The second Lyapunov method for *unstable* attractors

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Preliminaries. Invariance and attractivity

$\dot{x} = f(x), x \in R^n$ $\quad f(\cdot)$ *is locally Lipschitz in* D *(open in* R^n*)*

A set $S \subset D$ *is **forward (positively) invariant** if for every* $x_0 \in S$ $x(\cdot, x_0)$ *is defined on* $[0, \infty)$ *and* $x(t, x_0) \in S$ *for all* $t > 0$

A set $S \subset D$ *is **invariant** if for every* $x_0 \in S$ $\quad x(\cdot, x_0)$ *is defined on* $(-\infty, \infty)$ *and* $x(t, x_0) \in S$ *for all* t.

*A closed invariant set is **weakly attracting** if there exists a set V of strictly positive measure such that for all* $x_0 \in V$ *the solution* $x(.,x_0)$ *is defined on* $[0, \infty)$ *and*

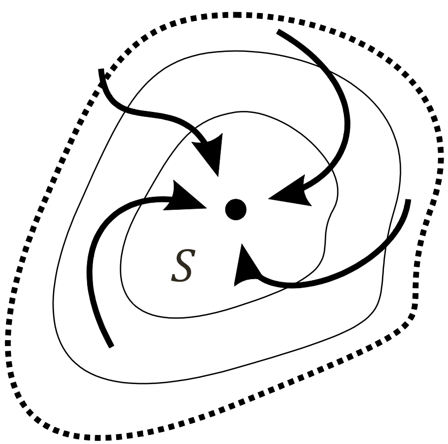
$$\lim_{t \to \infty} dist\big(S, x(t, x_0)\big) = 0$$

Preliminaries. Attractivity and stability

*The set $S \subset D$ (closed, invariant) is **attracting** if*

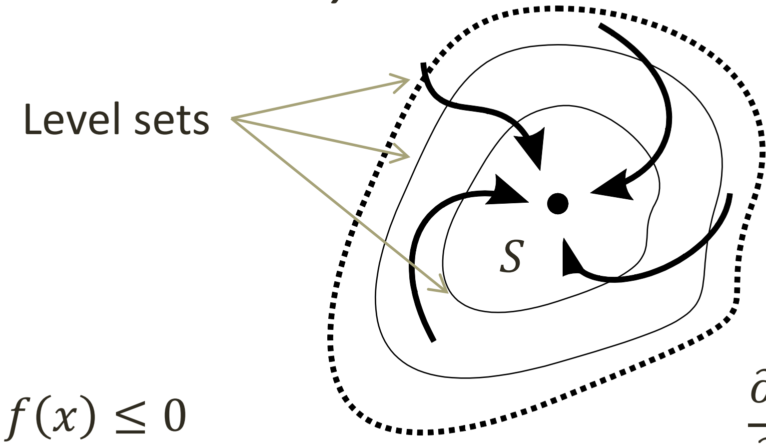
1) *it is weakly attracting*
2) *V is a neighbourhood of S and*
3) *V is forward-invariant*

*The set **is stable in the sense of Lyapunov** if for any neighbourhood V of S there is a forward invariant neighbourhood $W \subset D$ of S such that $W \subset V$*



Preliminaries. Motivation

This nested structure offers a remarkably convenient tool: the second Lyapunov method

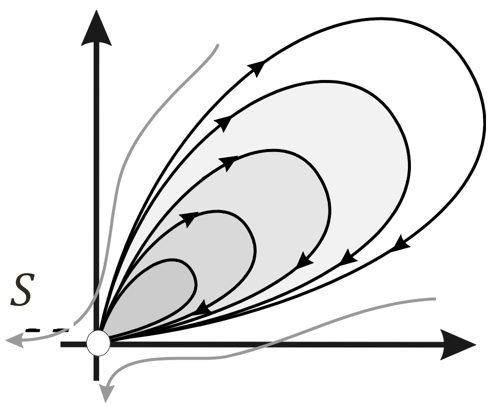


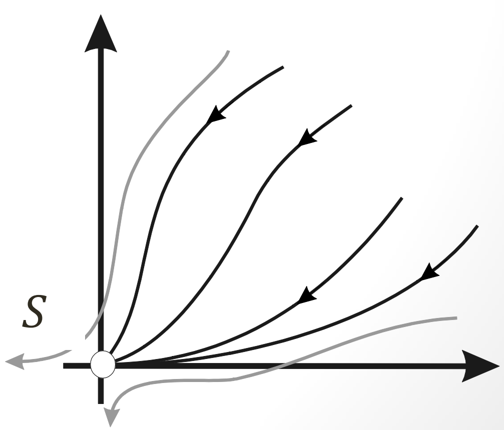
Tangential condition:

