Partial synchronization in networks of systems with linear time-delay coupling

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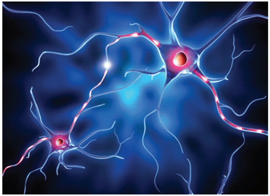
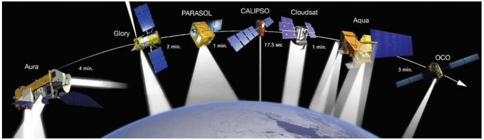
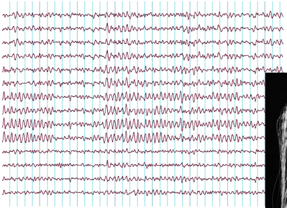
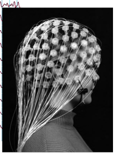
Network synchronization

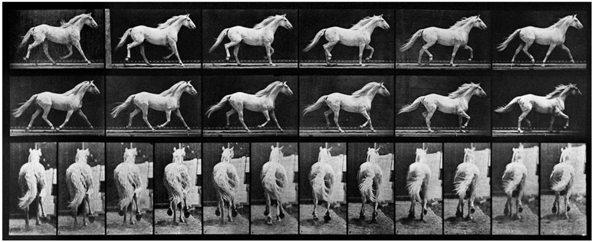




Partial synchronization







Outline

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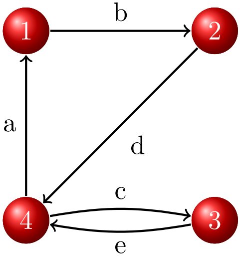
Networks

Weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$

- finite node set $\mathcal{V} = \{1, \ldots, k\}$
- edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$; $(i, j) \in \mathcal{E}$ denotes an edge *from* j *to* i
- weighted adjacency matrix $A \in \mathbb{R}^{k \times k}$



$$\mathcal{V} = \{1, 2, 3, 4\}$$

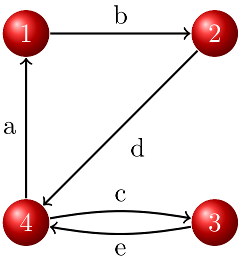
$$\mathcal{E} = \{(1, 4), (2, 1), (3, 4), (4, 2), (4, 3)\}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & d & e & 0 \end{pmatrix}$$

Networks

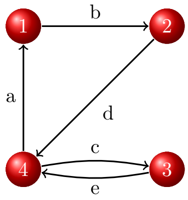
Weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$

- $\mathcal{G}$ is assumed to be simple, i.e. $(i,i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$, and strongly connected
- neighborsets $\mathcal{N}_i = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}\}$, $i \in \mathcal{V}$
- weighted in-degree matrix $D \in \mathbb{R}^{k \times k}$



$$\mathcal{N}_1 = \{4\} \quad \mathcal{N}_2 = \{1\}$$
$$\mathcal{N}_3 = \{4\} \quad \mathcal{N}_4 = \{2,3\}$$
$$D = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d+e \end{pmatrix}$$

Node dynamics and coupling functions



Each node $i \in \mathcal{V}$ is assigned the dynamics

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases}$$

with

- state $x_i(t) \in \mathbb{R}^n$
- input(s) $u_i(t) \in \mathbb{R}^m$, $1 \leq m \leq n$
- output(s) $y_i(t) \in \mathbb{R}^m$
- locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$
- matrices $B, C^\top \in \mathbb{R}^{n \times m}$ with

$$\text{rank}(BC) = \text{rank}(CB) = m$$

Node dynamics and coupling functions

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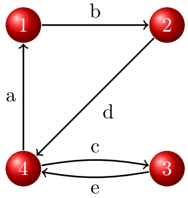


The interaction between the systems is given by the linear time-delay coupling law

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij} [y_j(t-\tau) - y_i(t)]$$

with

- a_{ij} the entries of the weighted adjacency matrix A
- constant time-delay $\tau > 0$
- constant coupling strength $\sigma > 0$

Partial Synchronization

Given

- a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$
- dynamical systems

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases}$$

- coupling $u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t)]$

Partial synchronization = the asymptotic match of the state of some, but not all, systems

$$x_i(t; t_0, \phi) \xrightarrow{t \to \infty} x_j(t; t_0, \phi) \text{ for some } i, j \in \mathcal{V}$$

