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STRUCTURE AND APPROXIMATIONS OF THE CHAPMAN-ENSKOG EXPANSION FOR THE LINEARIZED GRAD EQUATIONS

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ABSTRACT

A detailed structure of the Chapman-Enskog expansion for the linearized Grad moment equations is determined. A method of partial summing of the Chapman-Enskog series is introduced, and is used to remove short-wave instability of the Burnett approximations.

1. INTRODUCTION

Obtaining hydrodynamics from the Boltzmann equation is a classical problem. At the same time, a number of questions arising here are still unanswered. Thus, it is not quite clear which equations should follow the Navier-Stokes approximation. The classical Chapman-Enskog method [1] gives, in principle, the possibility to correct the Navier-Stokes approximation. However, as it has been shown in [2], the first corrections (the Burnett and the super-Burnett corrections) result in a catastrophe: short-wave instability of sonic waves occurs. As it was stated in [2], this result contradicts the H-theorem. It is commonly known that the entropy growth results in a decrease of any small perturbation of the equilibrium.

The "ultra-violet catastrophe" of the Burnett approximations makes us think that we have to take into account the very remote terms of the Chapman-Enskog expansion. This situation is similar to that which occurs in quantum field theory and statistical mechanics. Singularities of expansions which occur there can be removed by approximations of the whole series (partial summing of infinite subsequences of diagrams, Pade approximations, etc.).

In this paper we make an attempt to correct the Burnett approximations by partial summing of the Chapman-Enskog series. The difficulties in obtaining the terms of these series from the Boltzmann equation are common knowledge. For example, the Burnett and the super-Burnett approximations for the simplest case of Maxwell molecules have only recently been completely obtained [3]. Therefore, we deal with the Chapman-Enskog expansion for the linearized Grad hydrodynamic equations [4] rather than with the Boltzmann equation. At least in this (linear) case the difference between the results for Grad equations and the Boltzmann equation is negligible. Grad equations give us certain technical advantages, especially when working with the Chapman-Enskog series as a whole.

The structure of this paper is as follows: In section 2 the structure of the Chapman-Enskog coefficients for the linearized tension tensor and for the heat flow vector is set. The expression (2.7) particularizes the appropriate Grad's result for the linearized Boltzmann equation [5]. In section 3 we introduce a method of consecutive approximations, which approximates the Chapman-Enskog recurrent procedure.

This method involves partial summing of the Chapman-Enskog series. Some applications of this method to the linearized Grad equations are considered.

The results of this paper were presented at the CHISA International Congress (Praha, 1990) [6].

2. STRUCTURE OF THE CHAPMAN-ENSKOG EXPANSION

Denote as ρ_0, T_0, and $\vec{u}_0=0$ the equilibrium values of density, temperature and the flow velocity vector respectively. Notations ρ', T', and $\vec{u}'$ represent small deviations of hydrodynamic parameters from their equilibrium values.

Grad hydrodynamic equations [4], which will soon appear, contain the viscosity coefficient μ which relates Grad hydrodynamic equations to the Boltzmann equation. We use the representation $\mu(T)=\eta(T)T$. The function $\eta(T)$ depends upon the choice of the collision model in the Boltzmann equation. In particular, $\eta=const$ for Maxwell molecules, and η is proportional to $T^{-1/2}$ for rigid spheres. Everywhere below we use the scale system with the Boltzmann constant and the particle's mass equal to unity.

Let us introduce the dimensionless variables:

$$\vec{u}=T_0^{-1/2}\vec{u}', \quad \rho=\rho'/\rho_0, \quad T=T'/T_0, \quad \vec{x}=\eta(T_0)^{-1}T_0^{-1/2}\rho_0\vec{x}',$$

$$t=\eta(T_0)^{-1}\rho_0 t'. \tag{2.1}$$

Here $\vec{x}'$ represents the spatial vector, and t' is the time.

The linearized 13-moment Grad equations have the following form in terms of the variables (2.1):

$$\partial_t\rho=-\partial_i u_i; \qquad \partial_t T=-(2/3)(\partial_i u_i+\partial_i q_i);$$

$$\partial_t u_k=-\partial_k\rho-\partial_k T-\partial_i\sigma_{ik};$$

$$\partial_t\sigma_{ik}=-(\partial_i u_k+\partial_k u_i-\tfrac{2}{3}\delta_{ik}\partial_s u_s)-$$

$$-\tfrac{5}{2}(\partial_i q_k+\partial_k q_i-\tfrac{2}{3}\delta_{ik}\partial_s q_s)-\varepsilon^{-1}\sigma_{ik};$$

$$\partial_t q_k=-(5/2)\partial_k T-\partial_i\sigma_{ik}-(2/3)\varepsilon^{-1}q_k. \quad (2.2)$$

Here and further σ_{ik}, where $i,k=1,2,3$, represents the traceless part of the tension tensor, and q_k, where $k=1,2,3$, represents the heat flow vector. Notation ∂_i represents the partial derivative $\partial/\partial x_i$. In two repeated indices summation is assumed, and ε is a small parameter (the Knudsen number). The terms which are proportional to ε^{-1} appear from the Boltzmann collision integral.

