Partial synchronization in networks of systems with linear time-delay coupling

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Network synchronization
Partial synchronization
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Weighted directed graph $G = (\mathcal{V}, \mathcal{E}, A)$

- finite node set $\mathcal{V} = \{1, \ldots, k\}$
- edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$; $(i, j) \in \mathcal{E}$ denotes an edge from $j$ to $i$
- weighted adjacency matrix $A \in \mathbb{R}^{k \times k}$

$\mathcal{V} = \{1, 2, 3, 4\}$
$\mathcal{E} = \{(1, 4), (2, 1), (3, 4), (4, 2), (4, 3)\}$

$$A = \begin{pmatrix}
0 & 0 & 0 & a \\
b & 0 & 0 & 0 \\
0 & 0 & 0 & c \\
0 & d & e & 0
\end{pmatrix}$$
Networks

Weighted directed graph $G = (V, E, A)$

- $G$ is assumed to be simple, i.e. $(i, i) \notin E$ for all $i \in V$, and strongly connected
- Neighbor sets $N_i = \{j \in V | (i, j) \in E\}$, $i \in V$
- Weighted in-degree matrix $D \in \mathbb{R}^{k \times k}$

\[
\begin{align*}
N_1 &= \{4\} & N_2 &= \{1\} \\
N_3 &= \{4\} & N_4 &= \{2, 3\}
\end{align*}
\]

\[
D = \begin{pmatrix}
  a & b \\
  c & d + e
\end{pmatrix}
\]
Each node $i \in V$ is assigned the dynamics

\[
\begin{cases}
\dot{x}_i(t) = f(x_i(t)) + B u_i(t) \\
y_i(t) = C x_i(t)
\end{cases}
\]

with

- state $x_i(t) \in \mathbb{R}^n$
- input(s) $u_i(t) \in \mathbb{R}^m$, $1 \leq m \leq n$
- output(s) $y_i(t) \in \mathbb{R}^m$
- locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$
- matrices $B, C^\top \in \mathbb{R}^{n \times m}$ with

$$\text{rank}(BC) = \text{rank}(CB) = m$$
Node dynamics and coupling functions

Each node \( i \in V \) is assigned the dynamics

\[
\begin{align*}
\dot{x}_i(t) &= f(x_i(t)) + Bu_i(t) \\
y_i(t) &= Cx_i(t)
\end{align*}
\]

The interaction between the systems is given by the linear time-delay coupling law

\[
u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t - \tau) - y_i(t)]
\]

with

- \( a_{ij} \) the entries of the weighted adjacency matrix \( A \)
- constant time-delay \( \tau > 0 \)
- constant coupling strength \( \sigma > 0 \)
Partial Synchronization

Given

- a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$
- dynamical systems

\[
\begin{align*}
\dot{x}_i(t) &= f(x_i(t)) + B u_i(t) \\
y_i(t) &= C x_i(t)
\end{align*}
\]

- coupling $u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t - \tau) - y_i(t)]$

Partial synchronization = the asymptotic match of the state of some, but not all, systems

\[x_i(t; t_0, \phi) \xrightarrow{t \to \infty} x_j(t; t_0, \phi) \text{ for some } i, j \in \mathcal{V}\]
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Partial synchronization manifolds

A set

\[ \mathcal{P} = \left\{ \phi \in \mathcal{C} = \mathcal{C}([\tau, 0], \mathbb{R}^{nk}) \mid \phi = \text{col}(\phi_1, \phi_2, \ldots, \phi_k) \right\} \]

\[ \phi_i = \phi_j \text{ for some unordered pair(s) } (i, j) \in \mathcal{V} \times \mathcal{V} \]

is a partial synchronization manifold if it is positively invariant w.r.t. the coupled systems’ dynamics

\[ \dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t - \tau) \]

with

\[ x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_k(t) \end{pmatrix} \in \mathbb{R}^{kn} \quad \text{and} \quad F(x(t)) = \begin{pmatrix} f(x_1(t)) \\ \vdots \\ f(x_k(t)) \end{pmatrix} \]
Let $\Pi \in \mathbb{R}^{k \times k}$ a permutation matrix other than identity and

$$\mathcal{P}(\Pi) := \left\{ \phi \in C([-\tau, 0], \mathbb{R}^{kn}) \middle| \phi = \text{col}(\phi_1, \phi_2, \ldots, \phi_k) \right\}$$

$$\phi(\theta) \in \ker(I_{kn} - \Pi \otimes I_n), -\tau \leq \theta \leq 0$$

**Example with** $k = 3$

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Pi_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
Existence of partial synchronization manifolds

Given the coupled systems’ dynamics

\[ \dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t - \tau) \]

how to find a permutation matrix \( \Pi \neq I \) such that \( \mathcal{P}(\Pi) \) is a partial synchronization manifold?
Existence of partial synchronization manifolds

Given the coupled systems’ dynamics

\[ \dot{x}(t) = F(x(t)) - (D \otimes BC) x(t) + (A \otimes BC) x(t - \tau) \]

how to find a permutation matrix \( \Pi \neq I \) such that \( \mathcal{P}(\Pi) \) is a partial synchronization manifold?

