Reducing a family of attractors: parameter dependence in the reduced model

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Introduction

- Reduction across disciplines – variety of both applications and types of reduction.
- Applications: Complex systems and PDEs naturally give high-dimensional spaces.
- Types of reduction: Data reduction, Dynamical systems, Control systems.
- Key question: what structure are you trying to preserve?
Torus Attractor
Reduction of Attractors

- Smooth autonomous dynamical system, usually given in the form of a differential equation.
- Static parameters – constant w.r.t. time evolution, but may take different values in different instances of the system.
- Family of low-dimensional attractors.
- Challenge – Variation in attractors, e.g. bifurcations.
- Observation – The underlying vector field is a smooth function of both state and parameter.
Conventional Galerkin Approach

- Start with the differential equation:
  \[ \dot{x} = f(x) \]

- Apply projection, \( \hat{x} = Px \):
  \[ \dot{\hat{x}} = P \circ f(x) \]

- Use a choice of inverse to identify the low-dimensional states with high-dimensional states:
  \[ \dot{\hat{x}} = P \circ f \circ R(\hat{x}) = \hat{f}(\hat{x}) \]

- Problem 1 – To evaluate \( \hat{f} \) in general requires evaluation of the original vector field \( f \).

- Problem 2 – \( R \) will be approximated – the error in this approximation directly affects the quality of the reduced dynamics.
If the inverse approximation \( R \) is required to be linear, the only way to improve the reduced model is to increase the dimension of the reduced space.

This can result a higher-dimensional reduced model than is geometrically necessary.

Nonlinear Galerkin methods attempt to produce a nonlinear inverse approximation that enables lower-dimensional reduced models.

With parameters, even if the projection is parameter-independent, the inverse is still parameter-dependent.
Approach of Broomhead and Kirby

- Use the projection of the original model to determine requirements of the reduced vector field.
- Use optimization to find the best vector field that satisfies these requirements (from a space of candidates).
- Preserve the vector field along the attractor.
- Preserve (some of the) derivatives of the vector field on the attractor.
- By preserving the relevant parts of the vector field, the flow produces the same attractor in the low-dimensional space.
- This approach separates the concern of finding a good projection (the geometry of the attractor) from the concern of reproducing the dynamics (the vector field and its derivatives).
Formulation

- State space $X$ is a smooth manifold.
- Parameter space $U$, which we can think of as the product of intervals, $[a_1, b_1] \times \cdots \times [a_p, b_p]$.
- A smooth family of smooth vector fields, $V : U \to \mathcal{X}(X)$.

Look for relationship between the given original $(X, U, V)$ and the chosen reduced candidate family $(\hat{X}, \hat{U}, \hat{V})$.

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \hat{X} \\
U & \xrightarrow{Q} & \hat{U} \\
\mathcal{X}(X) & \xrightarrow{\mathcal{X}} & \mathcal{X}(\hat{X})
\end{array}
\]
Two classes of vector fields

- **Generic family**, e.g. constructed from radial basis functions:

\[ V_x = Lx + \sum_i w_i \phi(\|x - c_i\|) \]

- Can be used with a wide variety of examples.
- No knowledge of the dynamics required.

- **Specialised family** – tailored to the example system.
  - May require prior knowledge of the dynamics.
1. Embed the state space $X$ into an ambient $\mathbb{R}^n$.

2. Find an orthogonal projection that preserves the attractors (secant-based projection). The subspace is the reduced state space $\hat{X} = \mathbb{R}^d$.

3. Apply the projection to the original vector fields and their derivatives on their respective attractors.

4. Consider a radial basis family of vector fields on $\mathbb{R}^d$.

5. Use optimization to find the affine parameter map $U \rightarrow \hat{U}$ that best reproduces these aspects of the vector fields.
Secant-Based Projection