The application of the Chapman-Enskog method to the system (2.2) involves representing σ_{ik} and q_k in the form of the series:

$$\sigma_{ik}=\sum_{n=0}^{\infty}\varepsilon^{n+1}\sigma_{ik}^{(n)},\ q_k=\sum_{n=0}^{\infty}\varepsilon^{n+1}q_k^{(n)}. \quad (2.3)$$

The coefficients $\sigma_{ik}^{(n)}$ and $q_k^{(n)}$ are obtained from the following recurrent procedure:

$$\sigma_{ik}^{(0)}=-(\partial_i u_k+\partial_k u_i-(2/3)\delta_{ik}\partial_s u_s),\quad q_k^{(0)}=-(15/4)\partial_k T,$$

$$\sigma_{ik}^{(n)}=-\sum_{s=0}^{n-1}\partial_t^{(s)}\sigma_{ik}^{(n-s-1)}-\tfrac{2}{5}(\partial_i q_k^{(n-1)}+\partial_k q_i^{(n-1)}-\tfrac{2}{3}\delta_{ik}\partial_s q_s^{(n-1)}),$$

$$q_k^{(n)}=-(3/2)\{\sum_{s=0}^{n-1}\partial_t^{(s)}q_k^{(n-(s+1))}+\partial_i\sigma_{ik}^{(n-1)}\},\ \text{for } n\geqslant 1. \quad (2.4)$$

Operators $\partial_t^{(s)}$, where $s\geqslant 0$, act as follows:

$$\partial_t^{(0)}D\rho=-D\partial_s u_s,\quad \partial_t^{(0)}DT=-(2/3)D\partial_s u_s,\quad \partial_t^{(0)}Du_k=-D\partial_k(T+\rho),$$

$$\partial_t^{(s)}D\rho=0,\qquad \partial_t^{(s)}DT=-\tfrac{2}{3}D\partial_i q_i^{(s-1)},$$

$$\partial_t^{(s)} Du_k = -D\partial_i \sigma_{ik}^{(s-1)}, \quad s \geqslant 1. \tag{2.5}$$

Here and further D represents an arbitrary differential operator: $D=\partial_1^{l_1}\partial_2^{l_2}\partial_3^{l_3}$, where $l_i \geqslant 0$, and $\partial_i^0 = 1$.

According to (2.4) and (2.5) the coefficients in expansion (2.3) are expressed as spatial derivatives of the functions ρ, T, and u_k.

We now introduce the notations:

$$\langle \partial_i u_k \rangle = \partial_i u_k + \partial_k u_i - \tfrac{2}{3}\delta_{ik}\partial_s u_s,$$

$$\Gamma_{ik} = 2(\partial_i \partial_k - \tfrac{1}{3}\delta_{ik}\Delta), \quad \Delta = \partial_s \partial_s. \tag{2.6}$$

The main result of this section is as follows: coefficients $\sigma_{ik}^{(n)}$ and $q_k^{(n)}$ in expansions (2.3) have the form:

$$\sigma_{ik}^{(2n)} = c_n \Delta^n \langle \partial_i u_k \rangle + d_n \Delta^{n-1} \Gamma_{ik} \partial_s u_s,$$

$$\sigma_{ik}^{(2n+1)} = a_n \Delta^n \Gamma_{ik} \rho + b_n \Delta^n \Gamma_{ik} T,$$

$$q_k^{(2n)} = \alpha_n \Delta^n \partial_k \rho + \beta_n \Delta^n \partial_k T,$$

$$q_k^{(2n+1)} = \varphi_n \Delta^n \partial_k \partial_s u_s + \psi_n \Delta^{n+1} u_k; \tag{2.7}$$

for all $n \geqslant 0$. Here a_n, b_n, c_n, d_n, α_n, β_n, φ_n and ψ_n are the numerical coefficients.

