Long Wave Études

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Model Reduction across Disciplines
A celebration of Alexander Gorban's 60th birthday
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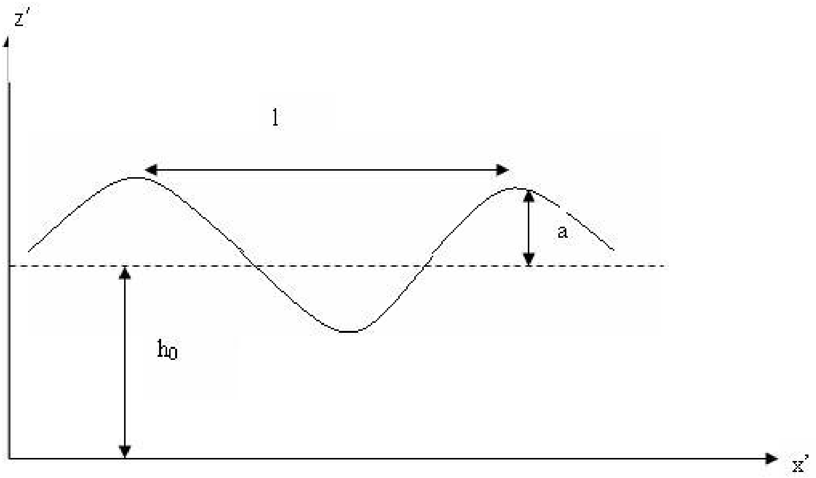
Plan of the talk

Long wave asymptotic expansion:

- Issues in hgher order terms.
- On the inconsistency of the Camassa-Holm model with the shallow water theory.



For potential incompressible flow $\mathbf{u}' = \nabla\phi'$, with zero surface tension and viscosity:

$$
\begin{aligned}
\phi'_{x'x'} + \phi'_{z'z'} &= 0 \\
\phi'_{z'} = 0 \quad &\text{at} \quad z' = 0 \\
\phi'_{z'} = \eta'_{x'}\phi'_{x'} + \eta'_{t'} \quad &\text{at} \quad z' = h_0 + \eta'(x', t') \\
\phi'_{t'} + \frac{1}{2}\left(\phi'^2_{x'} + \phi'^2_{z'}\right) + g\eta' = 0 \quad &\text{at} \quad z' = h_0 + \eta'(x', t')
\end{aligned}
$$

Shallow water waves asymptotic: We assume that

$$\epsilon = \frac{h_0^2}{L^2} \ll 1, \qquad \mu = \frac{a}{h_0} \ll 1,$$

$$\mu \geq \epsilon > \mu^2 \geq \mu\epsilon \geq \epsilon^2 > \mu^3 \geq \mu^2\epsilon \geq \mu\epsilon^2 \geq \epsilon^3 \ldots$$

In dimensionless variables x, z, t, η, ϕ

$$x' = Lx = \frac{h_0}{\sqrt{\epsilon}}x, \quad z' = h_0 z, \quad t' = \frac{L}{c_0}t = \frac{h_0}{\sqrt{\epsilon}c_0}t,$$

$$\eta' = a\eta = \mu h_0 \eta, \quad \phi' = \frac{\mu h_0 c_0}{\sqrt{\epsilon}}\phi, \quad c_0 = \sqrt{gh_0}$$

we assume that $\eta, \phi, \eta_t, \eta_x, \phi_x, \ldots = \mathcal{O}(1)$.

$$\begin{array}{lll} \epsilon\phi_{xx} + \phi_{zz} = 0 \quad \text{and} \quad \phi_z = 0 & & \text{at } z = 0, \\ \frac{1}{\epsilon}\phi_z = \mu\eta_x\phi_x + \eta_t & & \text{at } z = 1 + \mu\eta(x,t), \\ \phi_t + \frac{1}{2}\left(\mu\phi_x^2 + \frac{\mu}{\epsilon}\phi_z^2\right) + \eta = 0 & & \text{at } z = 1 + \mu\eta(x,t). \end{array}$$

$$\phi = \cos(\sqrt{\epsilon}z\frac{d}{dx})F(x,t) = F - \frac{\epsilon}{2}z^2F_{xx} + \frac{\epsilon^2}{24}z^4F_{xxxx} - \frac{\epsilon^3}{720}z^6F_{xxxxxx} + \mathcal{O}(\epsilon^4)$$