Outline

Problem setting

Partial synchronization manifolds

Stability of partial synchronization manifolds

Example

Concluding remarks

Partial synchronization manifolds

A set

$$\mathcal{P} = \Big\{ \phi \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^{nk}) \Big| \phi = \mathrm{col}(\phi_1, \phi_2, \ldots, \phi_k)$$

$$\phi_i = \phi_j \text{ for some unordered pair(s) } (i, j) \in \mathcal{V} \times \mathcal{V} \Big\}$$

is a partial synchronization manifold if it is positively invariant w.r.t. the coupled systems' dynamics

$$\dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t - \tau)$$

with

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_k(t) \end{pmatrix} \in \mathbb{R}^{kn} \quad \text{and} \quad F(x(t)) = \begin{pmatrix} f(x_1(t)) \\ \vdots \\ f(x_k(t)) \end{pmatrix}$$

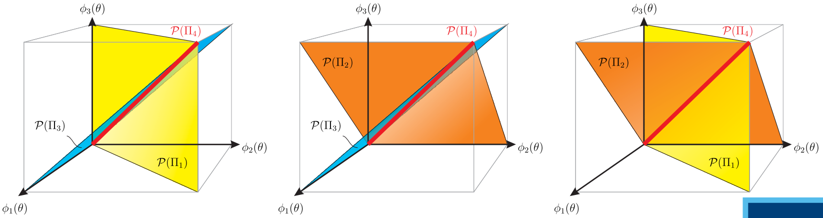
Partial synchronization manifolds

Let $\Pi \in \mathbb{R}^{k\times k}$ a permutation matrix other than identity and

$$\mathcal{P}(\Pi) := \Big\{ \phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^{kn}) \Big| \phi = \mathrm{col}(\phi_1, \phi_2, \ldots, \phi_k)$$
$$\phi(\theta) \in \ker(I_{kn} - \Pi \otimes I_n), -\tau \leq \theta \leq 0 \Big\}$$

Example with $k = 3$

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \Pi_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \Pi_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



Existence of partial synchronization manifolds

Given the coupled systems' dynamics

$$\dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t-\tau)$$

how to find a permutation matrix $\Pi \neq I$ such that $\mathcal{P}(\Pi)$ is a partial synchronization manifold?

Existence of partial synchronization manifolds

Given the coupled systems' dynamics

$$\dot{x}(t) = F(x(t)) - (D \otimes BC)x(t) + (A \otimes BC)x(t-\tau)$$

how to find a permutation matrix $\Pi \neq I$ such that $\mathcal{P}(\Pi)$ is a partial synchronization manifold?

Conditions for existence of partial synchronization manifolds independent of f

$\Rightarrow$ all information necessary to find a partial synchronization manifold is in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$

Structure in adjacency matrix

Let $K := \dim \ker(I - \Pi) \leq k - 1$. If all blocks of the block partitioned adjacency matrix

$$A = \begin{pmatrix} A_{11} & A_{2K} & \cdots & A_{1K} \\ A_{21} & A_{22} & \cdots & A_{2K} \\ \vdots & \ddots & \ddots & \vdots \\ A_{K1} & A_{K2} & \cdots & A_{KK} \end{pmatrix}$$

have constant row sums, then $\mathcal{P}(\Pi)$ with

$$\Pi = \begin{pmatrix} \Pi_1 & & \\ & \ddots & \\ & & \Pi_K \end{pmatrix},$$

Π_i are cyclic permutation matrices of dimension $\dim(A_{ii})$

is a partial synchronization manifold

Algebraic conditions

For a permutation matrix $\Pi \neq I$, if

- $\Pi D = D\Pi$ and
- $\ker(I - \Pi)$ is a right invariant subspace of A, i.e. $Av \in \ker(I - \Pi)$ for all $v \in \ker(I - \Pi)$,

then $\mathcal{P}(\Pi)$ is a partial synchronization manifold

For a permutation matrix $\Pi \neq I$, if

- $\Pi D = D\Pi$ and
- there exist a solution $X \in \mathbb{R}^{k \times k}$ to the matrix equation

$$(I - \Pi)A = X(I - \Pi)$$

then $\mathcal{P}(\Pi)$ is a partial synchronization manifold

Balanced coloring

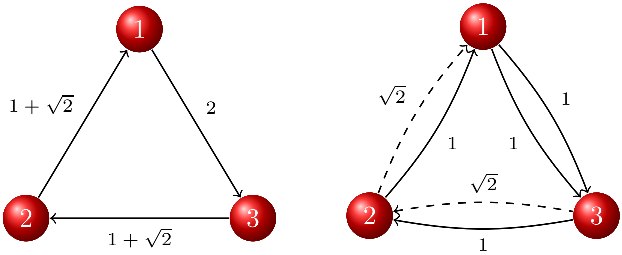
Let

$$a_{ij} = \sum_{\ell} \bar{a}_\ell k_{ij}^\ell, \quad \bar{a}_\ell \in \mathbb{R}_+, \quad k_{ij}^\ell \in \mathbb{Z}_+$$

with $\bar{a}_\ell$, $\ell = 1, \ldots, r$, rationally independent

Construct a multigraph $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}}_1, \bar{A}_1, \ldots, \tilde{\mathcal{E}}_r, \bar{A}_r)$ from $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ by replacing each edge $(i, j) \in \mathcal{E}$ of weight a_{ij} by k_{ij}^ℓ edges of weight $\bar{a}_\ell$.

