

*From Boltzmann to Euler: Hilbert's 6th problem
revisited*

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Hilbert 6th problem

In Hilbert's 1900 address to ICM in Paris, he proposed the problem for the limiting process which lead from the atomistic view to the laws of motion of continua. **Passage from the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number $\varepsilon \rightarrow 0$.**

As of this moment, success in the problem has been limited to cases where the limiting continuum equations possess smooth solutions, i.e. before shock formation. (An excellent up to date survey may be found in book of L. St. Raymond).

Is there an inherent mathematical reason that makes Hilbert's 6th problem unattainable and can we pinpoint this reason in a simple mathematical form ?.

The Chapman-Enskog expansion

- The Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla f = \frac{Q(f, f)}{\varepsilon} \quad (1)$$

where $f = f(t, x, \xi)$ is the probability of finding a molecule of gas at point $x \in \mathbb{R}^3$, at time t , moving with velocity $\xi \in \mathbb{R}^3$. We note each ξ_i varies from $-\infty$ to ∞ and hence particles are allowed to have infinite velocities. The function f determines macroscopic fluid variables of density, momentum and temperature via its moments. Denote the density, momentum and temperature by ρ , $\rho \mathbf{u}$ and Θ . The Chapman–Enskog expansion is a formal power series in the Knudsen number $\varepsilon > 0$ for the function f in terms of these macroscopic variables (which we denote by M).

$$f_{CE}(M, \xi) = f^{(0)}(M, \xi) + \varepsilon f^{(1)}(M, \xi) + \varepsilon^2 f^{(2)}(M, \xi) + \dots \quad (2)$$

$f^{(0)}(M, \xi)$ is the usual local Maxwellian. Truncation at 0th order yields the balance laws of mass, momentum and energy for an elastic fluid, i.e. compressible gas dynamics of an ideal gas. Truncation at order ε yields the Navier–Stokes–Fourier equations, truncation at order ε^2 yields the Burnett equations. Truncation at order yields the super-Burnett equations and so forth. The success of the Chapman–Enskog expansion to deliver the well known Navier-Stokes-Fourier theory at order has motivated many to view the Navier–Stokes–Fourier theory as fundamental in computing shock structure. But as seen from a simple one dimensional analogy, this justification is questionable.

If one wishes to determine the shock structure for the scalar conservation law

$$u_t + uu_x = 0$$

by imposition of a viscous term one gets Burgers' equation:

$$u_t + uu_x = \varepsilon u_{xx}.$$

Look for travelling wave solutions of the Burgers' equation

$$u = u\left(\frac{x - ct}{\varepsilon}\right)$$

and recover the ordinary differential equation

$$-cu' + uu' = u''.$$

Thus in any study of non-smooth solutions to our original scalar conservation laws via imposition of higher gradient terms we see **there is no concept of small derivative terms, i.e. all terms are a priori the same magnitude.**

The ε has been scaled out ! Hence any study of non-smooth hydrodynamics via a truncation of the Chapman–Enskog expansion, while convenient is an illegitimate use of the Boltzmann equation.

"VATICINIUM EX EVENTU"

(A pseudo-prophecy written after the event)

James Kugel, How to read the bible, (2007).

Nevertheless the Chapman–Enskog expansion is an appealing tool since it allows us to cast Hilbert's question in the language and tools of partial differential equations.

But if truncations of C-E are illegitimate for non-smooth solutions of the fluid equations that leaves us only one recourse, i.e. [summation of the entire Champan-Enskog expansion](#). This is exactly what A. Gorban and I. Karlin have done in a remarkable series of papers.

Results of Gorbán and Karlin

Take the first 13 moments of the Boltzmann equations and then close the system by Grad's closure rule for f .

