COROLLARY 2. Not every variety of commutative semigroups is contained in some minimal noncommutative variety of semigroups.

For example, the variety of all commutative semigroups is not contained in any of the five varieties indicated in Theorem 2 or in a variety of semigroups which is a variety of groups.

COROLLARY 3. The variety μ_3 is generated by the semigroup $\Pi = \{i, n, 0\}$ with the following multiplication table:

	i	n	0
i	i	0	0
n	n	0	0
0	0	0	0

Each proper subsemigroup of Π is commutative. The variety μ_2 is generated by its free semigroup of rank 2, which is a semigroup whose proper subsemigroups are all commutative.

Indeed, let F_2 be a free semigroup in μ_3 with free generators a and b. Then, putting $i = ab$, $n = b^2$, $0 = a^2b$, and using the identities of μ_3, we see that the indicated multiplication table holds for Π and that $\Pi \in \mu_3$. Since $\Pi \in \mu_3$, it follows that Π generates a subvariety of μ_3. But Π is noncommutative, hence it cannot generate a proper subvariety of μ_3.

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CERTAIN PROPERTIES OF FREDHOLM ANALYTIC SETS IN BANACH SPACES

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With the aid of the Lyapunov—Schmidt method of transition to a finite-dimensional equation (a modern treatment can be found, e.g., in [1, pp. 374-385]), we prove in this paper certain assertions about analytic sets in complex Banach spaces. The principal result is a counterpart of the finite-dimensional Remmert—Stein theorem (see, e.g., [2]), stating that an analytic set in an open set U is either discrete, or it contains points that are as close as desired to the boundary of U.

As an application we shall prove the nonnegativeness of the rotation of the vector field x—Ax with an analytic and completely continuous operator A; we also consider the finiteness of the number of solutions of an equation that depends on a parameter.

Some of the results of this paper were presented in [3] with an additional assumption about the existence of a basis in the Banach spaces under consideration. This assumption proved unnecessary.

1. Let E be a Banach space over the field of complex numbers C.

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Definition. A Fredholm analytic set in an open set $U \subset E$ is a subset S of the set U that has the following property: For any point $a \in U$ there exists an open connected neighborhood $V \subset U$ of the Banach space E_a and a holomorphic operator $F: V \to E_a$ whose derivative at the point a is a linear Noether operator such that

$$S \cap V = \{x \in V \mid F(x) = \theta\}.$$

LEMMA 1. A Fredholm analytic set is locally isomorphic to a finite-dimensional analytic set.

Proof. Let $a \in S$. According to the definition, the set $S \cap V$ coincides in an open connected neighborhood V of the point a with the set of solutions of the equation

$$F(x) = \theta, \tag{1}$$

where F is an operator that is analytic in V and has a Noether derivative. Now let us use the Lyapunov–Schmidt method of investigation of the solutions of Eq. (1) in a small neighborhood of the point a; according to it, there exists a bijective correspondence (realized by holomorphic functions) between the set of solutions of Eq. (1) and the set of solutions of a finite-dimensional system of holomorphic equations, and this proves the lemma.

Fredholm analytic sets satisfy the maximum modulus principle.

THEOREM 1. Let S be a connected Fredholm analytic set in an open set $U \subset E$. If the functional f is holomorphic in U and its modulus restricted to S reaches a maximum, then f will be a constant on S.

Proof. A finite-dimensional counterpart of this theorem is well known [4], and the possibility of reduction to the finite-dimensional case has been proved in the above lemma.

LEMMA 2. Let S be a compact Fredholm analytic set in an open set U of the space E. Then S will be finite.

Proof. By virtue of Lemma 1, S is locally connected. By virtue of compactness it consists of finitely many connected components. Let P be any of these components. Let us consider a continuous linear functional l restricted to P. Since P is compact, the modulus of l restricted to P will reach a maximum. By virtue of Theorem 1, l is constant on P. But for any two points of the space E there exists a continuous linear functional separating these points; therefore P cannot contain more than one point. This proves the finiteness of S.

Remark. For analytic sets in infinite-dimensional spaces, compactness does not in general imply finiteness.

THEOREM 2 (Counterpart of Remmert–Stein Theorem). Let U be a bounded open set in E and let H be a neighborhood of the set $\overline{U}$ (the closure of U); let the operator $A: H \to E$ be analytic and completely continuous in H, and suppose that the equation

$$x = Ax \tag{2}$$

does not have solutions on the boundary of U. Then the set S of solutions of this equation in U will be finite.

