Exponential Energy Growth in adiabatically changing Hamiltonian Systems

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Energy Transfer from

slow massive bodies
to
fast light bodies

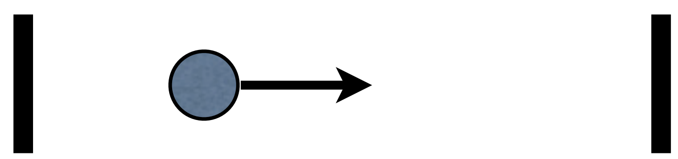
Fermi Acceleration

Fermi (1949)

Model for cosmic rays

Fermi - Ulam Model

1-d billiard with oscillating boundary



v_n velocity just before the nth collision

v_{n+1} velocity just after the nth collision

u_{n+1} velocity of the wall at the collision

Elastic reflection in the moving frame

$$v_{n+1} - u_n = -(v_n - u_n)$$

$$v_{n+1} = 2u_n - v_n$$

$$(-1)^{n+1} v_{n+1} = (-1)^n v_n + 2(-1)^{n+1} u_n$$

$(-1)^n v_n$ performs a random walk

Elastic reflection in the moving frame

$$\mathbb{E}\left(E_n\right) = \mathbb{E}\left(\frac{v_n^2}{2}\right) \sim n$$

Linear in the number of collisions

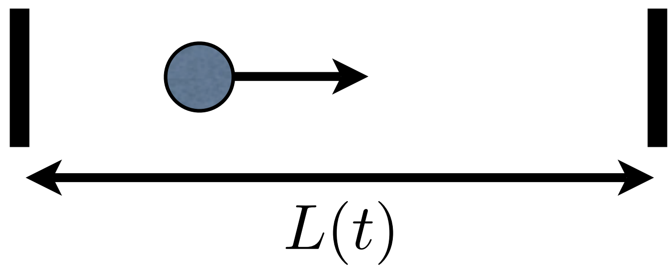
Quadratic in time

$$t_{n+1} - t_n \sim \frac{1}{\sqrt{n}}$$

$$t_n \sim \sqrt{n}$$

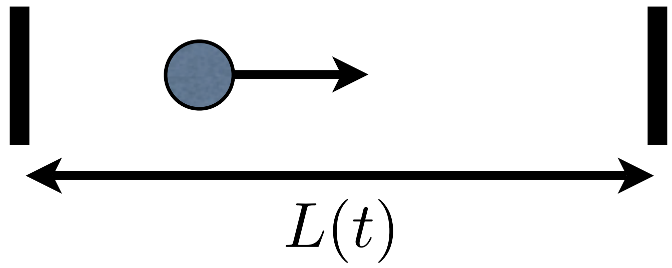
Smoothly moving boundary

No Accelaration



Smoothly moving boundary

No Accelaration



If *L(t)* is (quasi)periodic (and at least C^4) then KAM-theory implies that $L(t)v$ stays close to its initial value for *all* times

(Pustylnikov, Kamphorst, Sylva, Markaryan)

Chaos leads to Polynomial Energy Growth

In d-dimensional ($d \geq 2$) billiards with smoothly oscillating boundaries chaotic motion of the particles creates randomness necessary for the acceleration

Loskutov, Ryabov, Akinshin (numerics)

Polynomial Energy growth is fragile

arbitrarily weak linear dissipation stops the acceleration

Leonel and Bunimovich, Phys. Rev. Lett. 104, 224101 (2010)

Obstruction for Fast Growth

Ergodicity of the particle motion (frozen)