- Motivated by a proof of the Whitney Embedding theorem.
- For each pair of points on the manifold, the secant is the unique straight line through the pair.
- An orthogonal projection that preserves all the secants smoothly embeds the manifold into the subspace.
- Write a cost function and perform optimization to find an orthogonal projection $P$ that preserves all the secants:

$$\mathcal{F}(P) = \frac{1}{|K|} \sum_{k \in K} \|Pk\|^{-1}$$

where each $k$ is a unit vector describing a secant.
- This is a cost function on the Grassmann manifold, $\mathcal{F} : Gr_d(\mathbb{R}^n) \to [1, \infty]$.
- The Grassmann manifold has closed form expressions for its geodesics and parallel translation along geodesics!
The Brusselator is a reaction-diffusion equation describing an autocatalytic chemical reaction:

\[
\begin{align*}
\partial_t u &= 1 + u^2 v - (\lambda + 1) v + \nabla^2 u \\
\partial_t v &= \lambda u - u^2 v + \nabla^2 v.
\end{align*}
\]

- Parameter region \( \lambda \in [2.1, 3.9] \) – limit cycles.
- 2D physical space with a 32 \( \times \) 32 discretization and periodic boundaries. State space is \( \mathbb{R}^{2048} \).
- Look for projection from 2048 to 2 dimensions.
- Use 30 radial basis functions of type \( \phi(r) = r^2 \log r \).
Brusselator – Original
Brusselator – Reduced
Brusselator – $\lambda = 2.1$
Brusselator – $\lambda = 3.9$
Brusselator – Errors in the period

![Graph showing % Error vs Parameter](image-url)
Brusselator – Error in the time series

Parameter vs. RMS

- Parameter values: 2.1, 2.4, 2.7, 3, 3.3, 3.6, 3.9
- RMS values: 0, 0.001, 0.002, 0.003, 0.004

Graph shows fluctuations in RMS error with respect to parameter values.
The Rössler system is a standard example in dynamical systems. It has a 3D state space and features period-doubling bifurcations and chaos.

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\]

- Parameter region \( a = b = 0.1 \) and \( c \in [4, 8.8] \) – period-doubling bifurcations.
- Use 40 radial basis functions of type \( \phi(r) = r^3 \).
Rössler
G-equivariance: \( f(gx) = gf(x) \ \forall g \in G \)

For an orthogonal group action, can preserve equivariance by orthogonally projecting onto an invariant subspace of the group action.

Modify the radial basis functions to produce a manifestly equivariant vector field. The RBF

\[
\psi(x) = w\phi(\|x - c\|)
\]

becomes

\[
\psi(x) = \sum_{g \in G} gw\phi(\|x - gc\|)
\]

i.e. replace both the weight \( w \) and centre \( c \) with their respective orbits under the group action and sum contributions.

Can extend to continuous groups by integrating (Lie groups have a unique volume form up to a scale factor). But these intergrals are hard even for SO(2).
The FitzHugh-Nagumo model is:

\[
\begin{align*}
\partial_t u &= \lambda u - u^3 - 3v + \nabla^2 u \\
\partial_t v &= \frac{1}{3} (u - v + \nabla^2 v).
\end{align*}
\]

It has a symmetry: \( \mathbb{Z}_2 \)-equivariant for the action \((u, v) \mapsto (-u, -v)\).

- Parameter region \( \lambda \in [0.5, 4] \) – limit cycles.
- 2D physical space with a \( 32 \times 32 \) discretization and periodic boundaries. State space is \( \mathbb{R}^{2048} \).
- Look for projection from 2048 to 2 dimensions.
- Use 30 \( \mathbb{Z}_2 \)-equivariant radial basis functions of type \( \phi(r) = r^2 \log r \).
FitzHugh-Nagumo – Reduced
For the FitzHugh-Nagumo example, we observe that $N$ equivariant RBFs are approximately as good as $2N$ standard RBFs:

<table>
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<th># RBFs</th>
<th>Std RBF</th>
<th>Equi RBF</th>
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<tr>
<td>10</td>
<td>0.421851</td>
<td>0.0447792</td>
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<td>15</td>
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</tr>
<tr>
<td>60</td>
<td>0.00329719</td>
<td></td>
</tr>
</tbody>
</table>

Max percentage error in the period over the parameter region.
Thanks!

- A library of code is available on GitHub: https://github.com/cwzx/DRDSP
- My thesis at Manchester: eprints.ma.man.ac.uk/2134/