Non-uniform Small Gain Theorems and Their Applications

Tyukin Ivan

Department of Mathematics

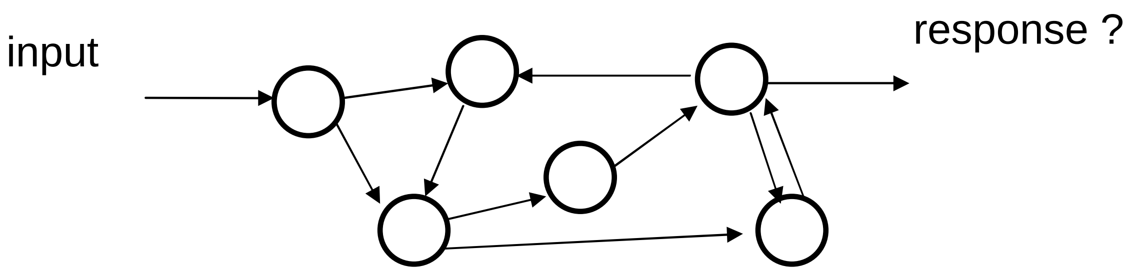
University of Leicester

Mathematics of Model Reduction, 2007

Outline

1. Introduction and brief review of the Small-Gain concepts
2. Motivation and statement of the problem
3. Main results (Non-uniform Small-Gain)
   3.1. Boundedness
   3.2. Convergence
4. Discussion (Pomet & Ilchman)
5. Some immediate applications
   5.1. Models with nonlinear parameterization in adaptive control (simple example)
   5.2. System identification and new observer canonic forms
6. Concluding remarks

1.1. Introduction. System

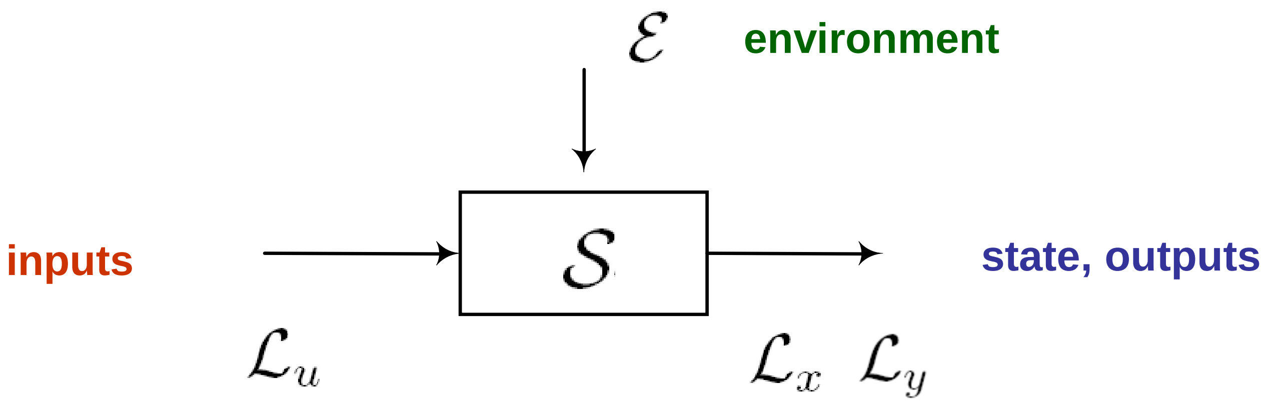


From Principles to Features

1	Input-output, input-state maps, their estimates, macro-variables	**Level 1. Basic Principles**
2	Differential equations	**Level 2. Laws and mechanisms**
3	Specific nonlinearities, specific models of processes	**Level 3. Specific Models**

reduction from the view point of prior information about the “building blocks” of the system

1. Introduction. Models



Models are (nonlinear) **mappings** of linear **normed** spaces

space of inputs

state space

$$\mathcal{S}_\mathcal{T}(\mathbf{u},\mathbf{e}):\ \mathcal{L}_u[t_0,T]\times\mathcal{E}\mapsto\mathcal{L}_x[t_0,T]$$

$$\mathcal{H}_\mathcal{T}(\mathbf{u},\mathbf{e}):\ \mathcal{L}_u[t_0,T]\times\mathcal{E}\mapsto\mathcal{L}_y[t_0,T]$$

space of environmental actions

space of outputs

$$\|\mathbf{f}\|_{p,[t_0,T]}=\left(\int_0^T\|\mathbf{f}(\tau)\|^p d\tau\right)^{1/p}<\infty$$

$\mathcal{T}=[t_0,T]\subseteq\mathcal{T}^*$

- is the set of all positive real numbers $\mathbb{R}_+$ - completeness
- non-empty interval - realizability

1. Introduction. Models

Available information: estimates of the mappings

input - state

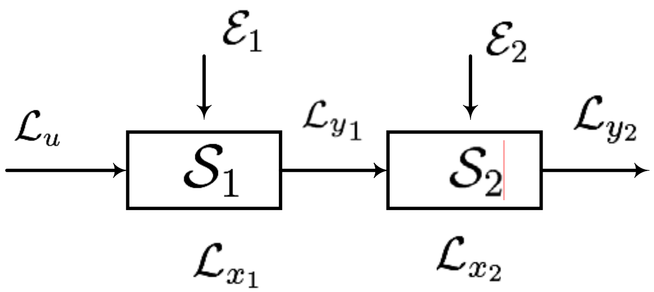
$$\|\mathbf{x}(t)\|_{\mathcal{L}_x,[t_0,T]} \le \gamma_{S,\mathcal{L}_x}(\mathbf{e}, \|\mathbf{u}(t)\|_{\mathcal{L}_u,[t_0,T]}, T)$$

input - output

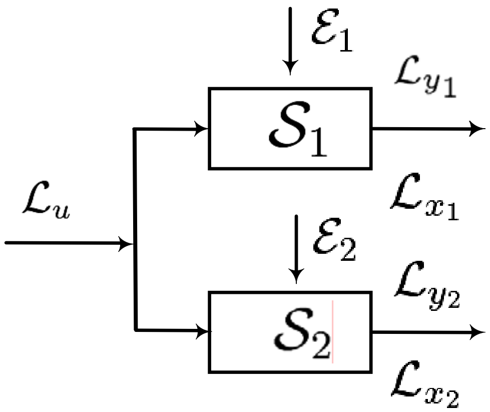
$$\|\mathbf{y}(t)\|_{\mathcal{L}_y,[t_0,T]} \le \gamma_{H,\mathcal{L}_y}(\mathbf{e}, \|\mathbf{u}(t)\|_{\mathcal{L}_u,[t_0,T]}, T)$$