- The Grad's 13 moment system linearized about the rest state:

$$\begin{aligned}
 \partial_t \rho &= -\nabla \cdot \mathbf{u}, \\
 \partial_t \mathbf{u} &= -\nabla \rho - \nabla \Theta - \nabla \cdot \boldsymbol{\sigma}, \\
 \partial_t \Theta &= -\frac{2}{3} (\nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q}), \\
 \partial_t \boldsymbol{\sigma} &= -\left((\nabla \mathbf{u}) + (\nabla \mathbf{u})^T - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbf{I} \right) \\
 &\quad - \frac{2}{3} \left((\nabla \mathbf{q}) + (\nabla \mathbf{q})^T - \frac{2}{3} \nabla \cdot \mathbf{q} \mathbf{I} \right) - \boldsymbol{\sigma}, \\
 \partial_t \mathbf{q} &= -\frac{5}{3} \nabla \Theta - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3} \mathbf{q}, \\
 \rho &= \rho + \Theta.
 \end{aligned}$$

- A simplified 10 moment theory in one space dimension:

$$\begin{aligned}\partial_t p &= -\frac{5}{3}\partial_x u, \\ \partial_t u &= -\partial_x p - \partial_x \sigma, \\ \partial_t \sigma &= -\frac{4}{3}\partial_x u - \frac{\sigma}{\varepsilon}.\end{aligned}$$

(Rescaled $x = x'/\varepsilon$, $t = t'/\varepsilon$ and then dropped the prime).

The role of the Knudsen number ε becomes apparent, i.e. it is an ordering tool for the Chapman–Enskog expansion. Write the C-E expansion

$$\sigma_{CE} = \varepsilon\sigma^{(0)} + \varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \dots$$

where $\sigma^{(n)}$ depend on p , u and their space derivatives.

$$\sigma_{CE} = -\frac{4}{3} \left(\varepsilon\partial_x u + \varepsilon^2\partial_x p + \frac{\varepsilon^3}{3}\partial_x^3 u + \dots \right).$$

Truncation as orders $\varepsilon, \varepsilon^2, \varepsilon^3$, yields dispersion relations

- (Navier-Stokes order):

$$\omega_{\pm} = -\frac{2}{3}k^2 \pm \frac{i|k|}{3} \sqrt{4k^2 - 15}.$$

- (Burnett order):

$$\omega_{\pm} = -\frac{2}{3}k^2 \pm \frac{i|k|}{3} \sqrt{8k^2 + 15}.$$

- (Super-Burnett order): we have a **Bobylev instability** for $k^2 > 3$, k being the frequency in Fourier space.

$$\omega_{\pm} = \frac{2}{9}k^2(k^2 - 3) \pm \frac{i|k|}{9} \sqrt{4k^6 - 24k^4 - 72k^2 - 135}.$$

The goal of Gorban and Karlin was summation, NOT truncation.

Take the Fourier transform of σ_{CE} and sum the series.

$$\begin{aligned}\hat{\sigma}_{CE} &= \sum_{n=0}^{\infty} -ika_n(-k^2)^n \hat{u} + \sum_{n=0}^{\infty} -k^2 b_n(-k^2)^n \hat{p} \\ &= -ikA(k^2)\hat{u} - k^2B(k^2)\hat{p}\end{aligned}$$

where

$$A(k^2) = \sum_{n=0}^{\infty} -ika_n(-k^2)^n, \quad B(k^2) = \sum_{n=0}^{\infty} b_n(-k^2)^n.$$

The good fortune in this example is that the sums A and B are related via the formula

$$A = \frac{B}{1 - k^2 B},$$

and that if B is written as $C = k^2 B$, then C satisfies the fundamental cubic equation

$$-\frac{5}{3}(1 - C)^2 \left(C + \frac{4}{5} \right) - \frac{C}{k^2} = 0. \quad (3)$$

Eq. (3) has one real and two complex roots. The real root is the one of interest to us and is negative for $k^2 > 0$, $C(0) = 0$, and monotone decreasing in k^2 , with asymptotic limit $C \rightarrow -4/5$ as $k^2 \rightarrow \infty$.

Hence A, B are known functions $A, B < 0$ for $k^2 > 0$.

In Fourier space, the hydrodynamics becomes

$$\begin{aligned}\hat{p}_t &= \frac{5}{3}ik\hat{u}, \\ \hat{u}_t &= ik\hat{p} + ik(-ikA(k^2)\hat{u}(t, k) - k^2B(k^2)\hat{p}),\end{aligned}$$

We set

$$\hat{p}(t, k) = e^{\omega t}P(k), \hat{u}(t, k) = e^{\omega t}U(k).$$

Then P and U satisfy

$$\begin{bmatrix} -\omega & \frac{5}{3}ik \\ ik - ik^3B & k^2A - \omega \end{bmatrix} \begin{bmatrix} P \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Set the determinant of the coefficient matrix to zero and find the dispersion relation (spectrum in Fourier space)

$$\omega^2 - \omega k^2 A + \frac{5}{3}(k^2 - k^4 B) = 0, \quad \text{i.e.,}$$

$$\omega = \frac{1}{2} \left(\frac{C}{1-C} \right) \pm |k| \left(\left(\frac{C}{1-C} \right)^2 \frac{1}{k^2} - \frac{20}{3}(1-C) \right)^{1/2}$$