$$\frac{\partial V}{\partial x} f(x) \leq 0$$

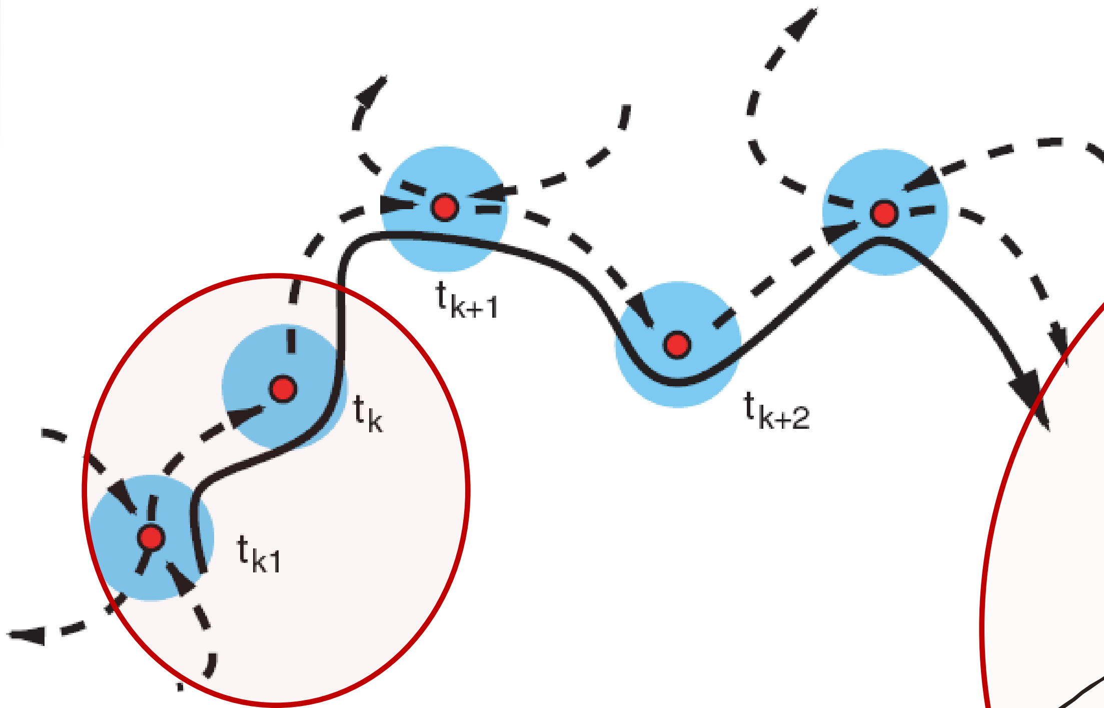
$$\frac{\partial V}{\partial x} f(x) < 0$$

Unstable attractors?





- *Decision-making*
- *Optimization*
- *Adaptive Control*
- *Aircraft dynamics, generators, models of species*



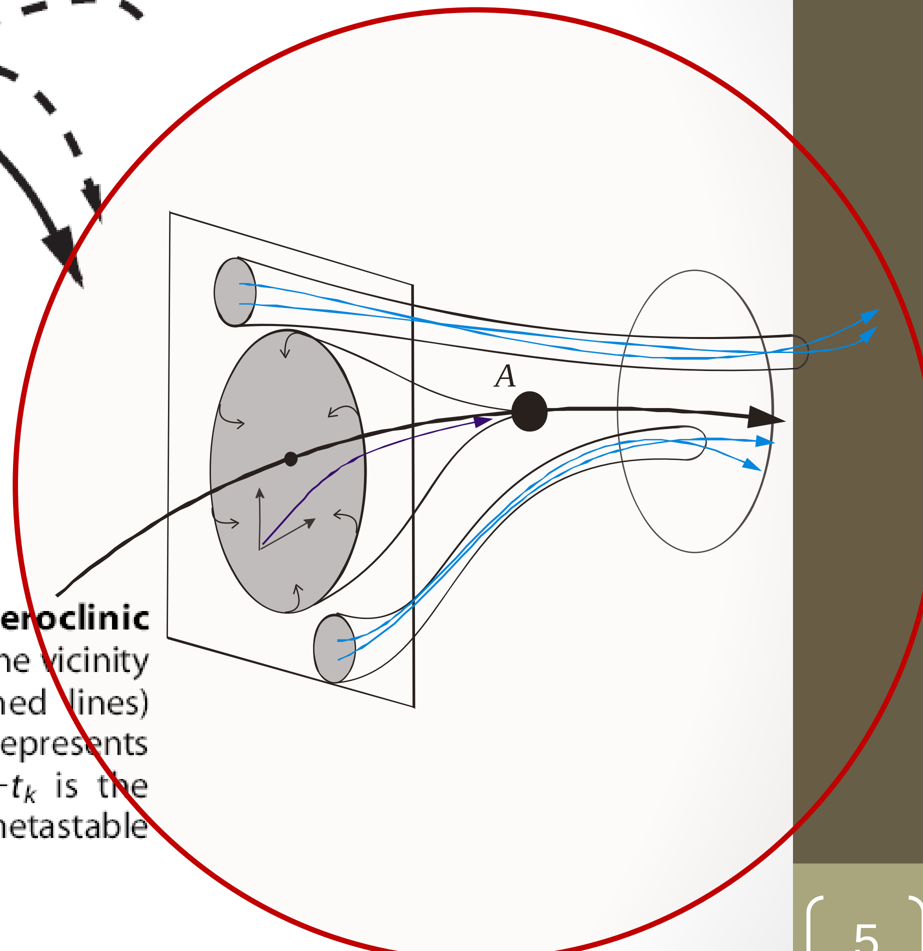
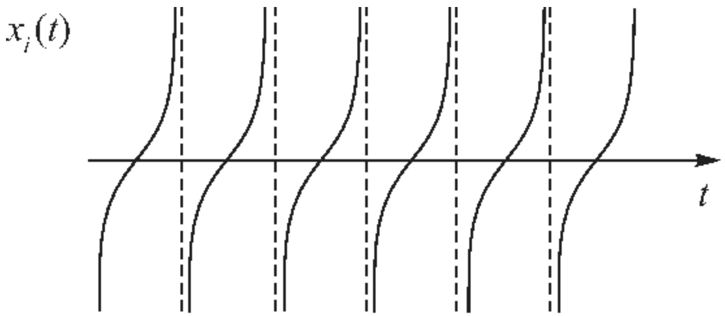
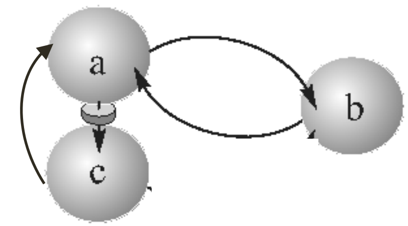


Figure 1. Schematic representation of a stable heteroclinic channel. The SHC is built with trajectories that condense in the vicinity of the saddle chain and their unstable separatrices (dashed lines) connecting the surrounding saddles (circles). The thick line represents an example of a trajectory in the SHC. The interval $t_{k+1} - t_k$ is the characteristic time that the system needs to move from the metastable state k to the k+1.

