On practical stability of a network of coupled nonlinear limit-cycle oscillators

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- Consider a nonlinear oscillator described by the Stuart-Landau equations

$$\dot{z} = -|z|^2 z + \mu z = f(z, \mu), \tag{1}$$

where $z \in \mathbb{C}$ is the state of the oscillator and $\mu = \beta + \mathrm{i}\omega \in \mathbb{C}$ is a complex parameter which defines asymptotic behavior of the oscillator.

- Dynamics of a network of N diffusively-coupled nonlinear oscillators is described by

$$\dot{\underline{z}} = F(\underline{z}) - \gamma \; L \; \underline{z}, \tag{2}$$

where constant parameter $\gamma \in \mathbb{R}_+$ defines the coupling strength, matrix $L \in \mathbb{R}^{N \times N}$ is a Laplacian matrix, $\underline{z} = \mathrm{col}(z_1, \ldots, z_N) \in \mathbb{C}^N$ and $F(\underline{z}) = [f(z_i, \mu_i)]_{i \in \{1, \ldots, N\}}$.

- Consider a nonlinear oscillator described by the Stuart-Landau equations

$$\dot{z} = -|z|^2 z + \mu z = f(z, \mu), \tag{3}$$

where $z \in \mathbb{C}$ is the state of the oscillator and $\mu = \beta + \mathrm{i}\omega \in \mathbb{C}$ is a complex parameter which defines asymptotic behavior of the oscillator.

- Dynamics of a network of N diffusively-coupled nonlinear oscillators is described by

$$\dot{\underline{z}} = F(\underline{z}) - \gamma \; L \; \underline{z}, \tag{4}$$

where constant parameter $\gamma \in \mathbb{R}_+$ defines the coupling strength, matrix $L \in \mathbb{R}^{N \times N}$ is a Laplacian matrix, $\underline{z} = \mathrm{col}(z_1, \ldots, z_N) \in \mathbb{C}^N$ and $F(\underline{z}) = [f(z_i, \mu_i)]_{i \in \{1, \ldots, N\}}$.

Formulation of the synchronization problem

Known fact : With the increase of the coupling strength the oscillators tend to synchronize their frequencies and their trajectories converge to a common limit cycle.

Two possible frameworks for synchronization analysis :

- Analysis of relative errors
  - Asymptotic stability : $z_i(t) - z_j(t) \to 0$ as $t \to \infty$
  - Practical stability : $\overline{\lim}_{t\to\infty} \|z_i(t) - z_j(t)\| \leq \delta(\gamma)$ and $\delta(\gamma) \to 0$ as $\gamma \to \infty$
- *Consensus* type analysis of the error relative to the "*averaged*" dynamics.
  - Asymptotic stability : $z_i(t) - z_m(t) \to 0$ as $t \to \infty$
  - Practical stability : $\overline{\lim}_{t\to\infty} \|z_i(t) - z_m(t)\| \leq \delta(\gamma)$ and $\delta(\gamma) \to 0$ as $\gamma \to \infty$

$\Rightarrow$ **Explicit formulation and stability analysis of the averaged dynamics $z_m(t)$.**

From static to dynamic consensus

	Static case	Dynamic case
• System model :	$\dot{\mathbf{x}} = -L\ \mathbf{x}$	$\Longrightarrow \dot{\mathbf{z}} = f(z, \lambda) + \tilde{L}\ \mathbf{z}$
• Averaged system : • State • Dynamics	$x_m = \frac{1}{N}\mathbf{1}^\top \mathbf{x}$ $\dot{x}_m = 0$	$\Longrightarrow z_m = \vartheta_{1l}^\top \mathbf{z}$ $\Longrightarrow \dot{z}_m = (\lambda_1 - c\lvert z_m\rvert^2)\ z_m + f_2(z_m, \mathbf{e})$
• Synchronization analysis : • Error • Error dynamics	$\mathbf{e} = \mathbf{x} - \mathbf{1}x_m$ $\dot{\mathbf{e}} = -L\mathbf{e}$	$\Longrightarrow \mathbf{e} = \mathbf{z} - \vartheta_r z_m$ $\Longrightarrow \dot{\mathbf{e}} = \tilde{L}\mathbf{e} + (\lambda_1 \mathbf{e} - PC(\mathbf{e}, z_m)(\mathbf{e} + z_m\mathbf{1}))$
• Stability properties :	Exponential stability $\mathbf{e} \to \mathbf{0}$	$\Longrightarrow$ Practical stability $\overline{\lim}\lVert f_2(z_m, \mathbf{e})\rVert \leq \delta(\gamma)$ and $\delta(\gamma) \to 0$ as $\gamma \to \infty$