Rom Kedar, Gelfreich, Turaev, MacKay

Obstruction for Fast Growth

Adiabatic Invariance

$$E^{d/2}V \approx constant$$

Anosov-Kasuga invariant

$$\frac{dV}{dt} = 0$$

Fast Growth

Adiabatic Invariance

$$E^{d/2}(0)V(0) \approx E^{d/2}(T)V(T)$$

Ergodicity Breaking

$$V(0) \neq V(T)$$

Fast Growth

After one period

$$E(T) = E(0) \times \left(\frac{V(0)}{V(T)} \right)^{2/d}$$

Numerical Results

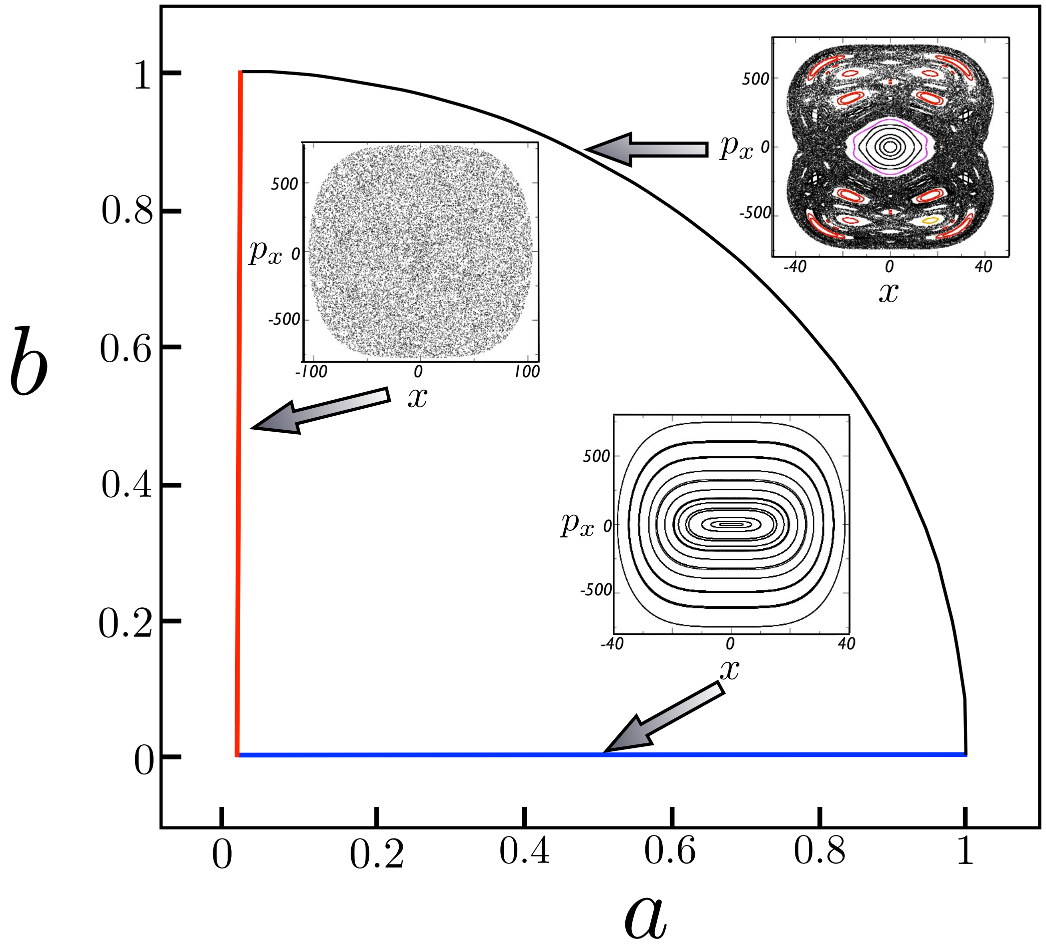
$$H(\boldsymbol{p}, \boldsymbol{q}, \tau(t)) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{a(t)}{4}\left(q_1^4 + q_2^4\right) + \frac{b(t)}{2}q_1^2 q_2^2$$