M. Rabinovich et al. 2008, PLOS Comp. Biology

A phase synchronization example





$$\dot{x}_a = 1 + x_a^2 + \epsilon(\delta(t - t_b) + \delta(t - t_c))$$

$$\dot{x}_b = 1 + x_b^2 + \epsilon(\delta(t - t_a))$$

$$\dot{x}_c = 1 + x_c^2 + \epsilon_1(x_a - x_c)$$

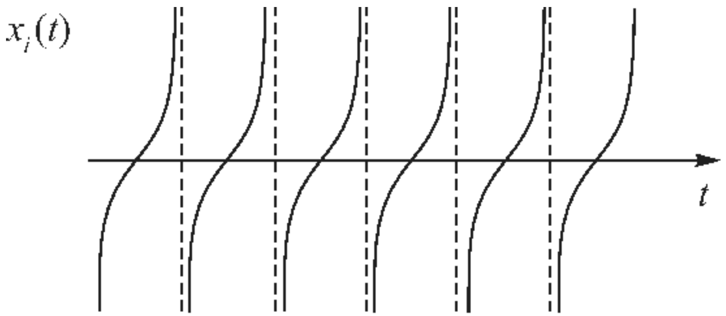
$$\dot{\phi}_a = \epsilon/\pi(\sin^2(\phi_b - \phi_a) + \sin^2(\phi_c - \phi_a))$$

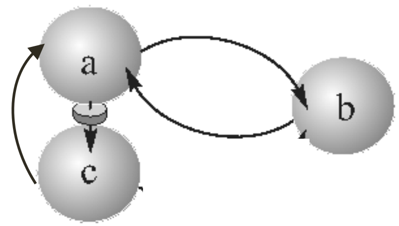
$$\dot{\phi}_b = \epsilon/\pi \sin^2(\phi_a - \phi_b)$$

$$\dot{\phi}_c = \epsilon_1/2 \sin(2(\phi_a - \phi_c))$$

$$\lambda = \phi_b - \phi_a, x = \phi_a - \phi_c$$

A phase synchronization example





$$\dot{\phi}_a = \epsilon/\pi(\sin^2(\phi_b - \phi_a) + \sin^2(\phi_c - \phi_a))$$

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$$\lambda = \phi_b - \phi_a, x = \phi_a - \phi_c$$

$$\begin{cases} \dot{x} = \frac{\varepsilon}{\pi}[\sin^2(\lambda) + \sin^2(x)] - \frac{\varepsilon_1}{2}\sin(2x) \\ \dot{\lambda} = -\frac{\varepsilon}{\pi}\sin^2(x) \end{cases}$$

Preliminaries. Issues

Is origin an attractor?

$$\dot{x} = -x + \lambda$$
$$\dot{\lambda} = -\gamma|x|^3$$

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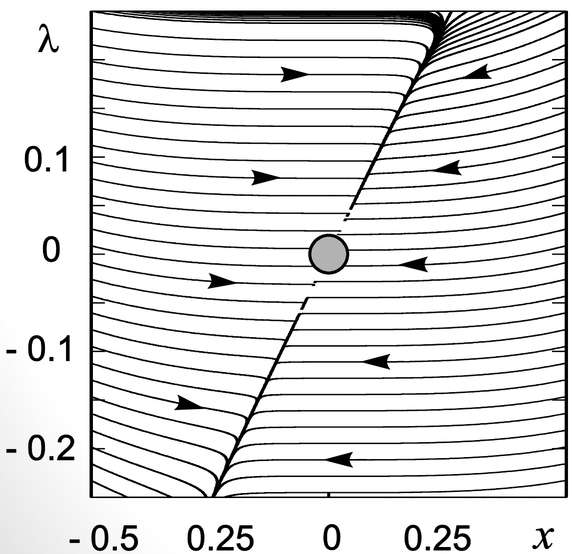
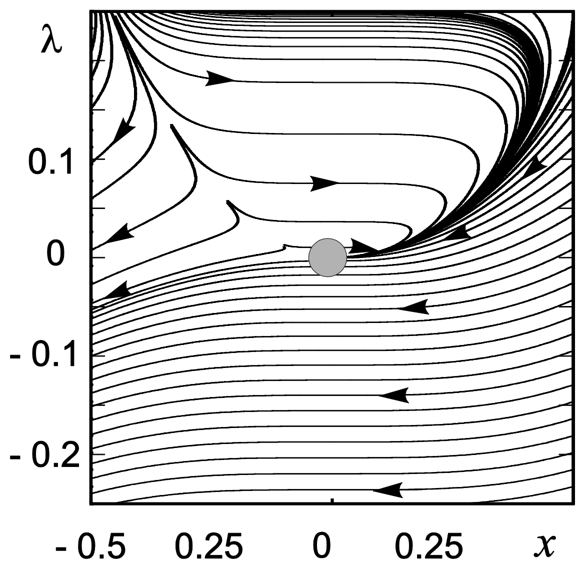
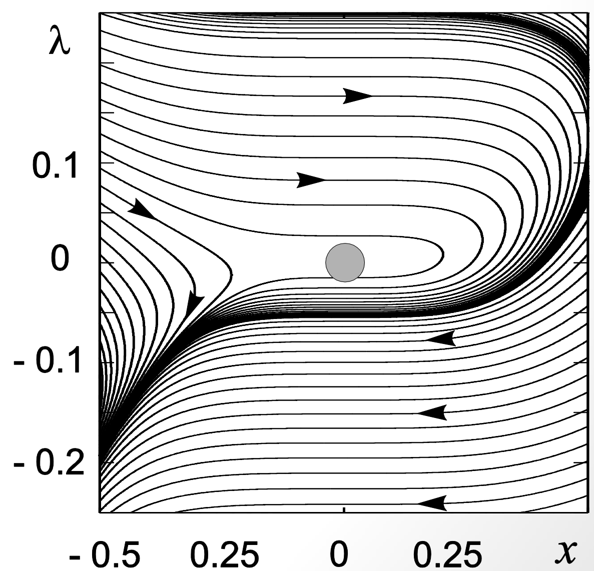