We represent a sketch of a proof by induction. Immediate calculations show:

$$\sigma_{ik}^{(1)} = -\Gamma_{ik}\rho + (1/2)\Gamma_{ik} T, \quad q_k^{(1)} = -(13/4)\partial_k \partial_s u_s + (3/2)\Delta u_k. \tag{2.8}$$

After taking into account $\sigma_{ik}^{(0)}$ and $q_k^{(0)}$ (2.4), we see that the statement (2.7) is proved for $n=0$. Let the structure (2.7) be set for a certain $n>0$. Then for $n+1$ we have

$$\sigma^{(2(n+1))} = -\{\sum_{s=0}^{2n+1} \partial_t^{(s)} \sigma^{(2(n+1)-(s+1))} + \frac{2}{3}(\partial_i q_k^{(2n+1)} +$$

$$+\partial_k q_i^{(2n+1)} - \frac{2}{3}\delta_{ik}\partial_s q_s^{(2n+1)}\} = \{\sum_{p=0}^{n} c_p c_{n-p} - \frac{2}{5}\psi_n\}\Delta^{n+1} < \partial_i u_k > +$$

$$+\Delta^n \Gamma_{ik} \partial_m u_m \{a_n + \frac{2}{3}b_n - \frac{2}{5}\varphi_n + \frac{1}{3}\sum_{p=0}^{n}(c_{n-p}(c_p + 4d_p)) +$$

$$+ \frac{2}{3}\sum_{p=1}^{n} b_{n-p}(\varphi_{p-1} + \psi_{p-1}) + 4d_{n-p}(c_p + d_p)\} . \qquad (2.9)$$

It is obvious that the structure of this expression coincides with the the structure of the corresponding coefficient $\sigma_{ik}^{(2m)}$ in (2.7). It is easy to check that the coefficients $\sigma_{ik}^{(2m+1)}$, $q_k^{(2m)}$ and $q_k^{(2m+1)}$ also have the structure (2.7).

The break of the series (2.3) at some certain $n \geqslant 0$ yields a closed set with respect to ρ, T, u_k. The stationary solution of this set for $n=2m$ is obtained from equations:

$$A_m(\Delta)\rho + B_m(\Delta)T = 0 , \quad C_m(\Delta)\rho + D_m(\Delta)T = 0 , \quad \partial_i u_i = 0 ,$$

$$A_m(\Delta) = \sum_{p=0}^{m} \alpha_p \varepsilon^{2p+1}\Delta^{p+1} , \qquad B(\Delta) = \sum_{p=0}^{m} \beta_p \varepsilon^{2p+1}\Delta^{p+1} ,$$

$$C_m(\Delta) = \Delta + \frac{4}{3}\sum_{p=0}^{m-1} a_p \varepsilon^{2(p+1)}\Delta^{p+1} , \quad D_m(\Delta) = \Delta + \frac{4}{3}\sum_{p=0}^{m-1} b_p \varepsilon^{2(p+1)}\Delta^{p+1} .$$

$$\sum_{p=0}^{m} \varepsilon^{2p+1} c_p \Delta^{p+2} u_k = 0 , \; k=1,2,3 . \qquad (2.10)$$

For $n=2m+1$ stationary equations are obtained from equations (2.10) by replacing $m-1$ with m in the operators $C_m(\Delta)$ and $D_m(\Delta)$.

For $n=0$ (the Navier-Stokes approximation) the stationary equations are:

$$\Delta T = 0 , \quad \Delta\rho = 0 , \quad \partial_s u_s = 0 , \quad \Delta^2 u_k = 0 . \qquad (2.11)$$

These equations result in degeneration of the time-independent Chapman-Enskog expansion (2.7): the only non-zero terms are $q_k^{(0)}$, $q_k^{(1)}$, $\sigma_{ik}^{(0)}$, $\sigma_{ik}^{(1)}$ and $\sigma_{ik}^{(2)}$. Due to the equality

$$\partial_i\sigma_{ik}^{(1)}=\partial_i\sigma_{ik}^{(2)}=\partial_k q_k^{(1)}=0,$$

the account of the terms $\sigma_{ik}^{(1)}$, $\sigma_{ik}^{(2)}$ and $q_k^{(1)}$ does not change equations (2.11). This degeneration of the Chapman-Enskog series was obtained recently by V.S. Galkin [7] for the linearized Boltzmann equation.

In a non-stationary case, the break of the series (2.3) for arbitrary finite order $n>0$ can result in short-wave instability of the equilibrium point, as it was shown in [2] for $n=1$ and $n=2$. As it was mentioned in the Introduction, an account of all the orders in the Chapman-Enskog expansion is necessary. Expression (2.9) shows that the recurrent procedure (2.4) for obtaining numerical coefficients (2.7) is rather complicated. In the next section we introduce a method of consecutive approximations of equations (2.4). This method approximates the series (2.3) in whole.

3. APPROXIMATIONS OF THE CHAPMAN-ENSKOG EXPANSION

Our approximation of the Chapman-Enskog recurrent system (2.4) is as follows.