Homogeneous Potential

$a = b$ angular-like

$b = 0$ periodic

Parameter Space



Exponential Rate

$$E_{n+1} = E_n \zeta_n,$$

Homogeneity of the potential

ζ_n independent random variable

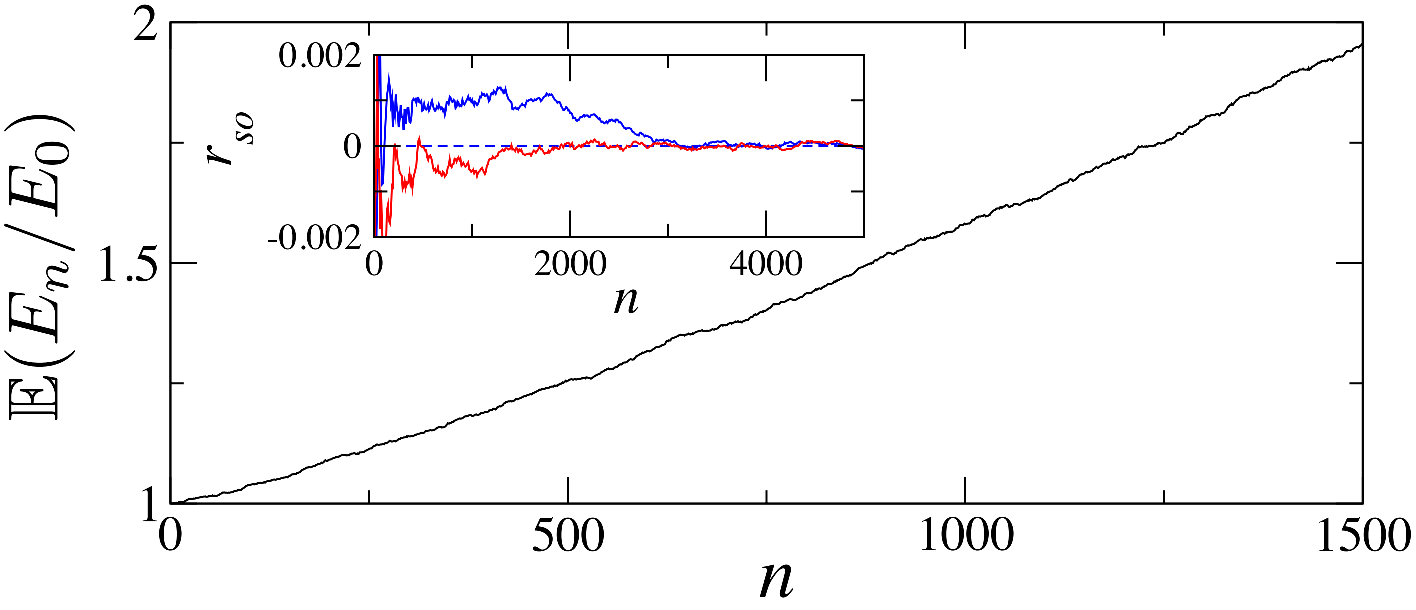