Questions:

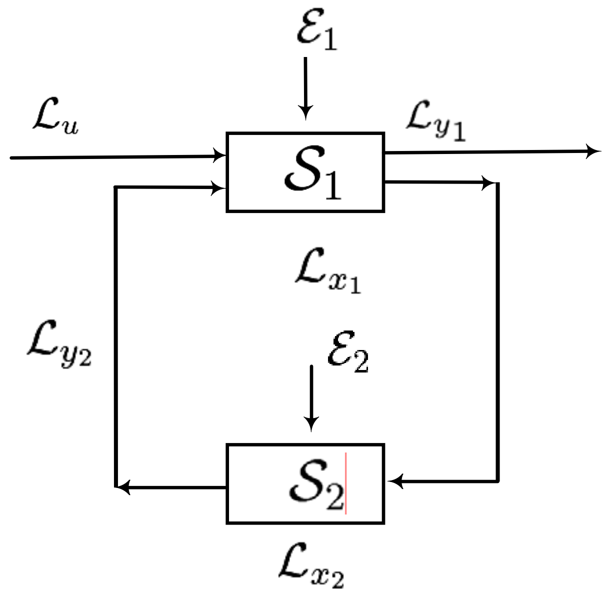
Serial interconnection



Parallel interconnection



Feedback interconnection



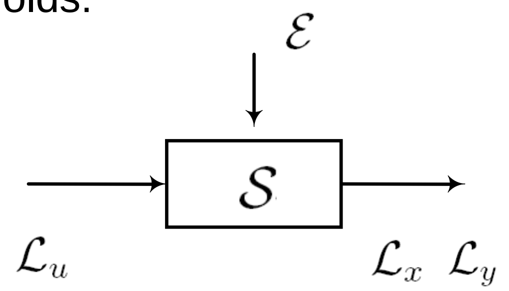
Realizability (completeness) of interconnections with feedbacks is not trivial and depends on properties of each systems

1. Introduction. Tools. Small-gain theorems

Definition 1. (*Zames 1966, input-to-output stability*) System S is **input-output stable** iff there is a positive constant β such that the following property holds:

$$\boxed{\|y\|_{\infty,[0,T]} \leq \beta + \underline{\gamma}\, \|u\|_{\infty,[0,T]}}$$

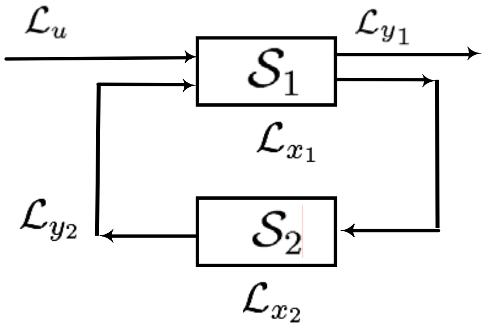
"gain" of the system



Theorem 1. (*Small-gain*) Interconnection of two input-output stable systems S_1 and S_2 is input-output stable (e.g. bounded outputs for bounded inputs) if

$$\boxed{\gamma_1 \gamma_2 < 1}$$

overall "gain" of the loop



$$\|y_1\|_{\infty,[0,T]} \leq \beta_1 + \gamma_1 \|y_2\|_{\infty,[0,T]} + \gamma_1 \|u_1\|_{\infty,[0,T]} \leq \beta_1 + \gamma_1\beta_2 + \gamma_1\gamma_2 \|y_1\|_{\infty,[0,T]} + \gamma_1 \|u_1\|_{\infty,[0,T]}$$

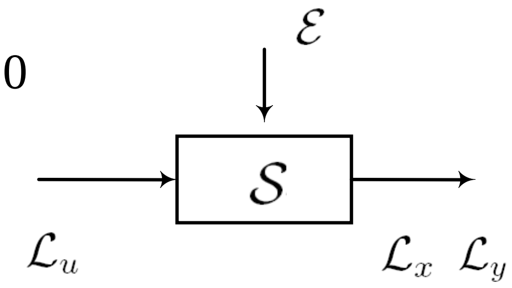
$$\|y_1\|_{\infty,[0,T]} \leq \boxed{(1-\gamma_1\gamma_2)^{-1}(\beta_1 + \gamma_1\beta_2)} + \boxed{(1-\gamma_1\gamma_2)^{-1}\gamma_1} \|u_1\|_{\infty,[0,T]}$$

1. Introduction. Tools. Small-gain theorems

Definition 2. (*Sontag 1989, input-to-state stability*) System S is **input-to-state stable (ISS)** iff there is a KL-function β and function γ of class K such that:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t) + \gamma(\|u\|_p), \quad \gamma \in C^0, \gamma(0) = 0$$

"gain" of the system



Results are available:

Jiang Z.-P., Teel A. R., Praly L. Small-gain theorems for ISS systems and applications. *Math. Control Signals Systems*.— 1994.— no. 7.— Pp. 95–120

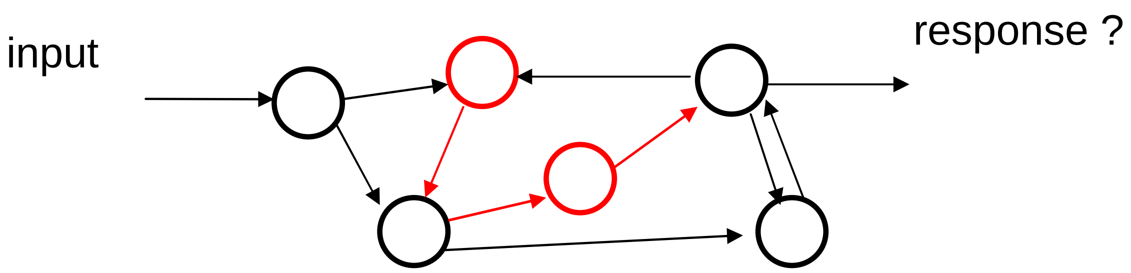
These are important because:

1. Small-gain conditions allow to state boundedness/completeness of the interconnection
2. Both positive and negative feedbacks are allowed
3. Level of initial uncertainty about the system's component is high (only estimates of the gains are required)
4. Numerous equivalences between the ISS and other properties such as Lyapunov stability, asymptotic gain etc. are available (Sontag, 1996)

Yet … global asymptotic Lyapunov stability of the origin of unperturbed system is required for Input-to-State Stability

1. Introduction. Goals

What if not all of the "building blocks" are globally asymptotically stable ?