Fix an arbitrary integer k_0, where $k_0\geqslant 1$. Equations (2.4) are replaced by:

$$\sigma_{ik}^{(n)}=-\{\sum_{m=0}^{n-1}\partial_t^{(m)}\sigma_i^{(n-(m+1))}+\frac{2}{5}(\partial_i q_k^{(n-1)}+\partial_k q_i^{(n-1)})-$$

$$-\frac{2}{3}\delta_{ik}\partial_s q_s^{(n-1)})\},\qquad \text{where } n=0,\ldots,k;$$

$$\sigma_{ik}^{(k_0+m)} \approx -\Big(\sum_{s=0}^{k_0-1} \partial_t^{(s)} \sigma_{ik}^{(k_0+m-(s+1))} + \tfrac{2}{5}\big(\partial_i q_k^{(k_0+m-1)} + \partial_k q_i^{(k_0+m-1)} - \tfrac{2}{3}\delta_{ik}\partial_l q_l^{(k_0+m-1)}\big)\Big), \quad \text{where } m \geqslant 1; \qquad (3.1)$$

and, by analogy, for $q_k^{(n)}$. The operators $\partial_t^{(s)}$, where $s=0,\dots,k_0-1$, are defined according to (2.5). Thus, we restrict ourselves to a finite set of operators $\partial_t^{(s)}$ according to the choice of k_0. Therefore, all terms up to the k_0-th order in expansion (2.3) are taken into account without any "cut-offs", and all of the other terms with $k>k_0$ are approximated. For $k_0\to\infty$ the system (3.1) coincides with the system (2.4). One can prove that the system (3.1) preserves the structure of (2.7). The approximation introduced results only in a changing of the values of numerical coefficients a_n, b_n, c_n, d_n, α_n, β_n, φ_n and ψ_n

Generally speaking, the equations (3.1) are simpler, than the system (2.4). In some cases equations (3.1) can be solved exactly. The last step of the algorithm consists of summing the series (2.3) with approximate coefficients (3.1).

The procedure introduced here is of a recipe character. We call this method "regularization" for short. Now we will consider some examples of its application.

We start with the simplest case of linearized one-dimensional 10-moment Grad equations:

$$\partial_t\rho = -\partial_x u, \qquad \partial_t T = -\tfrac{2}{3}\partial_x u,$$

$$\partial_t u = -\partial_x(T+\rho+\sigma), \qquad \partial_t\sigma = -\tfrac{4}{3}\partial_x u - \varepsilon^{-1}\sigma. \qquad (3.2)$$

Here σ represents the xx-component of the tensor σ_{ik}, and x represents the one-dimensional coordinate.

Fix $k_0=1$, and introduce variables u and $\theta=T+\rho$. In this case the approximate recurrent procedure (3.1) is:

$$\sigma^{(0)}=-\tfrac{4}{3}\partial_x u\,,\quad \sigma^{(1)}=-\partial_t^{(0)}\sigma^{(0)}\,,\quad \sigma^{(n)}\approx-\partial_t^{(0)}\sigma^{(n-1)}\,,\ n\geqslant 2$$

$$\partial_t^{(0)}\theta=-\tfrac{5}{3}\partial_x u\,,\qquad \partial_t^{(0)}u=-\partial_x\theta\,. \tag{3.3}$$

The structure of coefficients $\sigma^{(n)}$ (3.3) is as follows:

$$\sigma^{(2n)}=a_n\partial_x^{2n+1}u\,,\qquad \sigma^{(2n+1)}=b_n\partial_x^{2n+2}\theta\,. \tag{3.4}$$

Coefficients a_n and b_n are determined by the recurrent rule:

$$a_n=b_n\,,\ a_{n+1}=\tfrac{5}{3}a_n\,,\ a_0=-\tfrac{4}{3}\,. \tag{3.5}$$

From (3.4) and (3.5) we obtain:

$$\sigma^{(2n)}=(\tfrac{5}{3}\partial_x^2)^n(-\tfrac{4}{3}\partial_x u)\,,\qquad \sigma^{(2n+1)}=(\tfrac{5}{3}\partial_x^2)^n(-\tfrac{4}{3}\partial_x^2\theta)\,. \tag{3.6}$$

Summing the series $\sum_{n=0}^{\infty}\varepsilon^{n+1}\sigma^{(n)}$ with coefficients (3.6) yields:

$$\sigma_{1R}=R_1\sigma_1\,,\quad R_1=(1-\tfrac{5}{3}\varepsilon^2\partial_x^2)^{-1}\,,\quad \sigma_1=-\tfrac{4}{3}(\varepsilon\partial_x u+\varepsilon^2\partial_x^2\theta)\,. \tag{3.7}$$

The expression σ_1 represents the Burnett approximation Oof the nonequilibrium tension tensor σ.