$\ln E_n$ performs a random walk

Exponential Rate

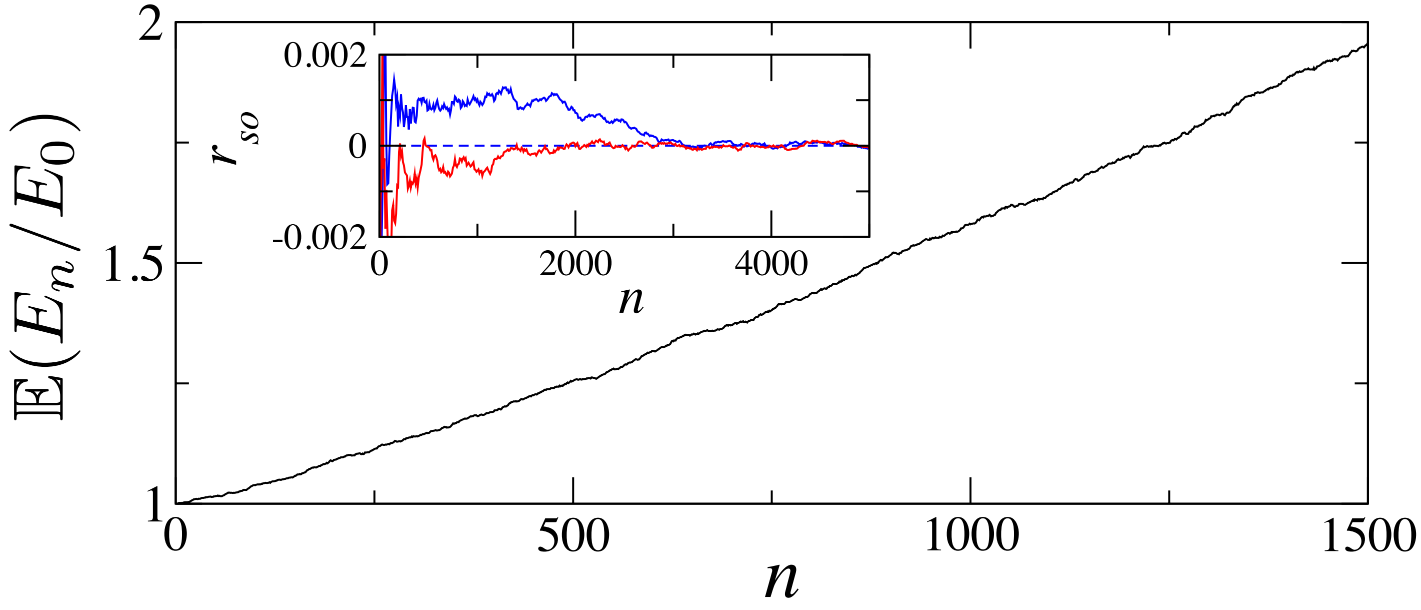
$$E_{n+1} = E_n \zeta_n,$$

$$r(n) = \frac{1}{n} \sum_{k=1}^{n} \ln \zeta_k$$

Ergodic Regime



Ergodic Regime



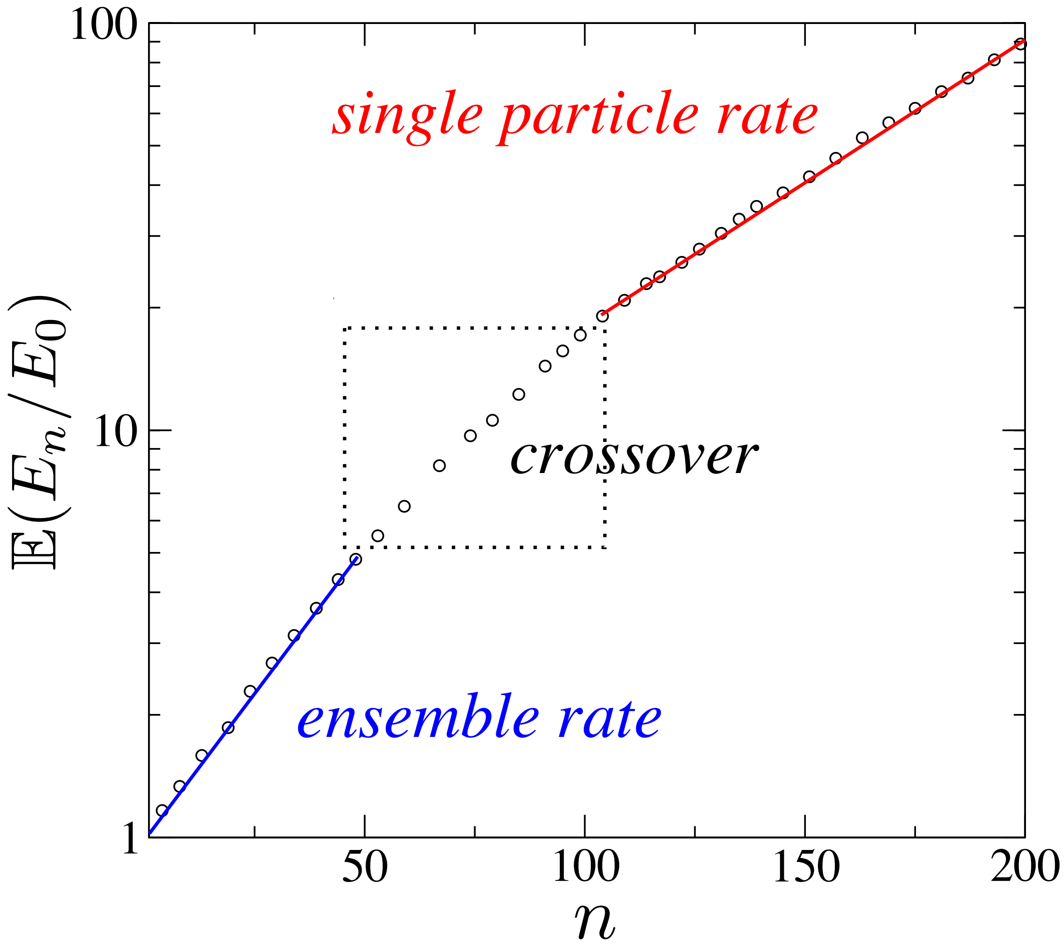