We wish

1. To have tools for analysis of interconnection of not necessarily globally asymptotically stable systems
2. The tools should allow sufficiently high degree of uncertainty about the models
3. Should preserve the small-gain spirit (**criteria are to be simple!**)
4. We shall be able to understand asymptotic behavior of the system as well

2. Motivation and Statement of the Problem

What kind of unstable interconnections to consider ? (contracting + exploring)

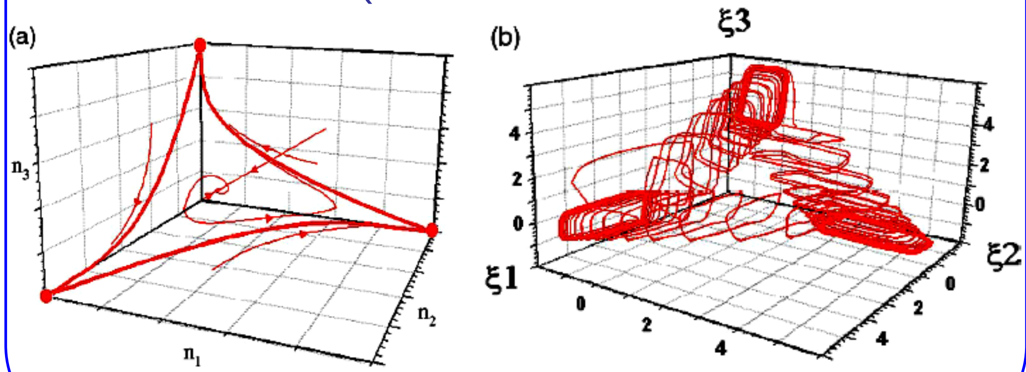
Systems at the point of bifurcation

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = \varepsilon + \gamma x_1^2, \gamma > 0$$

1) when calculating the central and stable manifolds is difficult

2) when global results are needed

Winner-less competition in neuroscience (Rabinovich, 2006)

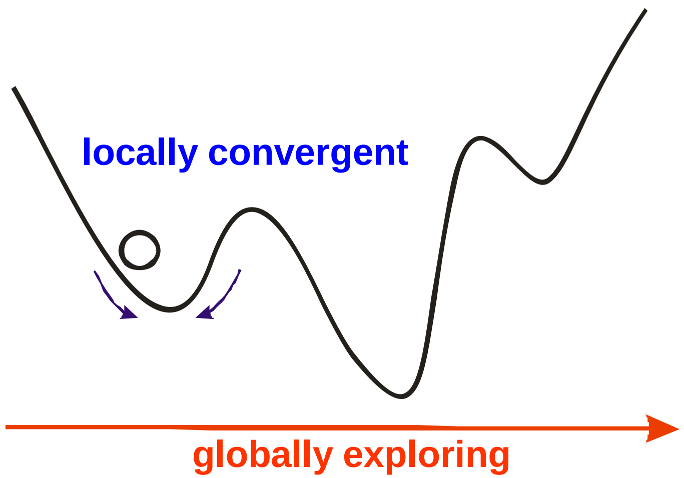


heteroclinic connections

Global optimization

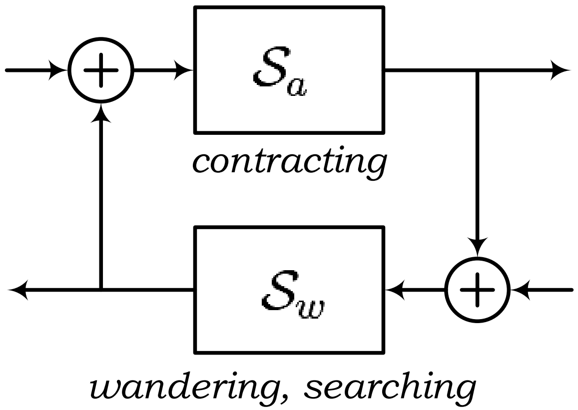
$$\dot{\mathbf{x}} = -\mu_x \frac{\partial Q(\mathbf{x})}{\partial \mathbf{x}} + \mu_t T(t), \;\; \mu_x, \mu_t \in \mathbb{R}_+$$
$$T(t) = h\{t, \mathbf{x}(t)\}, \;\; h : \mathbb{R}_+ \times L_\infty^n[t_0, t] \to L_\infty^n[t_0, t]$$

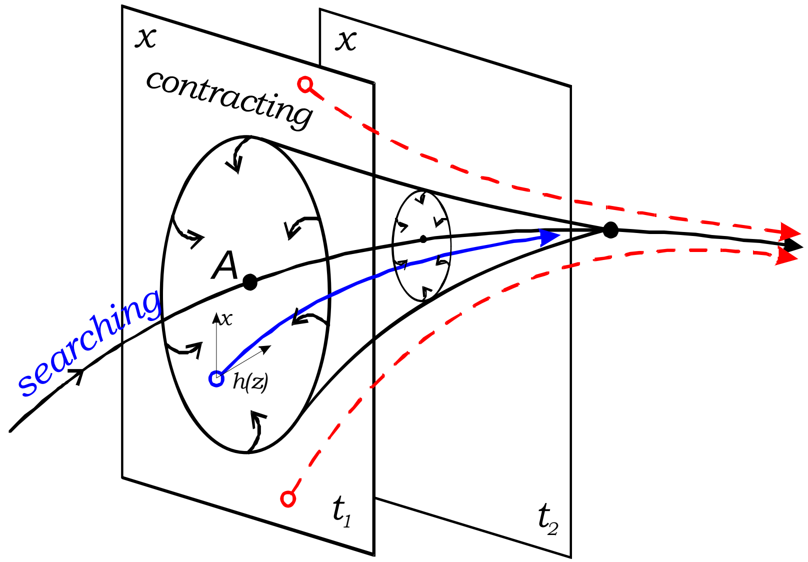
Interplay between locally convergent minimizers and exploring motions



2. Motivation and Statement of the Problem

Interconnections of systems with unbounded input-output maps





Contracting : $\mathcal{S}_a : \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|u_a(t)\|_{\infty,[t_0,t]}$

"Searching" : $\mathcal{S}_w : \int_{t_0}^{t} \gamma_1(u_w(\tau))d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^{t} \gamma_0(u_w(\tau))d\tau$

$$\gamma_0(a \cdot b) \leq \gamma_{0,1}(a) \cdot \gamma_{0,2}(b)$$

Interconnected as : $\int_{t_0}^{t} \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^{t} \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau$