Problem statement

(1)

$$\begin{cases} \dot{x} = f(x, \lambda, t) \\ \dot{\lambda} = g(x, \lambda, t) \end{cases}$$

where

$$f: R^n \times R \times R \to R^n,$$
$$g: R^n \times R \times R \to R$$

1) *continuous and locally Lipschitz uniformly in t*

2) g *is not allowed to change its sign (e.g. non-positive)*

3) $(0,0)$ *is an equilibrium and* 0 *is a weak attractor of* $\dot{x} = f(x, 0, t)$

4) *there is a* $p > 0$ *and a set* $\omega(p)$ *which is forward invariant for*

(2)

$$\dot{x} = f(x, \lambda, t)$$

for all $\lambda \in [0, p]$ *(for simplicity the set* $\omega(p)$ *can be set to* R^n*)*

Determine if (0,0) is an attractor for (1)?

Assumptions

Let D be an open subset of R^n, $\Lambda = [c_1, c_2], c_1 \le 0, c_2 > 0$ be an interval, and the closure of D contains the origin. Denote $D_\Omega = \overline{D} \times \Lambda \times R$

Assumption 1. *There is a continuous function $V: R^n \to R$ that is differentiable everywhere except for the origin, and five functions of one variable, $\underline{\alpha}, \bar{\alpha} \in K_\infty$, $\alpha, \beta: R_{\ge 0} \to R$, $\alpha, \beta \in C^0([0, \infty))$, $\alpha(0) = 0$, $\varphi \in K_0$ such that for every $(x, \lambda, t) \in (\overline{D} \setminus \{0\}) \times \Lambda \times R$ the following holds:*

$$\underline{\alpha}(\|x\|) \le V(x) \le \bar{\alpha}(\|x\|), \qquad \frac{\partial V}{\partial x} f(x, \lambda, t) \le \alpha\big(V(x)\big) + \beta\big(V(x)\big)\varphi(|\lambda|)$$

Assumption 2. *There exist functions $\delta, \xi \in K_0$ such that the following holds for all $(x, \lambda, t) \in D_\Omega$:*

$$-\xi(|\lambda|) - \delta(\|x\|) \le g(x, \lambda, t) \le 0$$

Results

Lemma 1 (The second Lyapunov method for (1))

Let Assumptions 1,2 hold for (1). Suppose that

(C1) *there exists a function* $\psi \in K \cap C^1(0,\infty)$ *and a number* a *such that for all* $V \in (0,a]$

(3) $$\frac{\partial\psi(V)}{\partial V}\Big(\alpha(V) + \beta(V)\varphi\big(\psi(V)\big)\Big) + \delta\big(\underline{\alpha}^{-1}(V)\big) + \xi\big(\psi(V)\big) \le 0$$

(C2) *the set* $\omega(\psi(a))$ *exists and either* $\overline{D}$ *contains* $\omega(\psi(a))$ *or the ball* $\{x|\ x \in R^n,$ $\|x\| \le \underline{\alpha}^{-1}(x)\}$ *is in* D

(C3) *the set* $\Omega_a \setminus \{(0,0)\}$ *where*

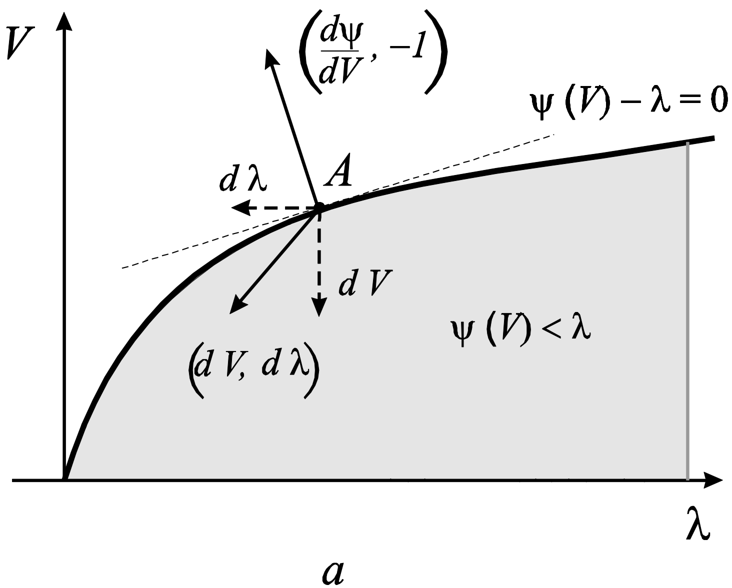
(4) $$\Omega_a = \{(x,\lambda)|x \in \omega\big(\psi(a)\big), \lambda \in R_{\ge 0},\ \psi(a) \ge \lambda \ge \psi\big(V(x)\big),\ V(x) \in (0,a]\}$$