Model description

Let us decouple network dynamics into linear and nonlinear parts

$$\dot{\mathbf{z}} = -C(\mathbf{z})\,\mathbf{z} + (\mathcal{M} - \gamma\, L)\mathbf{z} = -C(\mathbf{z})\,\mathbf{z} + A_\gamma \mathbf{z}, \qquad (5)$$

where $C(z) = diag\{|\mathbf{z}_1|^2, \ldots, |\mathbf{z}_N|^2\}$, $\mathcal{M} = diag\{\mu_1, \ldots, \mu_N\}$ and $A_\gamma = \mathcal{M} - \gamma\, L$ is a $N \times N$ complex symmetric matrix.
Let the eigenvalue of A_γ satisfy inequalities $Re(\lambda_1) \geq Re(\lambda_2) \geq \ldots \geq Re(\lambda_N)$.

Assumption 1

There exists $\gamma^* > 0$ such that for all $\gamma \geq \gamma^*$ there exists a complex orthogonal matrix V_γ such that matrix A_γ can be factorized as

$$A_\gamma = V_\gamma \Lambda_\gamma V_\gamma^\top,$$

where $\Lambda_\gamma \in \mathbb{C}^{N\times N}$ is a diagonal matrix whose diagonal elements are the eigenvalues of A_γ and we assume that $Re(\lambda_1(A_\gamma)) > Re(\lambda_2(A_\gamma))$.

Next, we present the matrix A_γ in the following form

$$A_\gamma = \lambda_1 \mathbb{I} + D$$

where $D = V_\gamma \tilde{\Lambda} V_\gamma^\top$ and $\tilde{\Lambda} = diag(0, \lambda_2 - \lambda_1, \ldots, \lambda_N - \lambda_1)$
$\Rightarrow$ The network dynamics can be written as

$$\dot{\mathbf{z}} = (\lambda_1 \mathbb{I} - C(\mathbf{z}))\,\mathbf{z} + D\,\mathbf{z}.$$

- Let $\vartheta_{\mathbf{r}}$ and $\vartheta_{\mathbf{l}}$ be the first columns of the matrices V_γ and $V_\gamma^\top$, *i.e* the right and left eigenvectors of A_γ (and D) corresponding to the largest eigenvalue $\lambda_1(A_\gamma)$ of the matrix A_γ (and $\lambda_1(D) = 0$).

- Define *averaged* Stuart-Landau oscillator as

$$\mathsf{z_m} = \vartheta_{\mathbf{l}}^* \mathbf{z}$$

- Define vector of synchronization errors $\mathbf{e} \in \mathbb{C}^N$ as

$$\mathbf{e} = \mathbf{z} - \vartheta_{\mathbf{r}} \mathsf{z_m} = \mathrm{P}\ \mathbf{z}$$

where $\mathrm{P} = I_{N\times N} - \vartheta_{\mathbf{r_1}} \vartheta_{\mathbf{l_1}}^*$ is a projection matrix.

Remark

Vector $\mathbf{e}$ *corresponds to the difference between the states* z_j *of the network oscillators and the scaled state* z_m *of the averaged system, while the eigenvector* $\vartheta_{\mathbf{r}} \in \mathbb{C}$ *corresponds to a rotation vector representing phase shifts of the oscillators relative to* z_m.

- Dynamics of the averaged oscillator of $\mathrm{z_m}$ is given by

$$\dot{\mathrm{z}}_\mathrm{m} = (\lambda_1 - \mathrm{c}|\mathrm{z_m}|^2)\, \mathrm{z_m} + \mathrm{f_2}(\mathrm{z_m}, \mathbf{e})$$

where $c = \sum_{i=1}^{N} \vartheta_{li}^* \; \vartheta_{ri}^* \; \vartheta_{ri}^2$ and

$$\begin{aligned}
\mathrm{f_2}(\mathrm{z_m}, \mathbf{e}) \; &= \\
&- \vartheta_{\mathbf{l1}}^* \begin{pmatrix} |z_1|^2 + z_1^* v_{r_{\mathbf{11}}} \mathrm{z_m} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |z_N|^2 + z_N^* v_{r_{\mathbf{1N}}} \mathrm{z_m} \end{pmatrix} \mathbf{e} \\
&- \vartheta_{l_\mathbf{1}}^* \begin{pmatrix} (v_{r1} \mathrm{z_m})^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (v_{rN} \mathrm{z_m})^2 \end{pmatrix} \bar{\mathbf{e}},
\end{aligned} \tag{7}$$