Theory of shallow water waves

We denote $w(x,t) = F_x$, then

$$
\begin{aligned}
0 = {} & \eta_t + w_x + (\eta_x w + \eta\, w_x)\,\mu - \frac{\epsilon}{6} w_{x,x,x} \\
& - \frac{\mu\epsilon}{2}\left(\eta_x w_{x,x} + w_{x,x,x}\eta\right) + \frac{\epsilon^2}{120} w_{x,x,x,x,x} \\
& - \frac{\mu^2\epsilon}{2}\left(2\eta_x w_{x,x}\eta + w_{x,x,x}\eta^2\right) + \frac{\mu\epsilon^2}{24}\left(w_{x,x,x,x,x}\eta + \eta_x w_{x,x,x,x}\right) - \frac{\epsilon^3}{5040} w_{x,x,x,x,x,x,x} +
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & w_t + \eta_x + \mu\, w w_x - \frac{\epsilon}{2}\left(w_{t,x,x} + 2\sigma\, \eta_{x,x,x}\right) \\
& - \frac{\mu\epsilon}{2}\left(2 w_{t,x}\eta_x + 2 w_{t,x,x}\eta + w w_{x,x,x} - w_x w_{x,x}\right) + \frac{\epsilon^2}{24} w_{t,x,x,x,x} \\
& + \mu^2\epsilon\left({w_x}^2\eta_x - 1/2\, w_{t,x,x}\eta^2 + w_x w_{x,x}\eta - w w_{x,x,x}\eta - w w_{x,x}\eta_x - w_{t,x}\eta\,\eta_x\right) \\
& + \frac{\mu\epsilon^2}{24}\left(4 w_{t,x,x,x,x}\eta + 2 w_{x,x} w_{x,x,x} - 3 w_x w_{x,x,x,x} + w_{x,x,x,x,x} w + 4 w_{t,x,x,x}\eta_x\right) \\
& - \frac{\epsilon^3}{720} w_{t,x,x,x,x,x,x} + \cdots
\end{aligned}
$$

Theory of shallow water waves

Reduction to unidirectional waves (Whitham, 1974)

$$
\begin{aligned}
&w = \eta + \mu\, a\, \eta^2 + \epsilon\, b\, \eta_{x,x} + \\
&\mu^2\, c\, \eta^3 + \mu\, \epsilon \left(d_1\, \eta_x^2 + d_2\, \eta \eta_{x,x} \right) + \epsilon^2\, e\, \eta_{x,x,x,x} \\
&+\mu^3\, f\, \eta^4 + \mu^2 \epsilon \left(g_1\, \eta^2 \eta_{x,x} + g_2\, \eta \eta_x^2 + g_3\, D_x^{-1}(\eta_x^3) \right) \\
&+\mu \epsilon^2 \left(h_1\, \eta \eta_{x,x,x,x} + h_2 \eta_x \eta_{x,x,x} + h_3\, \eta_{x,x}^2 \right) + \\
&\epsilon^3\, k\, \eta_{x,x,x,x,x,x} + \mathcal{O}(\mu^4, \mu^3 \epsilon, \mu^2 \epsilon^2, \mu \epsilon^3, \epsilon^4)
\end{aligned}
$$

$$
\{ a, b, c, d_1, d_2, e, f, g_1, g_2, g_3, h_1, h_2, h_3, k \} = \{ -\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{8}, \tfrac{3}{16}, \tfrac{1}{2}, \tfrac{1}{10}, -\tfrac{5}{64}, \tfrac{1}{8}, \tfrac{3}{32}, \tfrac{3}{16}, \tfrac{7}{20}, \tfrac{1091}{1440}, \tfrac{163}{360}, \tfrac{161}{1890} \}
$$

