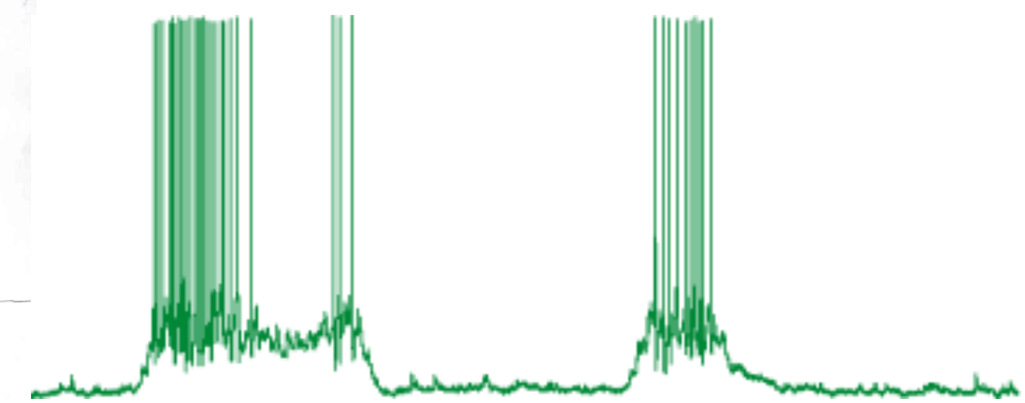












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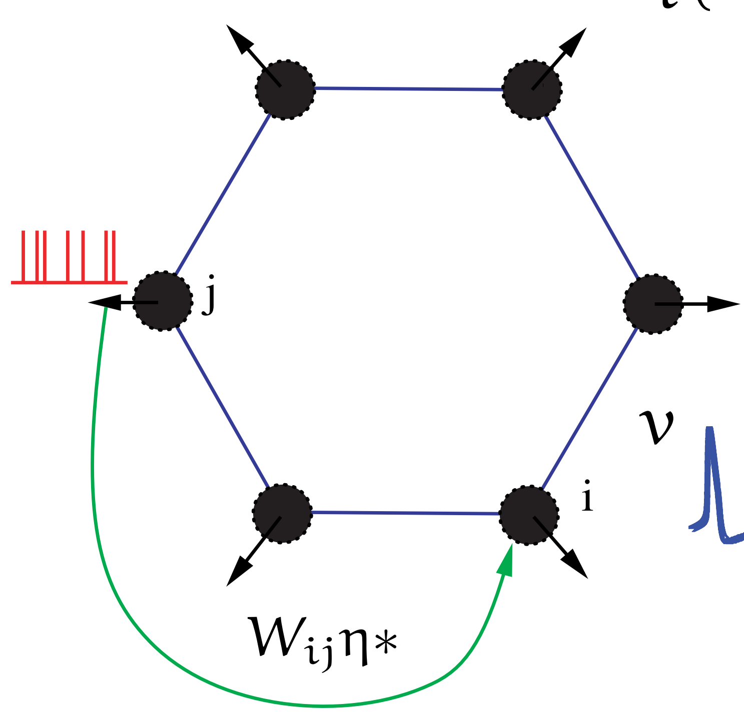
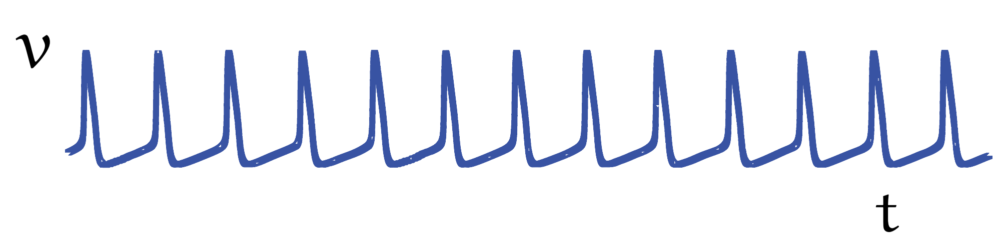
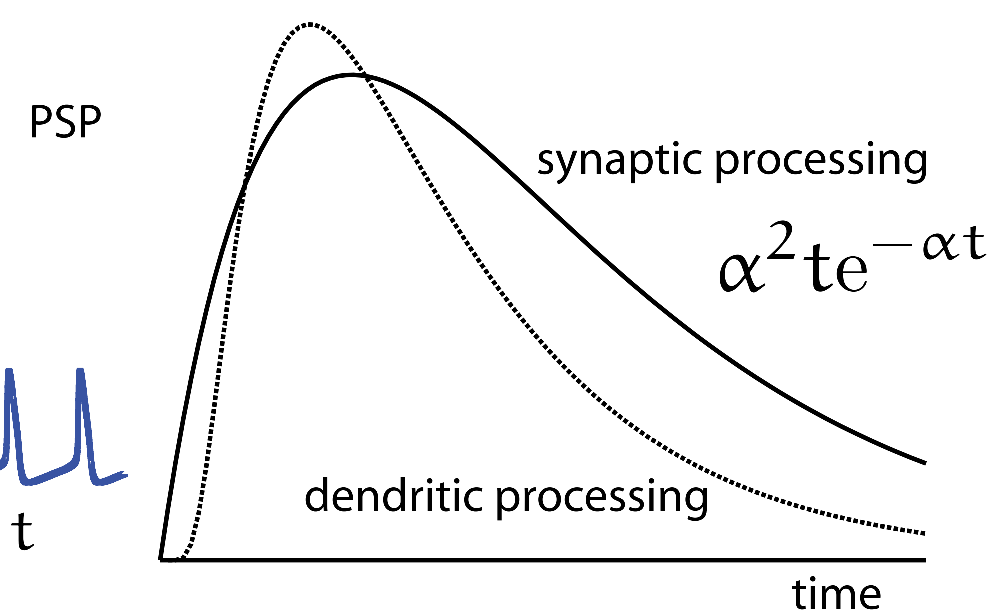


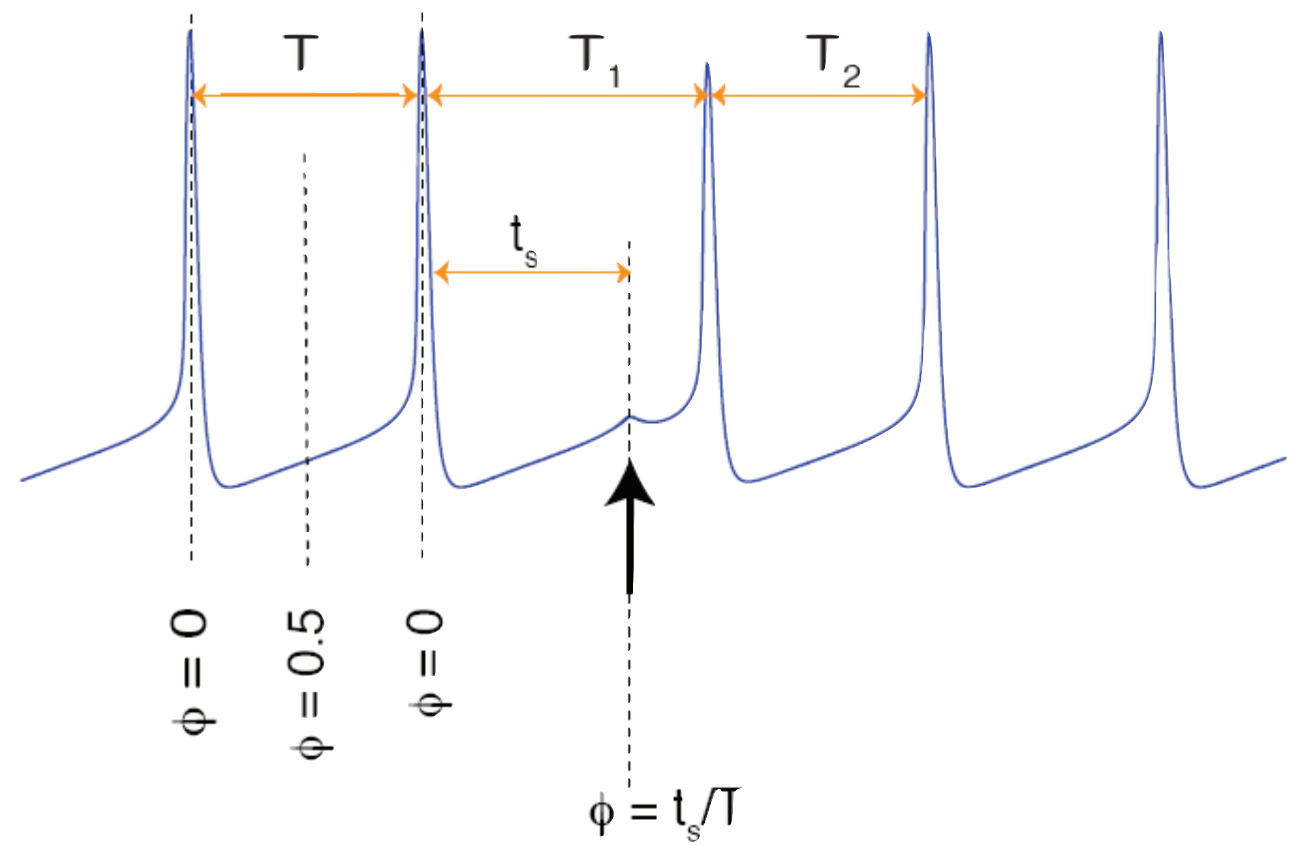
HH or other





Phase **R**esponse **C**urves

A PRC tabulates the transient change in the cycle period of an oscillator induced by a perturbation as a function of the phase at which it is received.



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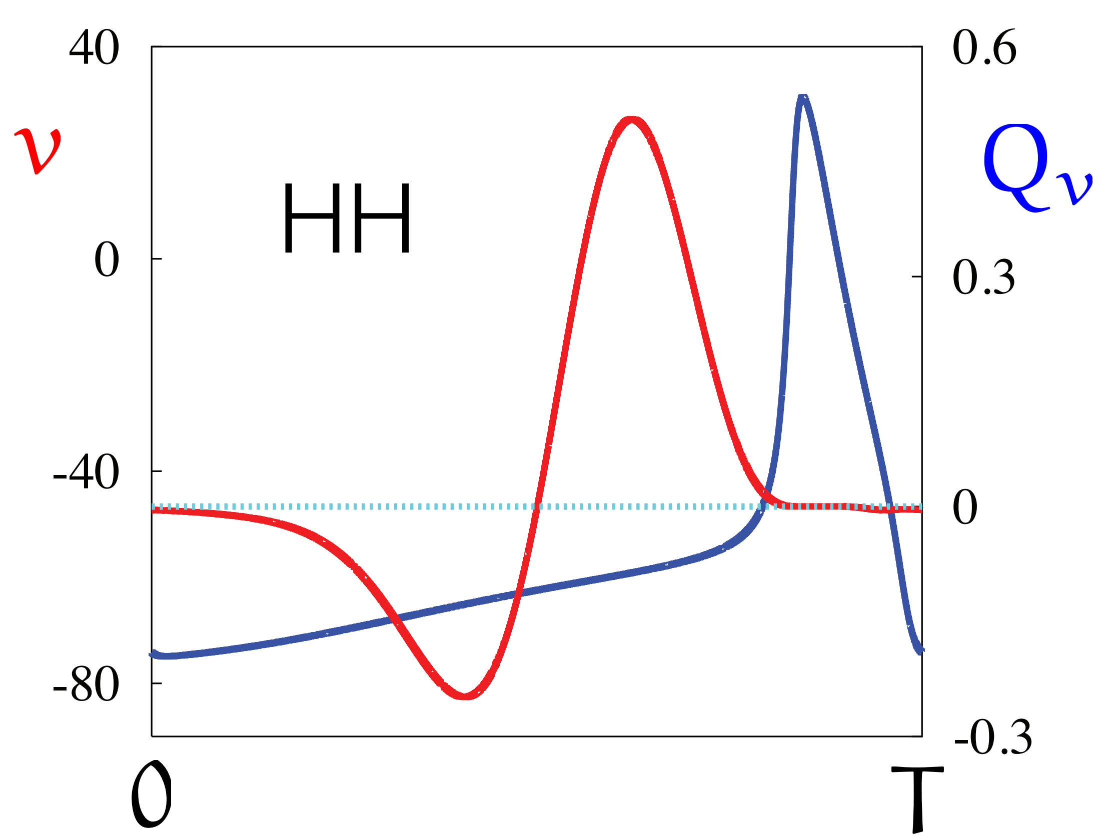
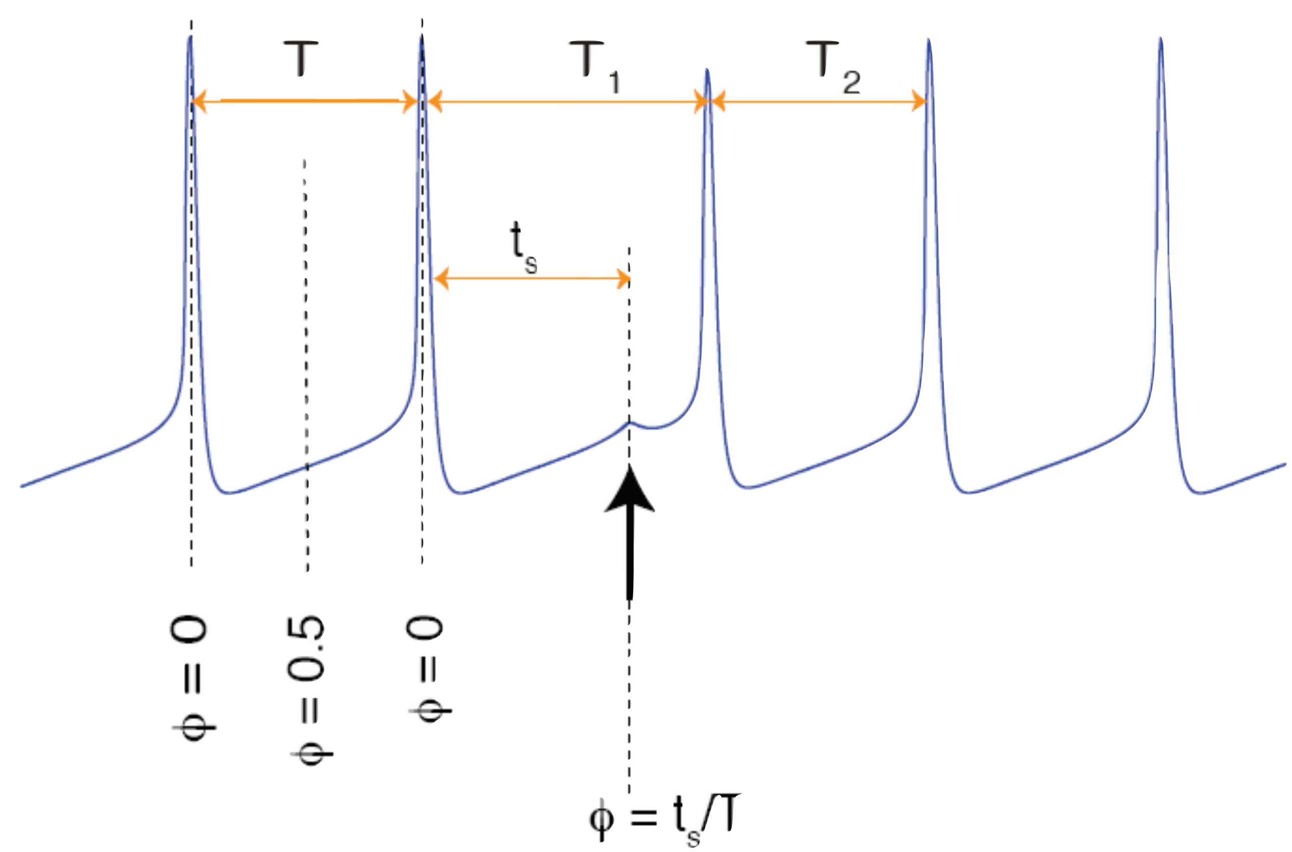
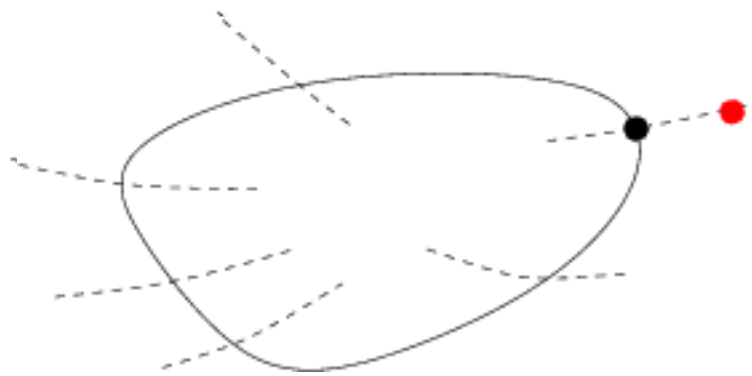


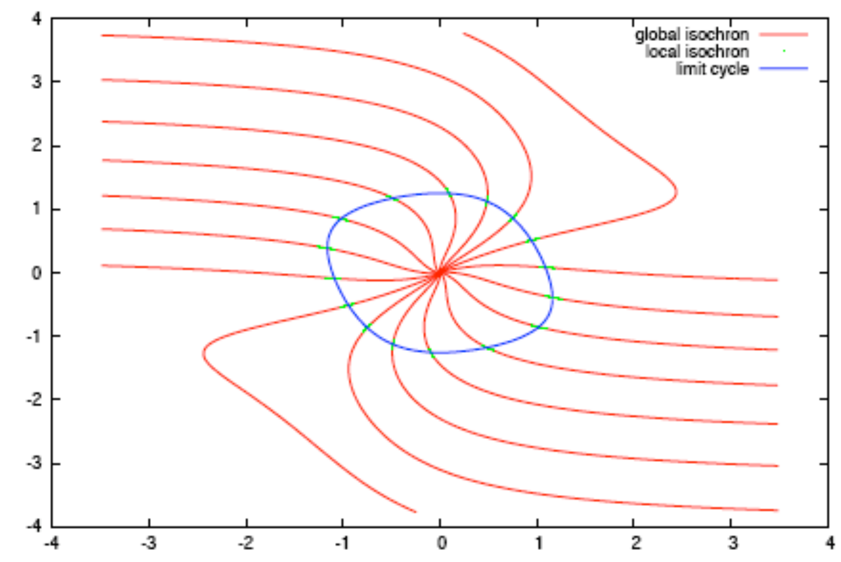


obtained numerically or experimentally

$$Q = \nabla_Z \theta$$



Isochrons as leaves of the stable manifold of a hyperbolic limit cycle

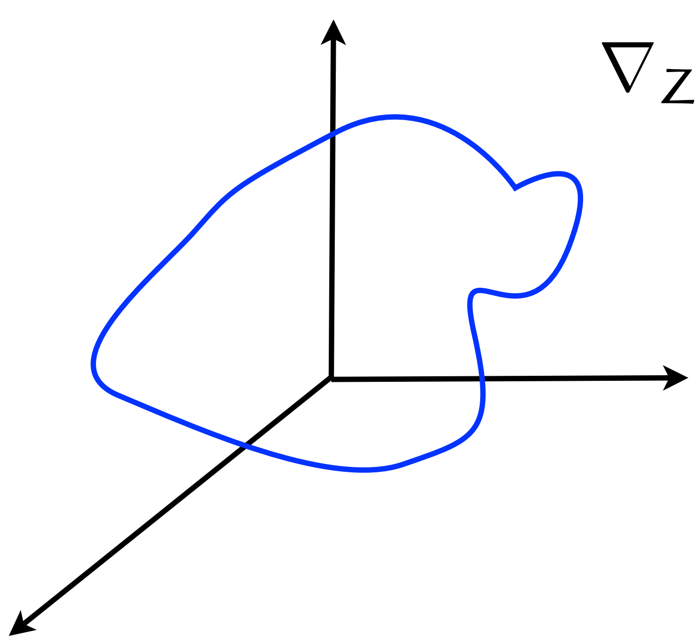


Call the orbit $z = Z(t)$ where $\dot{z} = F(z)$

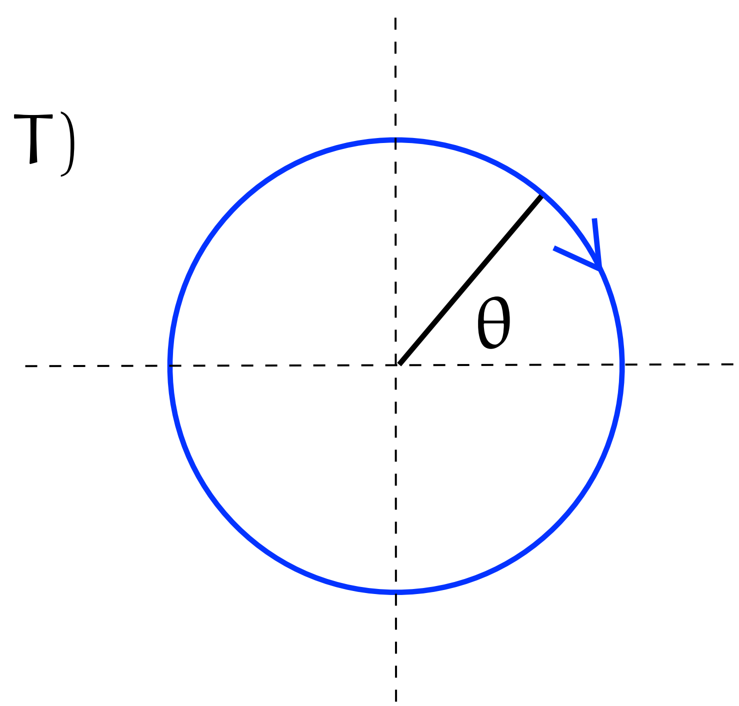
Introduce a phase (isochronal coordinates) θ

$$\frac{dQ}{dt} = D(t)Q, \qquad D(t) = -DF^T(Z(t))$$

$$\nabla_{Z(0)} \cdot F(Z(0)) = \frac{1}{T} \text{ and } Q(t) = Q(t+T)$$



$$\dot{\theta} = \frac{1}{T}$$



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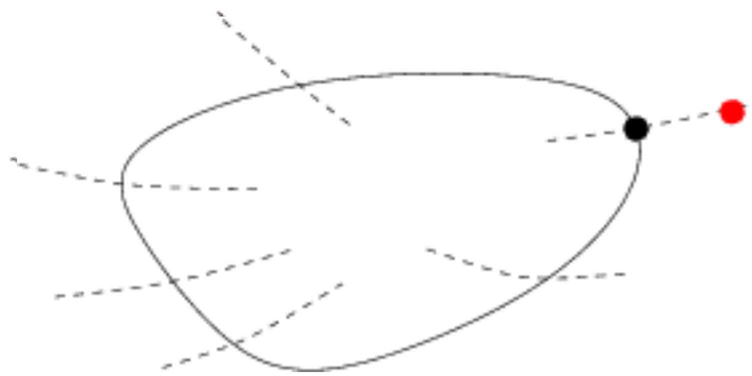

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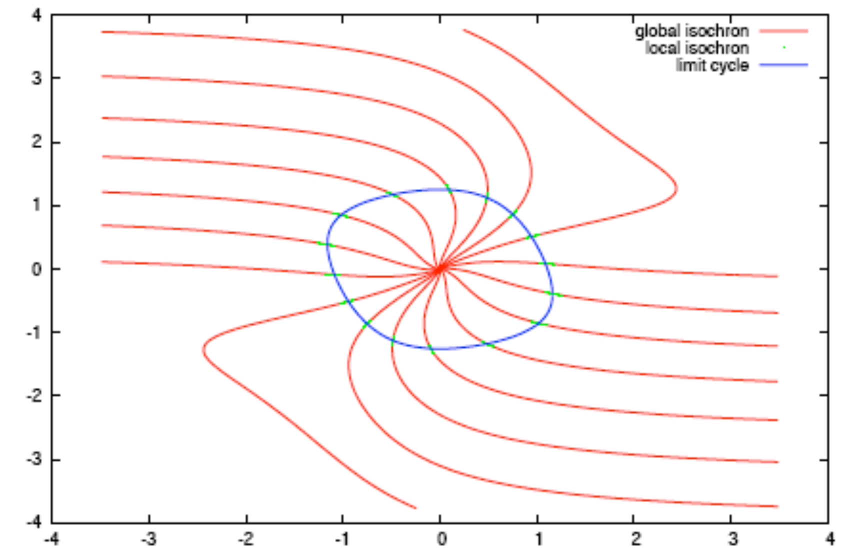

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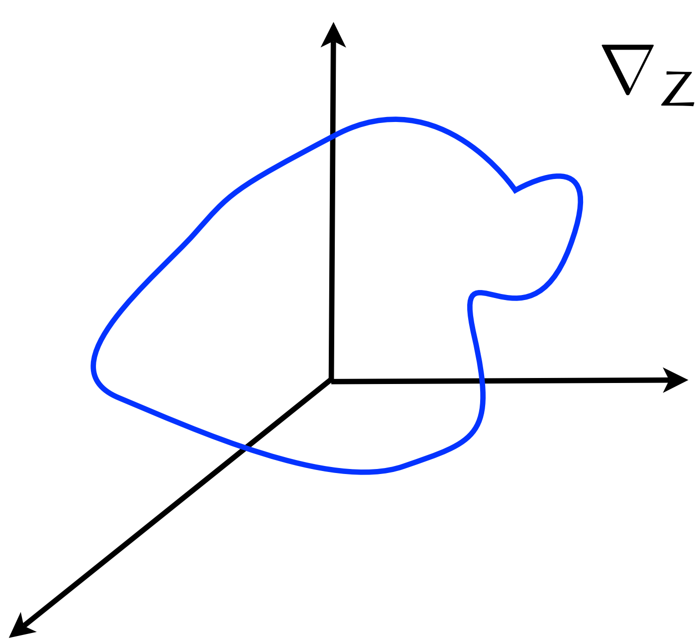

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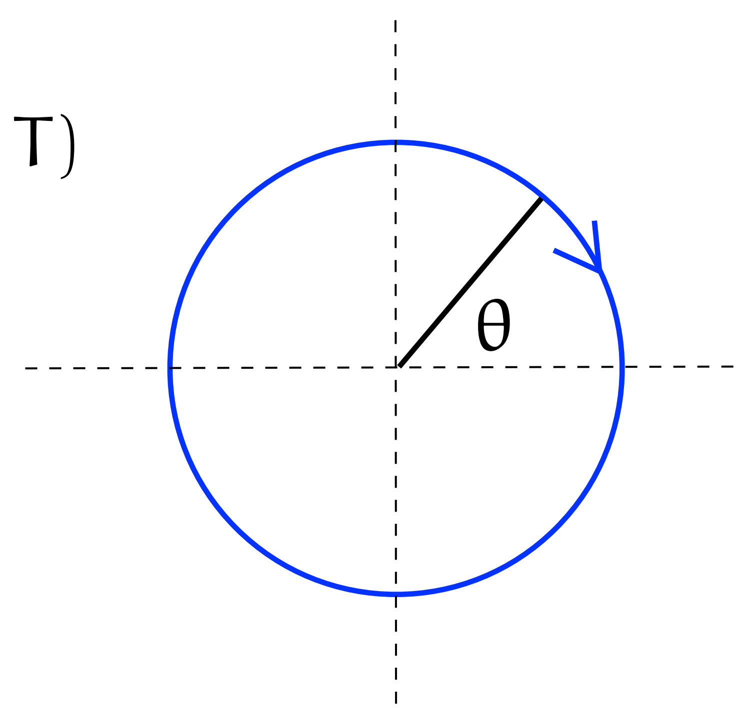
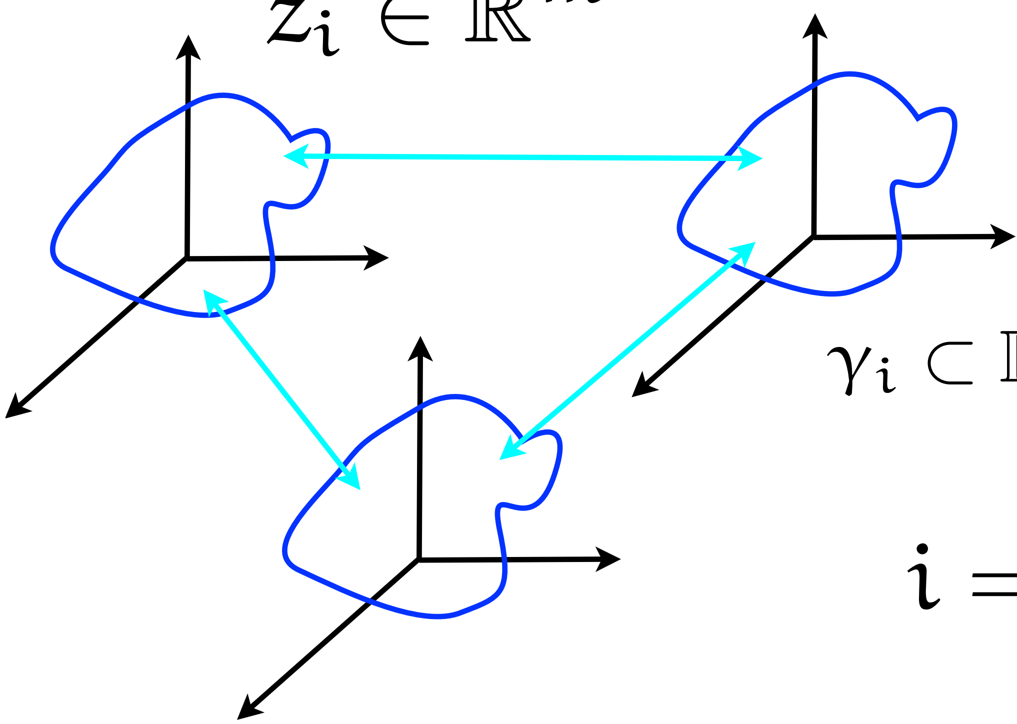


Weak Coupling

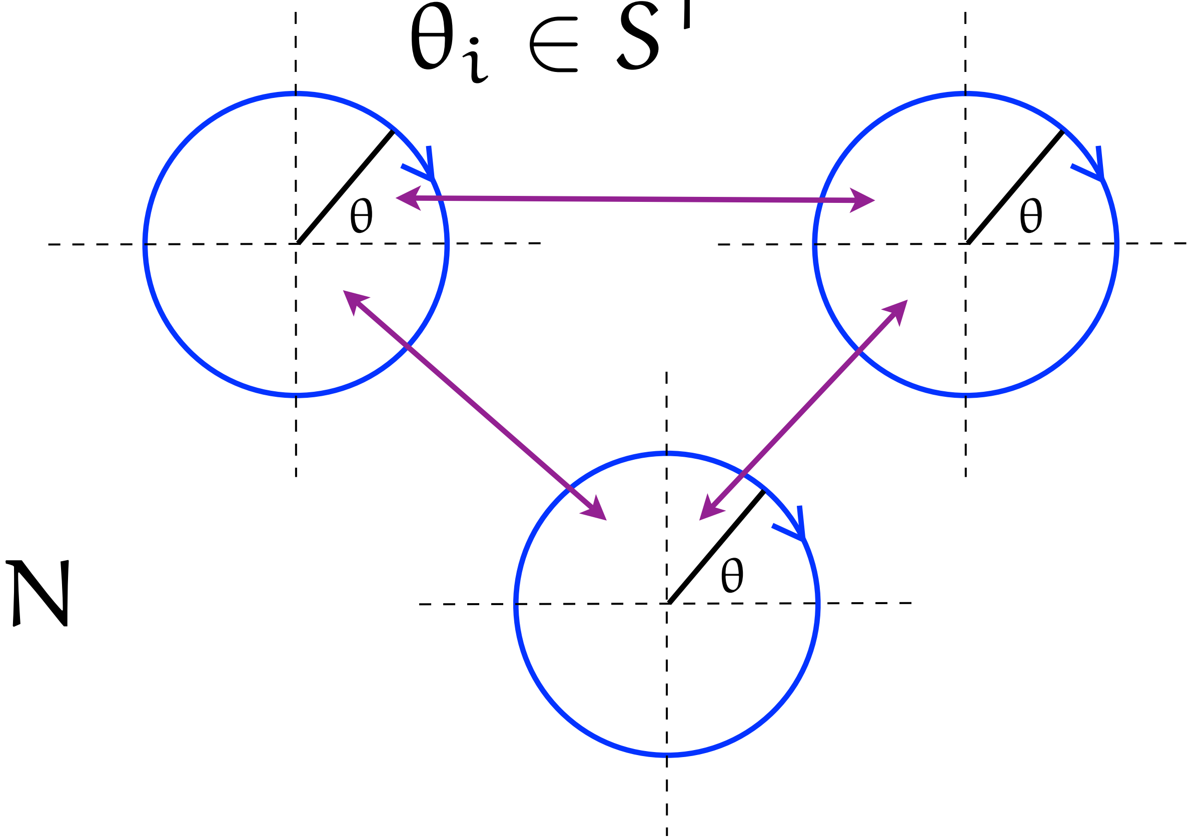
$z_i \in \mathbb{R}^m$



$\gamma_i \subset \mathbb{R}^m$

$i = 1, \ldots, N$

$\theta_i \in S^1$



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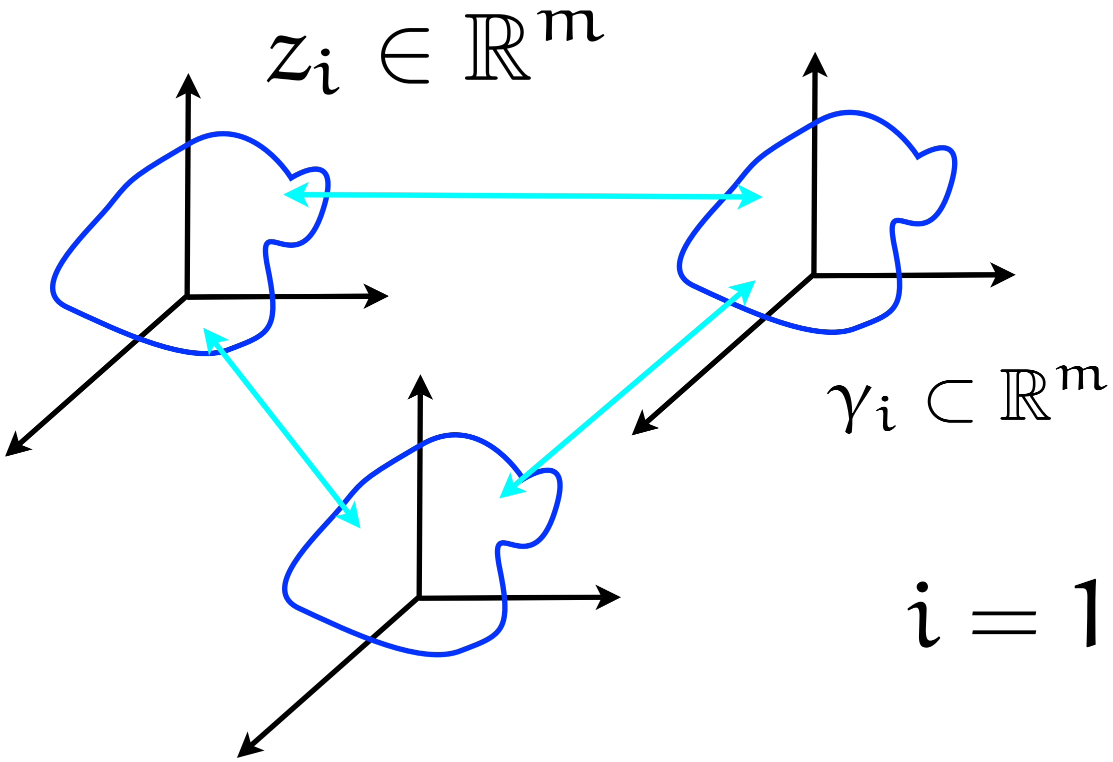

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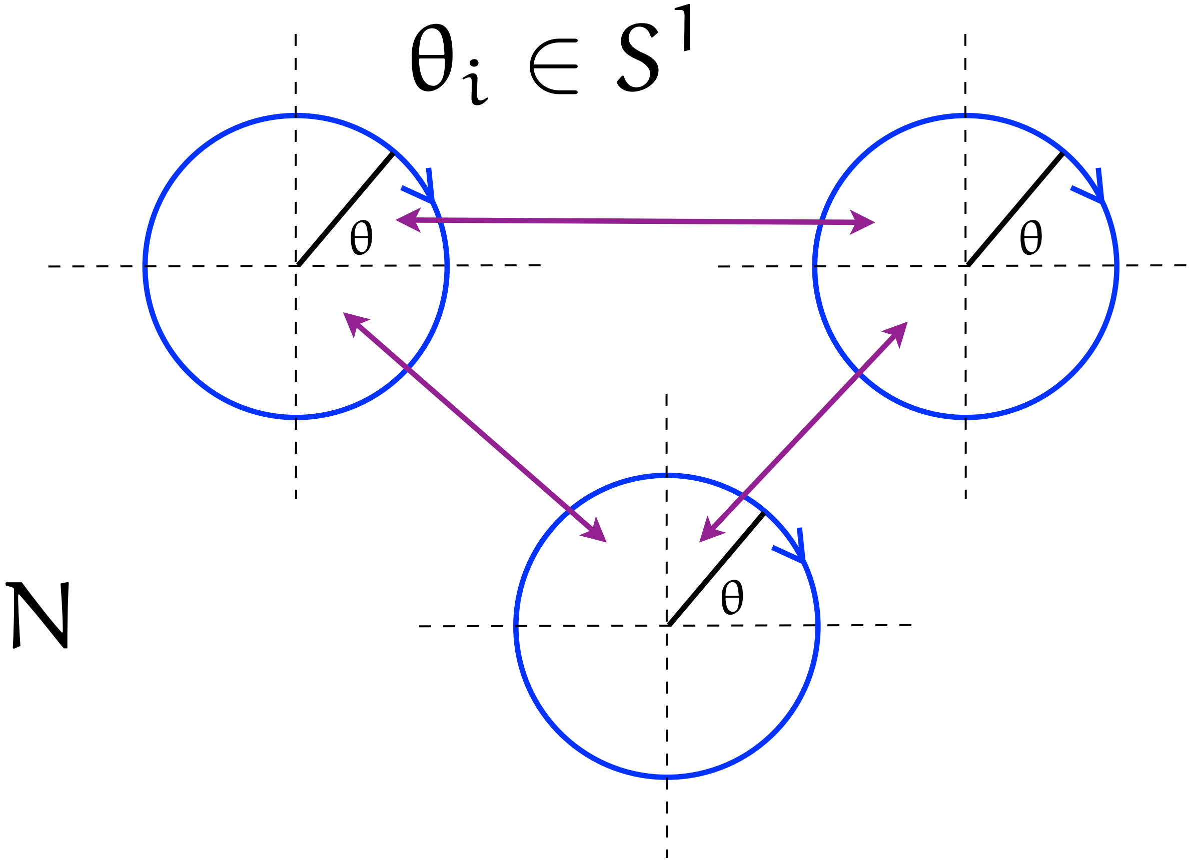
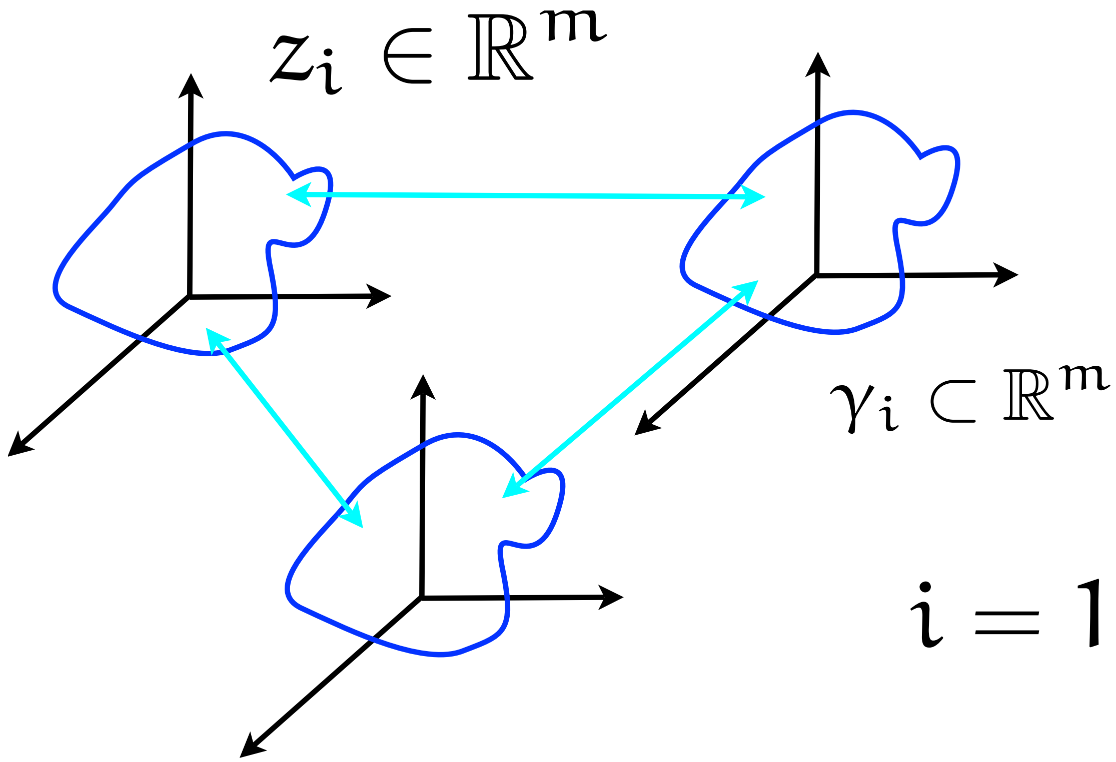


$$\dot{z}_i = F(z_i) + \epsilon G_i(z_1, \ldots, z_N)$$

Uncoupled system has an exponentially stable limit cycle γ_i

Weak Coupling



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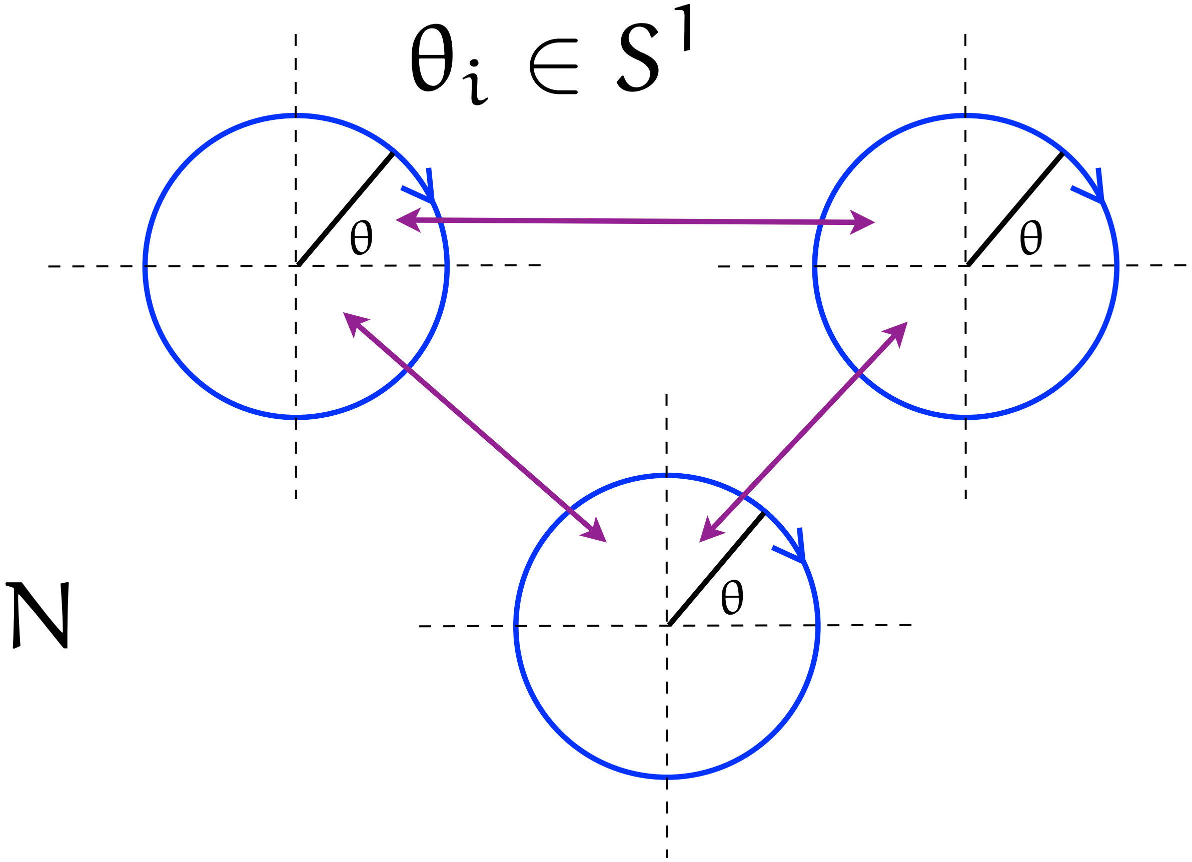