Recall C satisfies the cubic

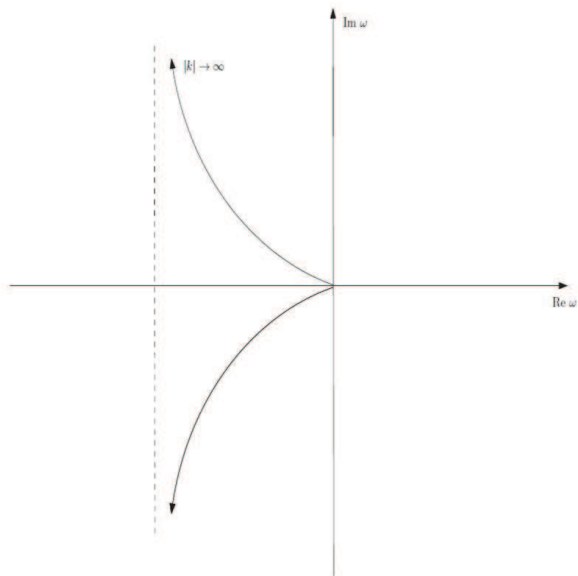
$$-\frac{5}{3}(C-1)^2 \left(C + \frac{4}{5} \right) = \frac{C}{k^2},$$

and hence $-\frac{5}{3}C \left(C + \frac{4}{5} \right) = \frac{C^2}{(1-C)^2} \frac{1}{k^2}$ and the formula for ω becomes

$$\omega = \frac{1}{2} \left(\frac{C}{1-C} \right) \pm i|k| \left(\frac{5C^2 - 16C + 20}{3} \right)^{1/2},$$

where we note that the quadratic form $5C^2 - 16C + 20$ is always positive. Since $C \rightarrow -\frac{4}{5}$ as $|k| \rightarrow \infty$ we see $Re\omega \rightarrow -\frac{2}{9}$ as $|k| \rightarrow \infty$

A sketch of the spectrum is given in Figure 1.



One issue not addressed by Gorban and Karlin in their papers was derivation of **an entropy equality**:

$$\begin{aligned} \frac{1}{2} \partial_t \left(\frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 \right) - ik(\bar{p}\hat{u} + \hat{p}\bar{u}) \\ = k^2 A(k^2) |\hat{u}|^2 + ik\bar{u}(-k^2 B(k^2)\hat{p}). \end{aligned}$$

But now use the relation $\frac{3}{5}\bar{p} = ik\bar{c}$ to write the equality as

$$\begin{aligned} \frac{1}{2} \partial_t \left(\frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 - \frac{3}{5} k^2 B(k^2) |\hat{p}|^2 \right), \\ - ik(\bar{p}\hat{u} + \hat{p}\bar{u}) = k^2 A(k^2) |\hat{u}|^2. \end{aligned}$$

This is the ‘entropy equality in Fourier space. Notice the entropy $\frac{1}{2} \left(\frac{3}{5} |\hat{p}| + |\hat{u}|^2 - \frac{3}{5} k^2 B(k^2) \right)$ and the dissipation $k^2 A(k^2) |\hat{u}|^2$ each have the desired signs (positive and negative, respectively). Since A and B are both negative for $k \neq 0$, integration yields

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 dk + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk \\ & \quad + \int_{-\infty}^{\infty} \left(\overline{\frac{\partial p}{\partial x}} \right) \hat{u} + \left(\overline{\frac{\partial u}{\partial x}} \right) \hat{p} dk \\ & = \int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}| dk \end{aligned}$$

or with an application of Parseval's identity

$$\begin{aligned} \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |p|^2 + |u|^2 dx + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk \\ = \int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk. \end{aligned}$$

The term

$$\int_{-\infty}^{\infty} -\frac{3}{5} k^2 B(k^2) |\hat{p}|^2 dk$$

represents a **non-local version of the capillarity**, where as the term

$$\int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk$$

is a **non-local version of the viscous dissipation**.