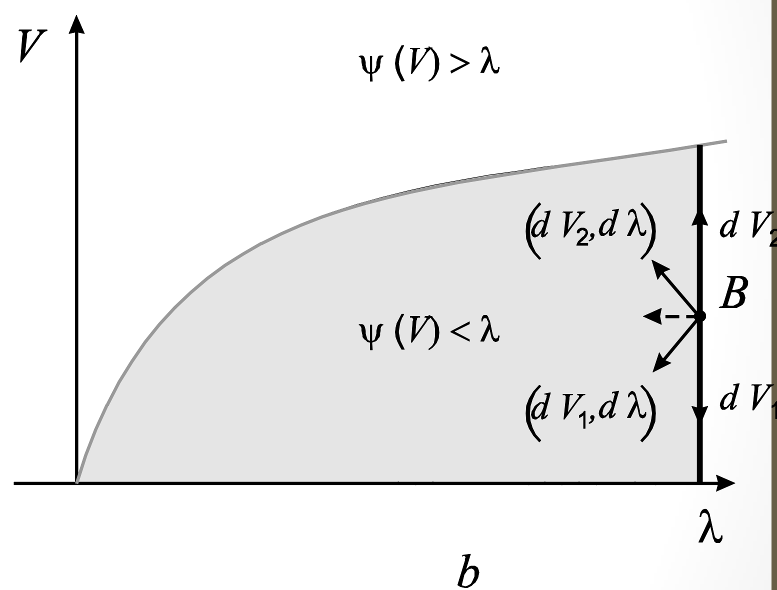
is in the interior of $\overline{D} \times \boldsymbol{\Lambda}$.

Then Ω_a *is forward invariant with respect to the dynamics of (1)*

Results. Sketch

$$\Omega_a = \{(x,\lambda) | x \in \omega(\psi(a)), \lambda \in R_{\geq 0}, \quad \psi(a) \geq \lambda \geq \psi(V(x)), \quad V(x) \in (0, a]\}$$





$$\frac{\partial \psi(V)}{\partial V}\Big(\alpha(V) + \beta(V)\varphi\big(\psi(V)\big)\Big) + \delta\big(\underline{\alpha}^{-1}(V)\big) + \xi\big(\psi(V)\big) \leq 0$$

Results.

Corollary (The second Lyapunov method for (1))

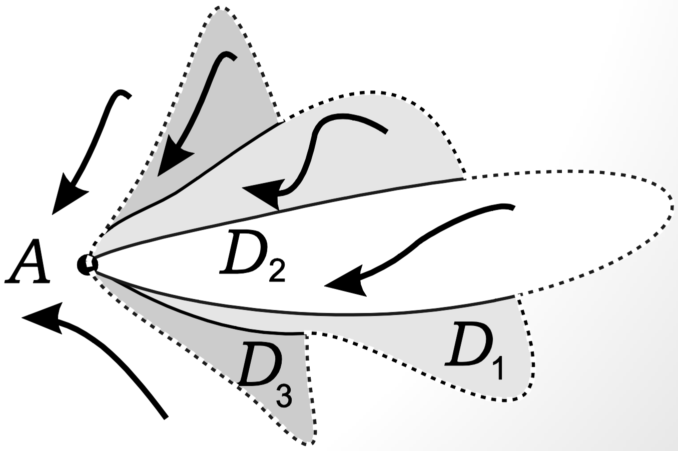
Let $D = R^n$, $\Lambda = R$ *and Assumptions 1,2 hold. Suppose that there exists a function* $\psi \in K \cap C^1(0,\infty)$ *and a number* a *such that for all* $V \in (0,a]$ *(3) holds:*

$$\frac{\partial \psi(V)}{\partial V}\Big(\alpha(V) + \beta(V)\varphi\big(\psi(V)\big)\Big) + \delta\big(\underline{\alpha}^{-1}(V)\big) + \xi\big(\psi(V)\big) \le 0$$

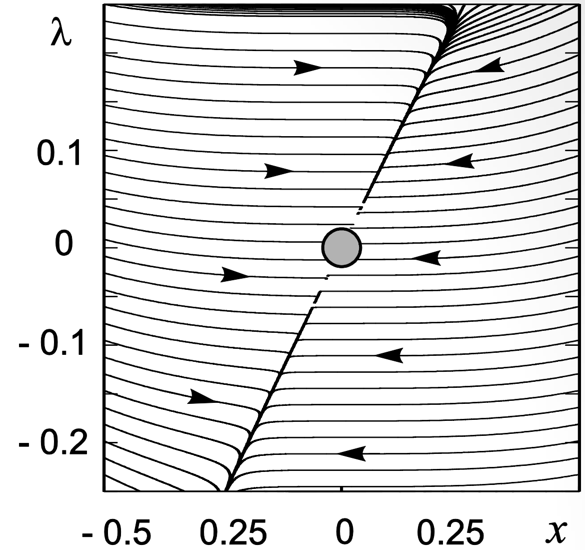
Then the set

$$\Omega_a = \{(x,\lambda) | x \in R^n, \lambda \in R_{\ge 0},\ \psi(a) \ge \lambda \ge \psi\big(V(x)\big),\ V(x) \in (0,a]\}$$

is forward invariant with respect to the dynamics of (1)



Will this help with issues?