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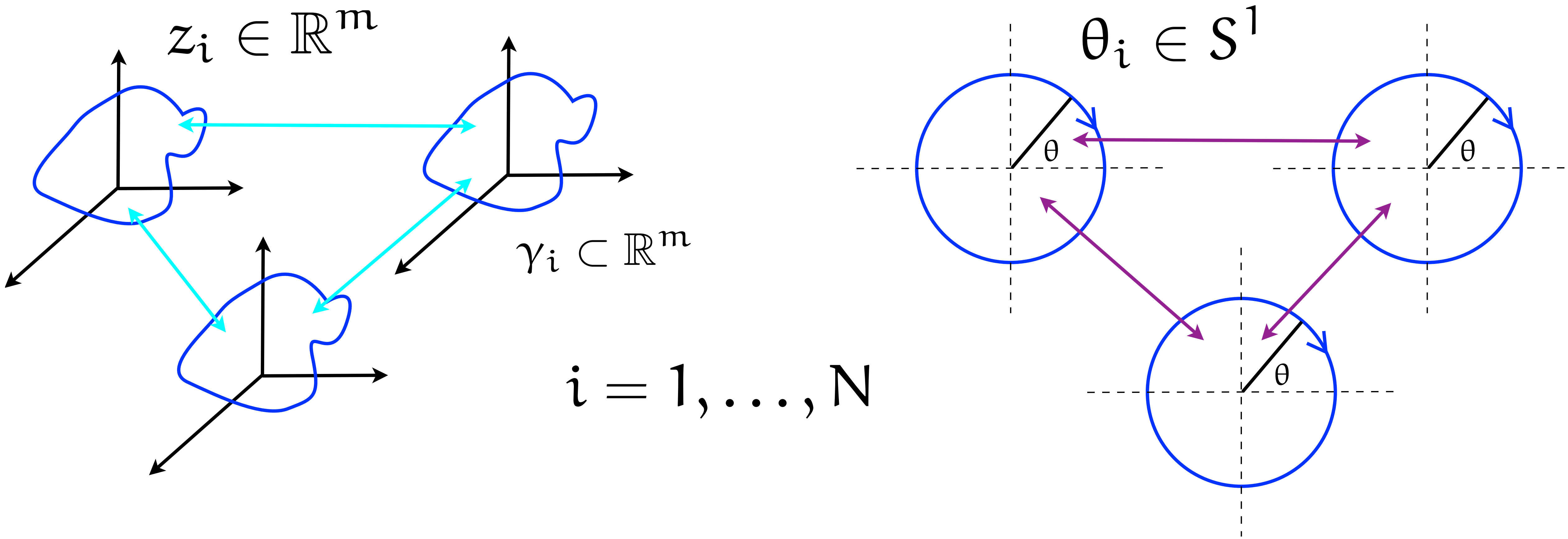
Uncoupled system has an exponentially stable limit cycle γ_i

Direct product of hyperbolic limit cycles is a normally hyperbolic invariant manifold

$$\dot{\theta}_i = \frac{1}{T} + \epsilon \left\langle Q(\theta_i), G_i(\Gamma(\theta)) \right\rangle$$

Weak Coupling



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Drive

PRC

Phase oscillator network

(after averaging)

$$\dot{\theta}_i = \frac{1}{T} + \epsilon \sum_j w_{ij} H(\theta_i - \theta_j)$$

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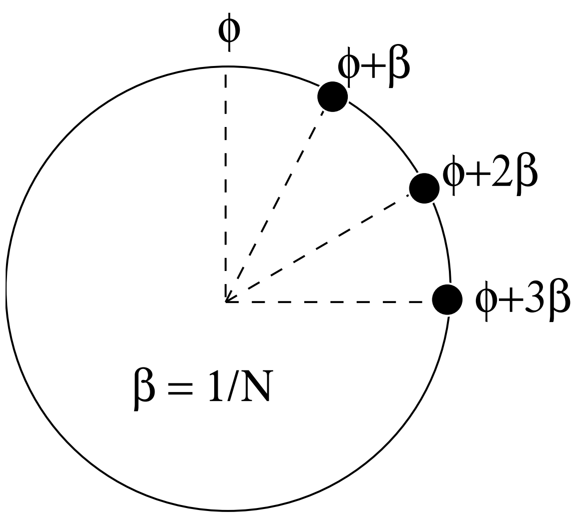
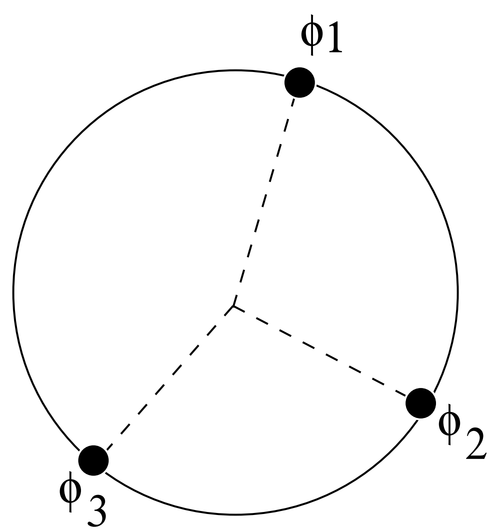


Dim(Fix(Γ)) = 1



Dim(Fix(Γ)) = 3

Bifurcations from maximally symmetric solutions
to ones with smaller isotropy groups.
eg. cluster states.

Phase oscillator network

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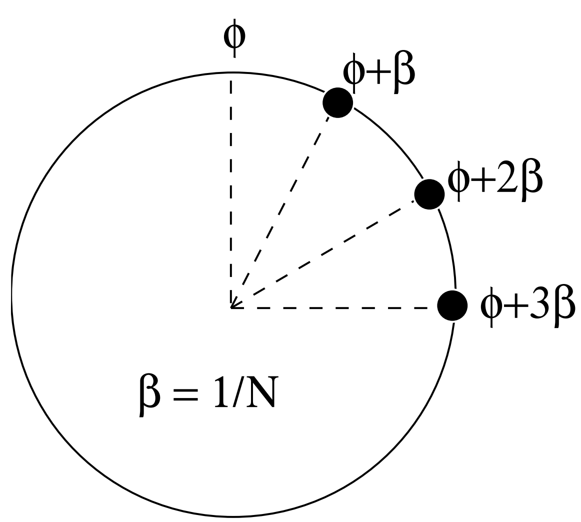

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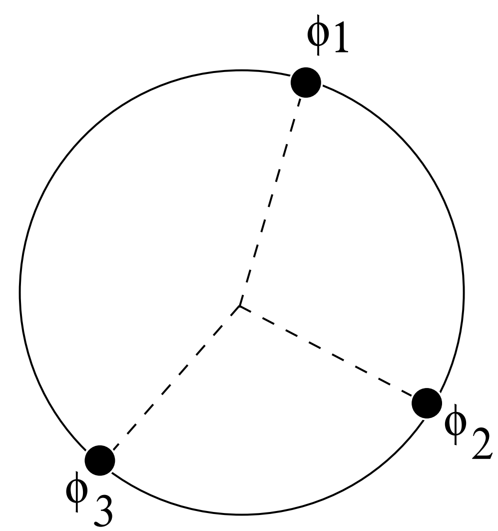


Dim(Fix(Γ)) = 3

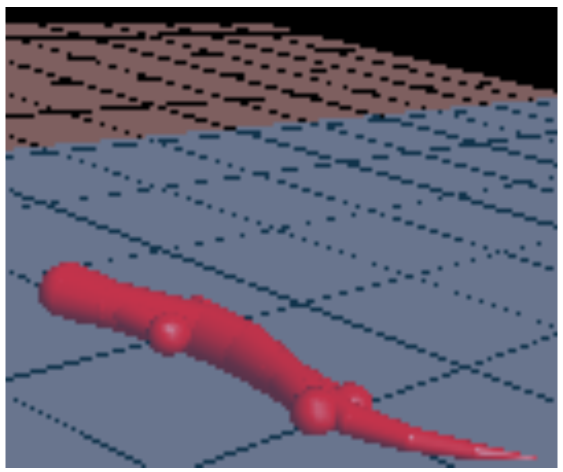
Bifurcations from maximally symmetric solutions
to ones with smaller isotropy groups.
eg. cluster states.

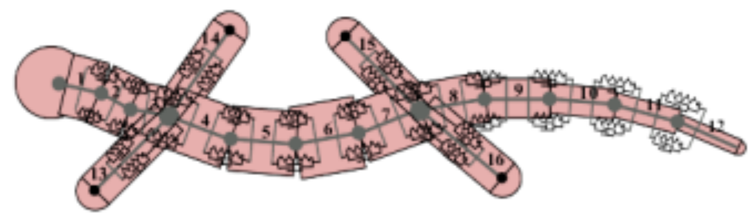
Stability of synchronous solution determined by eigenvalues of

$$\widehat{H}_{ij} = H'(0) \left[w_{ij} - \delta_{ij} \sum_k w_{ik} \right]$$

Oscillator networks

Applications of weakly coupled oscillator theory to CPGs, robot control, ...





Biorobotics lab at EPFL

Oscillator networks

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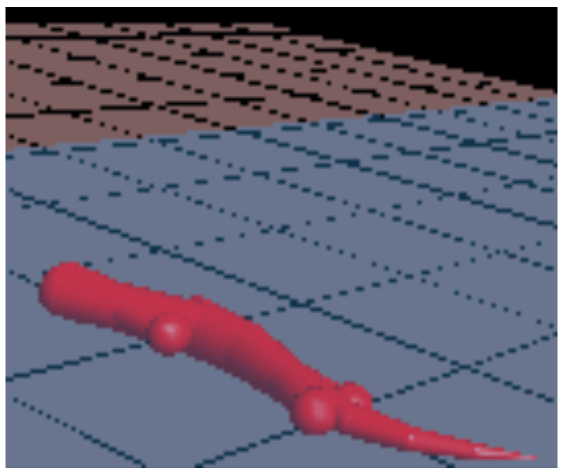

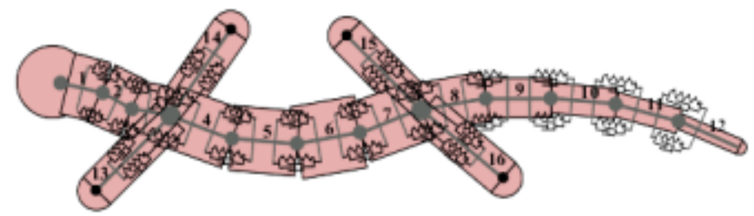

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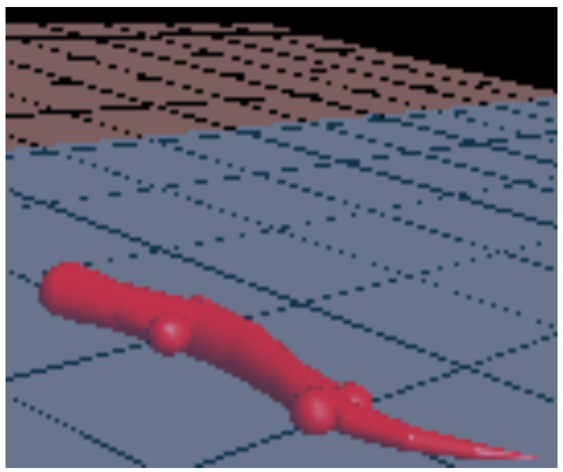

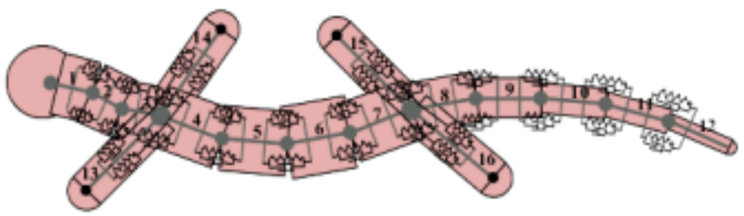

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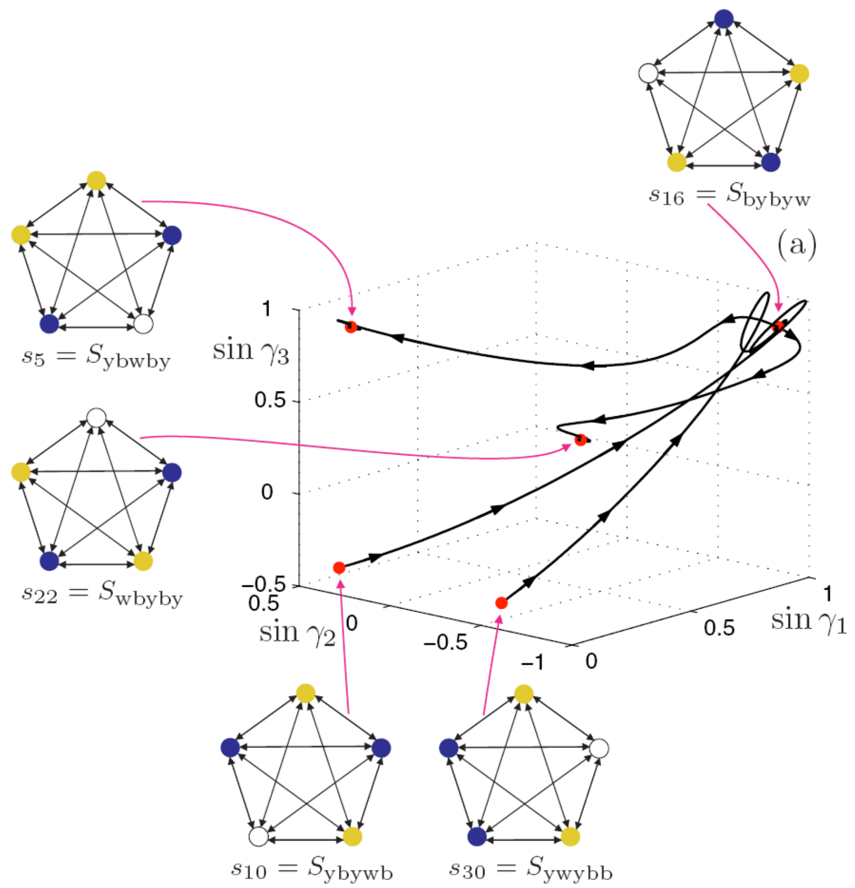

Rabinovich et al. Dynamical principles in neuroscience, Rev. Mod. Phys., 78, 2006.

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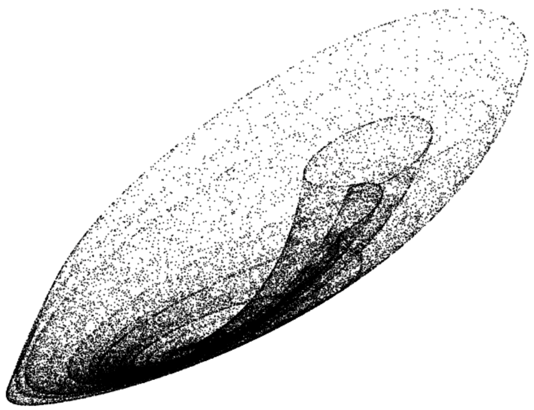
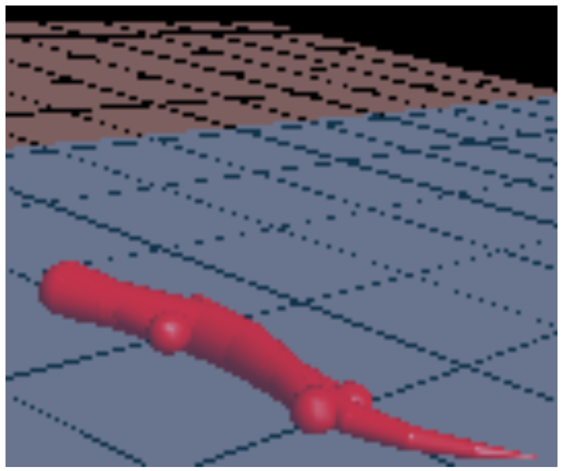
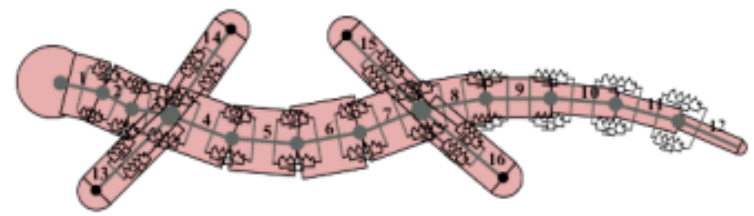


Oscillator networks

Applications of weakly coupled oscillator theory to CPGs, robot control, ...





Biorobotics lab at EPFL

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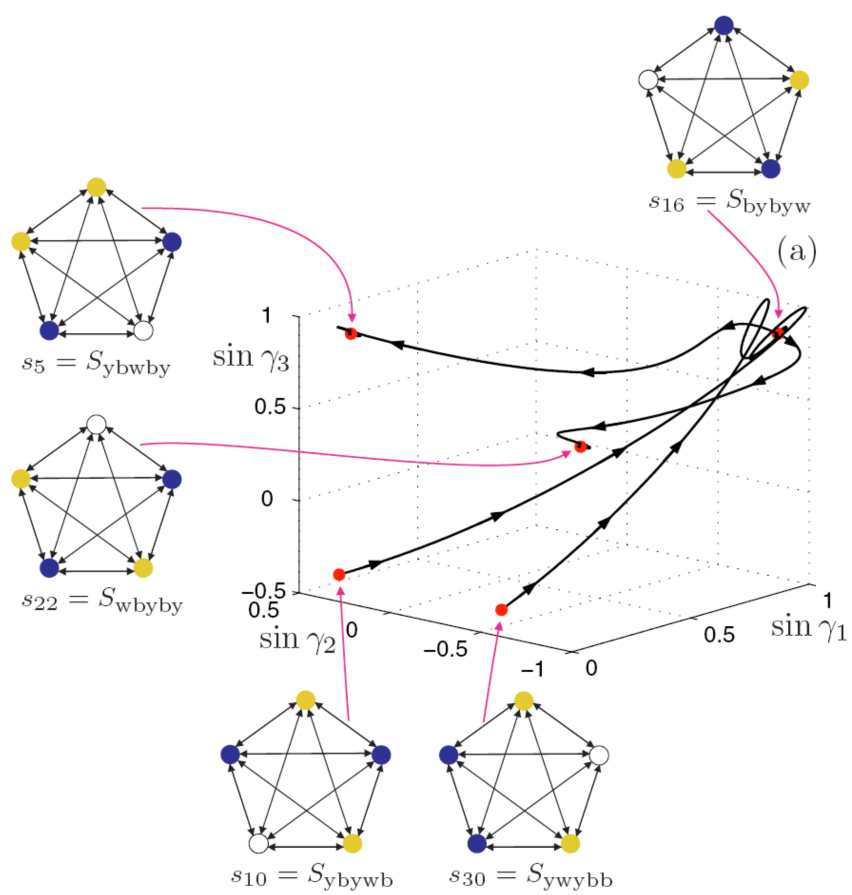
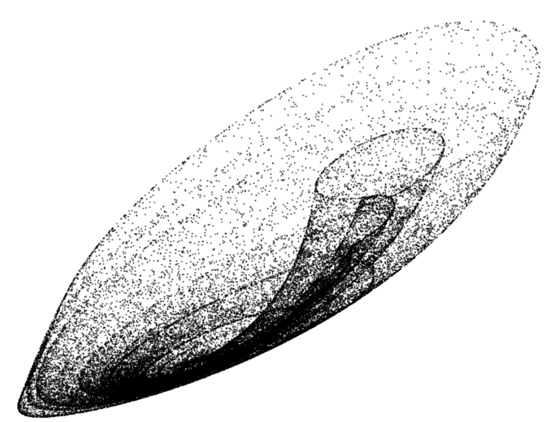
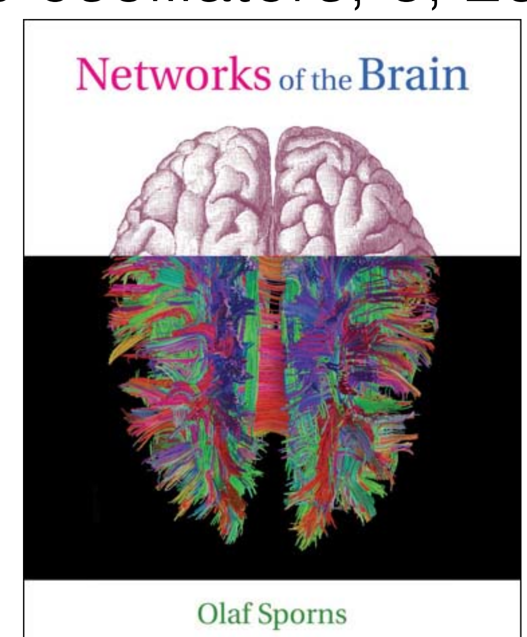