- Energy identity

$$\begin{aligned} \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{\rho} |p|^2 + |u|^2 dx + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{\rho} k^2 B(k^2) |\hat{p}|^2 dk \\ = \int_{-\infty}^{\infty} k^2 A(k^2) |\hat{u}|^2 dk. \end{aligned}$$

Since A, B are negative for $k^2 > 0$ we can interpret the above energy identity as

$$\begin{aligned} \partial_t(\text{MECHANICAL ENERGY}) + \partial_t(\text{CAPILLARITY ENERGY}) \\ = \text{VISCIOUS DISSIPATION} \end{aligned}$$

just as given in Korteweg's theory of capillarity. Exact summation of the Chapman–Enskog expansion has yielded a non-local version of Korteweg's theory and not Navier–Stokes theory.

Why is the theory non-local?

Non-locality not because of any physical mechanism but because of the truncation of our moment expansion. In fact Boillat and Ruggeri have shown that maximum waves speeds for moment truncation satisfy the inequality

$$V_{\max}^2 \geq \frac{6}{5} \sqrt{\frac{5}{3} \frac{k}{m}} \Theta(N - 1/2)$$

where N = number of moments.

Hence, we expect as the number of moments goes to infinity our non-local Korteweg theory would approach a linearization of Korteweg's local theory:

$$\begin{aligned}\partial_t \rho + \partial_i(\rho u_i) &= 0, \\ \partial_t(\rho u_i) + \partial_j(\rho u_i u_j) &= \partial_j T_{ij},\end{aligned}$$

where the Cauchy stress for Korteweg's theory is

$$\begin{aligned}T &= T^E + T^V + T^K, \\ T_{ij}^E &= -\rho \psi'(\rho) \delta_{ij}, \quad \rho^2 \psi'(\rho) = p(\rho), \\ T_{ij}^V &= \lambda(\text{tr} D) \delta_{ij} + 2\mu D_{ij} \\ D_{ij} &= \frac{1}{2}(\partial_j u_i + \partial_i u_j), \\ \lambda &= -\frac{2}{3}\mu, \quad \mu > 0, \\ T_{ij}^K &= \alpha \rho \partial_i(\rho \partial_j \rho) - \alpha \rho \partial_i \rho \partial_j \rho, \quad \alpha > 0.\end{aligned}$$

- **The full nonlinear energy balance equation:**

$$\begin{aligned}
 & \partial_t \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho \psi(\rho) + \frac{\alpha}{2} \rho \partial_i \rho \partial_i \rho \right) + \partial_j \left[u_j \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \rho \psi(\rho) - \frac{\alpha}{2} \rho \partial_i \rho \partial_i \rho \right) \right] \\
 & \quad + \partial_j [\alpha \rho (\partial_t \rho \partial_j \rho + u_i \partial_i \rho \partial_j \rho)] + \partial_j (u_i T_{ij}) + \mu (\partial_j (u_i \partial_i u_j) - \partial_i (u_i \partial_j u_j)) \\
 & = -(\lambda + \mu) (\partial_i u_i)^2 - \mu (\partial_j u_i) (\partial_j u_i) \\
 & \leq 0.
 \end{aligned}$$

(Here repeated indices imply summation.)

Summary

Gorban and Karlin's summation has shown us that we may reasonably conjecture that the sum of Chapman-Enskog expansion will yield a local version of Korteweg's theory of capillarity. Implication of Gorban and Karlin's summation for Hilbert's 6th problem. The whole issue may be seen in the energy balance. If we put the Knudsen number scaling into (reference "3.2") the coefficient α is actually a term $\alpha_0 \varepsilon^2$ and to recover the classical balance of energy of the Euler equation would require the sequence

$$\varepsilon^2 \rho^\varepsilon \partial_i \rho^\varepsilon \partial_i \rho^\varepsilon \rightarrow 0$$

in the sense of distributions of $\varepsilon \rightarrow 0$

This would require a strong interaction with viscous dissipation. The natural analogy is given by the use of the KdV-Burgers equation:

$$u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx}, \quad (4)$$

where at a more elementary level we see the competition between viscosity and capillarity. The result is known but far from trivial. In the absence of viscosity we have the KdV equation

$$u_t + uu_x = -K\varepsilon^2 u_{xxx}. \quad (5)$$

and we know from the results of Lax and Levermore that as $\varepsilon \rightarrow 0$, solution of (5) will not approach solution of the conservation law

$$u_t + uu_x = 0, \quad (6)$$

after the breakdown time of smooth solutions of (6). On the other hand, addition of viscosity which is sufficiently strong, i.e. K sufficiently small in (4) will allow passage as $\varepsilon \rightarrow 0$ to a solution of (6). This has been proven in the paper of M.E. Schonbek.