$$\begin{cases} \dot{x} = -x + \lambda \\ \dot{\lambda} = -\gamma |x|^3 \end{cases}$$

$$V(x) = x^2, \alpha(V) = -2V, \beta(V) = 2\sqrt{V}$$

$$\psi(V) = pV,\ p > 0,\ \varphi(|\lambda|) = |\lambda|,\ \xi(\cdot) = 0,\ \delta(|x|) = \gamma |x|^3$$

$$\frac{\partial \psi}{\partial V}\big(\alpha(V) + \beta(V)\varphi(|\lambda|)\big) + \delta\big(\sqrt{V}\big) \le 0$$

$$\big(-2p + (2p^2 + \gamma)\sqrt{V}\big)V \le 0$$

$$\Omega_a = \{(x, \lambda) | x \in R, \lambda \in R, \quad p\left(\frac{2p}{2p^2 + \gamma}\right)^2 \ge \lambda \ge px^2, \quad p \in R_{\ge 0}\}$$

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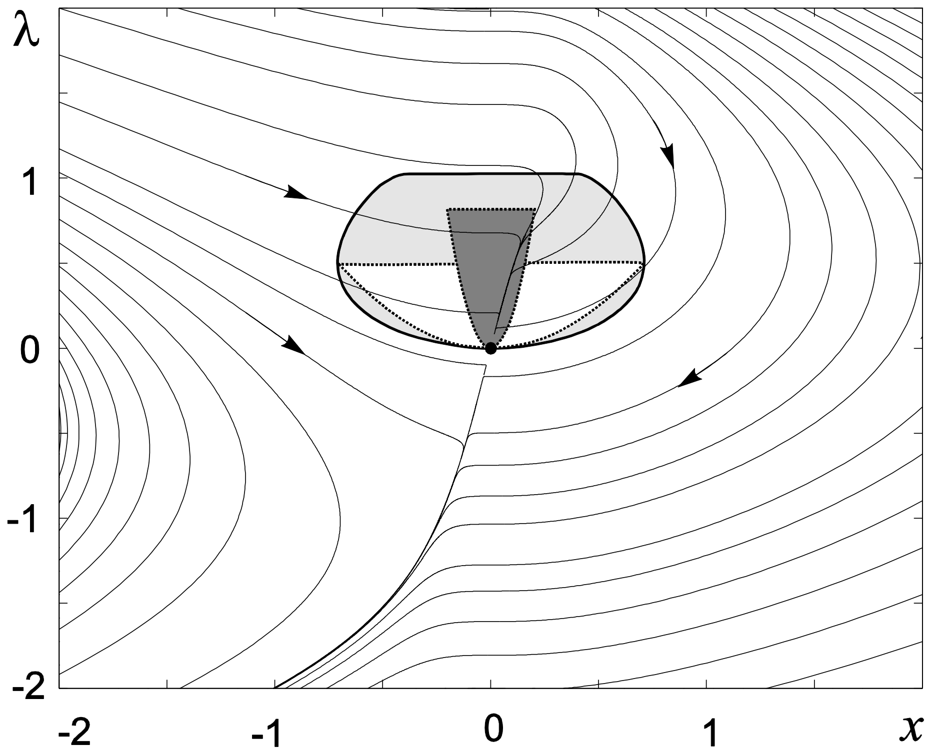

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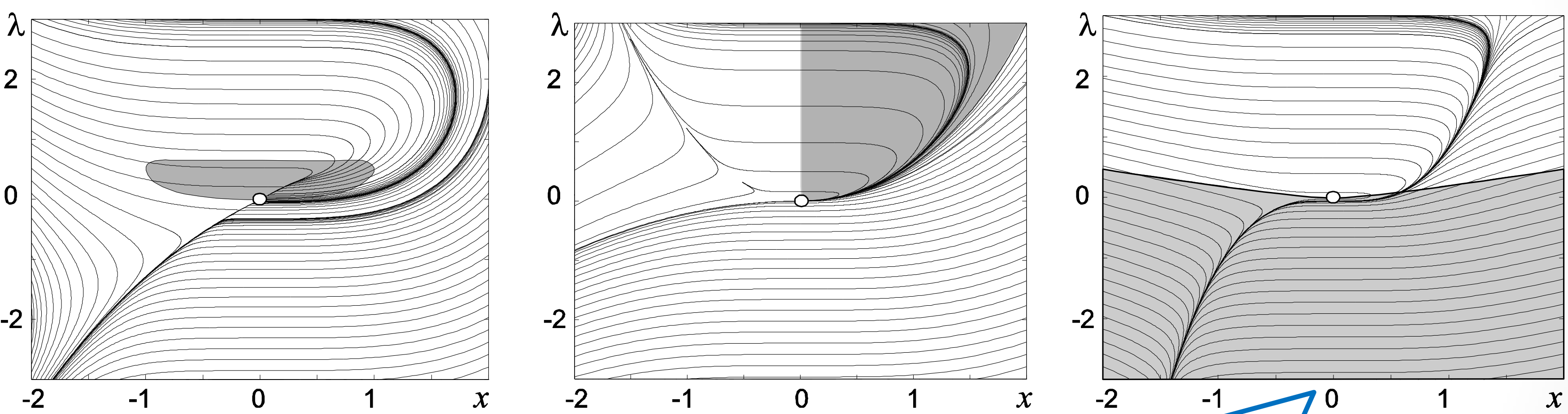