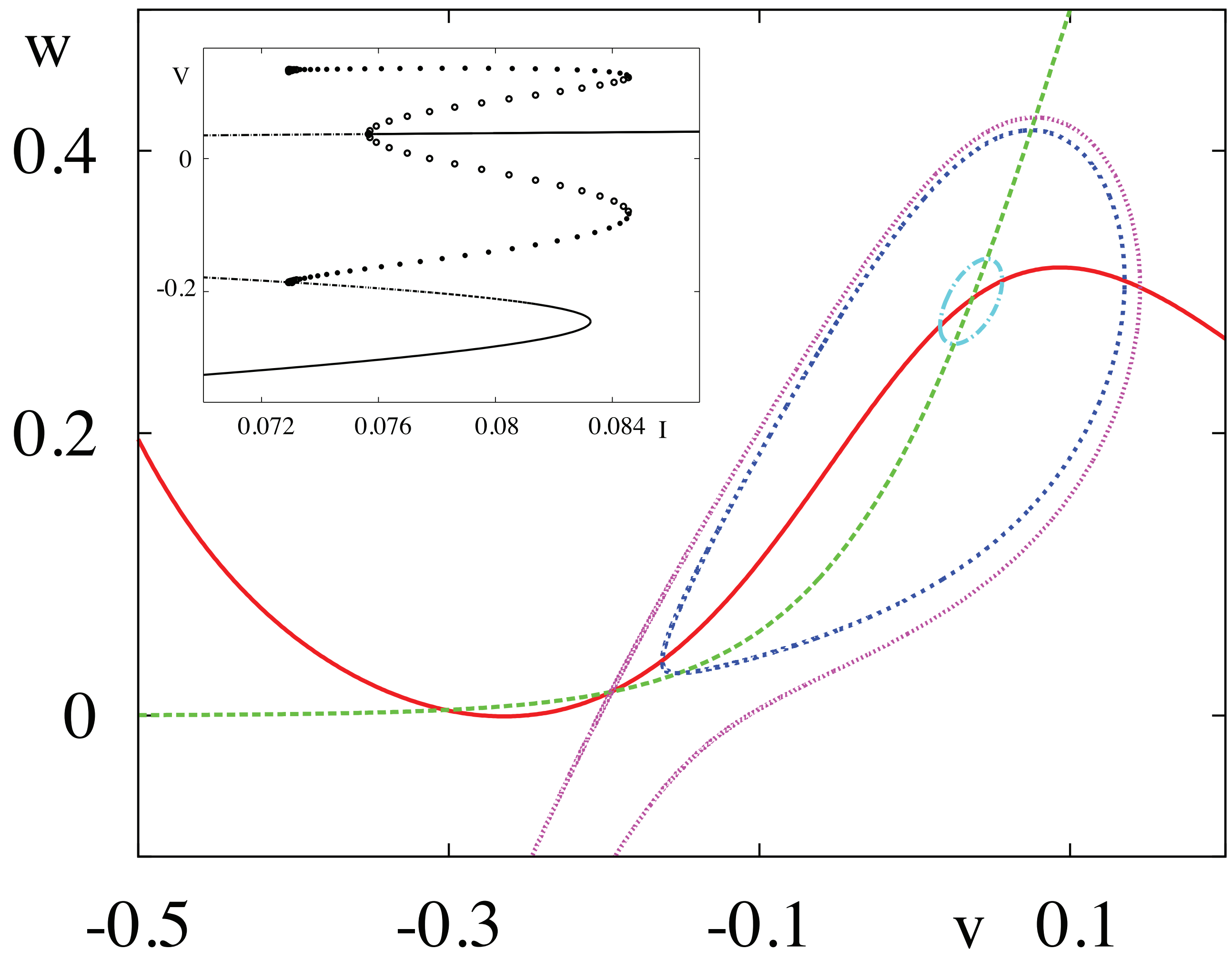




Breakdown of phase reduction

[the ‘i’ in iPRC]

Morris-Lecar example



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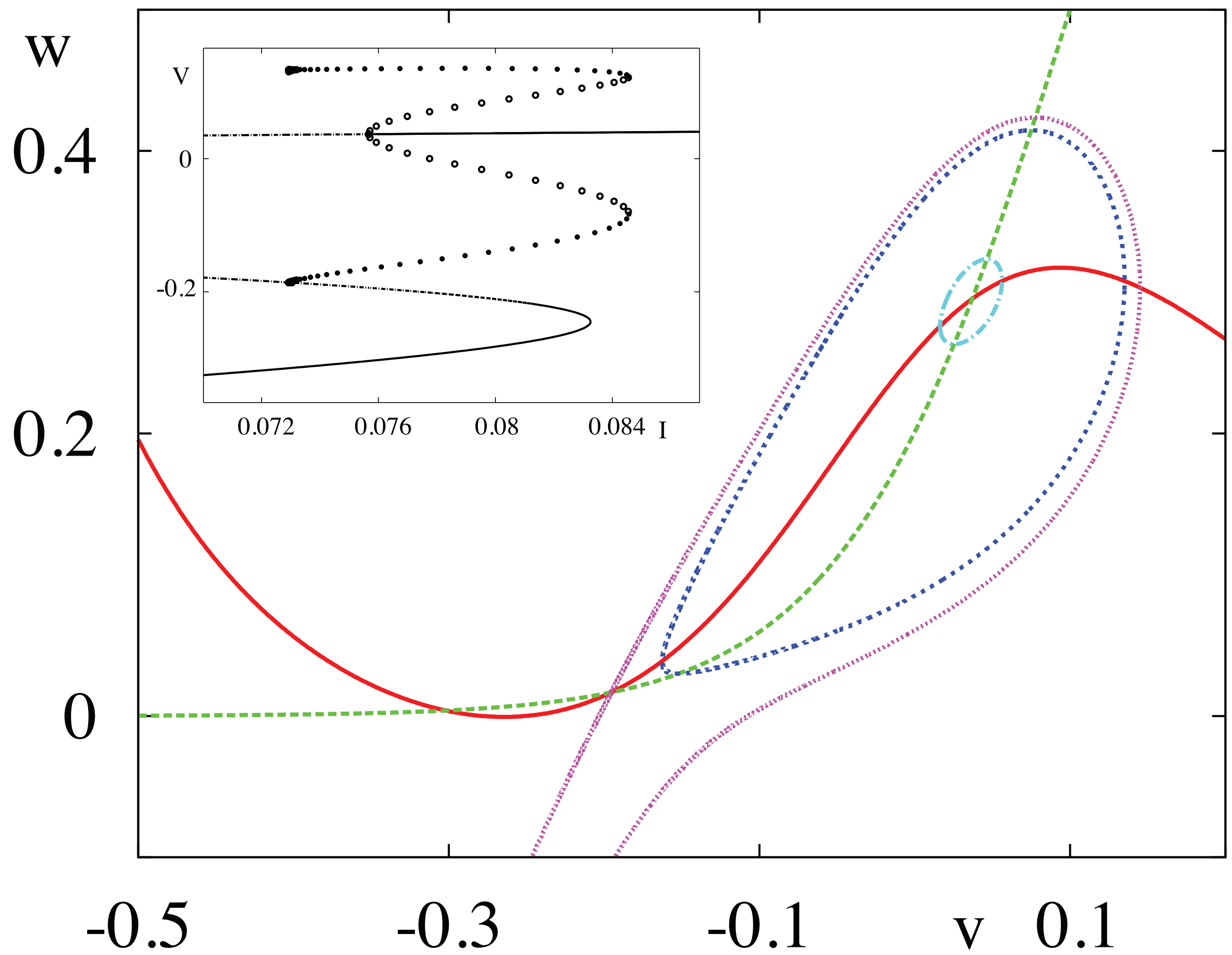
Leaving the basin of the periodic orbit



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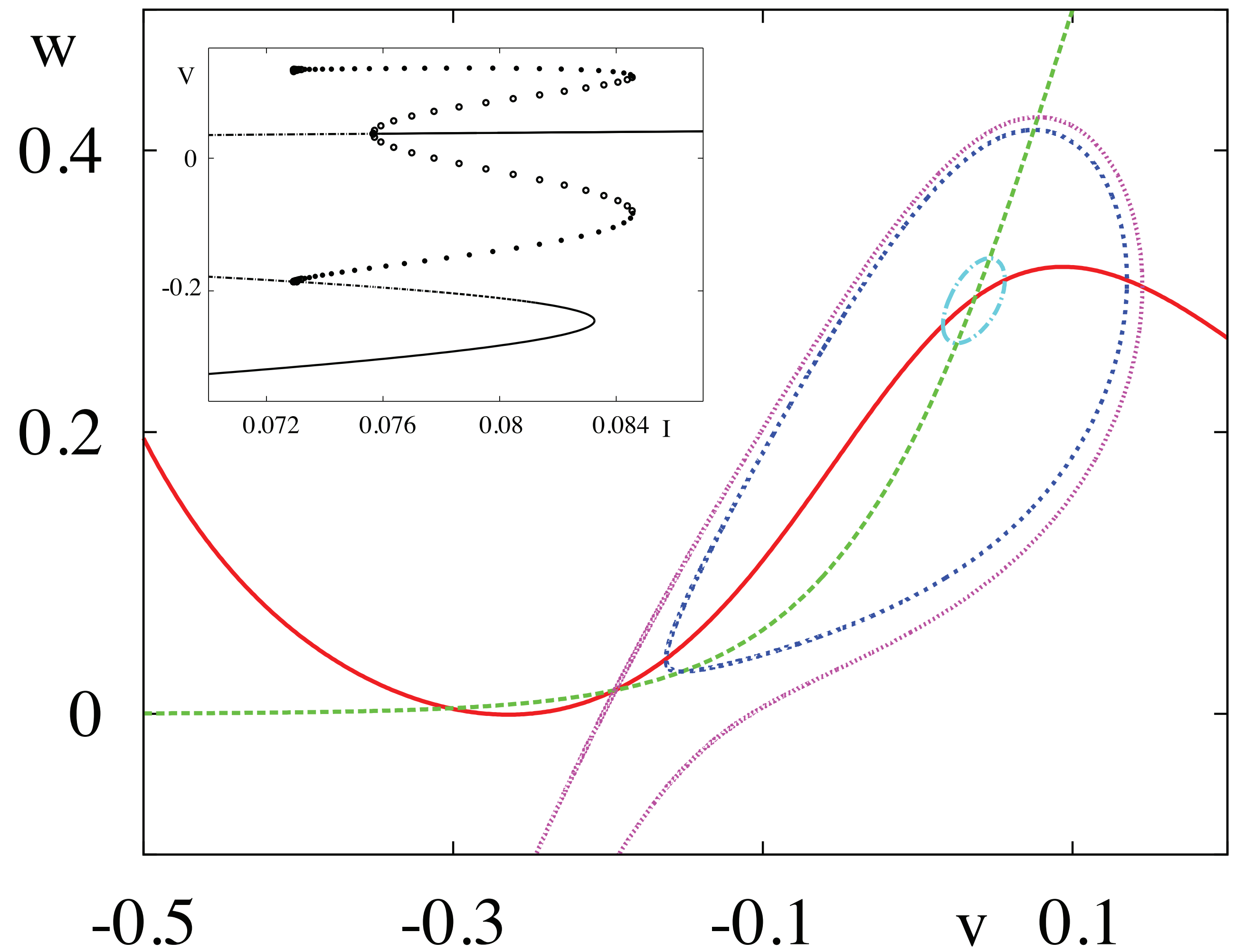

Invariant structures and *trapping*

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[the ‘i’ in iPRC]

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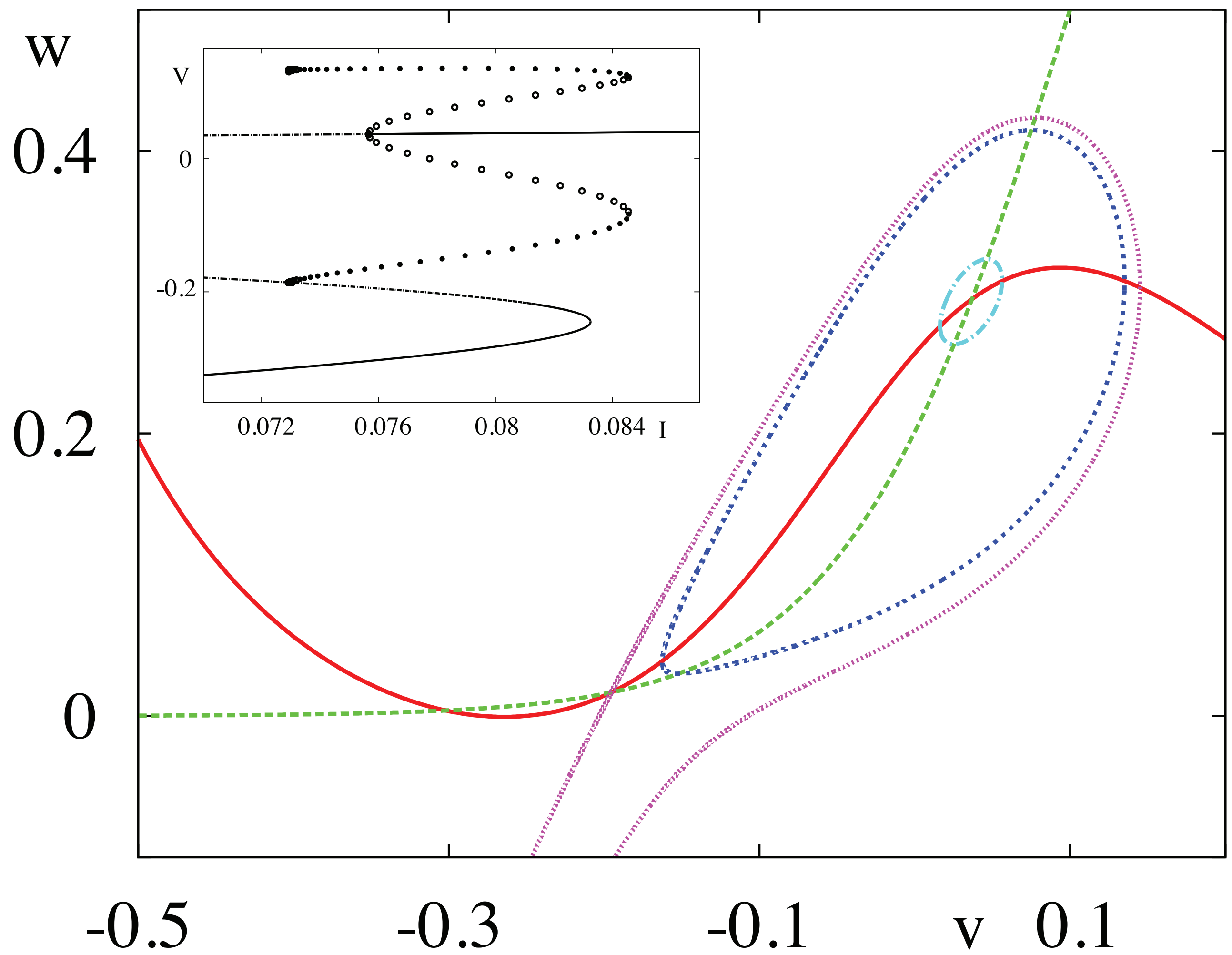
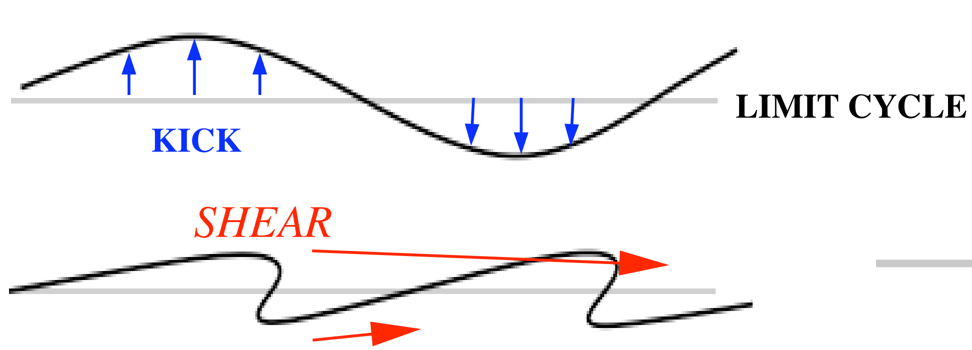
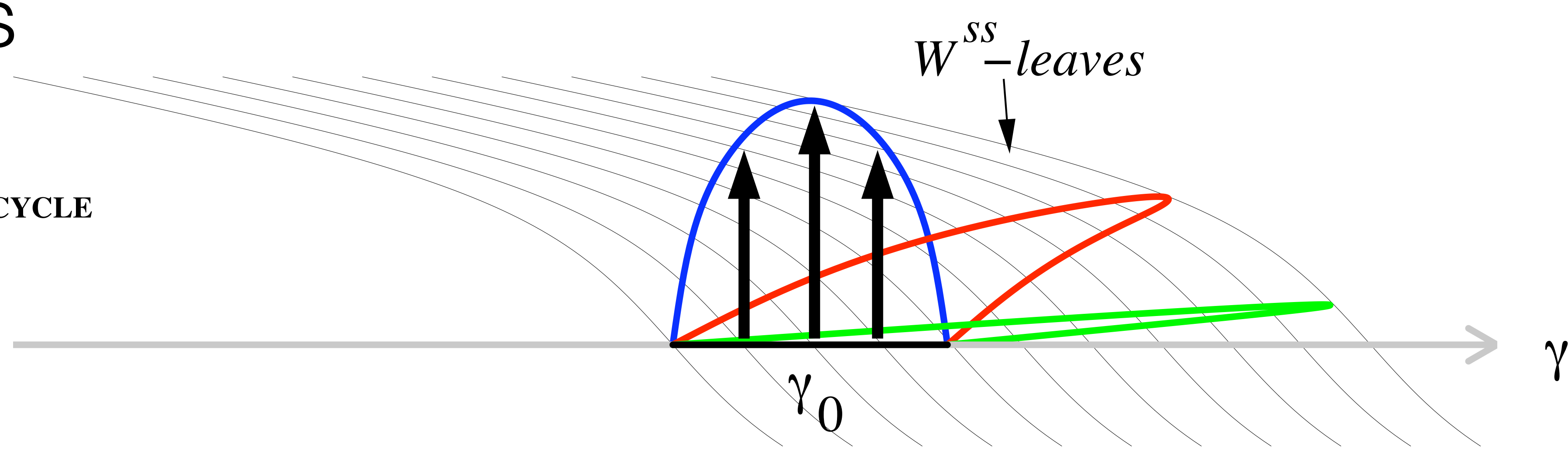


Invariant structures and *trapping*

Shear induced chaos





Aside - shear induced chaos

Q Wang and L-S Young "Strange attractors in periodically-kicked limit cycles and Hopf bifurcations", 2003 Comm. Math. Phys.

A toy model (that "breaks" the invariant circle)

$$\dot{\theta} = 1 + \sigma y, \qquad \dot{y} = -\lambda y + AH(\theta) \sum_{n=0}^{\infty} \delta(t - nT)$$

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$$\frac{\sigma}{\lambda} A \equiv \frac{\text{shear}}{\text{contraction rate}} \text{kick}$$

sufficiently large

[depends on 'bump' function H] $H(\theta) = \sin(2\pi\theta)$

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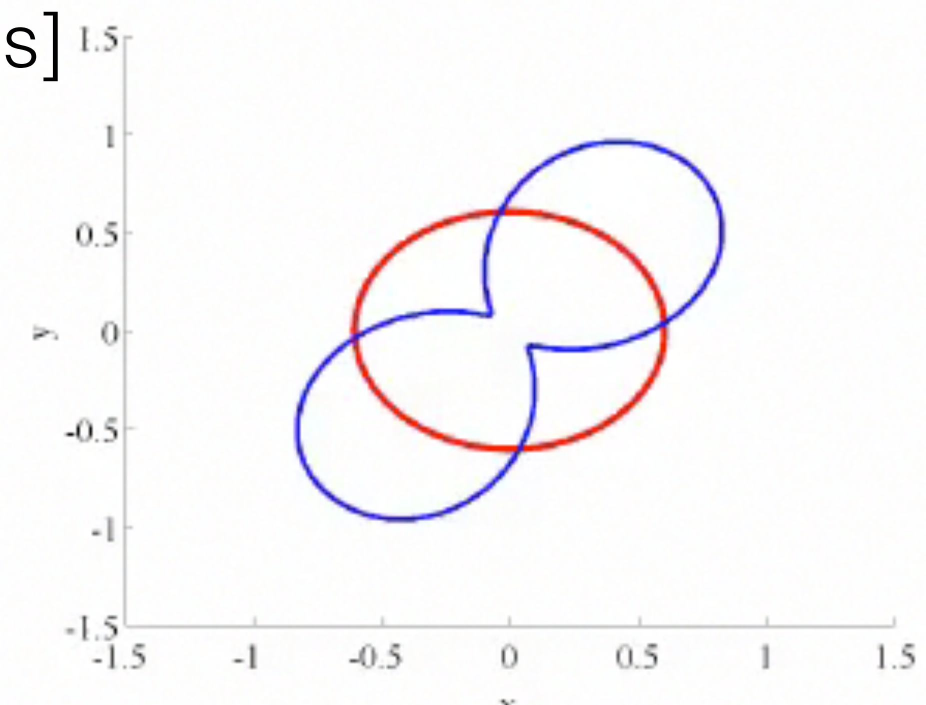