2. Motivation and Statement of the Problem

In case both subsystems are described by ODEs

contracting

$$\dot{\mathbf{x}} = \mathbf{f}_x(\mathbf{x}, u_a), \ \ \mathbf{f}_x(\cdot,\cdot) \in \mathcal{C}^1,$$

its solutions satisfy the estimate

$$\mathcal{S}_a : \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|u_a(t)\|_{\infty,[t_0,t]}$$

1) where the set $\mathcal{A}$ is Lyapunov-stable and globally attracting at $u_a(t) \equiv 0$ and

2) there exists a non-decreasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+ : \kappa(0) = 0$

$$\inf_{t \in [0,\infty)} \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \kappa(\|u_a(t)\|_{\infty,[t_0,\infty)})$$

Searching

$$\dot{\mathbf{z}} = \mathbf{f}_z(\mathbf{z}, u_w), \ \ \mathbf{f}_z(\cdot,\cdot) \in \mathcal{C}^1,$$

$$\mathcal{S}_w : \int_{t_0}^{t} \gamma_1(u_w(\tau))d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^{t} \gamma_0(u_w(\tau))d\tau, \ \forall\, t \geq t_0, \ t_0 \in \mathbb{R}_+$$

Interconnection

Interconnected systems

$$\mathcal{S}_a : \ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, h(\mathbf{z})),$$
$$\mathcal{S}_w : \ \dot{\mathbf{z}} = \mathbf{q}(\mathbf{z}, \mathbf{x})$$

output of the second subsystem evolves according to

2. Motivation and Statement of the Problem

The questions are:

1. Boundedness of the interconnection

2. Asymptotic behavior (allocation of the invariant sets)

Because the systems are not stable

we study (weak) trapping regions and (weak) attracting sets…

Definition 3. (*Milnor 1985, weakly attracting set*) A set $\mathcal{A}$ is weakly attracting (or Milnor attracting set) iff

1) It is closed, invariant and

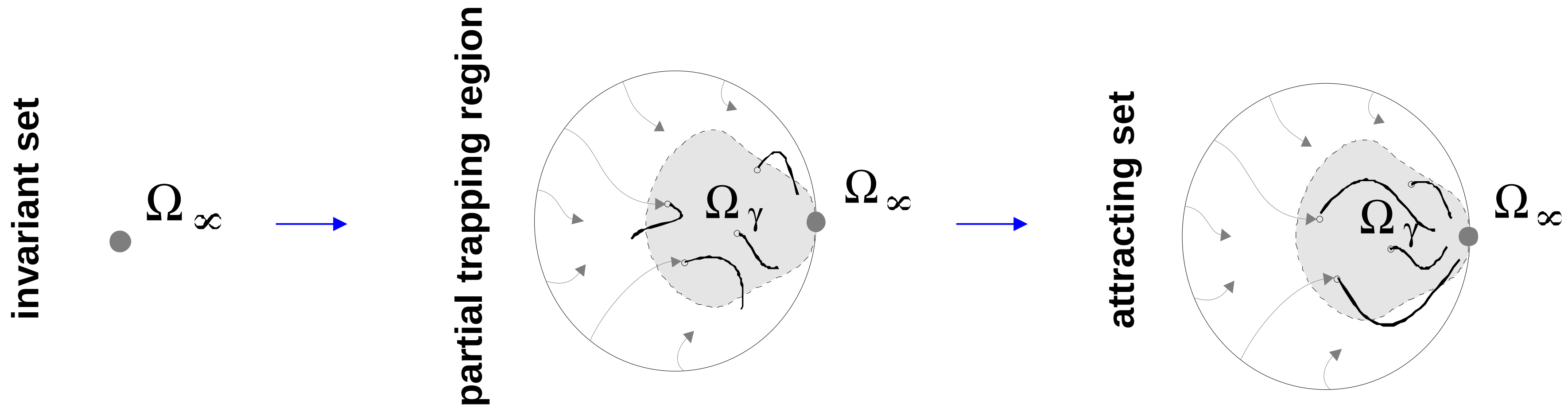
2) for some set $\mathcal{V}$ (not necessarily a neighborhood of $\mathcal{A}$) with strictly positive measure and for all $\mathbf{x}_0 \in \mathcal{V}$ the following limiting relation holds:

$$\lim_{t\rightarrow\infty} \|\mathbf{x}(t,\mathbf{x}_0)\|_{\mathcal{A}} = 0$$

Related concepts: Relaxation Times (A. Gorban, 1979, 2004)

3. Main Results

The program



Step 1

Step 2:
Trajectories starting in Ω_γ remain in a given region

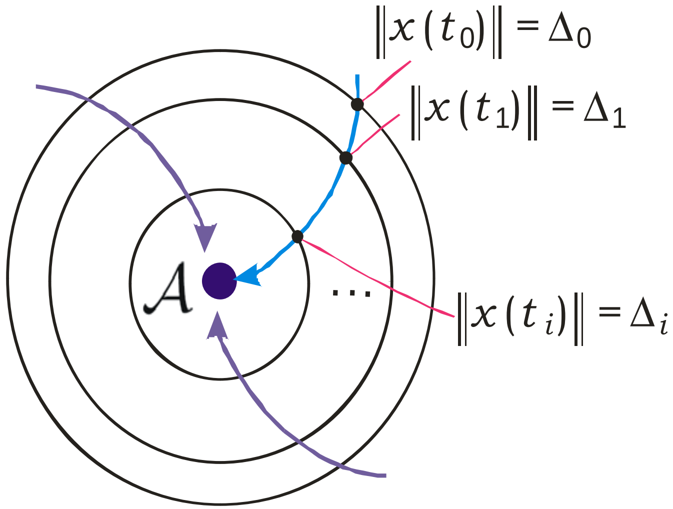
Step 3:
and eventually converge to the invariant set

We need a method for showing these for unstable systems

Standard approaches

Conventional notion of attracting set

1) domains of attraction – neighborhoods
2) for autonomous systems implies stability



Given: sequence of time instances t_i

Prove: sequence of distances Δ_i does not increase (i.e. converges)