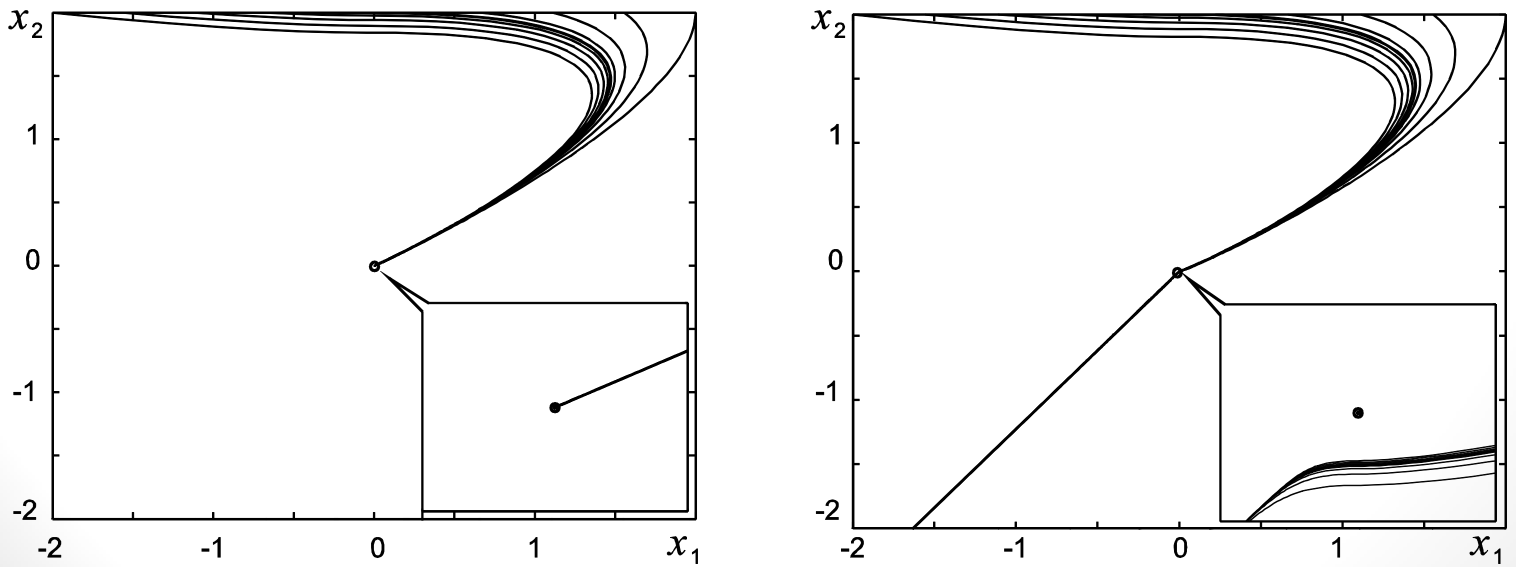
Sets corresponding to solutions that do not converge to the origin can be specified in the same manner (please see the ms on the table)

Are these estimates tight?

$$\begin{cases} \dot{x}_1 = -\tau x_1 + c_1 x_2 \\ \dot{x}_2 = -c_2 |x_1| \end{cases}, \quad \tau, c_1, c_2 > 0 \qquad \psi = p\sqrt{V}, p > 0, \qquad V = x_1^2$$

$c_2 \le p(\tau - p\, c_1)$ is the resulting condition

$p = \tau/(2c_1)$ maximizes the rhs → $c_2 \le \dfrac{\tau^2}{4\, c_1}$



Are these estimates tight?

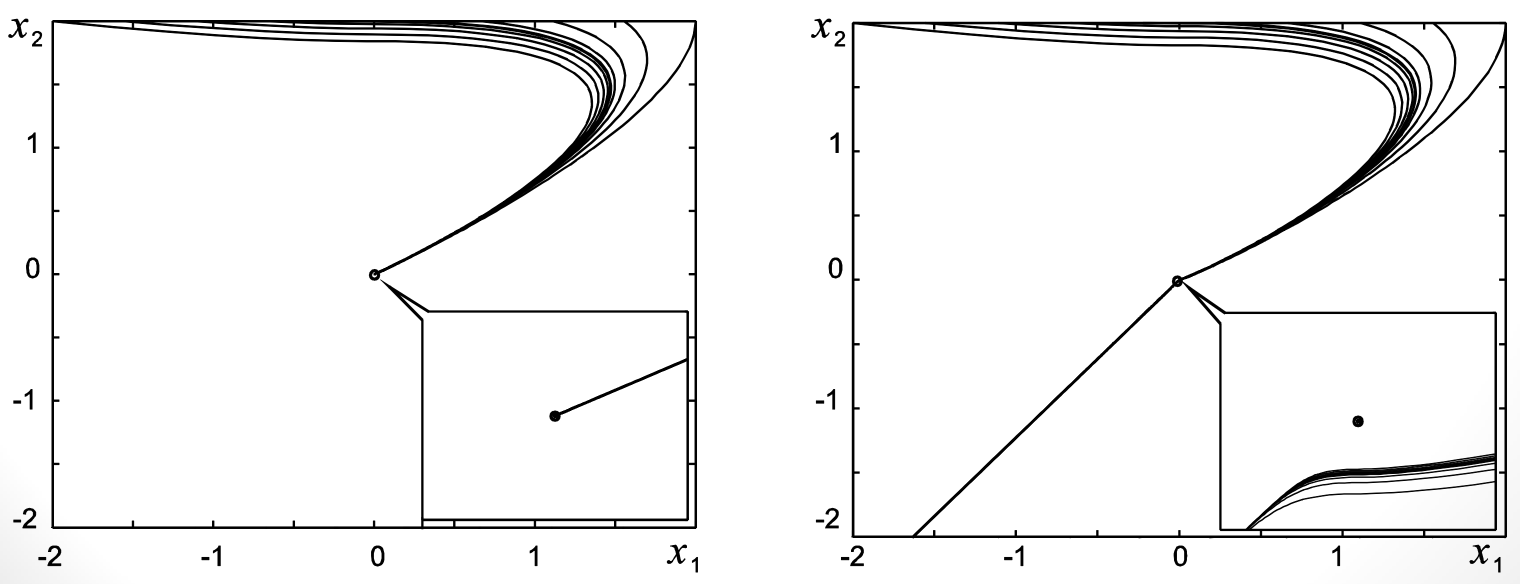
A sister system

$$\begin{cases} \dot{x}_1 = -\tau x_1 + c_1 x_2 \\ \dot{x}_2 = -c_2 x_1 \end{cases}, \quad \tau, c_1, c_2 > 0$$