Conditions for existence of partial synchronization manifolds independent of \( f \)

\[ \Rightarrow \text{all information necessary to find a partial synchronization manifold is in the graph } \mathcal{G} = (V, E, A) \]
Let $K := \dim \ker(I - \Pi) \leq k - 1$. If all blocks of the block partitioned adjacency matrix

$$A = \begin{pmatrix}
A_{11} & A_{2K} & \cdots & A_{1K} \\
A_{21} & A_{22} & \cdots & A_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
A_{K1} & A_{K2} & \cdots & A_{KK}
\end{pmatrix}$$

have constant row sums, then $\mathcal{P}(\Pi)$ with

$$\Pi = \begin{pmatrix}
\Pi_1 \\
\vdots \\
\Pi_K
\end{pmatrix}, \quad \Pi_i \text{ are cyclic permutation matrices of dimension } \dim(A_{ii})$$

is a partial synchronization manifold
Algebraic conditions

For a permutation matrix $\Pi \neq I$, if

- $\Pi D = D\Pi$ and
- $\ker(I - \Pi)$ is a right invariant subspace of $A$, i.e. $Av \in \ker(I - \Pi)$ for all $v \in \ker(I - \Pi)$,

then $\mathcal{P}(\Pi)$ is a partial synchronization manifold

For a permutation matrix $\Pi \neq I$, if

- $\Pi D = D\Pi$ and
- there exist a solution $X \in \mathbb{R}^{k \times k}$ to the matrix equation

$$(I - \Pi)A = X(I - \Pi)$$

then $\mathcal{P}(\Pi)$ is a partial synchronization manifold
Balanced coloring

Let

$$a_{ij} = \sum_{\ell} \bar{a}_{\ell} k_{ij}^{\ell}, \quad \bar{a}_{\ell} \in \mathbb{R}^+, \quad k_{ij}^{\ell} \in \mathbb{Z}^+$$

with $\bar{a}_{\ell}$, $\ell = 1, \ldots, r$, rationally independent

Construct a multigraph $\tilde{G} = (V, \tilde{E}_1, \tilde{A}_1, \ldots, \tilde{E}_r, \tilde{A}_r)$ from $G = (V, E, A)$ by replacing each edge $(i, j) \in E$ of weight $a_{ij}$ by $k_{ij}^{\ell}$ edges of weight $\bar{a}_{\ell}$.

Definition: Edges in $\tilde{G}$ with the same weight $\bar{a}_{\ell}$ are equivalent
Definition

A coloring of the nodes of $\tilde{G}$ with $K$ colors is a balanced coloring if and only if, for all $i, j = 1, \ldots, K$, every $c_i$-colored node receives edges of the same equivalence class (i.e. with same weight $\bar{a}_\ell$) from an equal number of nodes with color $c_j$.
For a permutation matrix $\Pi \neq I$, let $\sim_\Pi$ be the equivalence relations induced by $\Pi$:

$$i \sim_\Pi j \iff v_i = v_j \text{ for any } v = \col(v_1, \ldots, v_k) \in \ker(I - \Pi)$$

Color the multigraph $\tilde{G}$ by assigning the nodes the same color if and only if they belong to the same equivalence class $\sim_\Pi$.

If this coloring is a balanced coloring, then $\mathcal{P}(\Pi)$ is a partial synchronization manifold.
Theorem

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and a permutation matrix $\Pi \neq I$. The following statements are equivalent

- all blocks of the structured adjacency matrix have constant row sums
- $\Pi D = D \Pi$ and $\ker(I - \Pi)$ is a right invariant subspace of $A$
- $\Pi D = D \Pi$ and there exist a solution $X \in \mathbb{R}^{k \times k}$ to the matrix equation
  \[ (I - \Pi)A = X(I - \Pi) \]
- the coloring of the multigraph $\tilde{\mathcal{G}}$ according to the equivalence relations $\sim_\Pi$ is balanced
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Assumption: \( CB \) is (similar to) a positive definite matrix

There is a well-defined change of coordinates \( x_i \mapsto (z_i, y_i) \) such that

\[
\begin{align*}
\dot{z}_i(t) &= q(z_i(t), y_i(t)) \\
\dot{y}_i(t) &= a(y_i(t), z_i(t)) + CBu_i(t)
\end{align*}
\]
Convergent internal dynamics

Definition

A system

\[ \dot{z}_i(t) = q(z_i(t), \bar{y}(t)) \]

with state \( z_i(t) \in \mathbb{R}^p \) and input \( \bar{y}(t) \) that take values on some compact set \( \mathcal{Y} \subset \mathbb{R}^m \) is a convergent system if

- for any piece-wise continuous input \( \bar{y}(t) \) defined on \([t_0, \infty)\), all solutions \( z_i(\cdot) \) are defined and bounded for all \( t \in [t_0, \infty) \) for all initial conditions \( z_i(t_0) \in \mathbb{R}^p \);