$$\mathbb{E}(\zeta_i) = 0$$

Ergodicity Breaking

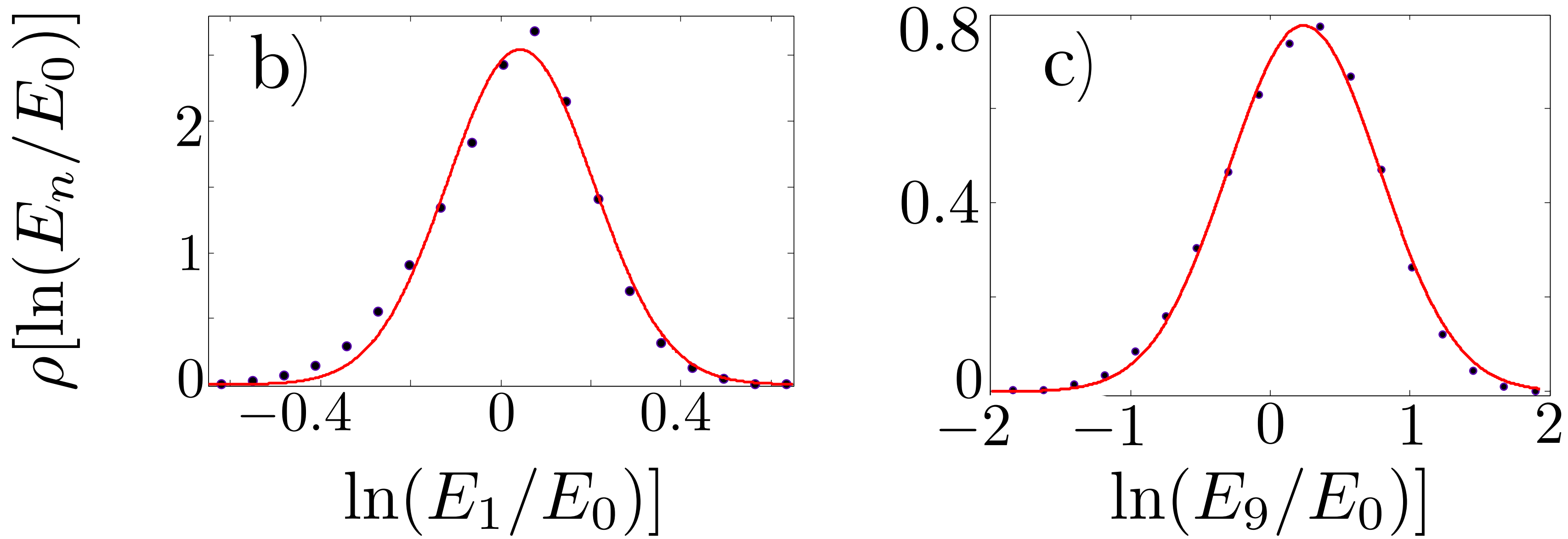
$$E_{n+1} = E_n \zeta_n,$$

$$\mathbb{E} \ln \zeta_i > 0,$$

Ergodicity Breaking



Ergodicity Breaking



Theory

Homogeneous Hamiltonians (few dof)