Are we in the Lax-Levermore case (5) or the Schonbek case (4) with K sufficiently small ? Rewrite hydrodynamics as the second order equation

$$\frac{3}{5}\hat{p}_{tt} + k^2\hat{p} + k^2 \left(-A(k^2) \left(\frac{3}{5} \right) \hat{p}_t - k^2 B(k^2)\hat{p} \right) = 0$$

and then attempt to write it in factored form

$$\left(\frac{3}{5} \frac{\partial}{\partial t} + \mu_1(k) \right) \left(\frac{\partial}{\partial t} + \mu_2(k) \right) \hat{p} = -k^2 \hat{p}.$$

A comparison yields

$$\begin{aligned}\mu_1(k) + \frac{3}{5}\mu_2(k) &= -\frac{3}{5}k^2A(k^2) \\ \mu_1(k)\mu_2(k) &= -k^4B(k^2)\end{aligned}$$

Define \hat{v} by the formula

$$\frac{\partial \hat{p}}{\partial t} + \mu_2(k)\hat{p} = ik\hat{v}$$

system now takes the form

$$\begin{aligned}\frac{3}{5}\partial_t \hat{v} &= ik\hat{p} - \mu_1(k)\hat{v} \\ \partial_t \hat{p} &= ik\hat{v} - \mu_2(k)\hat{p}\end{aligned}$$

or in physical space-time

$$\begin{aligned} \frac{3}{5} \partial_t \mathbf{v} &= -\partial_x \mathbf{p} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mu_1(k) \hat{\mathbf{v}}(k, t) dk, \\ \partial_t \mathbf{p} &= -\partial_x \mathbf{p} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mu_2(k) \hat{\mathbf{p}}(k, t) dk. \end{aligned}$$

Formulas for $\mu_1(k), \mu_2(k)$ are

$$\begin{aligned} 2\mu_1(k) &= -\frac{3}{5} \left(\frac{C}{1-C} \right) \pm \frac{3}{5} i |k| \left(\frac{5C^2 - 16C}{3} \right)^{1/2}, \\ 2\mu_2(k) &= -\left(\frac{C}{1-C} \right) \pm i |k| \left(\frac{5C^2 - 16C}{3} \right)^{1/2}. \end{aligned}$$

Since $5C^2 - 16C > 0$ (remember $C < 0$), the damping coefficients $\mu_1(k), \mu_2(k)$ are in fact COMPLEX valued reflecting the capillarity effect in the system.