- for any piece-wise continuous input \( \bar{y}(t) \) defined on \((\infty, \infty)\), there exists a unique globally asymptotically stable solution \( z_{\bar{y}}(\cdot) \) defined on \((\infty, \infty)\)

Convergent internal dynamics

Demidovich condition

If there is a matrix $W = W^\top > 0$ such that the matrix

$$\left( \frac{\partial q}{\partial z_i}(z_i, \bar{y}) \right)^\top W + W \left( \frac{\partial q}{\partial z_i}(z_i, \bar{y}) \right)$$

is uniformly negative definite on $\mathbb{R}^p \times \mathcal{Y}$, then the system

$$\dot{z}_i(t) = q(z_i(t), \bar{y}(t))$$

is a convergent system

A partial synchronization theorem

Theorem

Suppose that the solutions of the coupled systems are uniformly ultimately bounded and $\mathcal{P}(\Pi)$ with $\Pi \neq I$ is a partial synchronization manifold. If

- the subsystem $\dot{z}_i(t) = q(z_i(t), y_i(t))$ satisfies the Demidovich condition
- there is a constant $c > 0$ such that

\[
(I - \Pi)^\top (D - \frac{1}{2}(X + X^\top))(I - \Pi) \geq c(I - \Pi)^\top (I - \Pi)
\]

then there exist constants $\bar{\sigma}$ and $\gamma$ such that for

\[
\sigma \geq \bar{\sigma} \quad \text{and} \quad \sigma \tau \leq \gamma
\]

$\mathcal{P}(\Pi)$ is asymptotically stable
A partial synchronization theorem

\[ \sigma = \bar{\sigma} \]

\[ \mathcal{P}(\Pi) \text{ stable} \]

\[ \sigma \tau = \gamma \]
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A network of Hindmarsh-Rose model neurons

The Hindmarsh-Rose model neuron:

\[
\begin{align*}
\dot{z}_{1,i}(t) &= 0.001(4(y_i(t) + 0.795) - z_{1,i}(t)) \\
\dot{z}_{2,i}(t) &= 1 - 5y_i^2(t) - z_{2,i}(t) \\
\dot{y}_i(t) &= -y_i^3(t) + 3y_i^2(t) - z_{1,i}(t) + z_{2,i}(t) + u_i(t)
\end{align*}
\]

Properties:

▶ strictly semi-passive with a quadratic storage function
  ⇒ solutions of any network of Hindmarsh-Rose model neurons are uniformly bounded and uniformly ultimately bounded

▶ convergent \(z_i\)-dynamics

Remark: Many model neurons are strictly semi-passive and have convergent internal dynamics
Network 1

The network

- simple
- strongly connected
The network

- simple
- strongly connected

Balanced coloring 1

⇒ full sync manifold $\mathcal{S} = \mathcal{P}(\Pi_1)$

$$\Pi_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
The network
- simple
- strongly connected

Balanced coloring 2
⇒ partial sync manifold $\mathcal{P}(\Pi_2)$

$$\Pi_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
Stability of $\mathcal{P}(\Pi_1)$ and $\mathcal{P}(\Pi_2)$

- $(I - \Pi_1)A = X_1(I - \Pi_1)$ with

$$X_1 = \frac{1}{3} \begin{pmatrix}
3 & 3 & 3 & -9 & -6 & 6 \\
4 & -8 & -8 & 4 & 4 & 4 \\
-6 & 6 & 3 & 3 & 3 & -9 \\
4 & 4 & 4 & -8 & -8 & 4 \\
3 & -9 & -6 & 6 & 3 & 3 \\
-8 & 4 & 4 & 4 & 4 & 8
\end{pmatrix}$$

- $\Pi_2$ and $A$ commute, i.e. $\Pi_2 A = A \Pi_2$

$$(I - \Pi_2)A = A(I - \Pi_2) = X_2(I - \Pi_2)$$

- $(I - \Pi_i)\top(D - \frac{1}{2}(X_i + X_i\top))(I - \Pi_i) \geq c_i(I - \Pi_i)\top(I - \Pi_i)$, $i = 1, 2$, with

$$c_1 = 2.961 \quad \text{and} \quad c_2 = 4.297$$
Numerical simulation results
Network 2

Balanced coloring 1

Balanced coloring 2
Numerical simulation results

![Graph showing numerical simulation results with regions labeled Part Sync and Full Sync.](image-url)
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Summary

- we have presented four equivalent conditions for existence of partial synchronization manifolds
- we have presented conditions for a partial synchronization manifold to be asymptotically stable
- extensions to multiple time-delay case and coupling
  \[ u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t - \tau) - y_i(t - \tau)] \] are possible

Future research

- (numerically) efficient methods to determine (all) partial synchronization manifolds
- necessary conditions for asymptotic stability of partial synchronization manifolds
- robust/practical partial synchronization
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