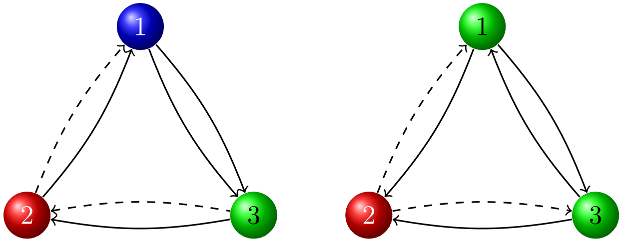
Definition: Edges in $\tilde{\mathcal{G}}$ with the same weight $\bar{a}_\ell$ are equivalent



Balanced coloring

Definition

A coloring of the nodes of $\tilde{\mathcal{G}}$ with K colors is a balanced coloring if and only if, for all $i, j = 1, \ldots, K$, every c_i-colored node receives edges of the same equivalence class (i.e. with same weight $\bar{a}_\ell$) from an equal number of nodes with color c_j



M. Golubitsky, I. Stewart and A. Török, SIAM J. Appl. Dyn. Syst. 4(1), 2005

Balanced coloring

For a permutation matrix $\Pi \neq I$, let $\sim_\Pi$ be the equivalence relations induced by Π:

$$i \sim_\Pi j \Leftrightarrow v_i = v_j \text{ for any } v = \mathrm{col}(v_1, \ldots, v_k) \in \ker(I - \Pi)$$

Color the multigraph $\tilde{\mathcal{G}}$ by assigning the nodes the same color if and only if they belong to the same equivalence class $\sim_\Pi$.

If this coloring is a balanced coloring, then $\mathcal{P}(\Pi)$ is a partial synchronization manifold.

Equivalent conditions for positively invariant $\mathcal{P}(\Pi)$

Theorem

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and a permutation matrix $\Pi \neq I$. The following statements are equivalent

- all blocks of the structured adjacency matrix have constant row sums
- $\Pi D = D\Pi$ and $\ker(I - \Pi)$ is a right invariant subspace of A
- $\Pi D = D\Pi$ and there exist a solution $X \in \mathbb{R}^{k \times k}$ to the matrix equation
$$(I - \Pi)A = X(I - \Pi)$$
- the coloring of the multigraph $\tilde{\mathcal{G}}$ according to the equivalence relations $\sim_\Pi$ is balanced

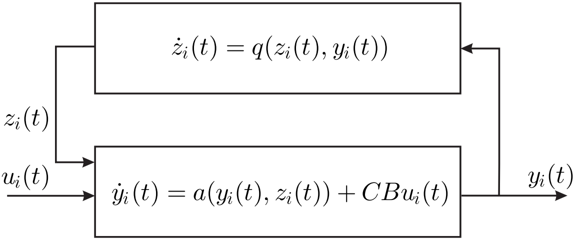
Outline

Change of coordinates

Assumption: CB is (similar to) a positive definite matrix

There is a well-defined change of coordinates $x_i \mapsto (z_i, y_i)$ such that

$$\begin{cases} \dot{z}_i(t) = q(z_i(t), y_i(t)) \\ \dot{y}_i(t) = a(y_i(t), z_i(t)) + CBu_i(t) \end{cases}$$



Convergent internal dynamics

Definition

A system

$$\dot{z}_i(t) = q(z_i(t), \bar{y}(t))$$

with state $z_i(t) \in \mathbb{R}^p$ and input $\bar{y}(t)$ that take values on some compact set $\mathcal{Y} \subset \mathbb{R}^m$ is a convergent system if

- for any piece-wise continuous input $\bar{y}(t)$ defined on $[t_0, \infty)$, all solutions $z_i(\cdot)$ are defined and bounded for all $t \in [t_0, \infty)$ for all initial conditions $z_i(t_0) \in \mathbb{R}^p$;
- for any piece-wise continuous input $\bar{y}(t)$ defined on $(-\infty, \infty)$, there exists a unique globally asymptotically stable solution $z_{\bar{y}}(\cdot)$ defined on $(-\infty,\ +\infty)$