Periodic and adiabatically changing parameters

Introduce an entropy (fast chaos)

$$S = \ln \Omega$$

volume bounded by the energy surface

Theory

Homogeneous Hamiltonians (few dof)

Periodic and adiabatically changing parameters

Then

$$S = \ln \Omega \text{``}= \alpha \ln E\text{''}$$

$$\Delta S \geq 0$$

Theory

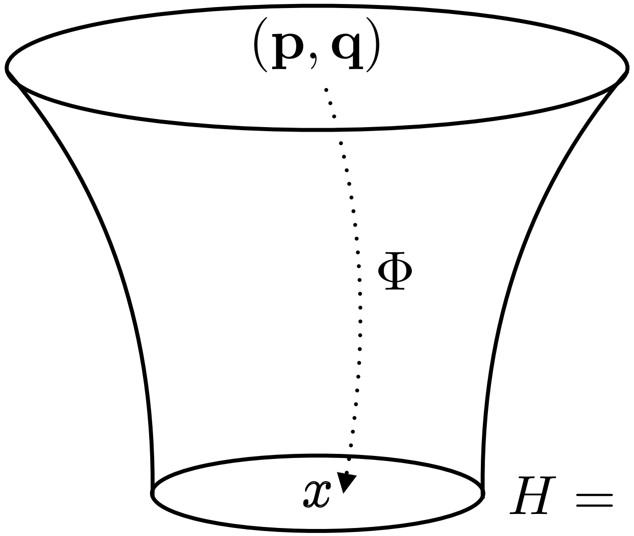
Ergodicity (Anosov-Kasuga)

$$\Delta S = 0$$

Generically

$$\Delta S = \theta > 0$$

New Coordinates



$v(M_1) + v(M_2) = 1$

$$J(E) = E^{\alpha},\ \alpha > 0$$

Slow-Fast System

Dynamics in the phase space is fast

Dynamics of energies is slow

$\ln E$ will be comparable with $\dot{\tau}$.

Dynamics of Energies

Starting

$$E = E_0,\ x \in M$$

After one period

$$E = \bar{E}_0 = e^{\lambda} E_0$$

Dynamics of Energies

Likewise

$$E = E_1,\ x \in M$$

move to

$$E = \bar{E}_1 = e^{\lambda} E_1,$$

Dynamics of Energies

The full system preserves the volume (Divergent free)

$$\alpha\lambda = \ln(v(M)/v(\bar{M}))$$

image of the set M by the flow of the full system after the period

Averaging Protocols

$$\dot{E} = \int \frac{\partial H}{\partial \tau} \delta(E - H) \mu_\tau(dx) \; \dot{\tau}$$

Entropy Formula

At a moment of strong chaos (relaxation to Liouville)

$$S = \alpha \int \langle \ln E \rangle dx$$

Changes in Entropy

$$\Delta S = \sum v(M_k) \ln \left[\frac{v(M_k)}{v(\bar{M}_k)} \right]$$

Changes in Entropy

Volumes $v(M_k) = v_k, \qquad v(\bar{M}_k) = \rho_k v_k$

Normalization $\sum \rho_k v_k = \sum v_k = 1$

$$\Delta S = -\ln\left(\prod \rho_k^{v_k}\right) \geq -\ln\left(\sum \rho_k v_k\right) = 0$$

Summing Up

Ergodicity (Anosov-Kasuga)

$$\Delta S = 0$$

Generically

$$\Delta S = \theta > 0$$

Exponential energy growth

Conclusion

Universal character (pumping at high order)

Hamiltonians with few dof are non-ergodic

Typical parameters changes leads to exp growth

Conclusion

Polynomial Potentials will "become"
homogeneous at high energies