Fix $k_0=2$. In this case equations (3.1) are:

$$\sigma^{(0)}=-\tfrac{4}{3}\partial_x u\,,\quad \sigma^{(1)}=-\partial_t^{(0)}\sigma^{(0)}\,,\quad \sigma^{(2)}=-\partial_t^{(0)}\sigma^{(1)}-\partial_t^{(1)}\sigma^{(0)}\,,$$

$$\sigma^{(n)}\approx-\partial_t^{(0)}\sigma^{(n-1)}-\partial_t^{(1)}\sigma^{(n-2)}\,,\quad \text{for } n\geqslant 3\,,$$

$$\partial_t^{(1)}\theta=0\,,\qquad \partial_t^{(1)}u=\tfrac{4}{3}\partial_x^2 u\,. \tag{3.8}$$

Coefficients $\sigma^{(n)}$ in (3.8) have the form (3.4). Coefficients a_n and b_n are defined by a recurrent procedure which is obtained from (3.5) by substituting 1/3 instead of 5/3. This results in:

$$\sigma_{2R}=R_2\sigma_1\,,\qquad R_2=(1-\tfrac{1}{3}\varepsilon^2\partial x^2)^{-1}\,. \tag{3.9}$$

The super-Burnett approximation σ_2 has the form:

$$\sigma_2=\sigma_1-\frac{4}{9}\varepsilon^3\partial_x^3 u\,. \qquad (3.10)$$

Note the appearance of the non-local operators R_1 and R_2 in hydrodynamics equations.

Substituting $\sigma^{(0)}$, σ_1, σ_{1R}, σ_2 and σ_{2R} instead of σ into the equations

$$\partial_t\theta=-\frac{5}{3}\partial_x u\,, \qquad \partial_t u=-\partial_x(\theta+\sigma)\,, \qquad (3.11)$$

then using the variables $t''=t/\varepsilon$ and $x''=x/\varepsilon$, and also using the representation

$$u=u_1\varphi\,, \qquad \theta=\theta_1\varphi\,, \qquad \varphi=exp(\omega t''+ikx'')\,, \qquad (3.12)$$

we obtain dispersion relationships $\omega(k)$ for sonic waves. These relationships are:

$$\omega_{1,2}=-\frac{2}{3}k^2\pm k\left(\frac{4}{9}k^2-\frac{5}{3}\right)^{1/2} \qquad (3.13)$$

for the Navier-Stokes approximation $\sigma^{(0)}$;

$$\omega_{1,2}=-\frac{2}{3}k^2\pm k\left(\frac{4}{9}k^2-\frac{5}{3}\left(1+\frac{4}{9}k^2\right)\right)^{1/2} \qquad (3.14)$$

for the Burnett approximation σ_1;

$$\omega_{1,2}=-\frac{2k^2}{3+5k^2}\pm i\frac{k}{2}\left[\frac{75k^4+41k^2+15}{25k^4+30k^2+9}\right]^{1/2} \qquad (3.15)$$

for the regularized Burnett approximation σ_{1R};

$$\omega_{1,2}=-\frac{2}{9}k^2(k^2-3)\pm\left\{\frac{1}{91}\left(4k^4(3-k^2)^2-45k^2(4k^2+3)\right)\right\}^{1/2} \qquad (3.16)$$

for the super-Burnett approximation σ_2;

$$\omega_{1,2}=-\frac{2k^2}{3+k^2}\pm i\frac{k}{2}\left[\frac{100k^4+312k^2+180}{3k^4+18k^2+21}\right]^{1/2} \qquad (3.17)$$

for the regularized super-Burnett approximation σ_{2R}.

Dispersion curves for the Burnett approximation (3.14) (dashed line) and for the regularized Burnett approximation (3.15) (smooth line) are shown in Fig.1. The directions of the arrows indicate increases of the