Proof. The set S is closed. Indeed, if $x_n \in S$, $n = 1, 2, \ldots$, and $x_n \to b$, then $b = Ab$ and $b \in U$, since Eq. (2) does not have solutions on $\overline{U} \setminus U$. Since S belongs to the compact set $A\overline{U}$, it follows that S is a compact set. By virtue of Lemma 2, the set S is finite.

Just as in the finite-dimensional case, we obtain from Theorem 2 the following

COROLLARY (see [2]). Let U be a bounded open set in E that contains θ, let H be a neighborhood of the set $\overline{U}$, and let the operator $A: H \to E$ be analytic and completely continuous in H, $A\theta = \theta$. Let $f_1, \ldots, f_m$ be continuous linear functionals such that θ is an isolated solution of the system

$$\begin{cases} x = Ax, \\ f_j(x) = 0, \quad j = 1, \ldots, m. \end{cases}$$

Then there exist neighborhoods W_j of elements f_j in the conjugate space E^* such that for any system

$$\begin{cases} x = Ax, \\ f_j'(x) = 0, \quad j = 1, \ldots, m, \end{cases}$$

where $f_j' \in W_j$, the point θ is an isolated solution.

Proof. It can be assumed that m = 1 (otherwise we go over from the space E to the subspace $L = \{x \in E | f_1(x) = \ldots f_m(x) = 0\}$). Let us assume that $f_1(x) \neq 0$ (otherwise the assertion is obvious), and consider the subspace $E_1 = \{x \in E | f_1(x) = 0\}$. Let us denote by P_r a sphere in E_1 that is centered at the origin and has a sufficiently small radius r, and on which the equation x = Ax does not have solutions. From the complete continuity of the operator A it evidently follows that there exists a positive number β such that for $x \in P_r$ we have

$$\|x - Ax\| \geqslant \beta. \tag{3}$$

Let $W_1 = \{f_1' \in E_1 | \|f_1' - f_1\|_{E^*} < \varepsilon\}$, where ε is a positive number. For $f_1' \in W_1$ let us denote by P_r' a sphere centered at θ that has a radius r which lies in the subspace $E_1' = \{x \in E | f_1'(x) = 0\}$. If $x \in P_r'$, then

$$\rho(x, E_1) = \frac{f_1(x)}{\|f_1\|} = \frac{|f_1(x) - f_1'(x)|}{\|f_1\|} \leqslant \frac{\varepsilon r}{\|f_1\|}.$$

With the aid of this inequality it is easy to show that $\rho(x, P_r)$ is smaller than a preassigned positive δ for a sufficiently small ε. It hence follows by virtue of (3) that the equation x = Ax does not have solutions on P_r'' for a sufficiently small ε. By virtue of Theorem 2, the interior of the sphere P_r' contains only finitely many solutions of the equation x = Ax, which completes the proof.

2. For a completely continuous vector field $\Phi(x) = x - Ax$ that does not vanish on the boundary D of a bounded open connected region U of the space E, a rotation on D has been defined in [5]. In the case that the field $\Phi(x)$ has finitely many fixed points in U, a rotation of $\Phi(x)$ on D is equal to the sum of the indices of these fixed points. If the operator A is not only completely continuous, but also analytic in U, and E is a complex Banach space, then the index of a fixed point of the field $\Phi(x) = x - Ax$ will be larger than zero. This has been proved in fact by Cronin in [6,7] (see [6, Theorem 5.1, pp. 228-230], and [7, Theorem 3, pp. 177-180]). From this result of Cronin and Theorem 2 we directly obtain

THEOREM 3. Suppose that the conditions of Theorem 2 are satisfied. Then the rotation γ of the field $\Phi(x) = x - Ax$ on the boundary of U will be nonnegative.

Remark. If $\gamma = 1$, then Eq. (2) will have a unique solution in U.

3. Under the conditions of Theorem 2, Eq. (2) has finitely many solutions in U. Let us consider the "perturbed" equation

$$x = Ax + Q(x, \lambda), \tag{4}$$

where λ is a complex parameter that varies in a circle M of the complex plane centered at the origin, and $Q: H \times M \to E$ is an analytic mapping, $Q(x, 0) = \theta$. Under certain general conditions, a finite number of solutions of Eq. (4) branch off from each solution of Eq. (2).).