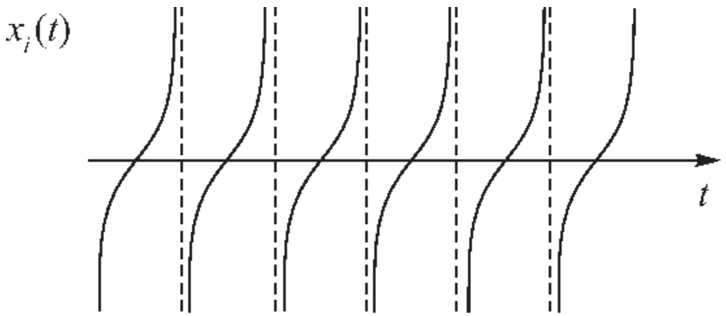
$$(-\tau - s)(-s) + c_1 c_2 = 0$$

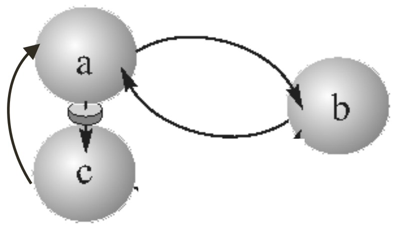
$$s^2 + \tau s + c_1 c_2 = 0$$

$\tau^2 - 4c_1c_2 < 0$ implies complex roots $c_2 \le \dfrac{\tau^2}{4\,c_1}$



A phase synchronization example





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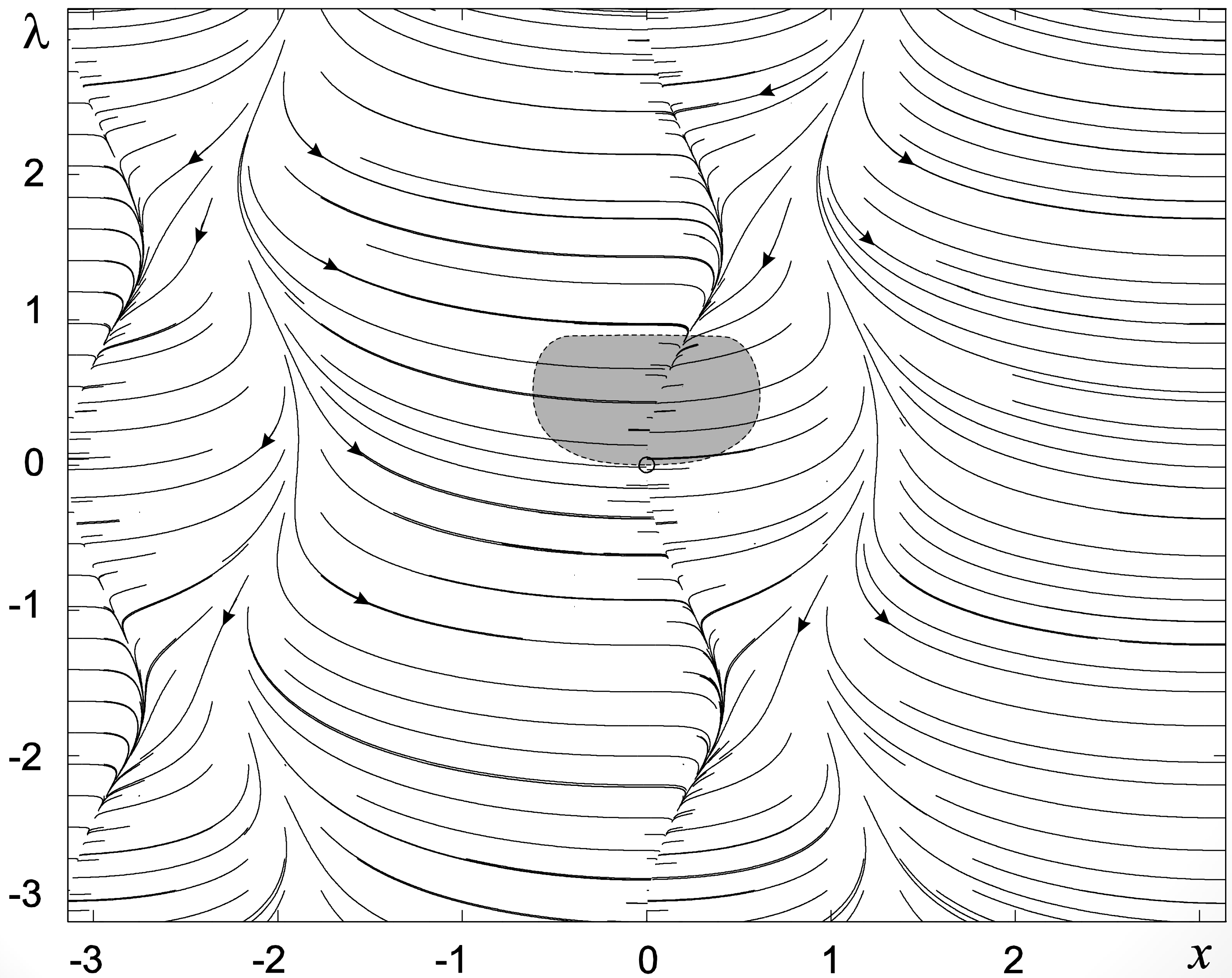

What's next?

- Convergence and convergence rates ?
- Analogue of the first Lyapunov method ?