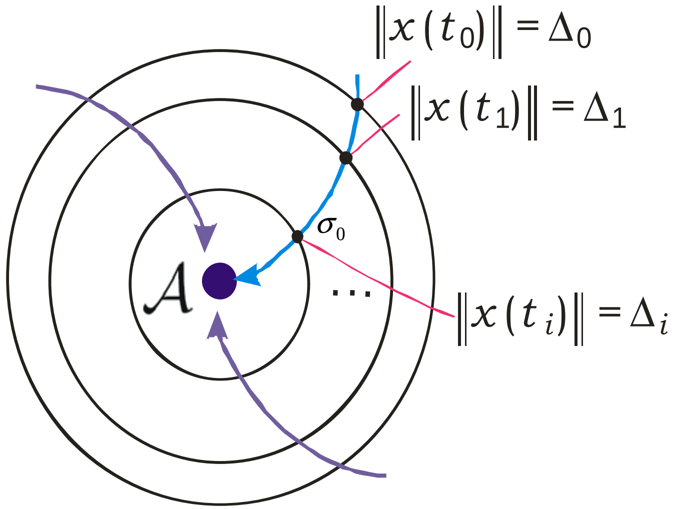
Mathematical framework

- contraction mapping theorems
- method of Lyapunov functions
- small-gain theorems

Standard approaches

Conventional notion of attracting set

1) domains of attraction – neighborhoods
2) for autonomous systems implies stability



Given: sequence of time instances t_i

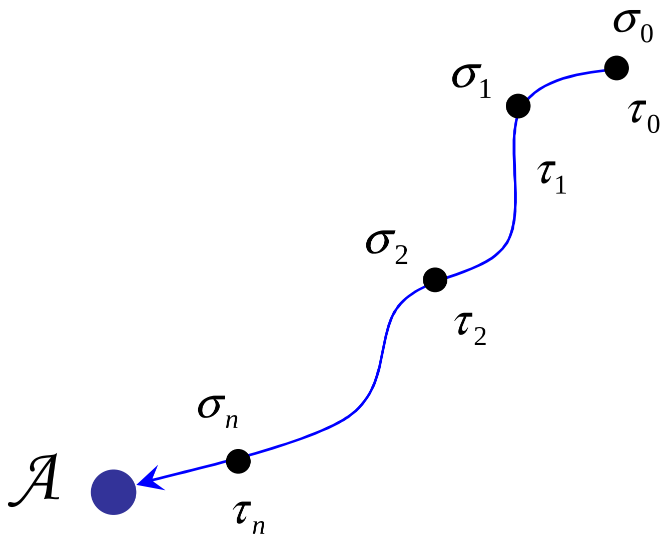
Prove: sequence of distances Δ_i does not increase (i.e. converges)

Mathematical framework

- contraction mapping theorems
- method of Lyapunov functions
- small-gain theorems

Possible unstable convergence

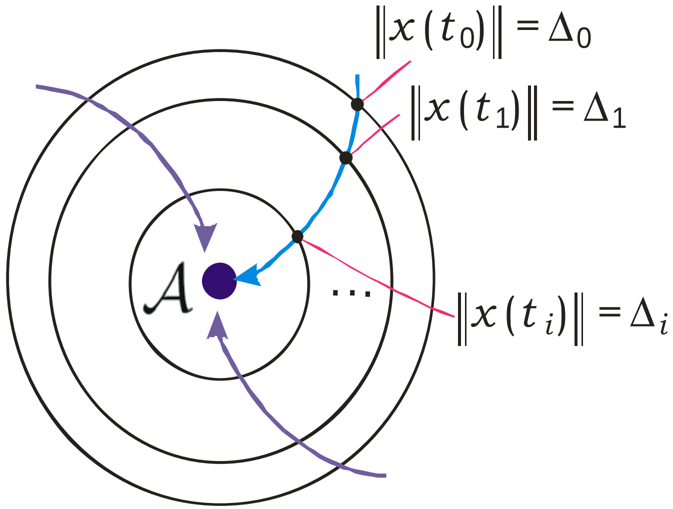
An asymptotically convergent trajectory that does not reach the target set in finite time …

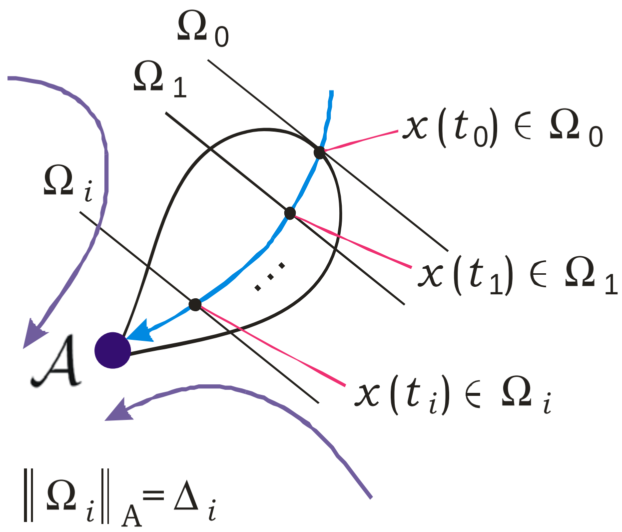


1) $\lim_{n\to\infty} \sigma_n = 0$

2) $\sum_n \tau_n = \infty$

Standard approaches	Proposed
Conventional notion of attracting set	**Weak attracting sets, concept of Milnor attracting sets**
1) domains of attraction – neighborhoods 2) for autonomous systems implies stability	1) domains of attraction – sets of positive measure 2) possible to analyze unstable systems





Given: sequence of time instances t_i	**Prove:** sequence of partial sums $\sum t_i$ diverges
Prove: sequence of distances Δ_i does not increase (i.e. converges)	**Given:** sequence of distances Δ_i

Mathematical framework (Standard approaches)

- contraction mapping theorems
- method of Lyapunov functions
- small-gain theorems

Mathematical framework (Proposed)

- to be seen in on the next slides …

3. Main Results. Boundedness

We introduce the following three sequences:

$$\mathcal{S} = \{\sigma_i\}_{i=0}^{\infty}, \quad \Xi = \{\xi_i\}_{i=0}^{\infty}, \quad \mathcal{T} = \{\tau_i\}_{i=0}^{\infty}$$

Property 1. The sequence $\mathcal{S}$ is strictly monotone and converging to the origin

$$\boxed{\lim_{n\to\infty} \sigma_n = 0, \ \ \sigma_0 = 1}$$

Property 2. For the given Ξ, $\mathcal{T}$ the following property holds (rate of contraction)

$$\boxed{\beta(\cdot, T_i) \le \xi_i \beta(\cdot, 0), \ \forall \ T_i \ge \tau_i}$$