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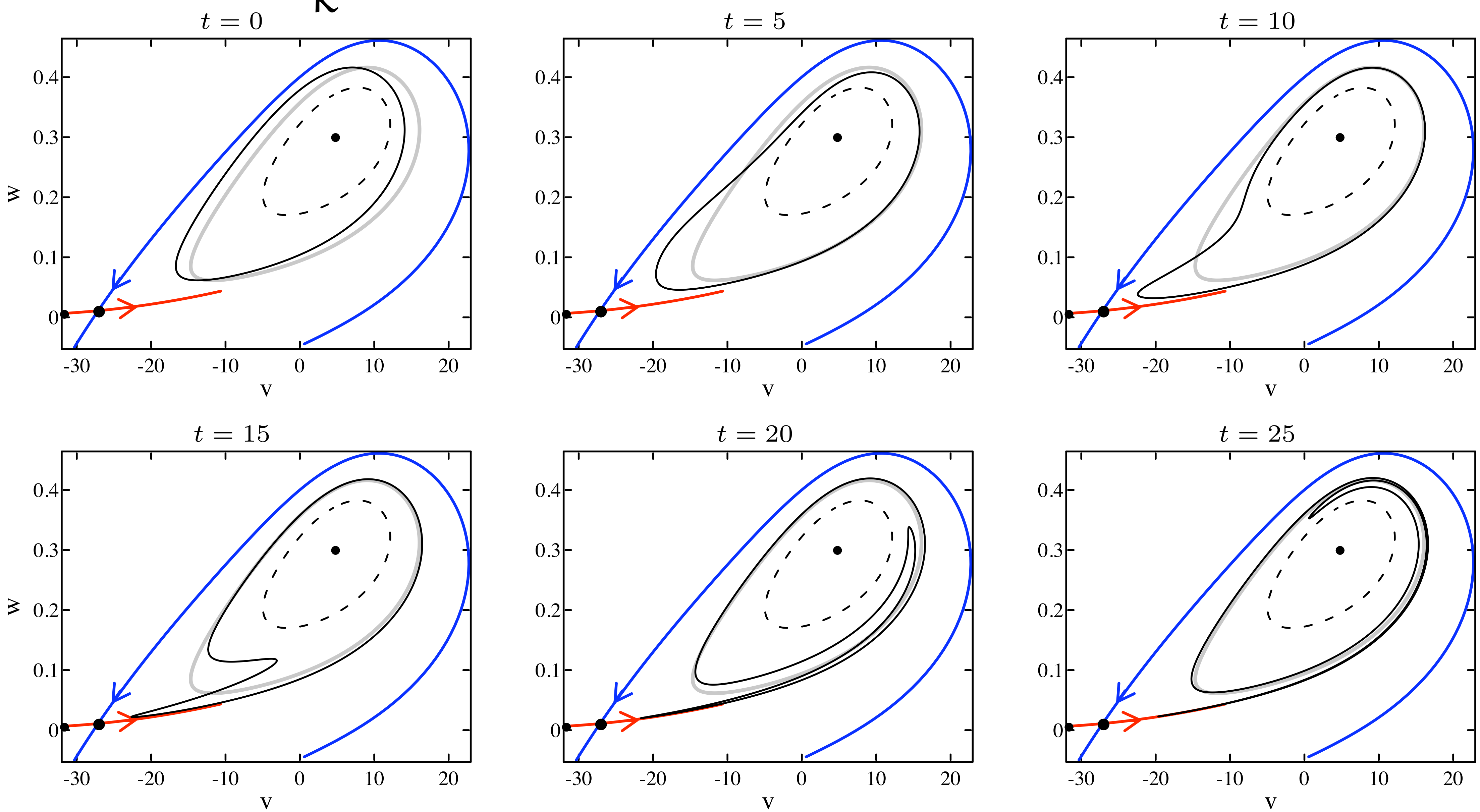
Chaotic response of ML to periodic kicking [not a toy model!]

$$\dot{v} \to \dot{v} + A \sum_k \delta(t - kT)$$

Shear induced folding one kick



iPRC/phase analysis would not predict chaos

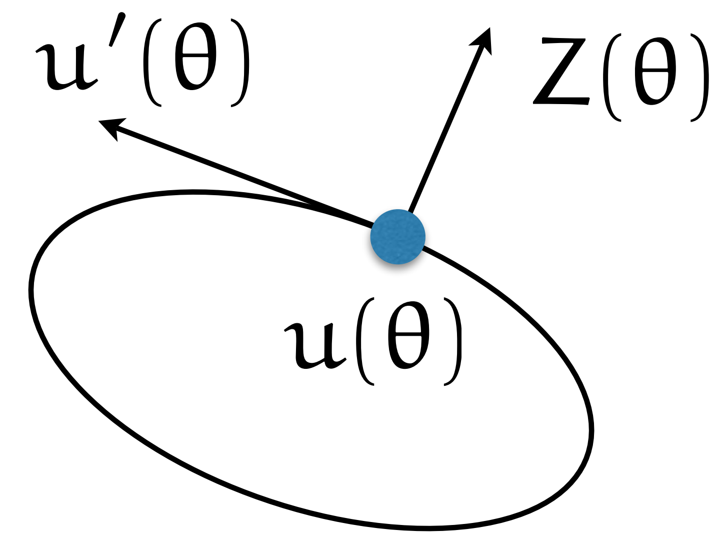
Phase-*amplitude*

Single cell - phase and amplitude (Hale 1969)

on cycle

orthogonal to cycle

$$z = u(\theta) + Z(\theta)\rho$$



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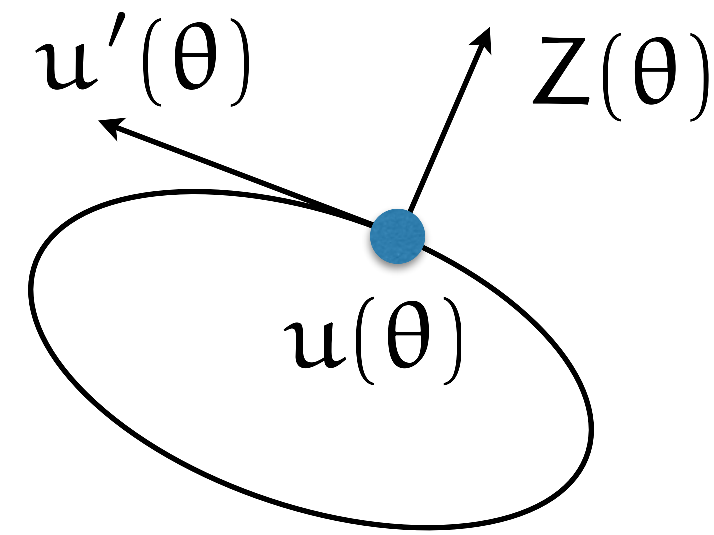

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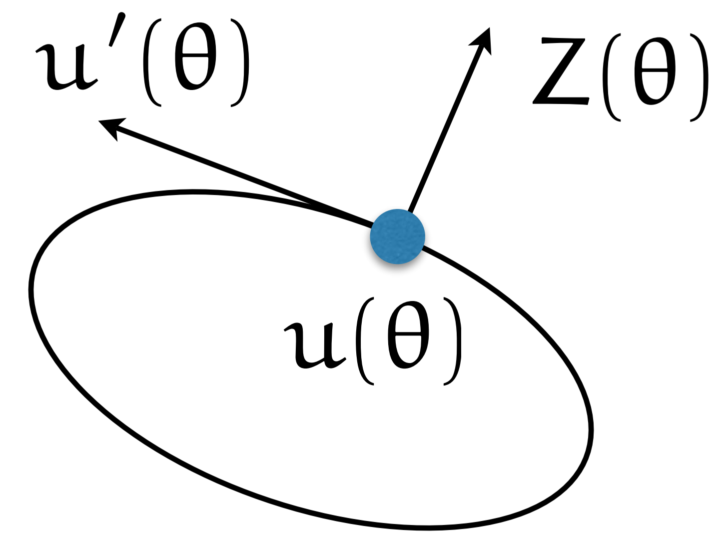

$$A(\theta) = Z^T \left[-\frac{dZ(\theta)}{d\theta} + \frac{\partial F(u(\theta))}{\partial z} Z(\theta) \right]$$

$$\dot{\theta} = 1 + f_1(\theta, \rho)$$
$$\dot{\rho} = A(\theta)\rho + f_2(\theta, \rho)$$

$$|f_1(\theta, \rho)| = O(|\rho|) \quad |\rho| \to 0$$

$$|f_2(\theta, 0)| = 0 \qquad \frac{\partial f_2(\theta, 0)}{\partial \rho} = 0$$

The perturbed system

$$\dot{z} = F(z) + \varepsilon g(x, t)$$

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$$\dot{\theta} = 1 + f_1(\theta, \rho) + \varepsilon h^T(\theta, \rho) g(u(\theta) + Z(\theta)\rho, t),$$
$$\dot{\rho} = A(\theta)\rho + f_2(\theta, \rho) + \varepsilon B(\theta, \rho) g(u(\theta) + Z(\theta)\rho)$$

hide the details for h and B [phase-amplitude responses]

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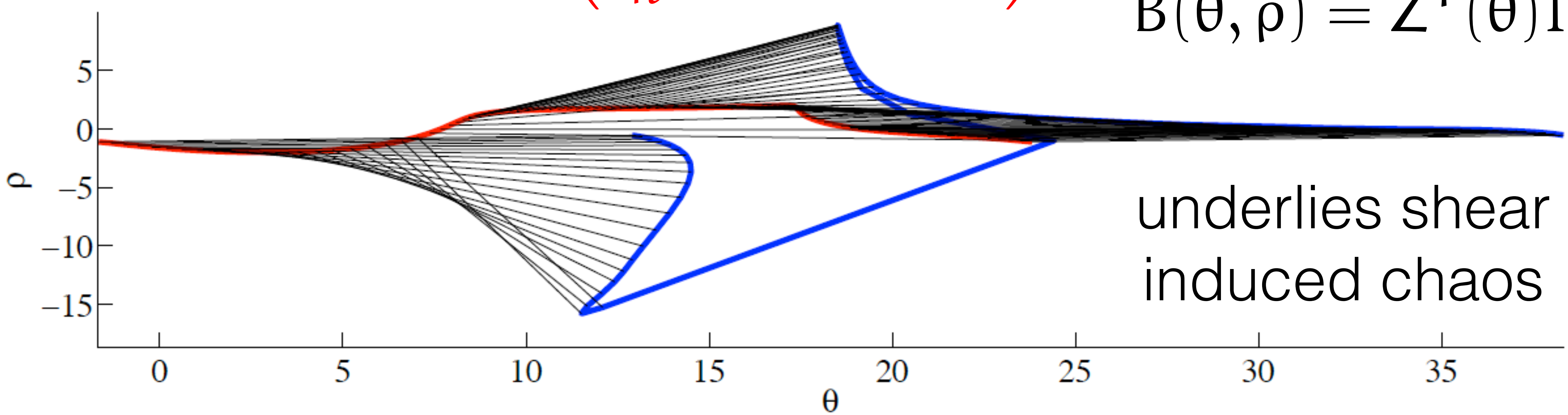
hide the details for h and B [phase-amplitude responses]

$$g(u(\theta) + Z(\theta), t) = \left(\sum_n \delta(t - nT), 0 \right)^T$$

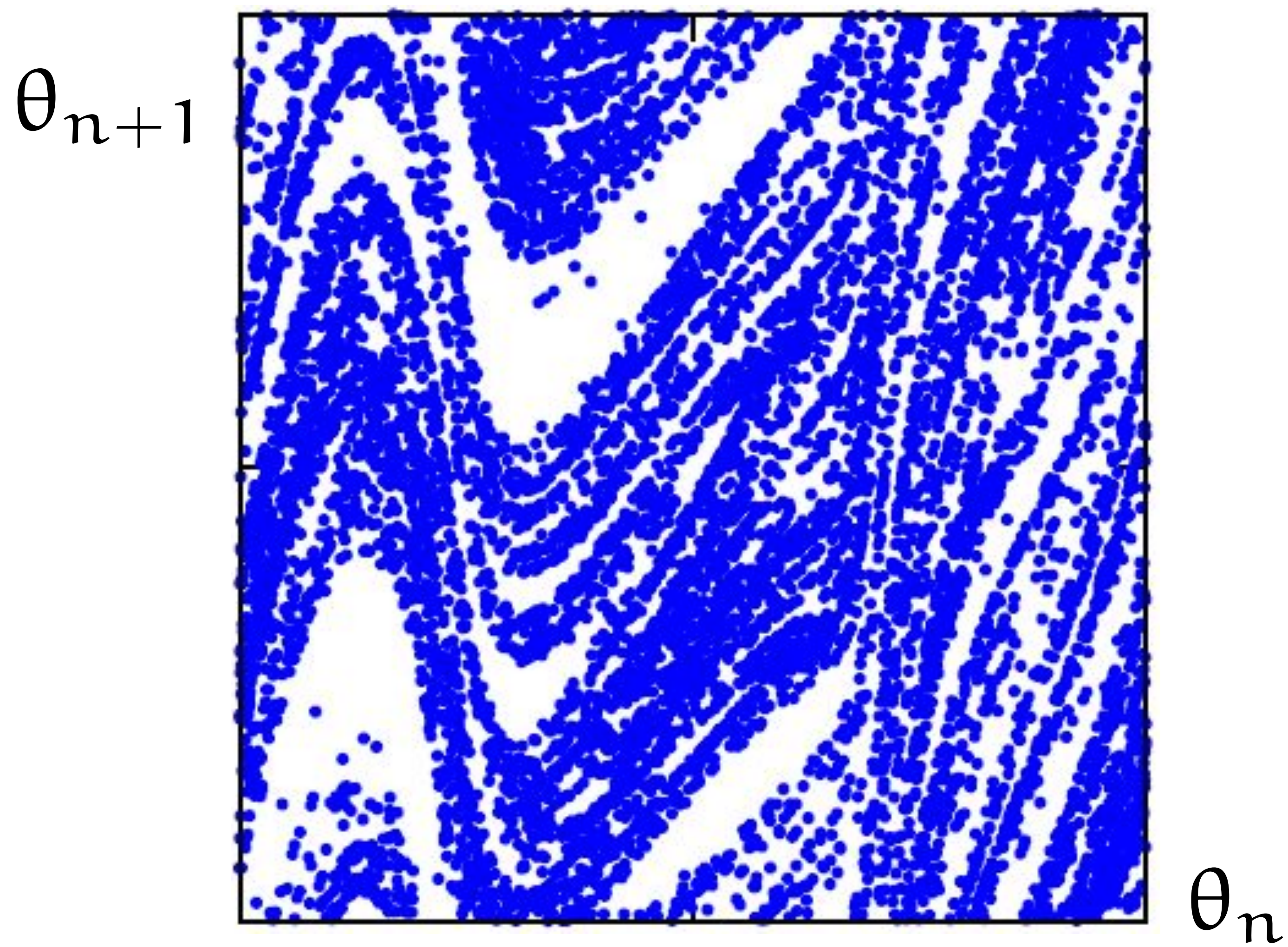
ML (2D):

$$B(\theta, \rho) = Z^T(\theta) I_2$$

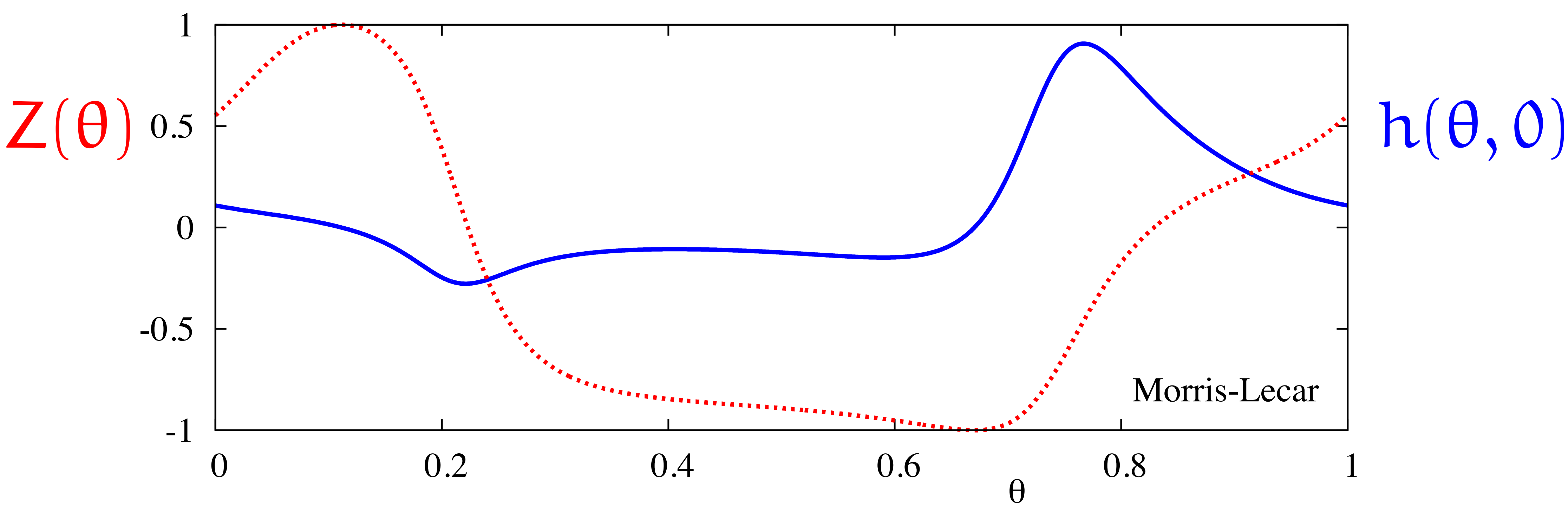


underlies shear induced chaos

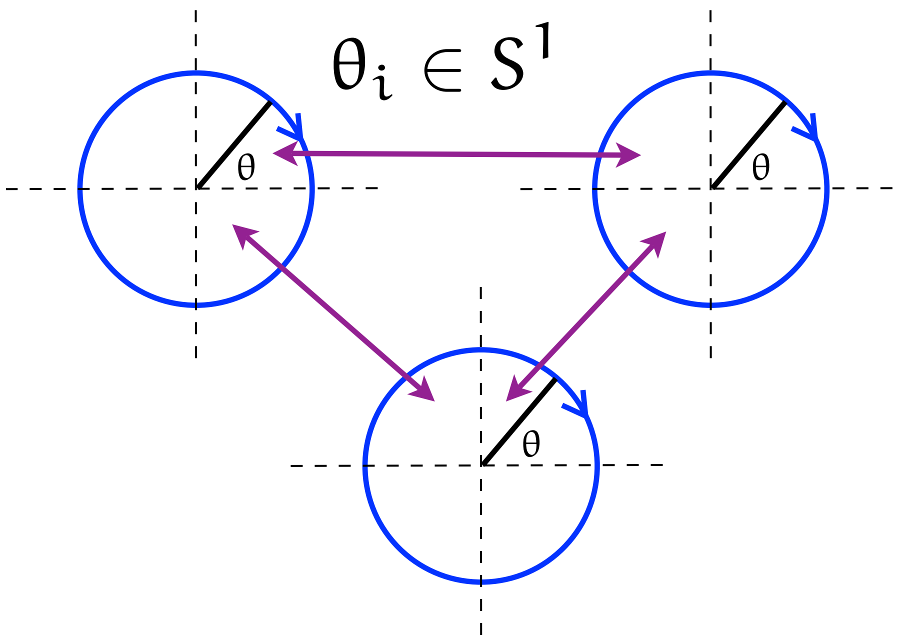
Positive Lyapunov exponent



$\theta_n = \theta(nT)$



Strongly coupled networks?



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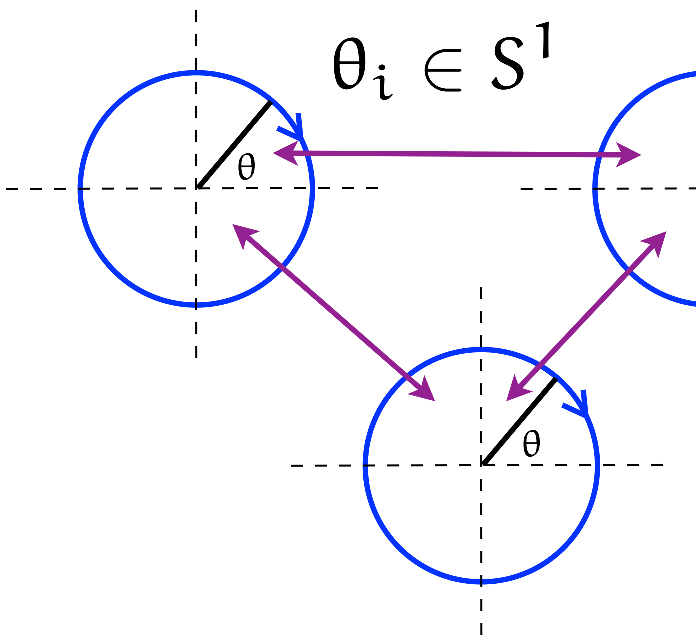

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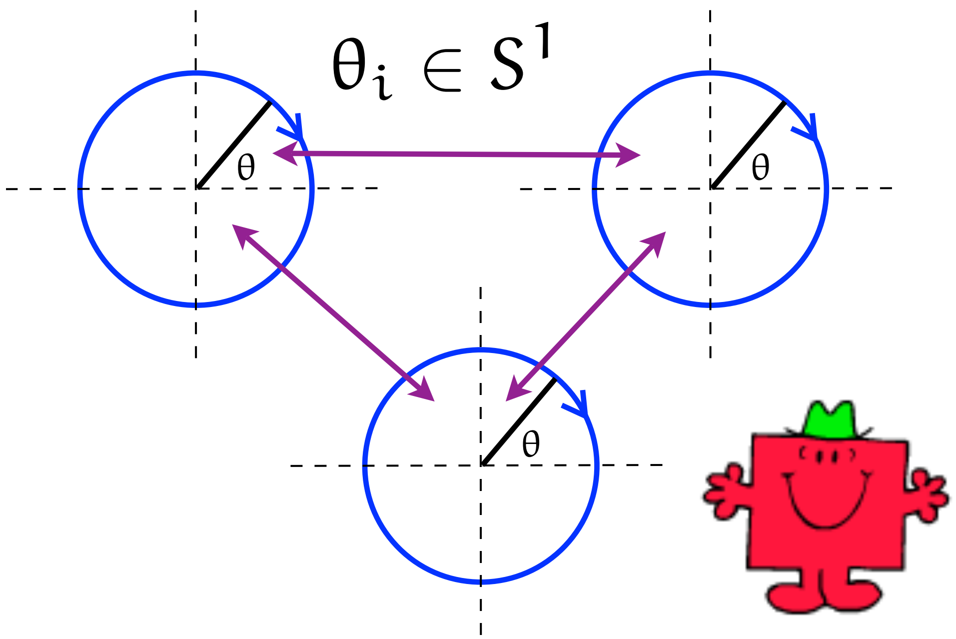

$$\dot{\theta}_i = f(\theta_i, \rho_i) + \varepsilon \sum_j w_{ij} H_1(\theta_i, \theta_j, \rho_i, \rho_j)$$