- Dynamics of synchronization errors $\mathbf{e}$ is given by

$$\dot{\mathbf{e}} = D\mathbf{z} + (\lambda_1 \mathbb{I} - P \; C(\mathbf{z}))\mathbf{z} \tag{8}$$

Practical stability of errors

Result 1

Consider the system (6) and let assumption 1 be satisfied. Then there exists a $\bar{\gamma} \geq \gamma^*$ such that the set $\mathcal{S}(\gamma) = \{\mathbf{z} \in \mathbb{C}^N : \mathbf{e}(\mathbf{z}) = 0\}$ is uniformly globally practically stable for all $\gamma \geq \bar{\gamma}$.

This result implies that for any arbitrary small $\delta > 0$ we can always find a coupling strength $\gamma^* > 0$ such that for the network of the Stuart-Landau oscillators with the coupling strength $\gamma \geq \gamma^*$ we have that synchronization errors errors $\mathbf{e}$ asymptotically satisfy the bound

$$|\mathbf{e}(t, \mathbf{z}_\circ)| \leq \delta \qquad \text{for all} \quad t \geq t^*.$$

Stability of the average dynamics

The dynamics of z_m can be seen as a perturbation of a Stuart-Landau oscillator with an input that depends on $\mathbf{e}$, that is

$$\dot{\mathrm{z}}_{\mathrm{m}} = \left(\lambda_1 - \mathrm{c}|\mathrm{z}_{\mathrm{m}}|^2\right) \mathrm{z}_{\mathrm{m}} + u, \tag{9}$$

where $u = \mathrm{f}_2(\mathrm{z}_{\mathrm{m}}, \mathbf{e})$.

Result 2

Consider a Stuart-Landau oscillator

$$\dot{z} = -a|z|^2 z + bz + u \tag{10}$$

with input $u \in \mathbb{C}$, state $z \in \mathbb{C}$ and $a, b \in \mathbb{C}$ are parameters such that $Re(a) > 0$. Then, the following statements hold :

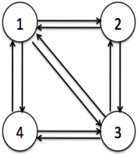
(1) if $Re(b) \leq 0$ then the origin $z \equiv 0$ is globally asymptotically stable (for $u \equiv 0$) and ISS for the system (8).

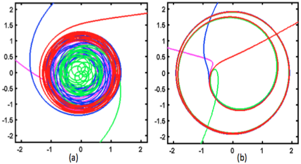
(2) If $Re(b) > 0$ then the limit cycle $\mathcal{A} = \{z : |z| = \sqrt{Re(a)/Re(b)}\}$ is almost globally asymptotically stable (for $u \equiv 0$) and almost ISS for the system (8).

Example

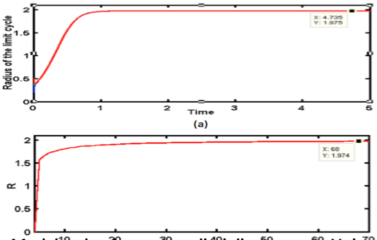
– Consider a symmetric network of four oscillators :

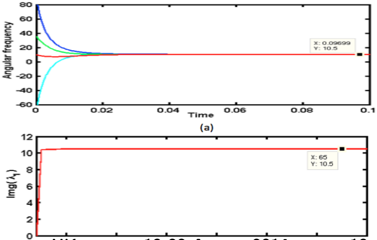
$$\frac{dz_j}{dt}=(\beta-|z_j|^2+i\ \omega_j)\ z_j+\gamma\ \sum_{i=1}^{4} a_{ji}\ (z_i-z_j).$$





– The synchronization frequency and the amplitude of synchronous limit cycle are defined by the eigenvalue $\lambda_1(A_\gamma)$ and its eigenvector.





Conclusions

- Synchronization of a network of diffusively coupled Stuart-Landau oscillators with symmetric interconnection graph was considered
- Practical stability of the limiting set was proven
- For general distribution of oscillators natural frequencies we gave approximate expressions for the limit cycle and synchronization frequency of the network.
- Practical stability of the limiting set was proven.