A. Pavlov, N. v.d. Wouw and H. Nijmeijer, "Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach", Birkhäuser, 2006

Convergent internal dynamics

Demidovich condition

If there is a matrix $W = W^\top > 0$ such that the matrix

$$\left(\frac{\partial q}{\partial z_i}(z_i, \bar{y})\right)^\top W + W \left(\frac{\partial q}{\partial z_i}(z_i, \bar{y})\right)$$

is uniformly negative definite on $\mathbb{R}^p \times \mathcal{Y}$, then the system

$$\dot{z}_i(t) = q(z_i(t), \bar{y}(t))$$

is a convergent system

A. Pavlov, N. v.d. Wouw and H. Nijmeijer, "Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach", Birkhäuser, 2006

A partial synchronization theorem

Theorem

Suppose that the solutions of the coupled systems are uniformly ultimately bounded and $\mathcal{P}(\Pi)$ with $\Pi \neq I$ is a partial synchronization manifold. If

- the subsystem $\dot{z}_i(t) = q(z_i(t), y_i(t))$ satisfies the Demidovich condition
- there is a constant $c > 0$ such that

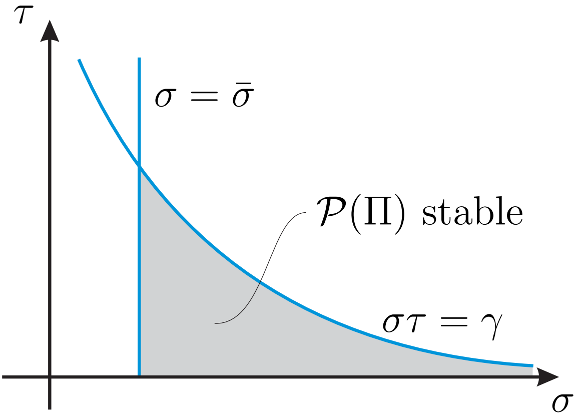
$$(I - \Pi)^\top (D - \tfrac{1}{2}(X + X^\top))(I - \Pi) \geq c(I - \Pi)^\top (I - \Pi)$$

then there exist constants $\bar{\sigma}$ and γ such that for

$$\sigma \geq \bar{\sigma} \quad \text{and} \quad \sigma\tau \leq \gamma$$

$\mathcal{P}(\Pi)$ is asymptotically stable

A partial synchronization theorem



Outline

Problem setting

Partial synchronization manifolds

Stability of partial synchronization manifolds

Example

Concluding remarks

A network of Hindmarsh-Rose model neurons

The Hindmarsh-Rose model neuron:

$$\begin{aligned}\dot{z}_{1,i}(t) &= 0.001(4(y_i(t)+0.795)-z_{1,i}(t))\\ \dot{z}_{2,i}(t) &= 1-5y_i^2(t)-z_{2,i}(t)\\ \dot{y}_i(t) &= -y_i^3(t)+3y_i^2(t)-z_{1,i}(t)+z_{2,i}(t)+u_i(t)\end{aligned}$$

Properties:

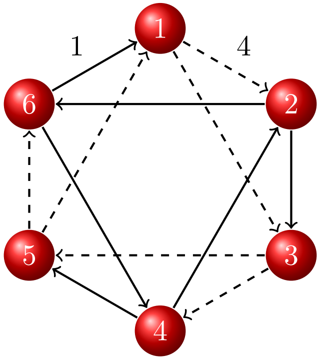
- strictly semi-passive with a quadratic storage function
  $\Rightarrow$ solutions of any network of Hindmarsh-Rose model neurons are uniformly bounded and uniformly ultimately bounded
- convergent z_i-dynamics

Remark: Many model neurons are strictly semi-passive and have convergent internal dynamics

Network 1

The network

- simple
- strongly connected



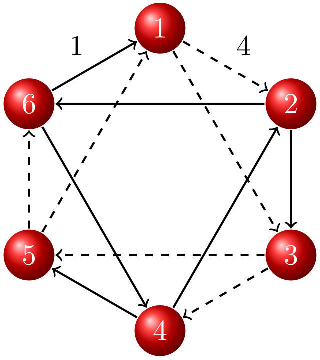
Network 1

The network

- simple
- strongly connected

Balanced coloring 1
$\Rightarrow$ full sync manifold $\mathcal{S} = \mathcal{P}(\Pi_1)$