Epilogue

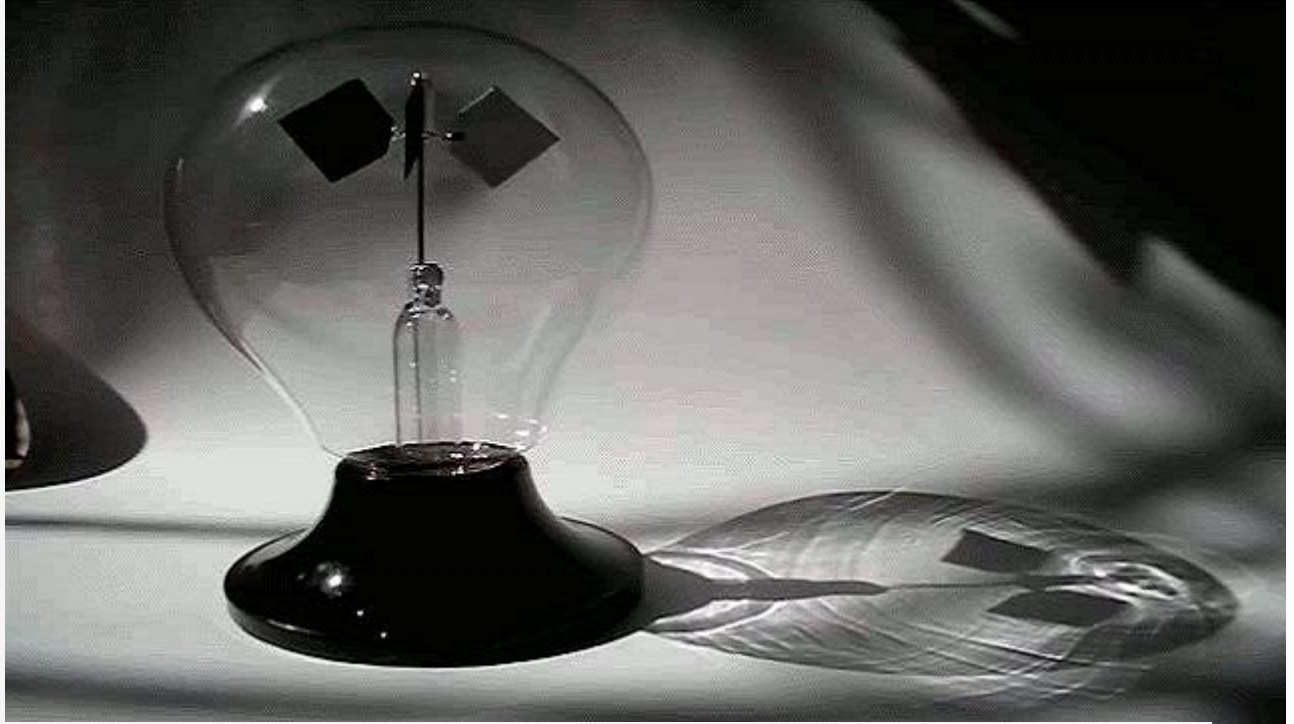
Things are not looking too promising for a possible resolution of Hilbert's 6th problem. It appears in the competition between viscosity and capillarity, capillarity has become a very dogged opponent, and the capillarity energy will not vanish in the limit as $\varepsilon \rightarrow 0$. Hilbert's hope may have been justified in 1900, but serious doubts are now apparent. Finally Boltzmann's theory has proven to be a useful model for rarefied gases; Euler's theory has proven to be a useful model for dense gases; maybe they must remain two INDEPENDENT theories and UNIFICATION will NOT succeed for non-smooth (shock) dynamics.

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The Light Mill, a toy that reveals how we might one day power space ships



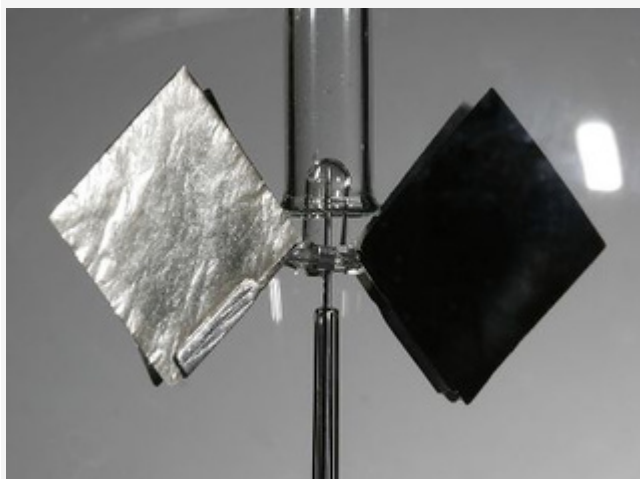
The Light Mill never reached the popularity of the Drinking Bird or Newton's Cradle, but chances are you've seen it. It looks like a light bulb with four spinning paddles inside. Although no power is pumped through the mill, it spins as long as light hits it. Find out how these little mills work, and how some day we might use them to jet around nebulae.

The Light Mill is one of the many toys that were invented well before anyone knew how they worked. And ultimately, they turned out to hold the key to a source of energy we might one day exploit when we're a fully space-going species.

The first Light Mill was built by chemist William Crookes in 1873. It's a small enclosed bubble of glass that has had most of the air pumped out of it, with four little squares of metal attached to a rotating weather-vane-like contraption inside. One side of each paddle is painted black while the other is silver. As soon as light hits the Light Mill, the paddles begin to spin. The black side of the paddle always turns away from the light and the silver side always turns into the light. Crookes

called it the Crookes Radiometer, but it became more well-known as the Light Mill, because he believed that the pressure of the light on the black side caused the paddles to turn.