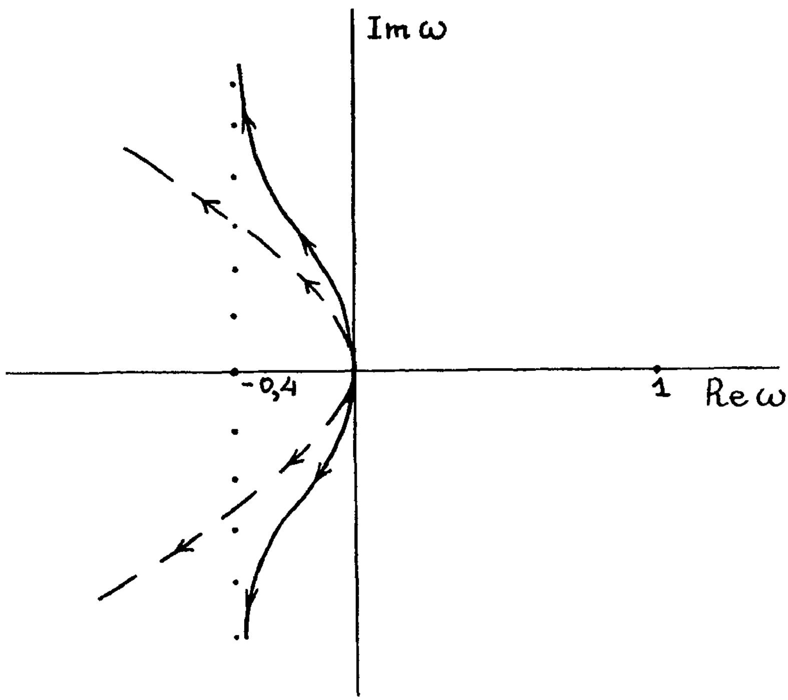

Fig. 1

wave vector square k^2. Dispersion curves for the super-Burnett approximation (3.16) (dashed line) and for the regularized super-Burnett approximation (3.17) (smooth line) are depicted in Fig.2. Regularization removes short-wave instability of the super-Burnett approximation (3.16).

The next example is the one-dimensional variant of the 13-moment system (2.2). The Burnett approximation is as follows:

$$\sigma_1=-\frac{4}{3}\varepsilon\partial_x u-\frac{4}{3}\varepsilon^2\partial_x^2\rho+\frac{2}{3}\varepsilon^2\partial_x^2 T, \quad q_1=-\frac{15}{4}\varepsilon\partial_x T-\frac{7}{4}\varepsilon^2\partial_x^2 u. \tag{3.18}$$

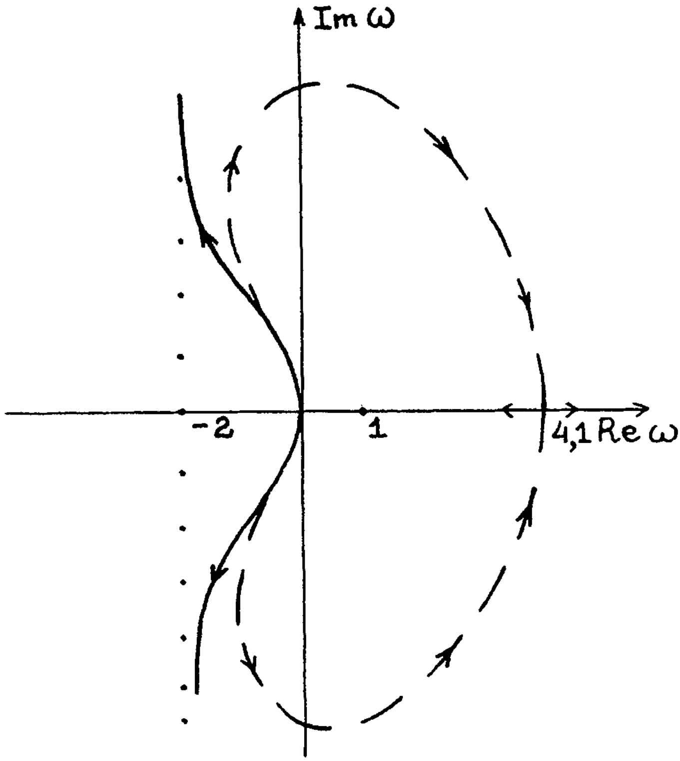

Fig. 2

Substituting the expression (3.18) into the one-dimensional equations (2.2), we obtain the dispersion equation:

$$18\omega^3+69\omega^2k^2+\omega k^2(30+97k^2-14k^4)+15k^4(3+k^2)=0. \quad (3.19)$$

Exactly in this case, short-wave instability was detected in [2].

Fix $k_0=1$ in the one-dimensional variant of equations (3.1). Then the approximate recurrent system

(3.1) takes the form:

$$\sigma^{(0)}=-\tfrac{4}{3}\partial_x u\,,\qquad \sigma^{(n)}\approx-\partial_t^{(0)}\sigma^{(n-1)}-\tfrac{8}{15}\partial_x q^{(n-1)}\,,\ n\geqslant 1\,,$$

$$q^{(0)}=-\tfrac{15}{4}\partial_x T\,,\qquad q^{(n)}\approx-\tfrac{3}{2}\partial_t^{(0)}q^{(n-1)}-\tfrac{3}{2}\partial_x\sigma^{(n-1)}\,,\ n\geqslant 1\,,$$