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



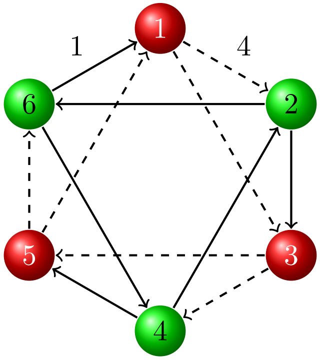
Network 1

The network

- simple
- strongly connected

Balanced coloring 2
⇒ partial sync manifold $\mathcal{P}(\Pi_2)$

$$\Pi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Stability of $\mathcal{P}(\Pi_1)$ and $\mathcal{P}(\Pi_2)$

- $(I - \Pi_1)A = X_1(I - \Pi_1)$ with

$$X_1 = \frac{1}{3}\begin{pmatrix} 3 & 3 & 3 & -9 & -6 & 6 \\ 4 & -8 & -8 & 4 & 4 & 4 \\ -6 & 6 & 3 & 3 & 3 & -9 \\ 4 & 4 & 4 & -8 & -8 & 4 \\ 3 & -9 & -6 & 6 & 3 & 3 \\ -8 & 4 & 4 & 4 & 4 & 8 \end{pmatrix}$$

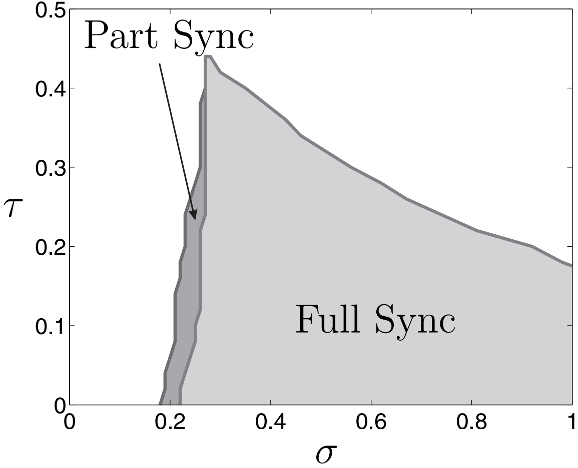
- Π_2 and A commute, i.e. $\Pi_2 A = A\Pi_2$

$$(I - \Pi_2)A = A(I - \Pi_2) = X_2(I - \Pi_2)$$

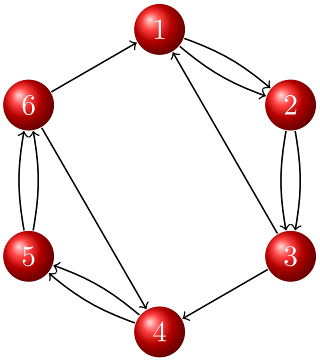
- $(I - \Pi_i)^\top(D - \frac{1}{2}(X_i + X_i^\top)(I - \Pi_i) \geq c_i(I - \Pi_i)^\top(I - \Pi_i)$, $i = 1, 2$, with

$$c_1 = 2.961 \quad \text{and} \quad c_2 = 4.297$$

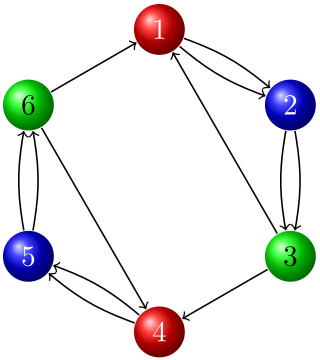
Numerical simulation results



Network 2

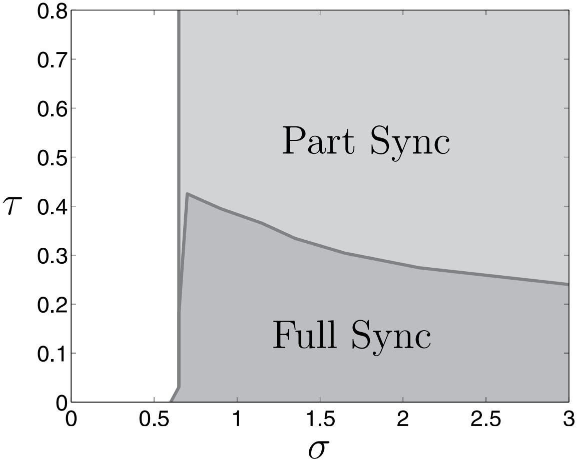


Balanced coloring 1



Balanced coloring 2

Numerical simulation results



Outline

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Concluding remarks

Summary

- we have presented four equivalent conditions for existence of partial synchronization manifolds
- we have presented conditions for a partial synchronization manifold to be asymptotically stable
- extensions to multiple time-delay case and coupling $u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t-\tau)]$ are possible

Future research

- (numerically) efficient methods to determine (all) partial synchronization manifolds
- necessary conditions for asymptotic stability of partial synchronization manifolds
- robust/practical partial synchronization

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