He was not correct. Since black absorbs photons and silver reflects it, the silver should turn away from the light. Photons exert pressure on a substance when they're absorbed, yes, but when they bounce off, they impart twice the pressure that absorbed photons do. If it were photon driven, the mill would turn the other way. Any debate on that question was silenced when it was found that, when all the air was pumped out of the mill and there was only a vacuum inside, the paddles stopped turning. If light were the medium turning the paddles, they would turn faster in a vacuum. Clearly the sparse amount of gas inside was turning the paddles, but how?



SEXPAND

It took six years and many wrong guesses for someone to finally come up with the answer. In 1879 Osborne Reynolds came up with a little thing he called 'thermal transpiration.' If a porous plate is suspended in an evenly-pressurized gas, and one side of that plate is heated, the gas flows from the cold side to the hot side. This accumulation of gas on the hot side of the plate causes the pressure to build up on the hot side, and the plate moves. But the squares of metal in the Light Mill aren't porous. Instead of thermal transpiration, the plates experience what's called thermal creep. At the very edges of the plate, the gas in contact creeps toward the hot side. Since black absorbs photons, it is hotter than the silver side, and the gas creeps to the black edge. The edge then gets a tiny

kick of pressure, and the mill spins. This is why the glass has to be evacuated of all but the barest traces of gas. The entire thing is powered by a few molecules of gas at the edge of each of the tiny paddles. If it had to fight through regular air pressure it wouldn't spin.

Thermal creep isn't just applicable in physics toys. Many people have seen liquid helium creep up the sides of its container. This and other superfluids, all of which have to be pretty cold to be fluid at all, will creep up walls towards heat. Thermal creep and thermal transpiration are the driving force of toys now, but they may not always be. Consider the vast nebulas that litter the universe, many of them filled with forming stars. The combination of dense gas and light are a great combination for thermal transpiration as a power system. A literal space ship, with giant porous, adjustable sails, might be able to push itself around by getting a star to heat its sails unevenly and gas to push its way through them. The Light Mill might be the ancient toy that inspires a legitimate form of transport.

VII. *On Stresses in Rarified Gases arising from Inequalities of Temperature.*

By J. CLERK MAXWELL, F.R.S., *Professor of Experimental Physics in the University of Cambridge.*

Received March 19,—Read April 11, 1878.

1. IN this paper I have followed the method given in my paper “On the Dynamical Theory of Gases” (Phil. Trans., 1867, p. 49). I have shown that when inequalities of temperature exist in a gas, the pressure at a given point is not the same in all directions, and that the difference between the maximum and the minimum pressure at a point may be of considerable magnitude when the density of the gas is small enough, and when the inequalities of temperature are produced by small* solid bodies at a higher or lower temperature than the vessel containing the gas.

2. The nature of this stress may be thus defined:—Let the distance from a given point, measured in a given direction, be denoted by h ; then the space-variation of the temperature for a point moving along this line will be denoted by $\frac{d\theta}{dh}$, and the space-variation of this quantity along the same line by $\frac{d^2\theta}{dh^2}$.

There will, in general, be a particular direction of the line h for which $\frac{d^2\theta}{dh^2}$ is a maximum, another for which it is a minimum, and a third for which it is a maximum-minimum. These three directions are at right angles to each other, and are the

* The dimensions of the bodies must be of the same order of magnitude as a certain length λ , which may be defined as the distance travelled by a molecule with its mean velocity during the time of relaxation of the medium.

The time of relaxation is the time in which inequalities of stress would disappear if the rate at which they diminish were to continue constant. Hence

$$\lambda = 2 \left(\frac{2p}{\pi\rho} \right)^{\frac{1}{2}} \frac{\mu}{p} = 2\mu \left(\frac{2}{\pi\rho p} \right)^{\frac{1}{2}}.$$

On the hypothesis that the encounters between the molecules resemble those between “rigid elastic” spheres, the free path of a molecule between two successive encounters has a definite meaning, and if l is its mean value,

$$l = \frac{3}{2} \mu \left(\frac{\pi}{2p\rho} \right)^{\frac{1}{2}} = \frac{3\pi}{8} \lambda = 1.178\lambda.$$

So that the mean path of a molecule may be taken as representing what we mean by “small.”

If the force between the molecules is supposed to be a continuous function of the distance, the free path of a molecule has no longer a definite meaning, and we must fall back on the quantity λ , as defined above.