$$\partial_t^{(0)}\rho=-\partial_x u\,,\qquad \partial_t^{(0)}T=-\tfrac{2}{3}\partial_x u\,,\qquad \partial_t^{(0)}u=-\partial_x(T+\rho)\,,\qquad (3.20)$$

and for $k_0=1$ the approximate equalities are exact. In the one-dimensional case the structure (2.7) is as follows:

$$\sigma^{(2n)}=c_n\partial_x^{2n+1}u\,,\qquad \sigma^{(2n+1)}=a_n\partial_x^{2n+2}T+b_n\partial_x^{2n+2}\rho\,,$$

$$q^{(2n)}=\alpha_n\partial_x^{2n+1}T+\beta_n\partial_x^{2n+1}\rho\,,\qquad q^{(2n+1)}=\varphi_n\partial_x^{2n+2}u\,.\qquad (3.21)$$

It is useful to introduce the objects: the space $X=R^3$ including the vectors x_n with components (a_n,b_n,φ_n); the space $Y=R^3$ including the vectors y_n with components (c_n,α_n,β_n). From equations (3.20) we obtain a vector analog of scalar equations (3.5):

$$x_n=Sy_n\,,\qquad y_{n+1}=Lx_n\,,\qquad y_0=(-\tfrac{4}{3},\ -\tfrac{15}{4},\ 0)\,,\qquad (3.22)$$

Here matrices S and L are:

$$S=\begin{vmatrix}1 & -8/15 & 0\\ 1 & 0 & -8/15\\ -3/2 & 1 & 3/2\end{vmatrix}\,,\qquad L=\begin{vmatrix}2/3 & 1 & -8/15\\ -3/2 & 0 & 3/2\\ 0 & -3/2 & 3/2\end{vmatrix}\,.\qquad (3.23)$$

The solution of the equations (3.22) is as follows:

$$y_n=K^n y_0\,,\qquad x_n=SK^n y_0\,,\qquad K=LS\,.\qquad (3.24)$$

The next steps of regularization involve nothing new in comparison with the examples suggested above. Finally we obtain:

$$\sigma_{1R}=P_c R_1 y_0\varepsilon\partial_x u+P_a SR_1 y_0\varepsilon^2\partial_x^2 T+P_b SR_1 y_0\varepsilon^2\partial_x^2\rho\,,$$

$$q_{1R}=P_\alpha R_1 y_0\varepsilon\partial_x T+P_\beta R_1 y_0\varepsilon\partial_x\rho+P_\varphi SR_1 y_0\varepsilon^2\partial_x^2 u\,,$$

$$R_1=(1-\varepsilon^2 K\partial_x^2)^{-1}\qquad (3.25)$$

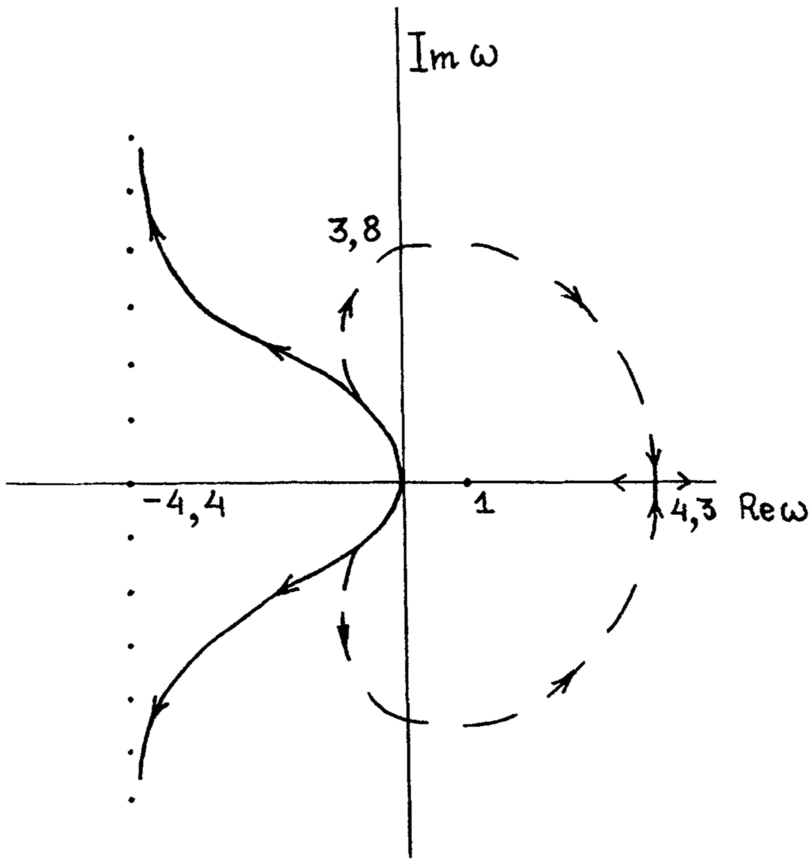

Fig. 3

Here P_μ represents the projector on the appropriate axis in the space X or in the space Y (e.g. $P_c y_n = c_n$). The dispersion equation for the approximation (3.25) is rather complicated, and therefore it is not represented here. Dispersion curves for the Burnett approximation (3.18) (dashed line) and for the regularized Burnett approximation (3.25) (smooth line) are depicted in

Fig.3. Only acoustic branches are shown, diffusive branches for both of the approximations (3.18) and (3.25) are practically the same.