$$
\begin{aligned}
0 &= \eta_x + \eta_t + \frac{3}{2}\,\mu\,\eta_x\eta + \frac{\epsilon}{6}\,\eta_{x,x,x} \\
&-\frac{3}{8}\,\eta^2\eta_x\mu^2 + \left(\frac{5}{12}\,\eta\,\eta_{x,x,x} + \frac{23}{24}\,\eta_x\eta_{x,x}\right)\epsilon\,\mu + \epsilon^2\,\frac{19}{360}\,\eta_{x,x,x,x,x} \\
&+ \left(\frac{23}{16}\eta\,\eta_x\eta_{x,x} + \frac{5}{16}\eta^2\eta_{x,x,x} + \frac{19}{32}\eta_x{}^3\right)\epsilon\,\mu^2 \\
&+ \left(\frac{1079}{1440}\,\eta_{x,x,x,x}\eta_x + \frac{317}{288}\,\eta_{x,x}\eta_{x,x,x} + \frac{19}{80}\,\eta_{x,x,x,x,x}\eta\right)\epsilon^2\mu \\
&+ \left(\frac{55}{3024}\,\eta_{x,x,x,x,x,x,x}\right)\epsilon^3 + \frac{3}{16}\,\eta^3\eta_x\mu^3 + \mathcal{O}(\mu^4, \mu^3\epsilon, \mu^2\epsilon^2, \mu\epsilon^3, \epsilon^4)
\end{aligned}
$$

$$
\Omega = \sqrt{gk\tanh(kh_0)} \to \sqrt{\frac{k}{\sqrt{\epsilon}}\tanh(\sqrt{\epsilon}k)} \simeq k - \frac{\epsilon k^3}{6} + \frac{19\epsilon^2 k^5}{360} - \frac{55\epsilon^3 k^5}{3024}
$$

1. Merkant's paradox:

Assuming $\eta, \eta_x, \ldots \to 0$ as $x \to \pm\infty$ we get:

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta\, dx = \frac{3\epsilon\mu^2}{16}\int_{-\infty}^{\infty} \eta_x^3\, dx \neq 0 \text{ in general!}$$

2. The coefficients found in orders $\epsilon^3,\ \epsilon^2\mu,\ \epsilon\mu^2,\ \mu^3$ do not satsfy the criteria of asymptotic integrability (Kodama) and thus the problem of gravity water waves is not integrable (does not have higher conservation laws and/or symmetries).

R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
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R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, J. Fluid Mech. 455 (2002) 63–82.
R.S. Johnson, On solutions of the Camassa–Holmequation, Proc. R. Soc. Lond. A, 495 (2003) 1687–1708.

Yahoo! - 63.100 results; Google - 36.900 results.

[R.Bhatt and AVM]

Camassa–Holm equation:

$$V_T - V_{TXX} + 2\omega V_X + 3VV_X - 2V_X V_{XX} - VV_{XXX} = 0$$

It has one-soliton solution ($\omega > 0,\ c > 2\omega$):

$$V = \frac{c(c - 2\,\omega)}{c + 2\omega \sinh^2 \theta}$$

$$X - cT - X_0 = 2\sqrt{\frac{1}{(1 - 2\,\frac{\omega}{c})}}\theta + \ln\left(\frac{\cosh\left(\theta - \operatorname{arctanh}\left(\sqrt{1 - 2\,\frac{\omega}{c}}\right)\right)}{\cosh\left(\theta + \operatorname{arctanh}\left(\sqrt{1 - 2\,\frac{\omega}{c}}\right)\right)}\right).$$

As $\omega \to 0$ it approaches to a peakon solution (Johnson):

$$V \to c\exp(-|X - cT - X_0|).$$

Camassa-Holm "model"

Reduction to the CH equation (Dullin-Gotwald-Holm)
Galilean transformation: $X = x - \delta t, \;\; T = t.$
Kodama Transformation

$$\eta\,(X, T) = u + \mu\,(\alpha_1 u^2 + \alpha_2 u_X D_X^{-1} u) + \epsilon\,\beta\, u_{XX}$$

and applying the Helmholtz operator

$$H = 1 - \nu\,\epsilon\, D_X^2$$

to achieve the following:

- to vanish coefficients at the terms $u_{XXX}, \;\; u_X u^2, \;\; u_{XXXXX};$
- the coefficients at the terms $u_T, u_{TXX}, uu_X, u_{XX} u_X, u_{XXX} u$ satisfy conditions

$$C(u_{XX} u_X) : C(u_{XXX} u) = 2:1, \quad C(u_{TXX}) C(uu_X) : C(uu_{XXX}) C(u_T) = 3:1.$$

Then

$$\delta = \frac{9}{19}, \quad \alpha_1 = \frac{7}{20}, \quad \alpha_2 = -\frac{1}{5}, \quad \beta = \frac{1}{30}, \quad \nu = \frac{19}{60}.$$

Thus we get

$$u_T + \frac{10}{19} u_X + \mu \frac{3}{2} uu_X - \epsilon \frac{19}{60} u_{TXX} - \mu\epsilon \frac{19}{120} (2u_X u_{XX} + uu_{XXX}) = \mathcal{O}(\mu^3, \mu^2\epsilon, \mu\epsilon^2, \epsilon^3)$$

Camassa-Holm "model"

Ignoring terms in orders $\epsilon^3, \epsilon^2\mu, \epsilon\mu^2, \mu^3$ and after the re-scaling

$$V = u, \quad y = \frac{2\sqrt{15}}{\sqrt{19\epsilon}} X, \quad \tau = \frac{\mu\sqrt{15}}{\sqrt{19\epsilon}} T$$

this equation takes the "standard Camassa-Holm" form:

$$V_\tau - V_{\tau yy} + 2\omega V_y + 3VV_y - 2V_y V_{yy} - VV_{yyy} = 0$$

Where $\omega = \frac{10}{19\mu}$. In the special case $\omega = 0$ this equation is called "the peakon equation".

The next neglected term in the Camassa-Holm "model"

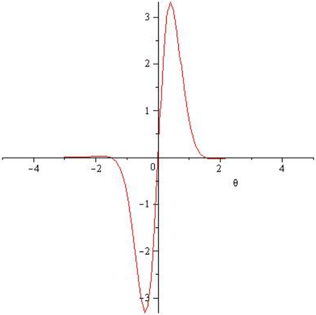
The correction to the equation due to the terms of orders $\epsilon^3, \epsilon^2\mu, \epsilon\mu^2, \mu^3$ is: (here $W_y = V$)

$$\frac{12}{361}\,\frac{V_{yy}V_y W^2}{\omega^2} + \frac{2440}{361}\,\frac{V_{yy}V_y V}{\omega} + \frac{2765}{722}\,\frac{{V_y}^3}{\omega} + \omega\frac{446}{2527}\,V_{yyyyyyy}$$
$$+\frac{1806}{361}\,V_y V_{yyyy} - \frac{24}{361}\,\frac{WV_{yyyy}V}{\omega} + \frac{24}{361}\,\frac{WV_{yy}^2}{\omega} - \frac{6}{361}\,\frac{W\;V_{yy}V^2}{\omega^2}$$
$$+\frac{610}{361}\,V_{yy}V_{yyy} + \frac{632}{361}\,V_{yyyyy}V + \frac{340}{361}\,\frac{V_{yyy}V^2}{\omega}$$

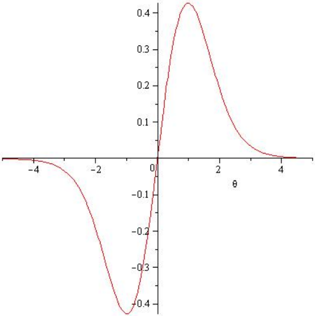
- The neglected terms do not have any small parameter (a power of ϵ, μ).
- Solutions of the C-H equation with size and amplitude of order 1 violate the long wave asymptotic assumption.
- The parameter ω, which should be zero for pickon solutions, appears in the denominator in the higher asymptotic terms.

The next neglected term in the Camassa-Holm "model"

Let we choose $\omega = 0.1$ and fix $c = 1$ in the solution.
Terms in the Camassa-Holm equation:



VV_{yyy}



V_τ

The next neglected term in the Camassa-Holm "model"

Neglected "correction" terms $\frac{446}{2527}\,\omega\, V_{y,y,y,y,y,y,y}$ and $\frac{632}{361}\, V V_{y,y,y,y,y}$: