Reducing a family of attractors: parameter dependence in the reduced model

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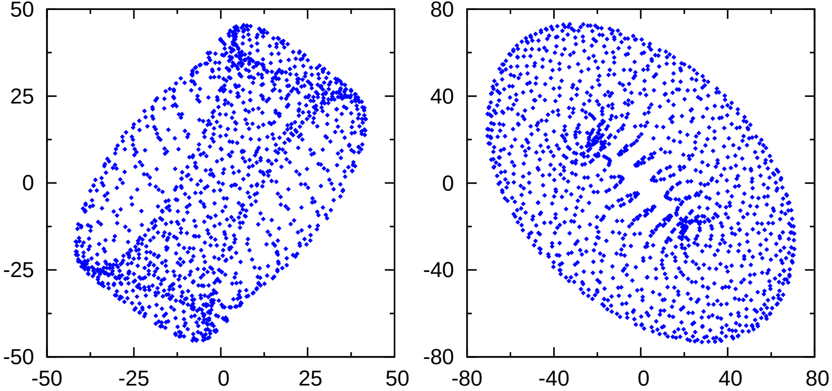
August 21, 2014

Introduction

- Reduction across disciplines – variety of both applications and types of reduction.
- Applications: Complex systems and PDEs naturally give high-dimensional spaces.
- Types of reduction: Data reduction, Dynamical systems, Control systems.
- Key question: what structure are you trying to preserve?

Torus Attractor



Reduction of Attractors

- Smooth autonomous dynamical system, usually given in the form of a differential equation.
- Static parameters – constant w.r.t. time evolution, but may take different values in different instances of the system.
- Family of low-dimensional attractors.
- Challenge – Variation in attractors, e.g. bifurcations.
- Observation – The underlying vector field is a smooth function of both state and parameter.

Conventional Galerkin Approach

- Start with the differential equation:

$$\dot{x} = f(x)$$

- Apply projection, $\hat{x} = Px$:

$$\dot{\hat{x}} = P \circ f(x)$$

- Use a choice of inverse to identify the low-dimensional states with high-dimensional states:

$$\dot{\hat{x}} = P \circ f \circ R(\hat{x}) = \hat{f}(\hat{x})$$

- Problem 1 – To evaluate $\hat{f}$ in general requires evaluation of the original vector field f.
- Problem 2 – R will be approximated – the error in this approximation directly affects the quality of the reduced dynamics.

Conventional Galerkin Approach

- If the inverse approximation R is required to be linear, the only way to improve the reduced model is to increase the dimension of the reduced space.
- This can result a higher-dimensional reduced model than is geometrically necessary.
- Nonlinear Galerkin methods attempt to produce a nonlinear inverse approximation that enables lower-dimensional reduced models.
- With parameters, even if the projection is parameter-independent, the inverse is still parameter-dependent.

Approach of Broomhead and Kirby

- Use the projection of the original model to determine requirements of the reduced vector field.
- Use optimization to find the best vector field that satisfies these requirements (from a space of candidates).
- Preserve the vector field along the attractor.
- Preserve (some of the) derivatives of the vector field on the attractor.
- By preserving the relevant parts of the vector field, the flow produces the same attractor in the low-dimensional space.
- This approach separates the concern of finding a good projection (the geometry of the attractor) from the concern of reproducing the dynamics (the vector field and its derivatives).

Formulation

- State space X is a smooth manifold.
- Parameter space U, which we can think of as the product of intervals, $[a_1, b_1] \times \cdots \times [a_p, b_p]$.
- A smooth family of smooth vector fields, $V : U \to \mathfrak{X}(X)$.

Look for relationship between the given original (X, U, V) and the chosen reduced candidate family $(\hat{X}, \hat{U}, \hat{V})$.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \hat{X} \\ U & \xrightarrow{Q} & \hat{U} \\ V \big\downarrow & & \big\downarrow \hat{V} \\ \mathfrak{X}(X) & & \mathfrak{X}(\hat{X}) \end{array}$$

Reduced Family of Vector Fields

Two classes of vector fields

- Generic family, e.g. constructed from radial basis functions:

$$V_x = Lx + \sum_i w_i \phi(\|x - c_i\|).$$

  - Can be used with a wide variety of examples.
  - No knowledge of the dynamics required.
- Specialised family – tailored to the example system.
  - May require prior knowledge of the dynamics.

Strategy

1. Embed the state space X into an ambient $\mathbb{R}^n$.
2. Find an orthogonal projection that preserves the attractors (secant-based projection). The subspace is the reduced state space $\hat{X} = \mathbb{R}^d$.
3. Apply the projection to the original vector fields and their derivatives on their respective attractors.
4. Consider a radial basis family of vector fields on $\mathbb{R}^d$.
5. Use optimization to find the affine parameter map $U \to \hat{U}$ that best reproduces these aspects of the vector fields.

Secant-Based Projection

- Motivated by a proof of the Whitney Embedding theorem.
- For each pair of points on the manifold, the secant is the unique straight line through the pair.
- An orthogonal projection that preserves all the secants smoothly embeds the manifold into the subspace.
- Write a cost function and perform optimization to find an orthogonal projection P that preserves all the secants:

  $$\mathcal{F}(P) = \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \|Pk\|^{-1}$$

  where each k is a unit vector describing a secant.
- This is a cost function on the Grassmann manifold, $\mathcal{F} : \mathrm{Gr}_d(\mathbb{R}^n) \to [1, \infty]$.
- The Grassmann manifold has closed form expressions for its geodesics and parallel translation along geodesics!

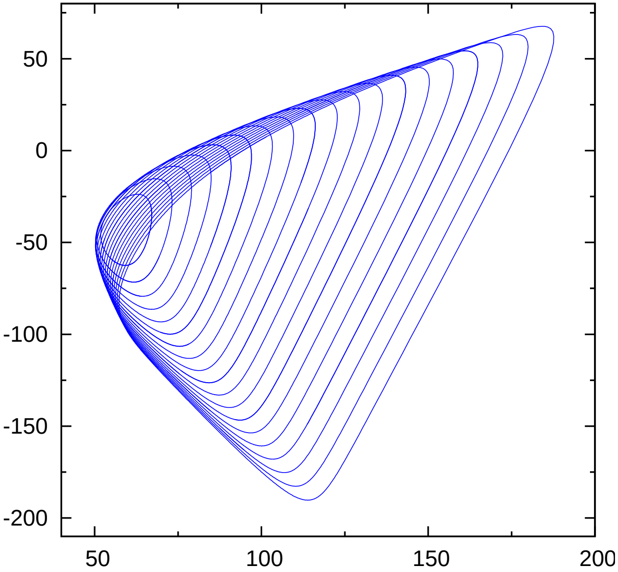
Example: Brusselator

The Brusselator is a reaction-diffusion equation describing an autocatalytic chemical reaction:

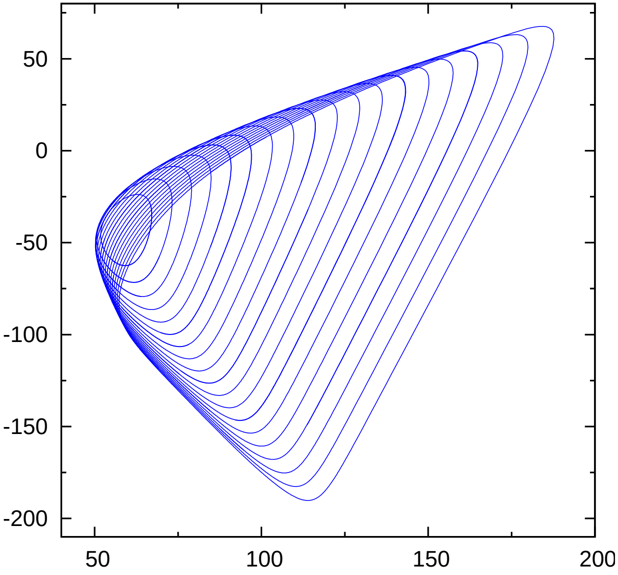
$$\partial_t u = 1 + u^2 v - (\lambda + 1)v + \nabla^2 u$$
$$\partial_t v = \lambda u - u^2 v + \nabla^2 v.$$

- Parameter region $\lambda \in [2.1, 3.9]$ – limit cycles.
- 2D physical space with a 32×32 discretization and periodic boundaries. State space is $\mathbb{R}^{2048}$.
- Look for projection from 2048 to 2 dimensions.
- Use 30 radial basis functions of type $\phi(r) = r^2 \log r$.

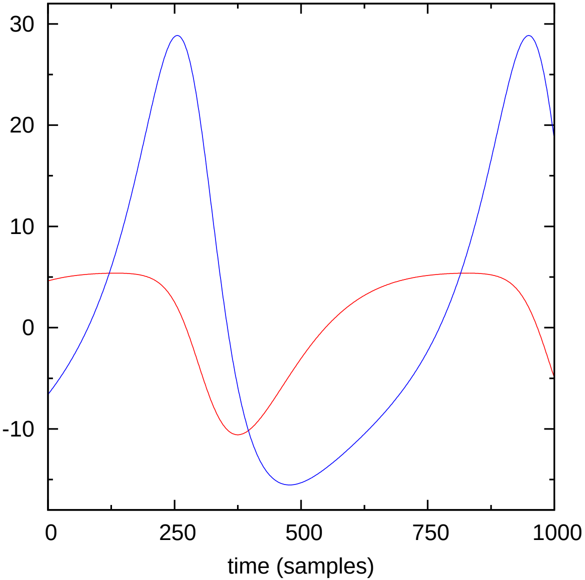
Brusselator – Original



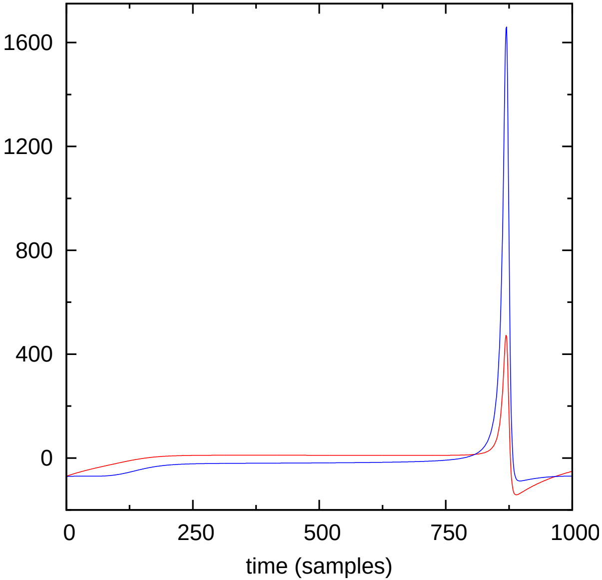
Brusselator – Reduced



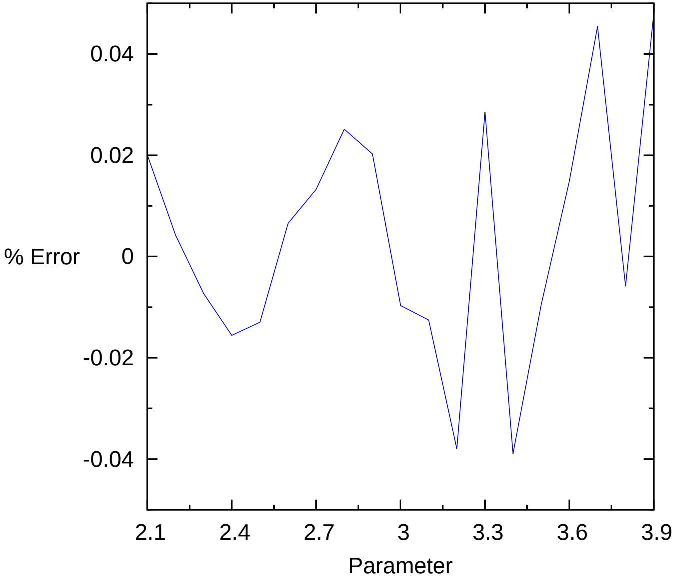
Brusselator – $\lambda = 2.1$



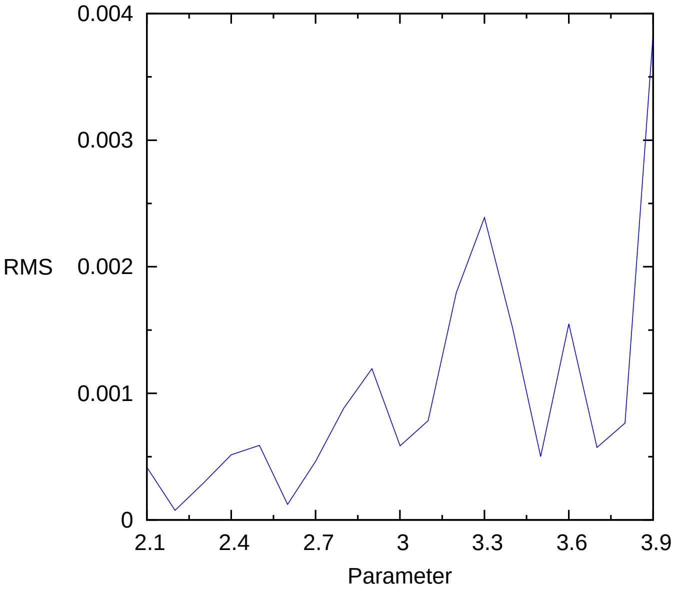
Brusselator – $\lambda = 3.9$



Brusselator – Errors in the period



Brusselator – Error in the time series



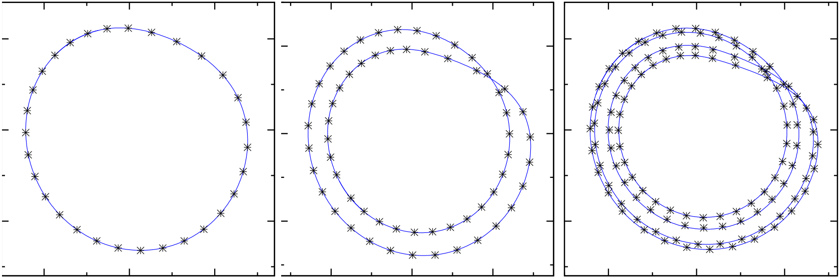
Example: Rössler

The Rössler system is a standard example in dynamical systems. It has a 3D state space and features period-doubling bifurcations and chaos.

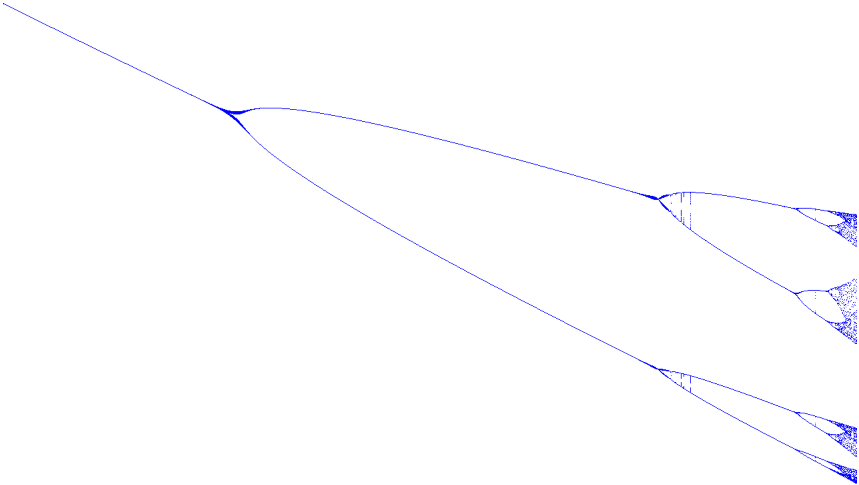
$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$

- Parameter region $a = b = 0.1$ and $c \in [4, 8.8]$ – period-doubling bifurcations.
- Use 40 radial basis functions of type $\phi(r) = r^3$.