Property 3. Consider

$$\Phi : \begin{array}{ll} \phi_j(s) & = \phi_{j-1} \circ \rho_{\phi,j}(\xi_{i-j} \cdot \beta(s,0)), \ j = 1, \ldots, i \\ \phi_0(s) & = \beta(s,0) \end{array} \qquad \Upsilon : \begin{array}{ll} \upsilon_j(s) & = \phi_{j-1} \circ \rho_{\upsilon,j}(s) \\ \upsilon_0(s) & = \beta(s,0) \end{array}$$

$$\phi_{j-1}(a+b) \le \phi_{j-1} \circ \rho_{\phi,j}(a) + \phi_{j-1} \circ \rho_{\upsilon,j}(b)$$

then functions

$$\boxed{\begin{array}{l} \sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}) \le B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) \\ \sigma_n^{-1} \cdot \left(\sum_{i=0}^{n} \upsilon_i(c|h(\mathbf{z}_0)|\sigma_{n-i}) \right) \le B_2(|h(\mathbf{z}_0)|, c) \end{array}}$$

are bounded from above

3. Main Results. Boundedness

Theorem 2. (Non-uniform convergence) Let there exist sequences

$$\mathcal{S} = \{\sigma_i\}_{i=0}^{\infty}, \quad \Xi = \{\xi_i\}_{i=0}^{\infty}, \quad \mathcal{T} = \{\tau_i\}_{i=0}^{\infty}$$

satisfying Properties 1 – 3. Furthermore, let

1. There exist a positive number $\Delta_0 > 0$ such that $\frac{1}{\tau_i}\frac{(\sigma_i - \sigma_{i+1})}{\gamma_{0,1}(\sigma_i)} \geq \Delta_0$
2. The set of points $\mathbf{x}_0, \mathbf{z}_0$ satisfying

$$\gamma_{0,2}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c) + c|h(\mathbf{z}_0)|) \leq h(\mathbf{z}_0)\Delta_0$$

   is not empty

3. Partial sums from $\mathcal{T}$ diverge: $\sum_{i=0}^{\infty} \tau_i = \infty$

Then the state of the interconnection converge into the following subset:

$$\boxed{\Omega_a = \{\mathbf{x} \in \mathcal{X},\ \mathbf{z} \in \mathcal{Z} |\ \|\mathbf{x}\|_{\mathcal{A}} \leq c \cdot h(\mathbf{z}_0),\ \mathbf{z}:\ h(\mathbf{z}) \in [0, h(\mathbf{z}_0)]\}}$$

3. Main Results. Boundedness

Separable contracting dynamics

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(t - t_0) + c \cdot \|h(\mathbf{z}(\tau, \mathbf{z}_0))\|_{\infty,[t_0,t]}$$

With Lipschitz nonlinearity in the searching part

$$|\gamma_0(s)| \leq D_{\gamma,0} \cdot |s|$$

$$\int_{t_0}^{t} \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^{t} \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau,$$

Corollary. (Non-uniform Small-Gain) There is a trapping region if the following holds

$$\boxed{D_{\gamma,0} \cdot c \cdot \mathcal{G} < 1,}$$

with

$$\mathcal{G} = \beta_t^{-1}\left(\frac{d}{\kappa}\right)\frac{k}{k-1}\left(\beta_t(0)\left(1 + \frac{\kappa}{1-d}\right) + 1\right)$$

for some $d \in (0,1),\ \kappa \in (1,\infty)$

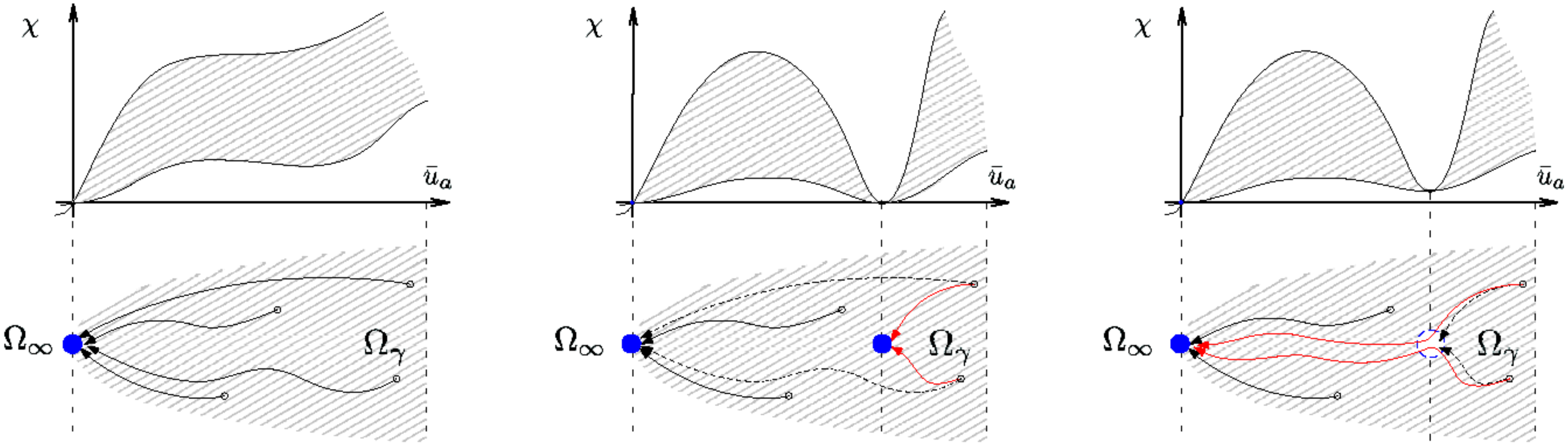
3. Main Results. Convergence

$$\mathcal{S}_a : \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|u_a(t)\|_{\infty,[t_0,t]}$$

steady-state characteristics

$$\forall\ u_a(t) \in \mathcal{U}_a : \ \lim_{t\to\infty} u_a(t) = \bar{u}_a \ \Rightarrow \ \lim_{t\to\infty} \|\mathbf{x}(t)\|_{\mathcal{A}} \in \chi(\bar{u}_a)$$

Attracting sets are determined by zeros of the steady-state responses of the contracting dynamics



And the dynamics convergence is determined by the relative timescale of “slow” unstable (fast, asymptotically stable) motions → <u>model reduction</u>