Finally, we introduce the result of regularization of the Burnett approximation ($k_0=1$) for equations (2.2):

$$\sigma_{1Rik}=P_c R_1 y_0 \varepsilon \langle \partial_i u_k \rangle + P_d R_1 y_0 \Delta^{-1}\Gamma_{ik}\varepsilon\partial_s u_s + P_a S R_1 y_0 \varepsilon^2 \Gamma_{ik}\rho +$$
$$+P_b S R_1 y_0 \varepsilon^2 \Gamma_{ik} T,$$

$$q_{1Rk}=P_\alpha R_1 y_0 \varepsilon \partial_k \rho + P_\beta R_1 y_0 \varepsilon \partial_k T + P_\varphi S R_1 y_0 \varepsilon^2 \partial_k \partial_s u_s + P_\psi S R_1 y_0 \varepsilon^2 \Delta u_k,$$

$$R_1=(1-\varepsilon^2 SL\Delta)^{-1}, \quad y_0=(0, -1, 0, -\tfrac{15}{4}) \qquad (3.26)$$

When obtaining expression (3.26) we used four-dimensional vectors $x_n=(a_n, b_n, \varphi_n, \psi_n)$ and $y_n=(d_n, c_n, \alpha_n, \beta_n)$. The analogous representation was used in the previous example. Notations S and L represent the matrices:

$$S=\begin{vmatrix} 1 & 1 & -2/5 & 0 \\ 1 & 1 & 0 & -2/5 \\ -2 & -1/2 & 3/2 & 1 \\ 0 & -3/2 & 0 & 0 \end{vmatrix}, \quad L=\begin{vmatrix} 1 & 3/2 & 2/5 & 0 \\ 0 & 0 & 0 & 2/5 \\ -2 & 0 & 3/2 & 3/2 \\ 0 & -2 & 3/2 & 3/2 \end{vmatrix}. \qquad (3.27)$$

4. CONCLUSIONS

The method of regularization introduced in section 3, is based on ideas of the Pade approximations [8] and on partial summing of series. One cannot predict whether or not this method indeed results in stability of the wave spectrum. This situation is typical for the methods which use the Pade approximations. A number of curious examples are collected in the monograph [9].

The examples suggested in section 3 predict limit of the decrement in short-wave asymptotic.

For the linearized Grad equations one can approach the problem of the wave spectrum stability without any break of the sequence of the operators $\partial_t^{(s)}$. For example, in the case of the 10-moment equations the regularized stress tensor σ_{nR} can be supposed to be as follows:

$$\sigma_{nR}=F_{1n}(\partial_x^2)\partial_x u+F_{2n}(\partial_x^2)\partial_x^2\theta .$$

The symbols of the operators $F_{1n}(-k^2)$ and $F_{2n}(-k^2)$ are defined as the series in powers of $-k^2$. The first n coefficients of these series are given. The rest of the coefficients are defined according to a more or less complicated recurrent rule. The functions $F_{1n}(-k^2)$ and $F_{2n}(-k^2)$ are inserted into the dispersion equation. We have to construct an approximation of the functions $F_{1n}(-k^2)$ and $F_{2n}(-k^2)$ so that this approximation would preserve the given segment of the Taylor series, and so that the H-theorem will be valid.

The dependence of the functions $F_{1n}(\partial_x^2)$ and $F_{2n}(\partial_x^2)$ only on ∂_x^2 is the result of the simple structure (2.7) and (3.4). The formal transfer of the procedure (3.1) to the nonlinear Grad equations is not difficult. However, in this case one cannot obtain any simple structure similar to (2.7) and (3.4). Nevertheless, one can select some of the similar summands in every term of expansion (e.g., the maximal nonlinear terms $(\partial_x u)^{n+1}$). The regularization in this case can eliminate the nonphysical "negative viscosity" of the Burnett approximations. Another paper will be devoted to these problems.

After this paper was prepared, we learned that a similar approach was developed recently by a group of mathematicians for the regularization of semi-classical expansion for the Schroedinger equation [9-12].

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