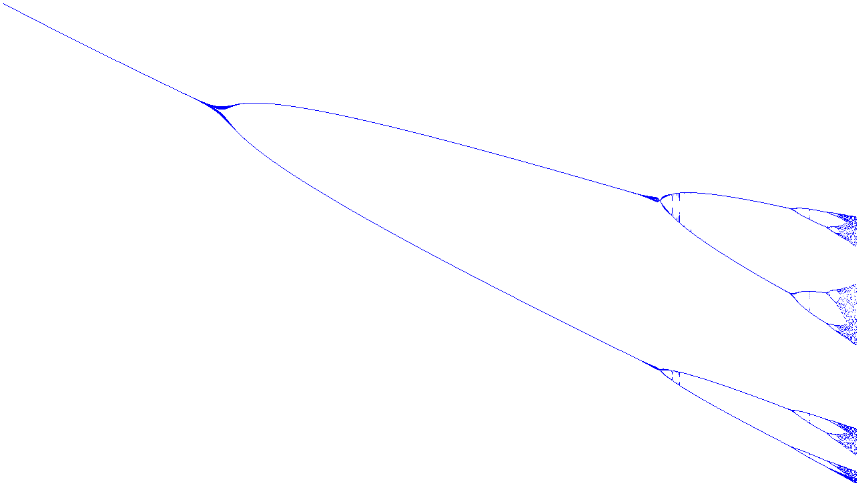
Rössler



Rössler – Original



Rössler – Reproduction



Bonus: Preserving Symmetry in the Reduced Model

- G-equivariance: $f(gx) = gf(x)\ \forall g \in G$
- For an orthogonal group action, can preserve equivariance by orthogonally projecting onto an invariant subspace of the group action.
- Modify the radial basis functions to produce a manifestly equivariant vector field. The RBF

  $$\psi(x) = w\phi(\|x - c\|)$$

  becomes

  $$\psi(x) = \sum_{g \in G} gw\phi(\|x - gc\|)$$

  i.e. replace both the weight w and centre c with their respective orbits under the group action and sum contributions.
- Can extend to continuous groups by integrating (Lie groups have a unique volume form up to a scale factor). But these intergrals are hard even for SO(2).

Example: FitzHugh-Nagumo

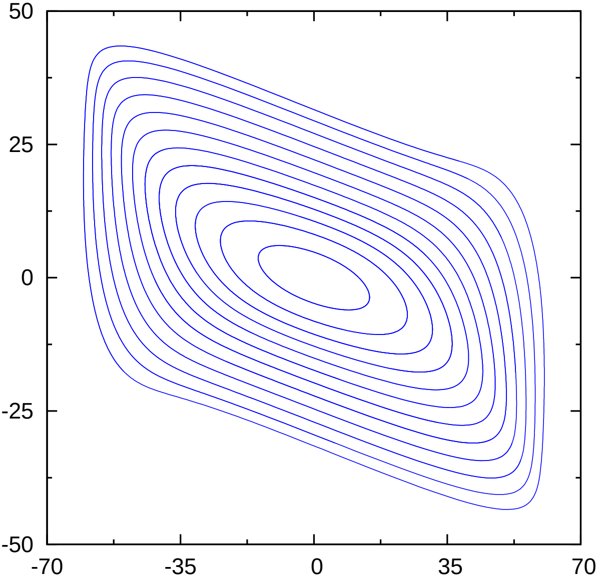
The FitzHugh-Nagumo model is:

$$\partial_t u = \lambda u - u^3 - 3v + \nabla^2 u$$
$$\partial_t v = \frac{1}{3}(u - v + \nabla^2 v).$$

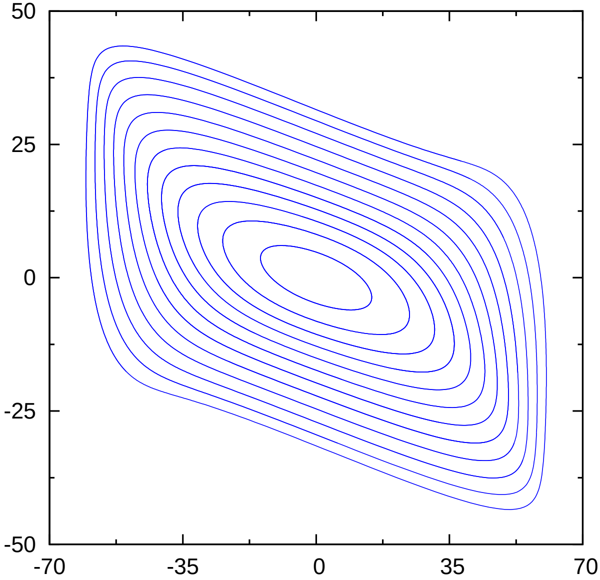
It has a symmetry: $\mathbb{Z}_2$-equivariant for the action $(u, v) \mapsto (-u, -v)$.

- Parameter region $\lambda \in [0.5, 4]$ – limit cycles.
- 2D physical space with a 32×32 discretization and periodic boundaries. State space is $\mathbb{R}^{2048}$.
- Look for projection from 2048 to 2 dimensions.
- Use 30 $\mathbb{Z}_2$-equivariant radial basis functions of type $\phi(r) = r^2 \log r$.

FitzHugh-Nagumo – Original



FitzHugh-Nagumo – Reduced



FitzHugh-Nagumo – Equivariant vs. Standard RBFs

- For the FitzHugh-Nagumo example, we observe that N equivariant RBFs are approximately as good as $2N$ standard RBFs:

# RBFs	Std RBF	Equi RBF
10		0.421851
15		0.0447792
20	0.543544	0.0101049
30	0.207695	0.00159639
40	0.0403139	
60	0.00329719	

Max percentage error in the period over the parameter region.

Thanks!

- A library of code is available on GitHub: `https://github.com/cwzx/DRDSP`
- C. Welshman and J. M. Brooke, **Dimensionality Reduction of Dynamical Systems with Parameters: A Geometric Approach**, SIAM Journal on Applied Dynamical Systems, 2014, 13(1):493–517. `dx.doi.org/10.1137/130913675`
- My thesis at Manchester: `eprints.ma.man.ac.uk/2134/`