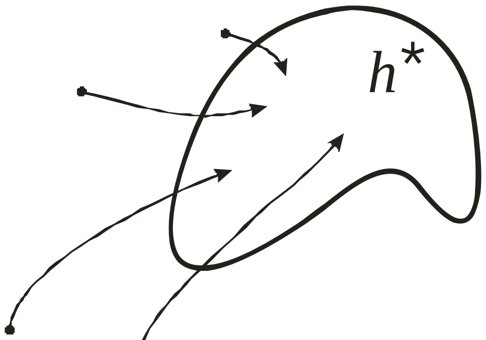
4. Discussion

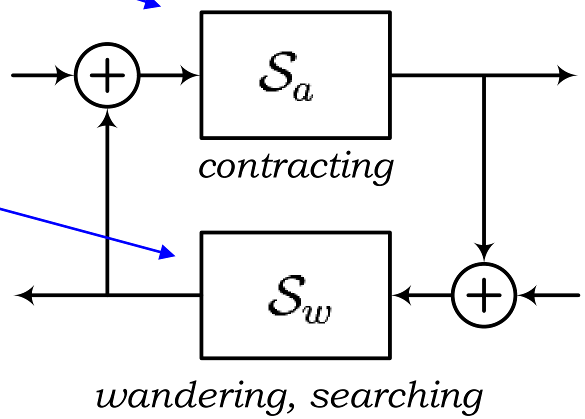
- **Universal adaptive stabilization (Ilchman, Pomet, Martenson)**

  "How much information one should know to adaptively stabilize a nonlinear system?"

$$\begin{aligned} \dot{x}(t) &= f(t,x(t)) + g(t,x(t))\,u(t) \\ y(t) &= h(t,x(t))\,, \end{aligned} \qquad \begin{aligned} \|g(t,x)\| &\leq \hat{g} \\ \|h(t,x)\| &\leq \hat{h}\|x\| \end{aligned}$$

$$\begin{aligned} u(t) &= (K\circ\beta\circ k)(t)\,y(t) \\ \dot{k}(t) &= \|y(t)\|^q \end{aligned}$$





there is a point h^* in $[0,h(0)]$ such that

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t-t_0)$$

for all $h(t)$ from **its neighborhood** $O(h^*)$ (e.g. asymptotic properties of the interconnection do not vary when $h(t)$ is from $O(h^*)$)

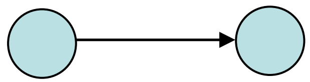
<u>Our results apply also to the case when $O(h^*)$ is empty!</u>

4. Discussion

- **Asymptotic properties of interconnections of integral Input-to-State Stable systems (Arcak, Sontag)**

$$\dot{x} = f(x),$$
contracting

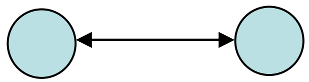
$$\dot{z} = q(x, z),$$
integrally ISS



<u>Our results apply to bi-directional connections</u>

$$\dot{x} = f(x, z),$$
contracting

$$\dot{z} = q(x, z)$$
partially integrally ISS



4. Discussion

- **We can look beyond conventional Small-gain framework …**
  "How small is **small** ?"

Consider the following interconnection

$$\begin{cases} \dot{x}_1 = -\lambda_1 x_1 + c_1 x_2 \\ \dot{x}_2 = -\lambda_2 x_2 - c_2 |x_1| \end{cases}$$

Conventional Small-gain condition

$$\boxed{\frac{c_1}{\lambda_1} \cdot \frac{c_2}{\lambda_2} < 1}$$

can be vary with time

When $\lambda_2 \rightarrow 0$

$$\begin{cases} \dot{x}_1 = -\lambda_1 x_1 + c_1 x_2 \\ \dot{x}_2 = -c_2 |x_1| \end{cases}$$

The loop gain is large

Our non-uniform small gain results, however, guarantee existence of the trapping region when

$$\boxed{\frac{c_1}{\lambda_1} \cdot \frac{c_2}{\lambda_1} < \frac{1}{16}}$$

$$|x_1(t_0)| \le \left[\frac{1}{c_2} \lambda_1 \left(\ln \frac{\kappa}{d} \right)^{-1} \frac{k-1}{k} - \frac{c_1}{\lambda_1} \left(2 + \frac{\kappa}{1-d} \right) \right] x_2(t_0)$$

5.1 Applications.

Adaptive control in presence of nonlinear parameterization

Class of systems: $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, t) + \mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\theta}) - \mathbf{f}(\boldsymbol{\xi}(t), \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)$

where $\boxed{\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, t) + \mathbf{u}(t)}$ **is input-state stable (contracting)**

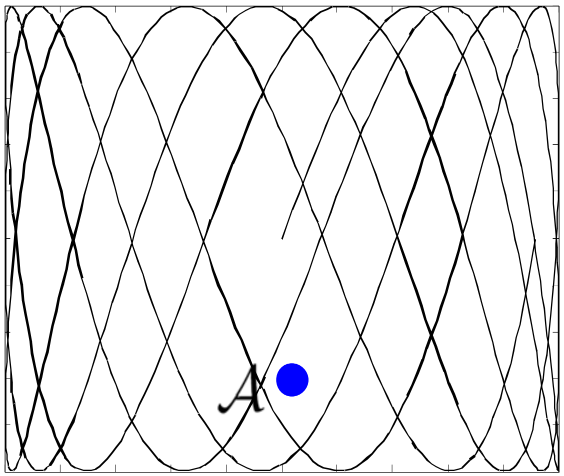
$\dot{\boldsymbol{\lambda}} = S(\boldsymbol{\lambda}), \ \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0 \in \Omega_\lambda \subset \mathbb{R}^\lambda,$ **Poisson-stable in the domain** $\Omega_\lambda \subset \mathbb{R}^n$ **and solutions are dense in** $\Omega_\lambda \subset \mathbb{R}^n$

$$\boxed{\begin{aligned}\hat{\boldsymbol{\theta}} &= \boldsymbol{\eta}(\boldsymbol{\lambda})\\ \dot{\boldsymbol{\lambda}} &= \gamma \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} S(\boldsymbol{\lambda}),\end{aligned}}$$

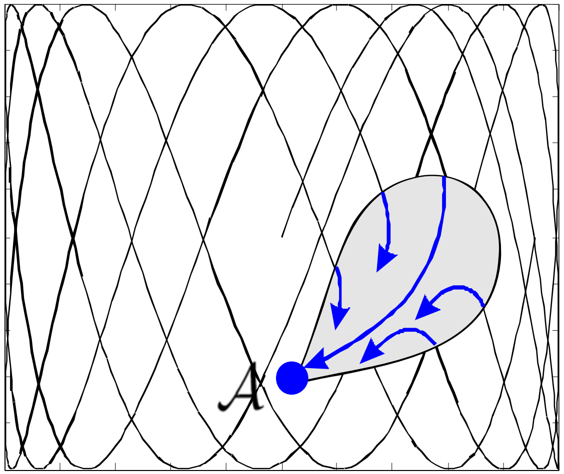
– searching, wandering dynamics

γ **is sufficiently small** → $\lim_{t\to\infty} \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} = 0, \ \lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}' \in \Omega_\theta$