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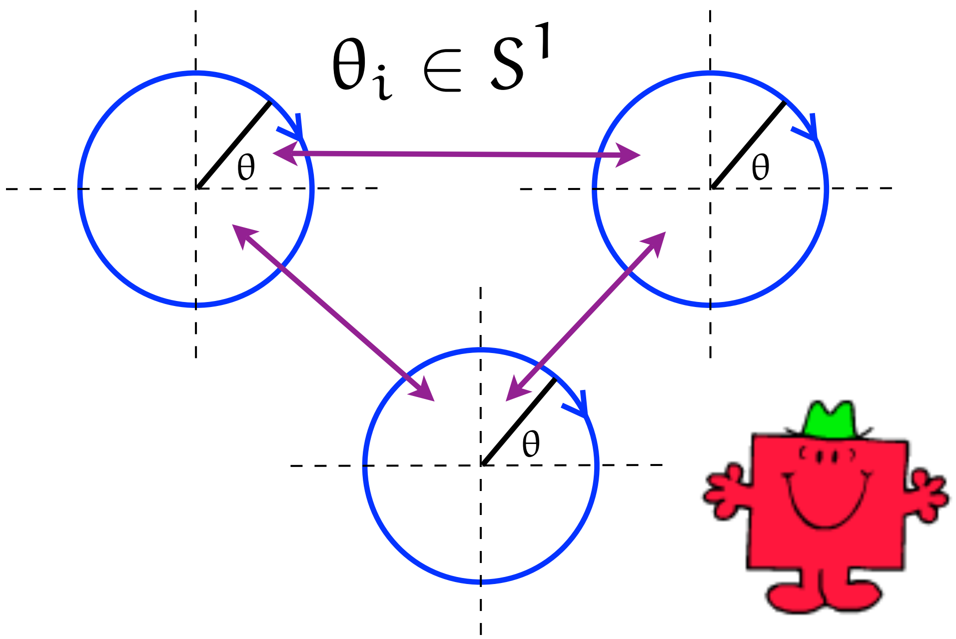

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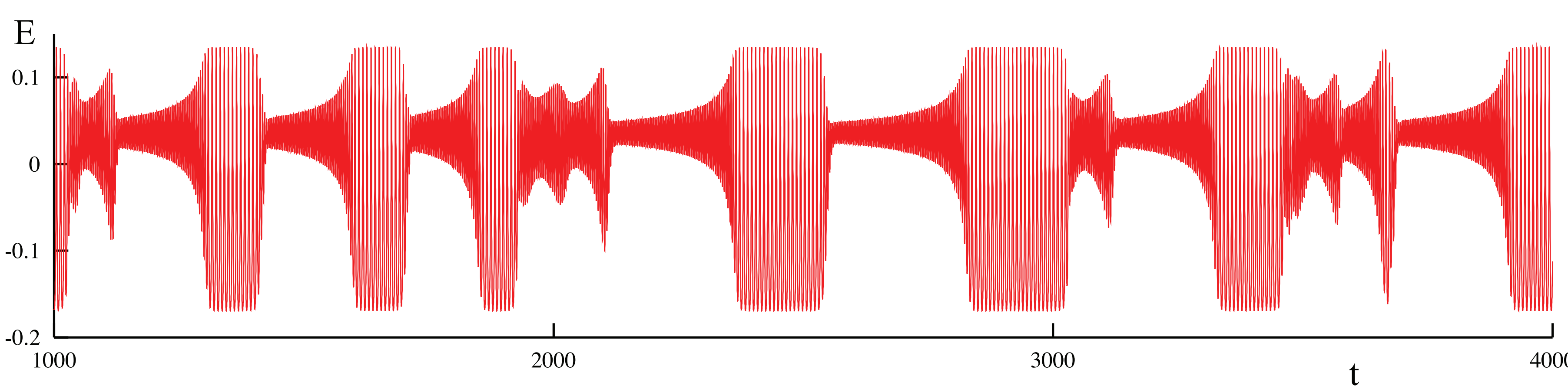
global linear coupling of ML - mean field signal as average membrane potential



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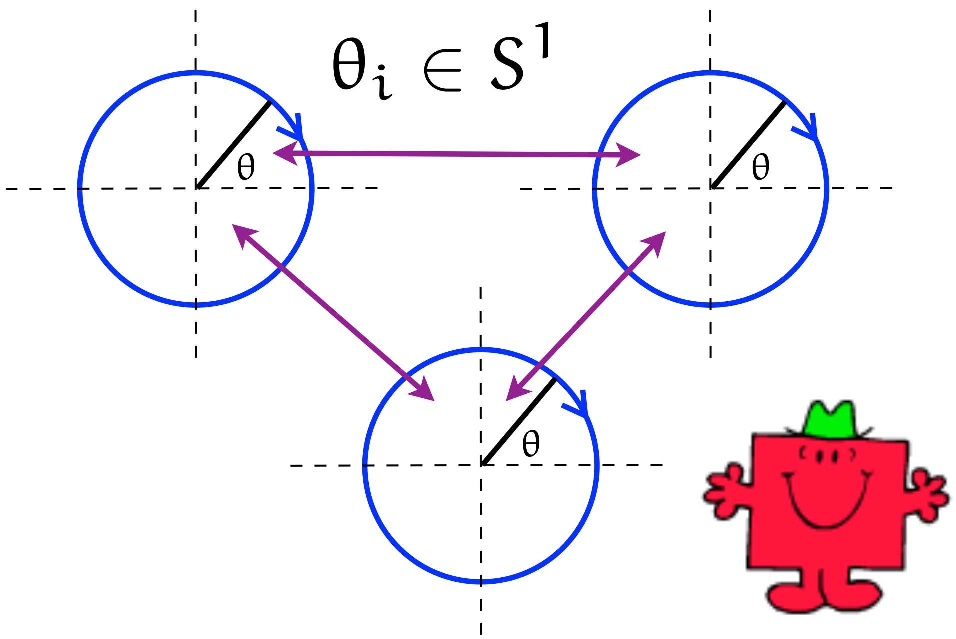

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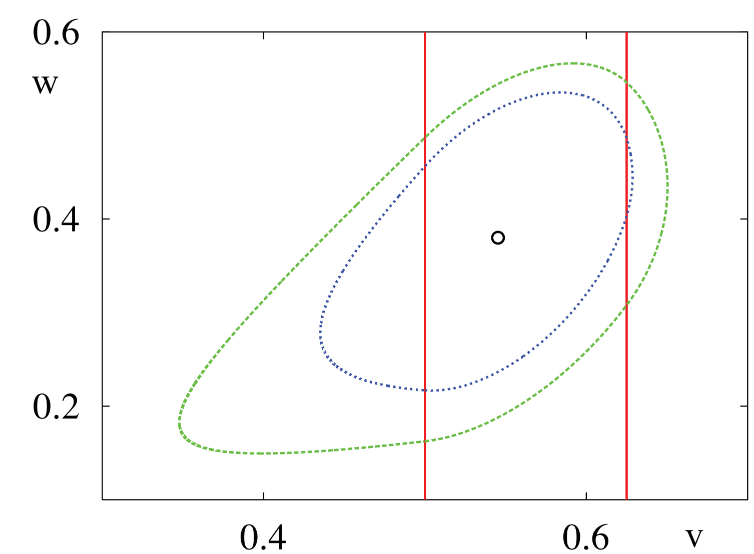

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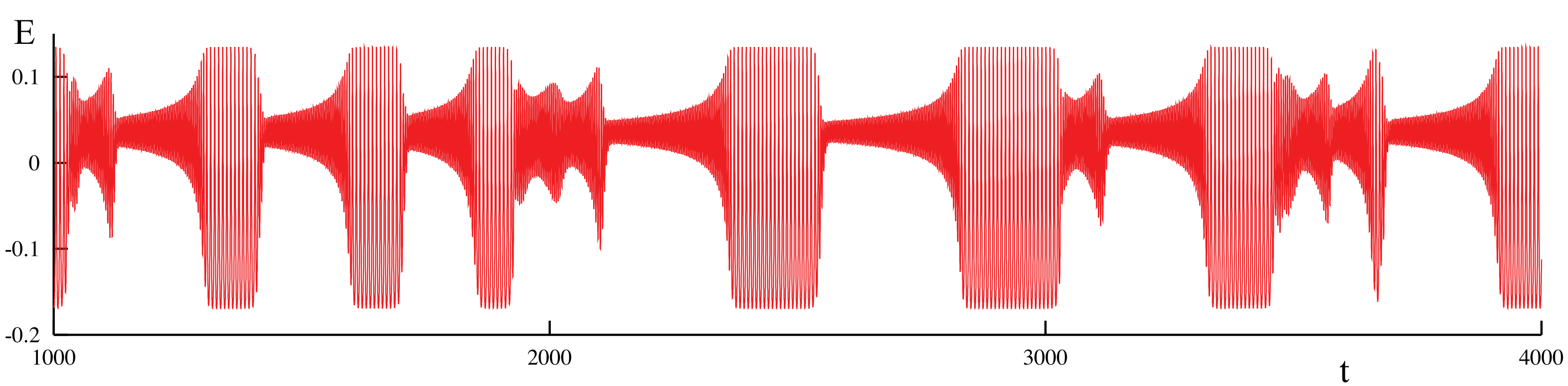
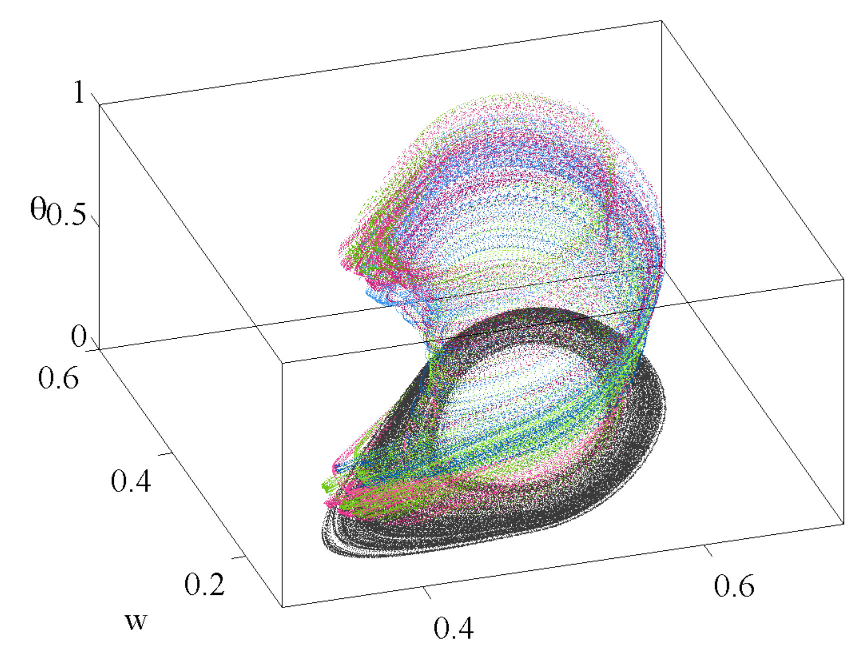
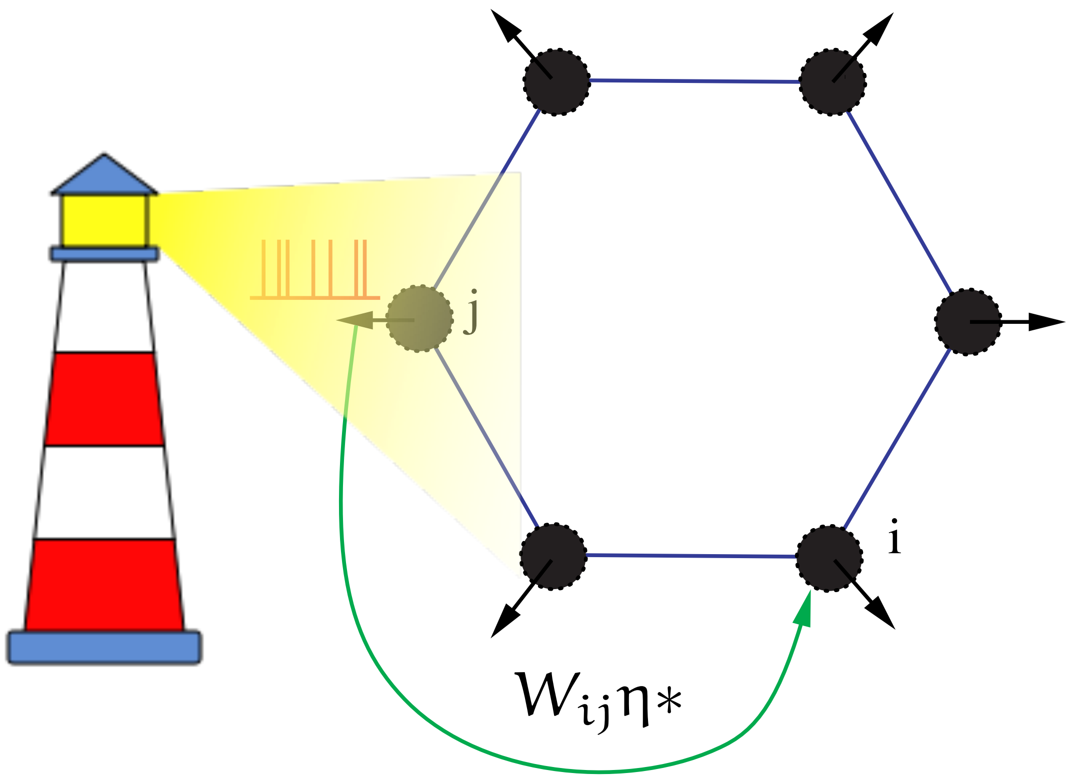




S Coombes 2008 Neuronal networks with gap junctions: A study of piece-wise linear planar neuron models, SIAM Journal on Applied Dynamical Systems, Vol 7, 1101-112

A solvable spiking neurodynamics?



Haken Lighthouse model

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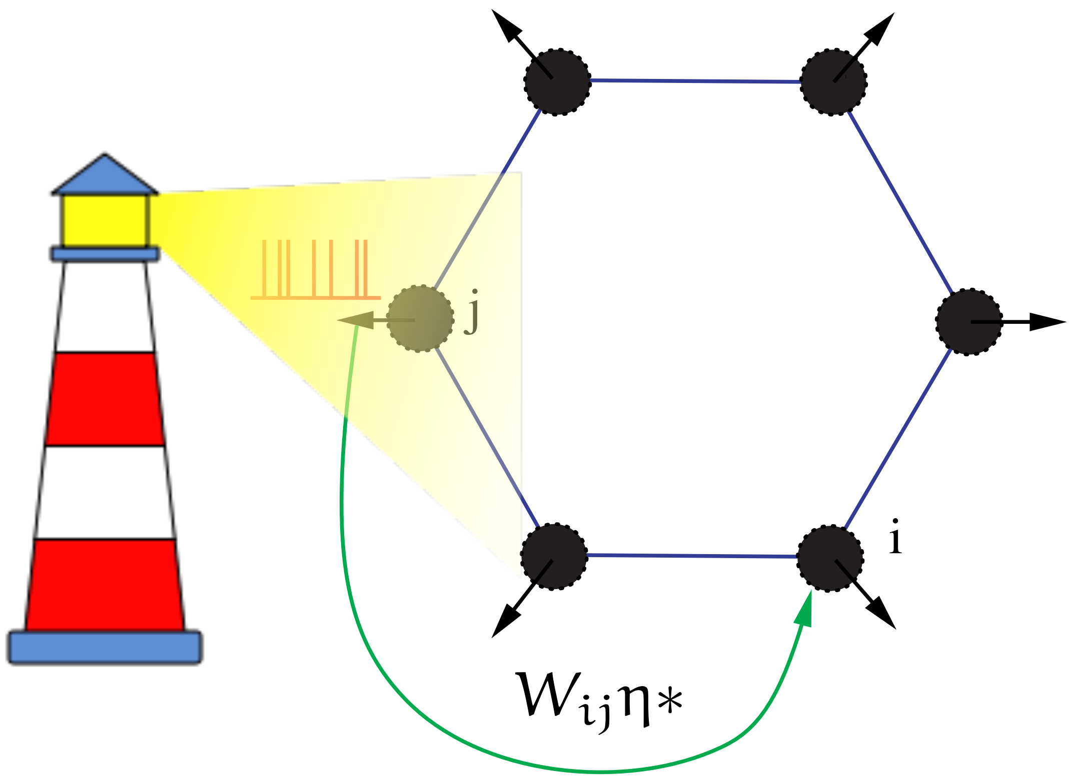

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$$\frac{\partial \theta(x,t)}{\partial t} = H(u(x,t) - h)$$

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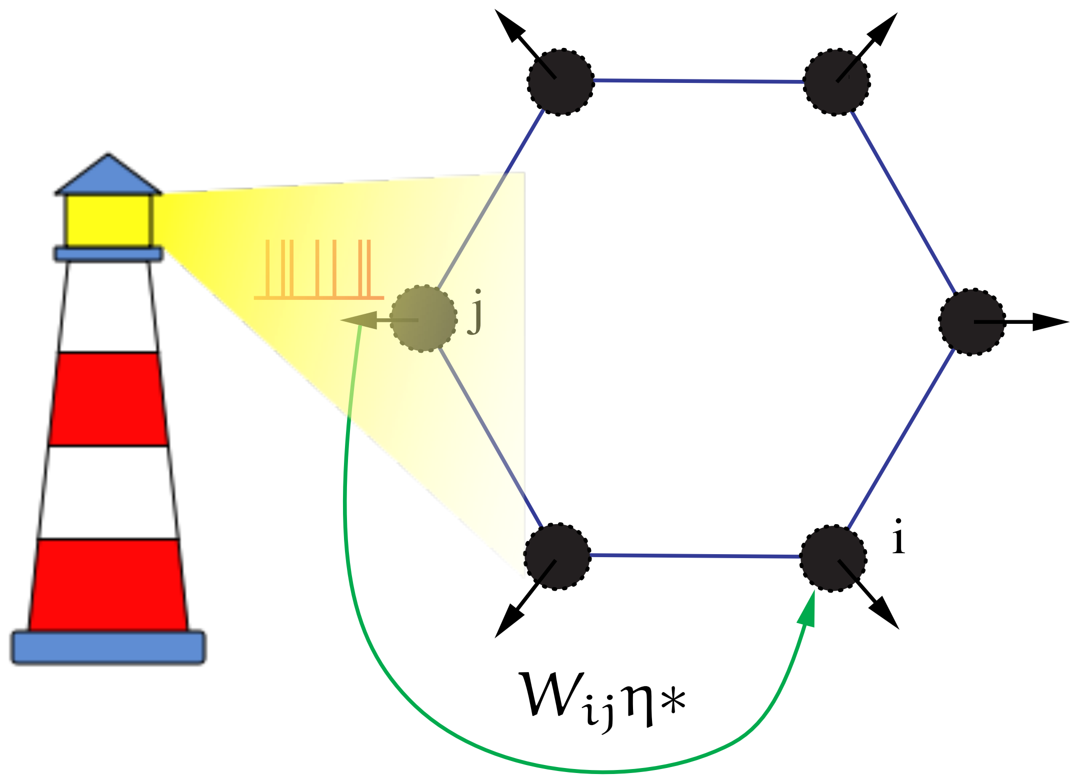

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$$u(x,t) = \sum_m \int_0^\infty ds \eta(s) \int_{-\infty}^\infty dy w(x-y) \delta(s - t + T^m(y))$$

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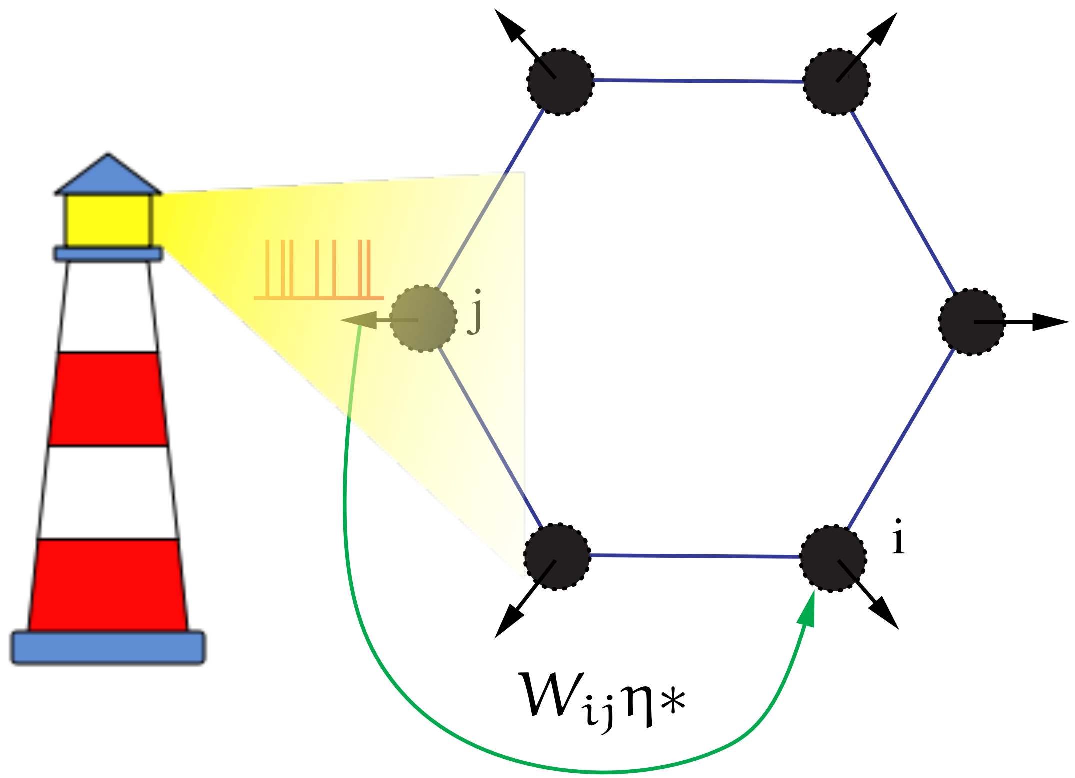