THEOREM 4. Suppose that the conditions of Theorem 2 are satisfied. Let $\|Q(x, \lambda)\| \to 0$ for $\lambda \to 0$ uniformly in $x \leq \overline{U}$. Then there exists a positive α such that for $|\lambda| < \alpha$ the equation (4) has finitely many solutions in U.

Proof. At first let us impose on the operator $Q(x, \lambda)$ the additional condition of complete continuity in $\overline{U}$ for any $\lambda \in M$. As we noted in obtaining the corollary of Theorem 2, from the complete continuity of the operator A there follows the existence of a positive number β such that for $x \in \overline{U} \setminus U$ we have

$$\|x - Ax\| > \beta.$$

On the other hand, there exists a positive α such that for $x \in \overline{U}$ and $|\lambda| < \alpha$,

$$\|Q(x, \lambda)\| < \beta.$$

It follows from these two inequalities that for $|\lambda| < \alpha$ Eq. (4) does not have solutions on the boundary of U. But in this case it follows from Theorem 2 that for $|\lambda| < \alpha$ Eq. (4) has finitely many solutions in U.

Now let us consider the general case. Suppose that the set S of solutions of Eq. (2) in U consists of the points $h_1, \ldots, h_n$ (the case of S empty can be considered in exactly the same way). In a small neighborhood of the point h_k (k = 1, ..., n), Eq. (2) is equivalent to the Lyapunov—Schmidt branching equation in finite-dimensional space. Since the branching equation is an equation with completely continuous operators, it is possible to apply to it the above analysis. Thus there exists a positive α and open neighborhoods V_j of the points h_j (j = 1, ..., n) such that for $|\lambda| < \alpha$ Eq. (4) has finitely many solutions in V_j. Let $W = \overline{U} \setminus \bigcup_{j=1}^{n} V_j$. Let us show that for sufficiently small λ Eq. (4) does not have solutions in W. Let us assume the contrary. Then there exists a sequence $\{\lambda_i\}_1^\infty$ that converges to zero, and a sequence $\{x_i\}_1^\infty$ of points in W such that

$$x_i = Ax_i + Q(x_i, \lambda_i).$$

The sequence $\{Q(x_i, \lambda_i)\}_1^\infty$ converges to θ, and from the sequence $\{Ax_i\}_1^\infty$ it is possible to select a convergent subsequence. By going over to the limit with respect to this sequence, we obtain the equation h = Ah for an h∈W, which is impossible.

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A BOUNDARY-VALUE PROBLEM FOR AN ELLIPTIC–PARABOLIC EQUATION

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Let Ω be a bounded domain of an n-dimensional Euclidean space such that Ω is the same sum of two domains Ω_1 and Ω_2, where Ω_1 is a strictly interior subdomain of the domain Ω. Let S be the boundary of Ω and S_k the boundary of Ω_k, k = 1, 2. In what follows, we put $\Gamma = S_1$, $\Gamma_T = \Gamma \times [0, T]$, $S_T = S \times [0, T]$, $Q_T^{(k)} = \Omega_k \times [0, T]$, k = 1, 2.

Let the function $K(x, t) \geq 0$; further, we introduce the notation

$$\Gamma_+ = \{(x, 0) : x \in \Omega_2,\ K(x, 0) > 0\},\ \Gamma_1 = \bar{\Gamma}_+.$$

Our concern here is the following boundary-value problem.

Problem. Determine a function u(x, t) satisfying in the domain $Q_T^{(1)}$ an elliptic equation of the form

$$\mathscr{L}_1 u = f_1(x, t), \tag{1}$$

and in the domain $Q_T^{(2)}$ a parabolic equation of the form

$$\mathscr{M}_2 u \equiv K(x, t) u_t + \mathscr{L}_2 u = f_2(x, t), \tag{2}$$

where $\mathscr{L}_k = -\frac{\partial}{\partial x_i}\left[a_{ij}^{(k)}(x, t)\frac{\partial}{\partial x_j}\right] + b_i^{(k)}(x, t)\frac{\partial}{\partial x_i} + c^{(k)}(x, t)$, k = 1, 2; at t = 0 the function u(x, t) is to satisfy the initial condition

$$u|_{\Gamma_1} = 0, \tag{3}$$

on the boundary S_T, one of the classical boundary conditions, e.g., the first boundary condition

$$u|_{S_T} = 0. \tag{4}$$

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