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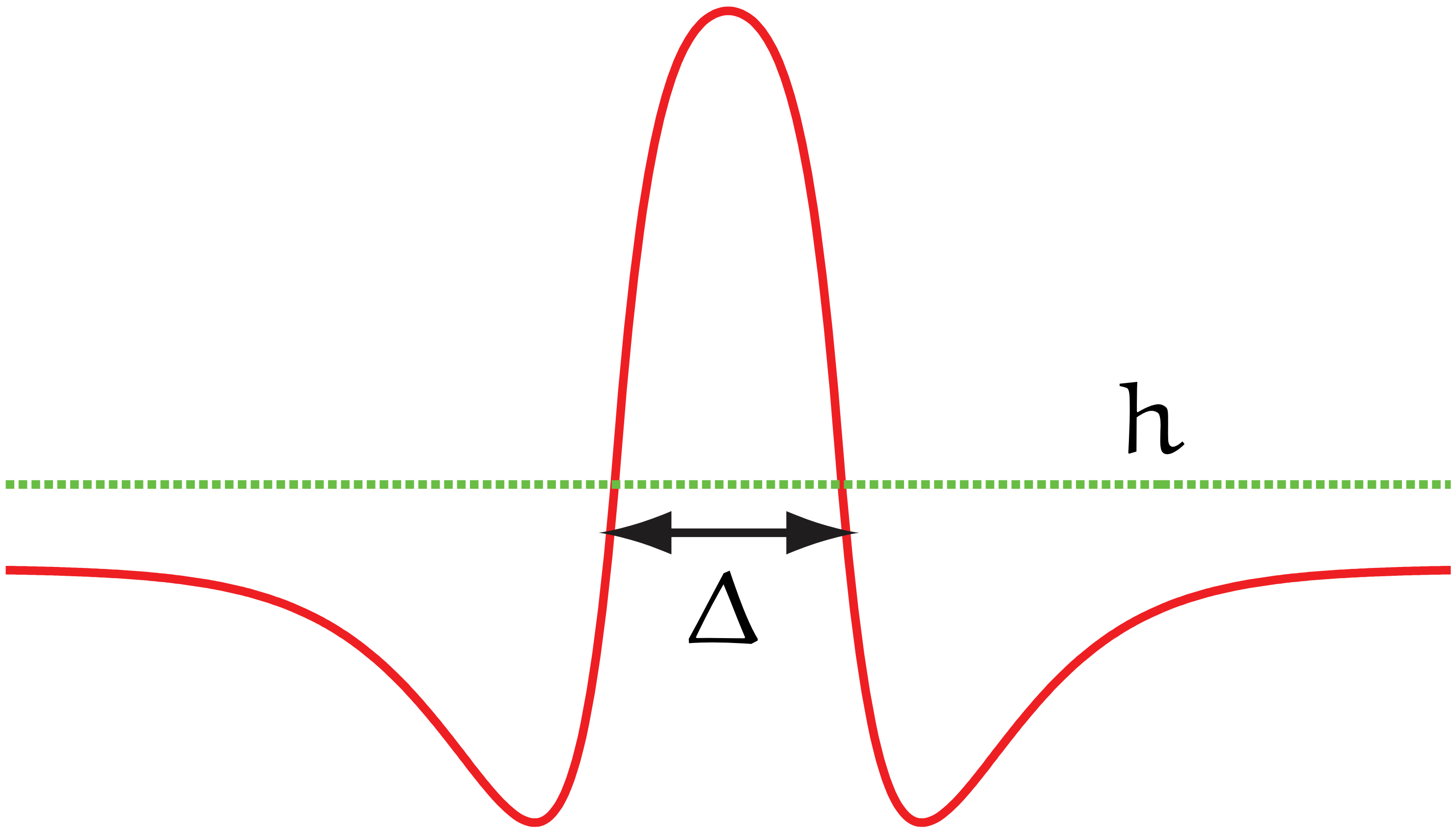
$\eta(t) = \alpha e^{-\alpha t} H(t)$ and slow synapses recovers Amari

$$\frac{1}{\alpha} \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^\infty dy w(x-y) H(u(y,t) - h)$$

Amari bumps with Mexican-hat

$$q(x) = \int_0^{\Delta} dy\, w(x-y)$$



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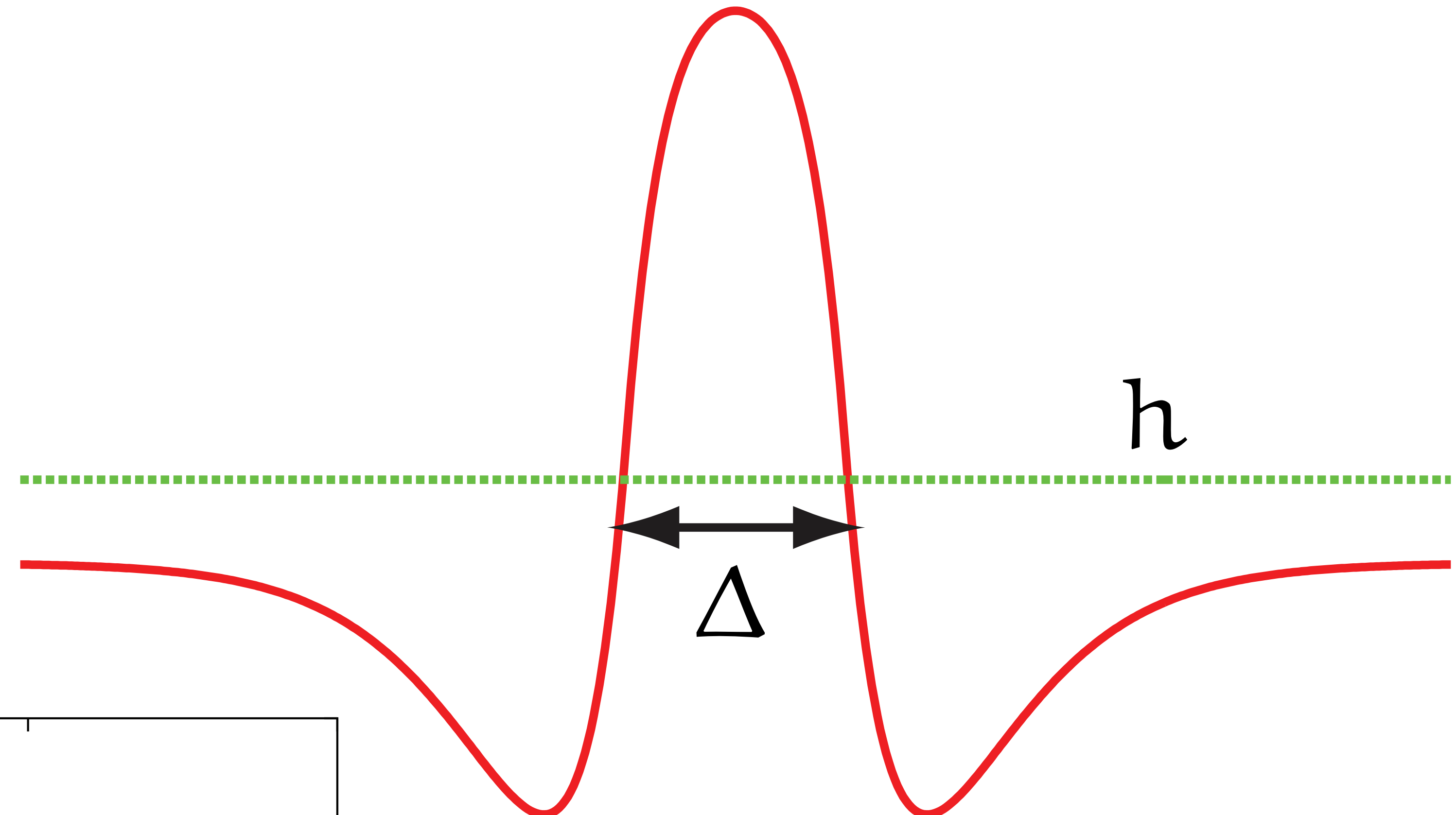

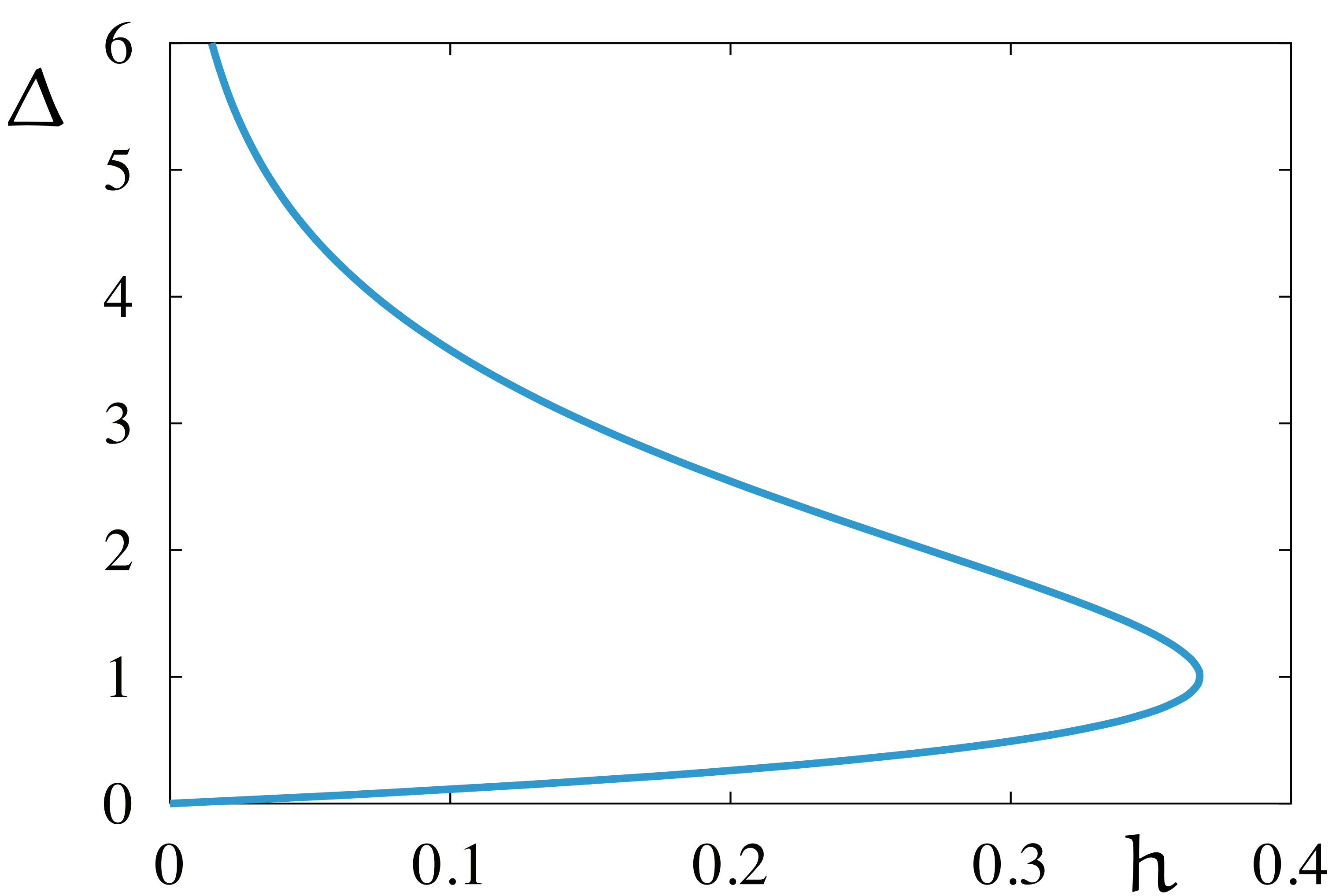

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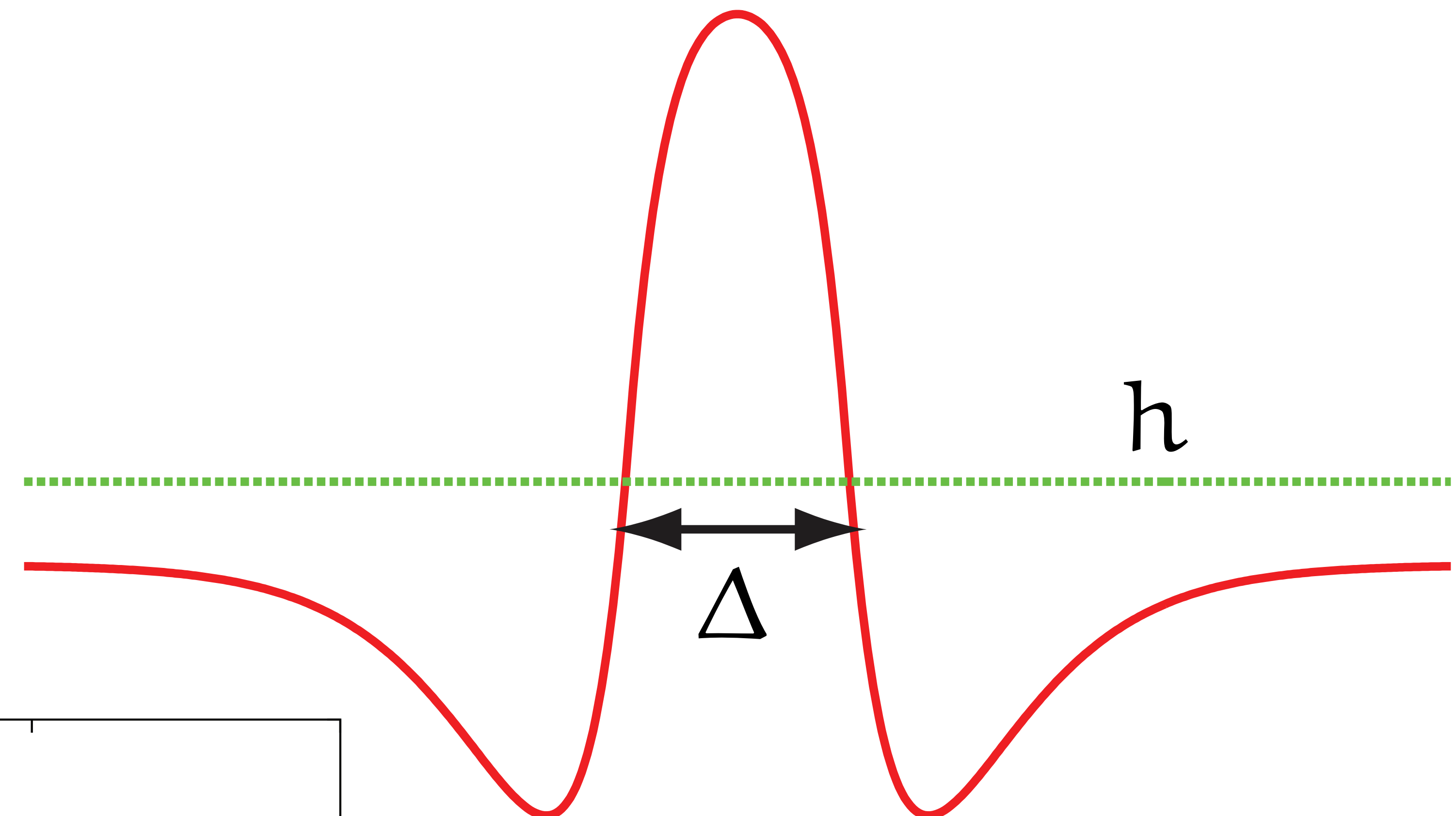
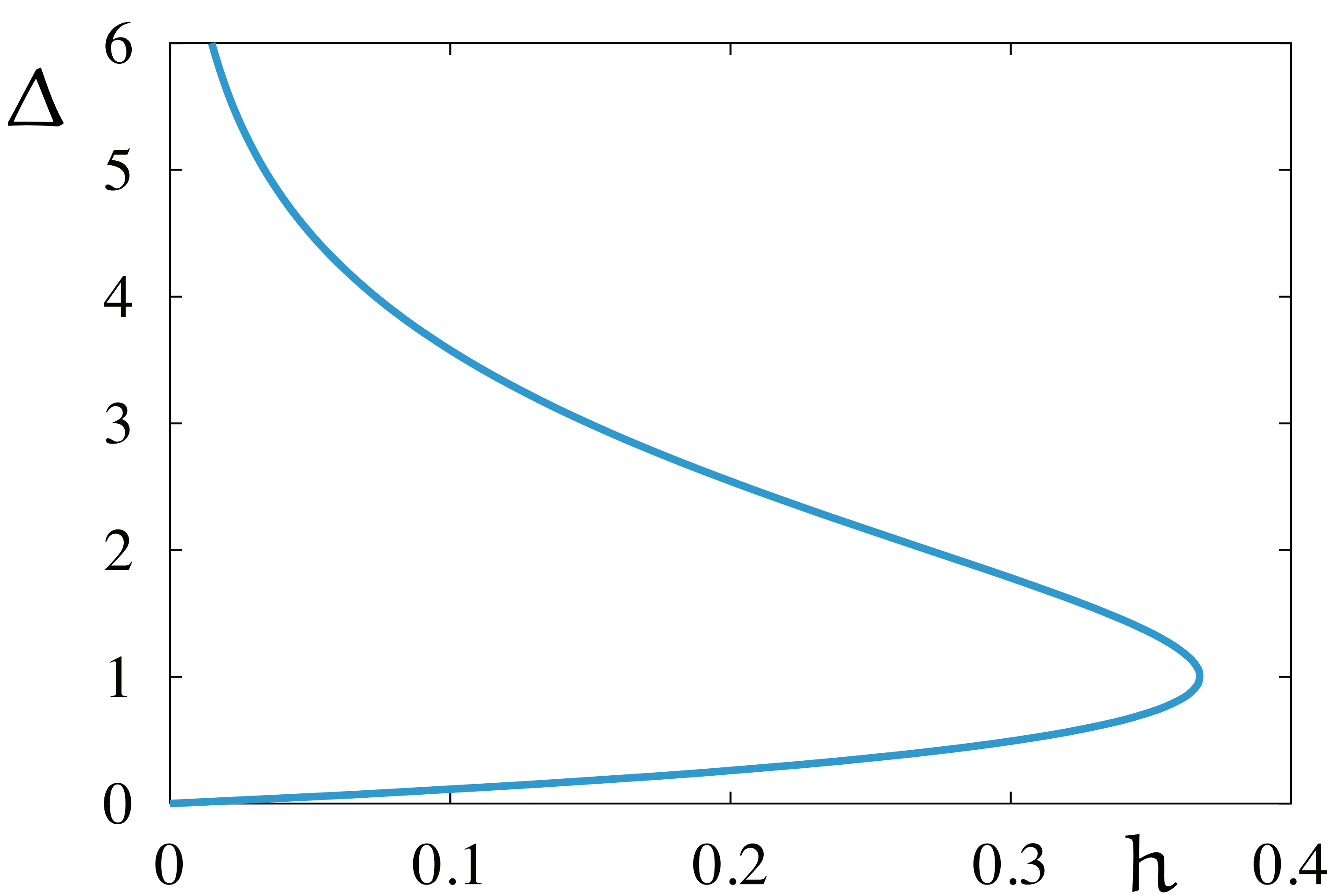


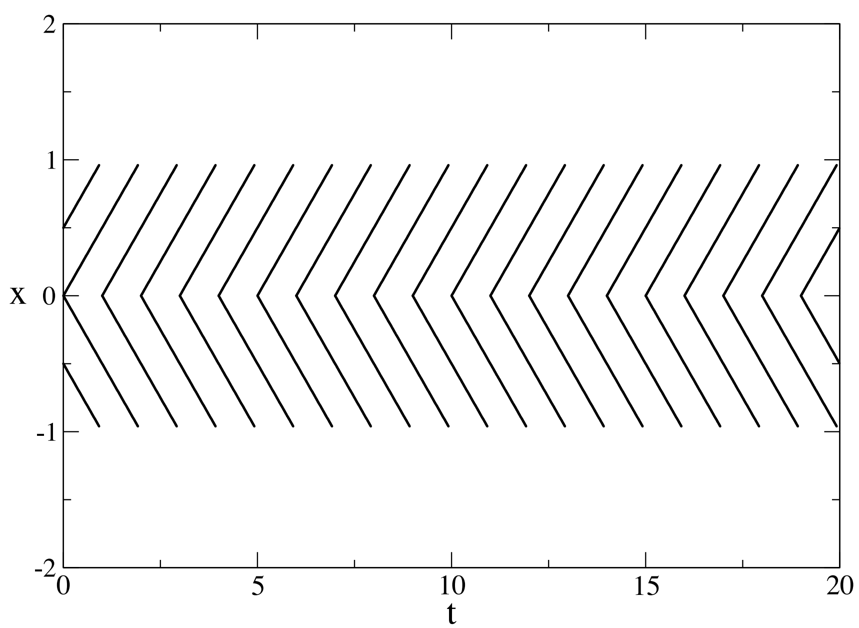


$$q(0) = h = q(\Delta)$$

working memory

Spiking bumps

$$h = \int_0^{\Delta} w(y)dy + 2\mathrm{Re}\left(\sum_{n>0} \widehat{\eta}(2\pi n)G(2\pi n)e^{2\pi i n t^*}\right) \quad *$$



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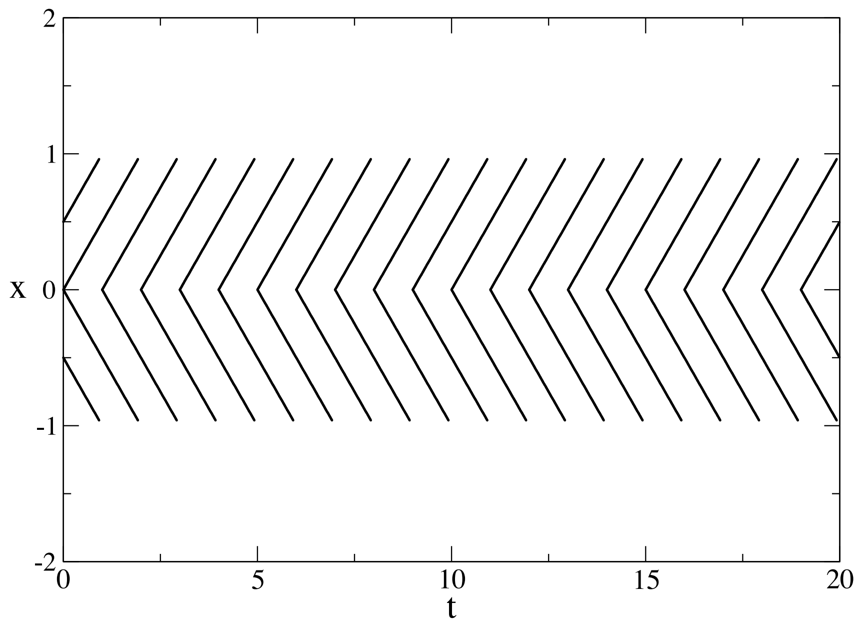

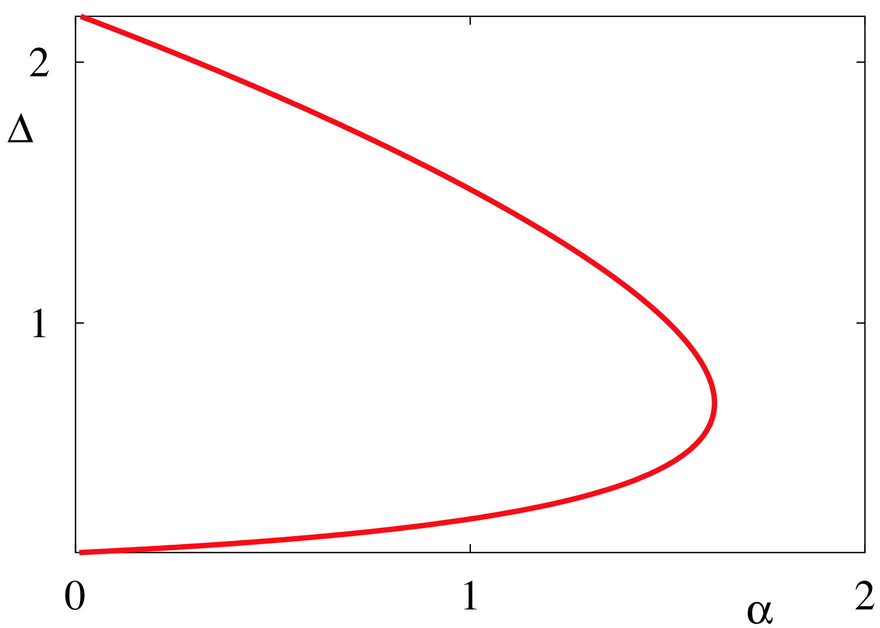

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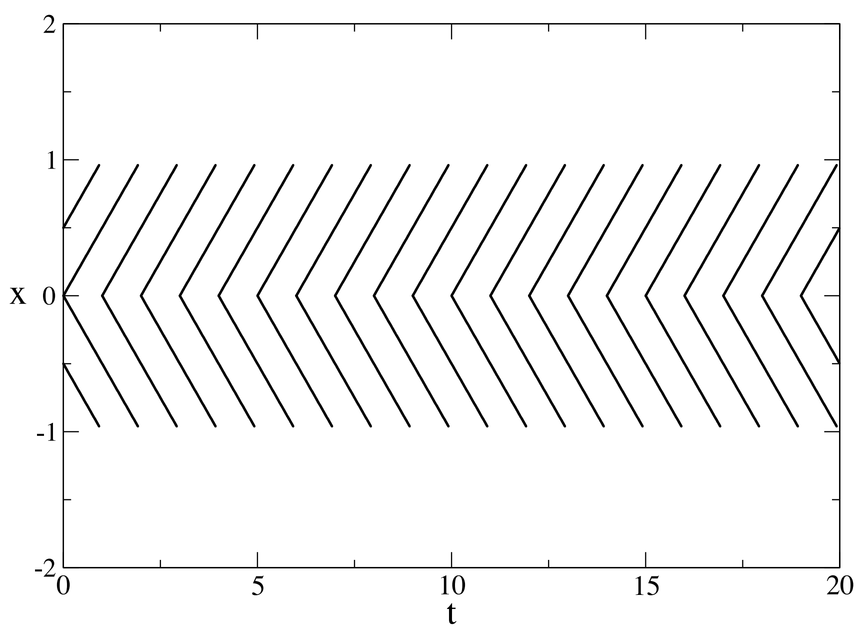

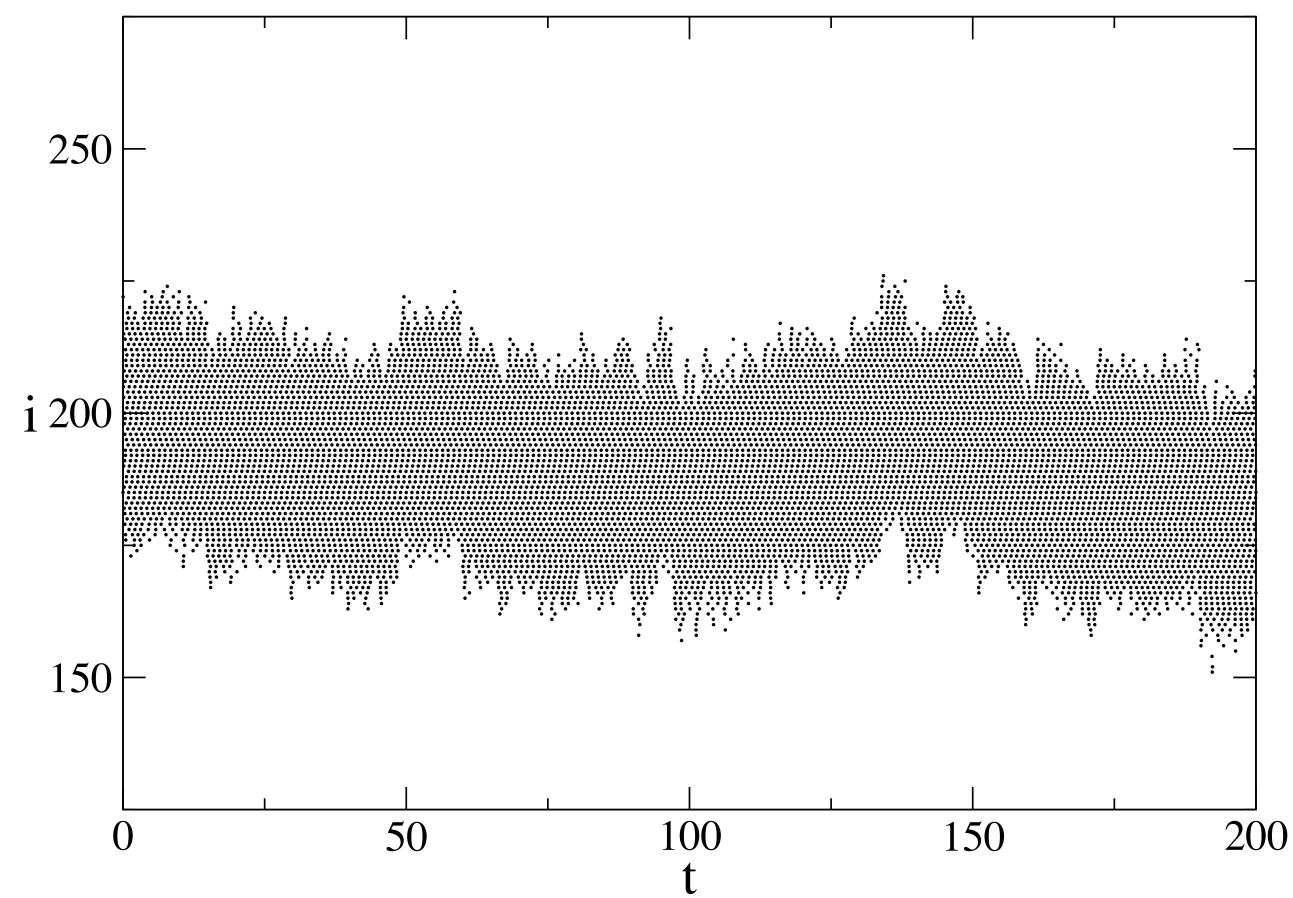

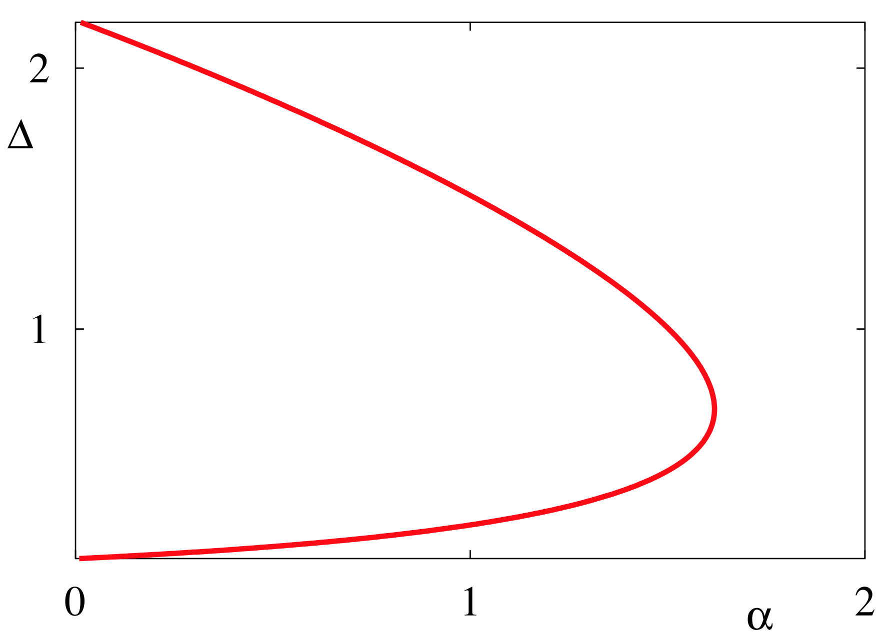

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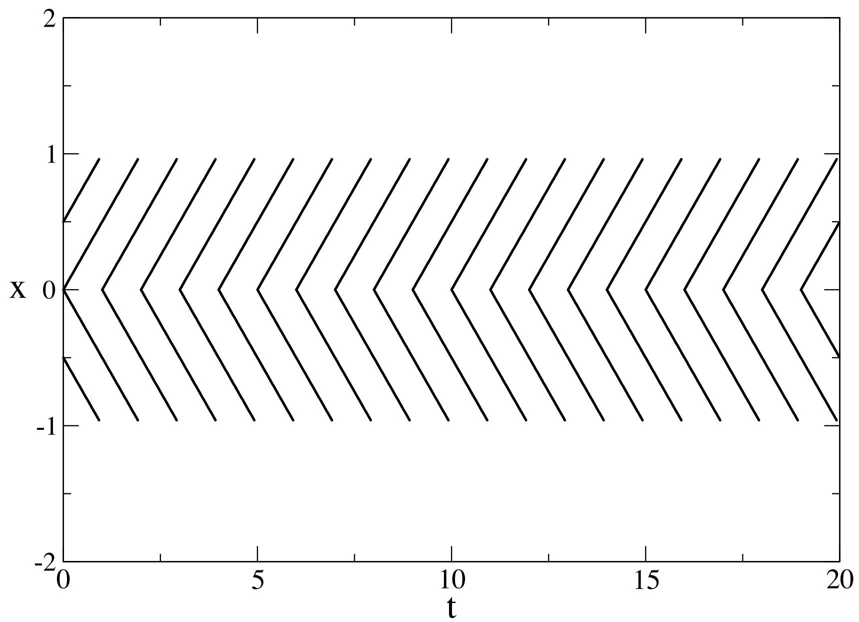
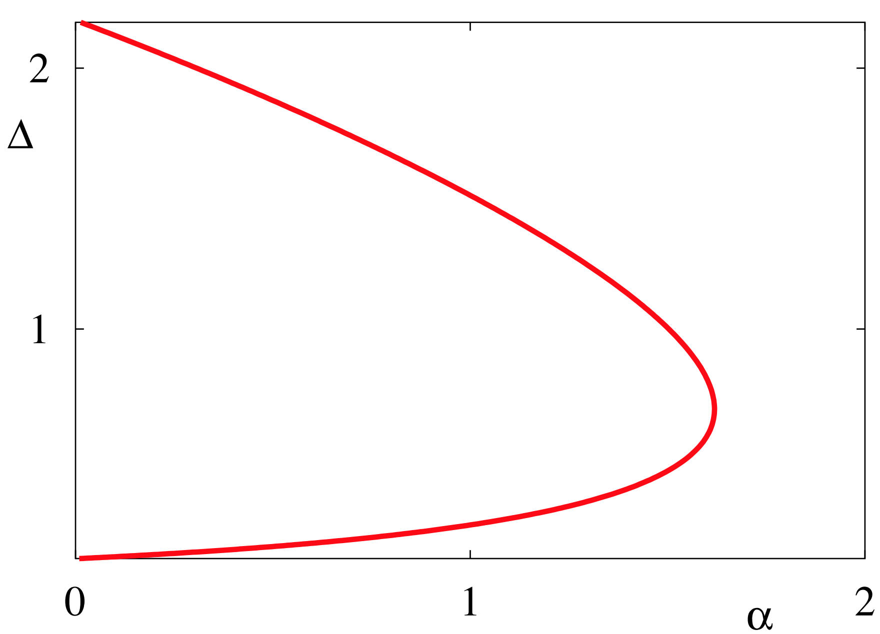
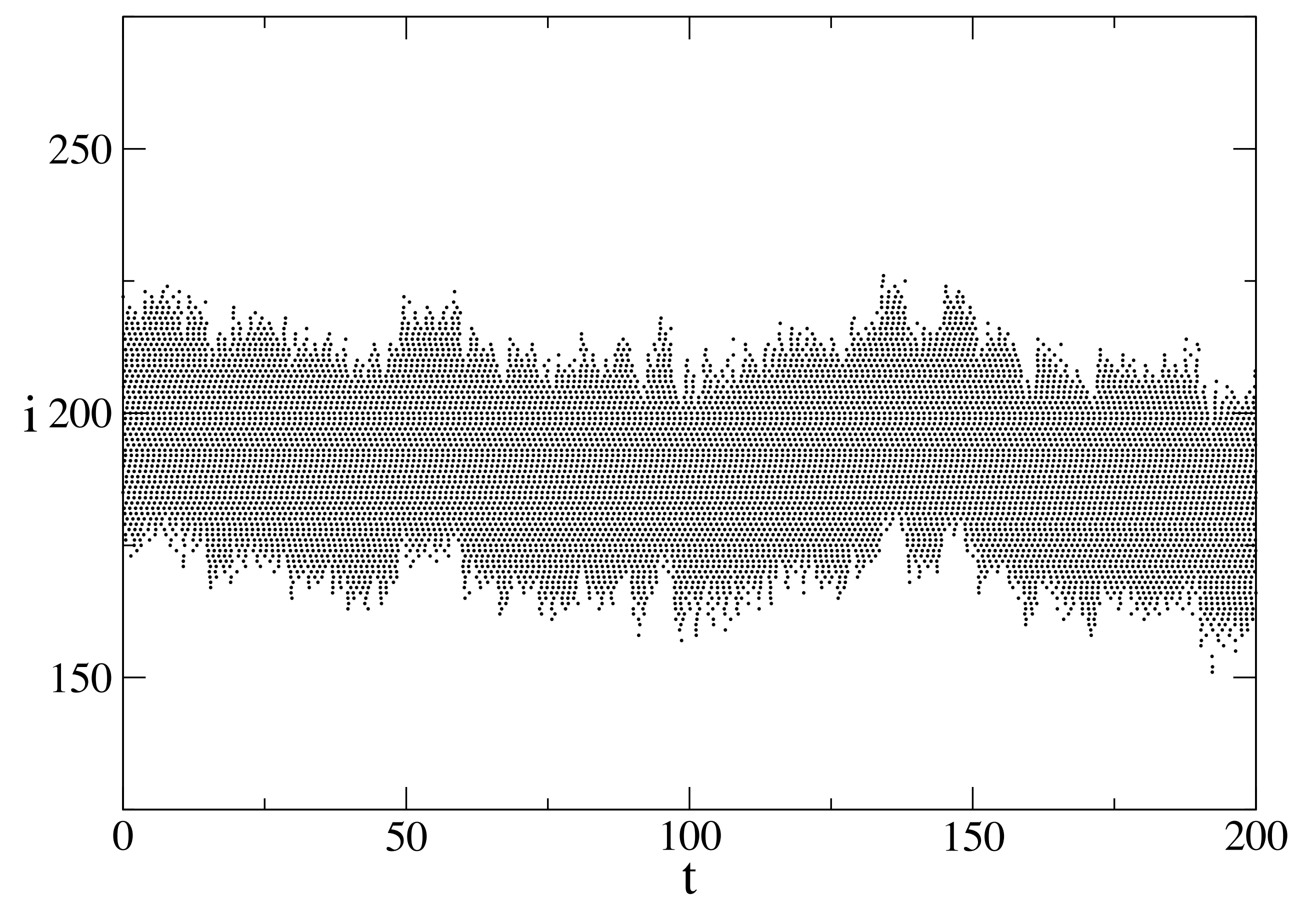
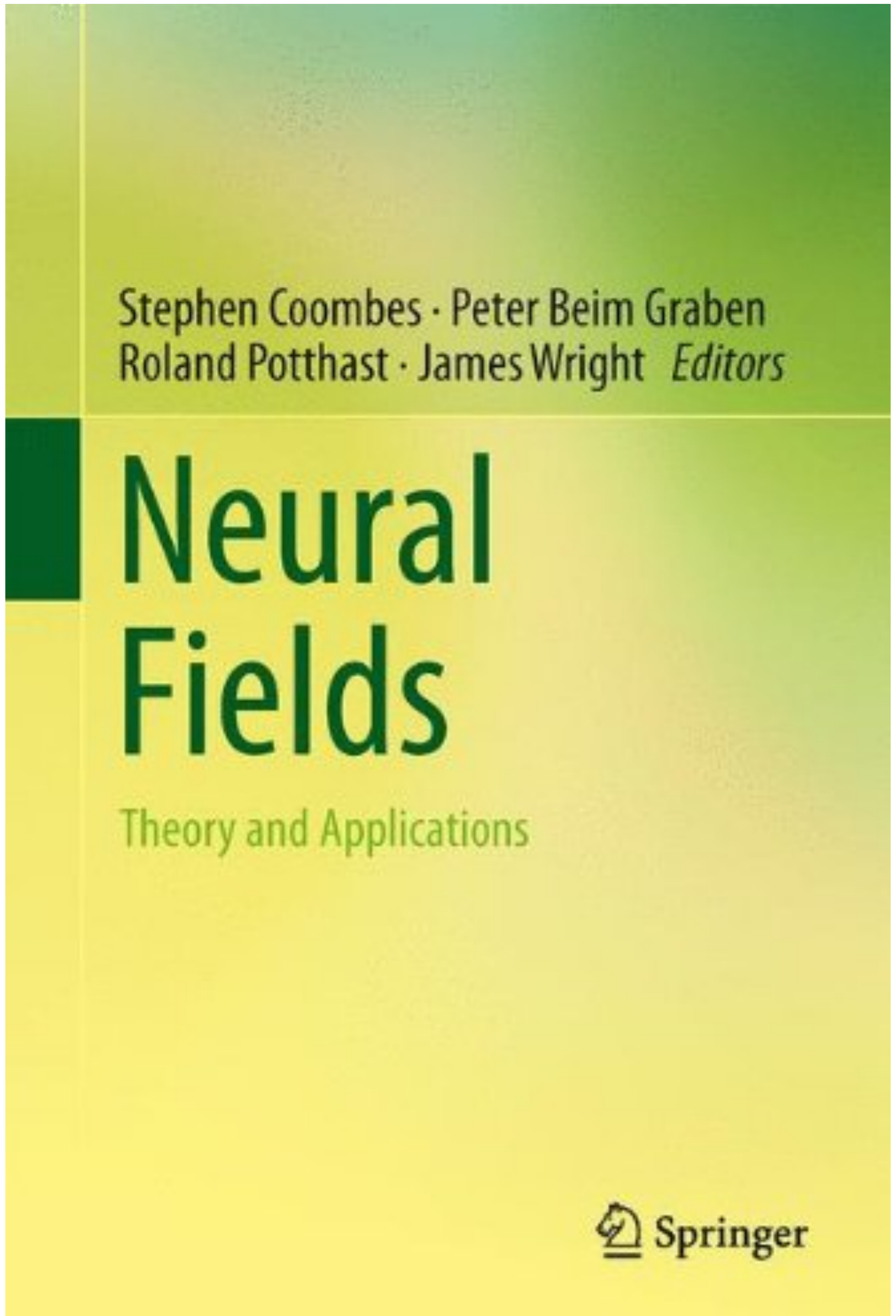
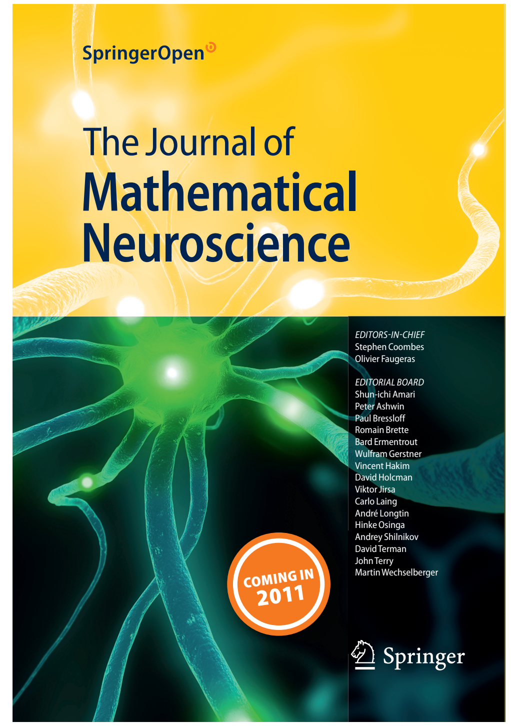






Is there an effective beyond MF model that has * as a solution ?





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