Poisson stability and density



Small gamma



5.1 Applications

$$\dot{x} = -kx + \sin(x\theta + \theta) - \sin(x\hat{\theta} + \hat{\theta})$$

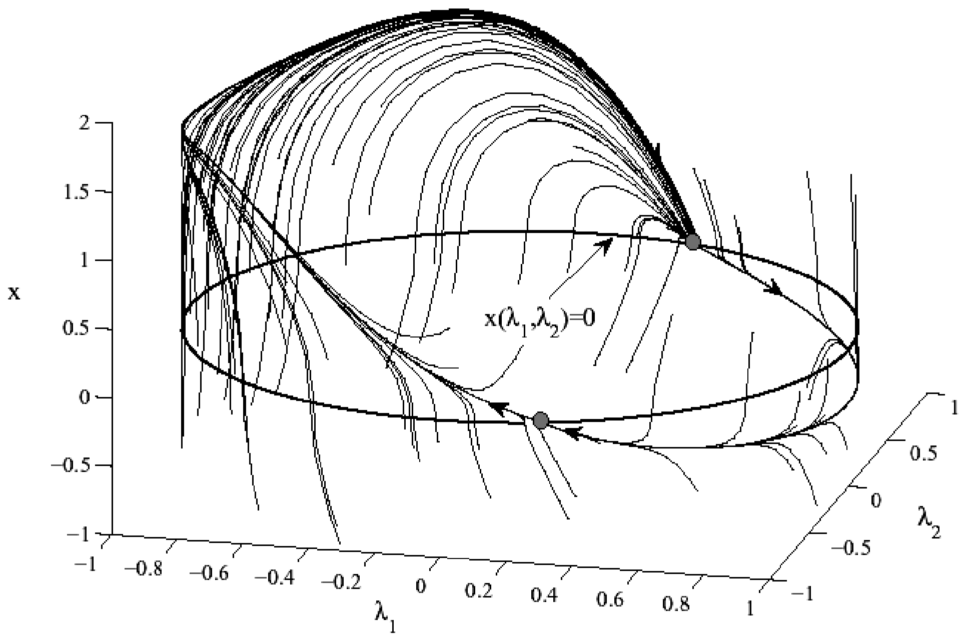
contracting

nonlinearly parameterized uncertainty

compensator

$$\dot{\lambda}_1 = \gamma |x| \lambda_1$$
$$\dot{\lambda}_2 = -\gamma |x| \lambda_2, \ \lambda_1^2(t_0) + \lambda_2^2(t_0) = 1 \qquad \hat{\theta} = (1, \ 0)^T \boldsymbol{\lambda}$$

searching dynamics



Non-uniform small gain condition:

$$0 < \gamma < -\ln\left(\frac{0.5}{2}\right)\frac{1}{2}\cdot\frac{1}{6} = 0.1155$$

5.2 Applications

State and parameter estimation for systems in non-canonic adaptive observer form

Canonic form

$$\dot{\mathbf{x}} = \mathbf{R}\mathbf{x} + \boldsymbol{\varphi}(y(t), t)\boldsymbol{\theta} + \mathbf{g}(t)$$

$$\mathbf{R} = \begin{pmatrix} 0 & \mathbf{k}^T \\ 0 & \mathbf{F} \end{pmatrix}, \quad \mathbf{x} = (x_1, \ldots, x_n)$$

$$y(t) = x_1(t)$$

unknown

a-priori known

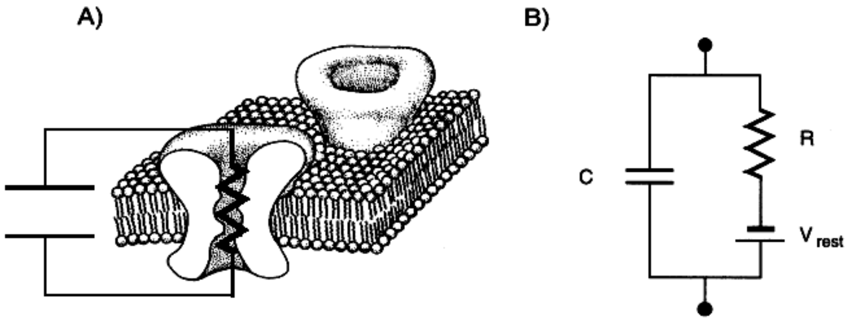
Relevant models

$$\dot{x}_0 = \theta_1^T \boldsymbol{\phi}_0(x_0, t) + \sum_{i=1}^{n} x_i$$

$$\dot{x}_i = -\lambda_i x_i + \theta_i^T \boldsymbol{\phi}_i(x_0, t), \quad x_i(t_0) = x_{0,i}$$

unknown

Passive Membrane Model



$C \approx 1\mu F / cm^2$

Fitzhug-Nagumo, Hindmarsh-Rose, Hodgkin-Huxley

Too many uncertain parameters in the “wrong” places !

5.2 Applications

The main idea is to use the following auxiliary system

$$\begin{cases} \dot{\hat{x}}_0 = -\alpha \hat{x}_0 + \hat{\theta}^T \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t) \\ \dot{\hat{\theta}} = -\gamma_\theta (\hat{x}_0 - x_0) \bar{\boldsymbol{\phi}}(x_0, \hat{\boldsymbol{\lambda}}, t), \;\; \gamma_\theta, \alpha \in \mathbb{R}_{>0} \end{cases}$$

that turns the original equations

$$\dot{x}_0 = \theta_1^T \boldsymbol{\phi}_0(x_0, t) + \sum_{i=1}^{n} x_i$$

$$\dot{x}_i = -\lambda_i x_i + \theta_i^T \boldsymbol{\phi}_i(x_0, t), \;\; x_i(t_0) = x_{0,i}$$

into:

$$\dot{\mathbf{q}} = \mathbf{A}(\hat{\boldsymbol{\lambda}}(t), t)\mathbf{q} + \mathbf{b}\; u(\boldsymbol{\lambda}, \hat{\boldsymbol{\lambda}}, t)$$

low-dimensional nonlinearity, **searching**

uniformly globally asymptotically stable (in the sense of Lyapunov), **contracting**

So the actual reduction is achieved!

5.2. Identification of the cell dynamics (membrane potential)

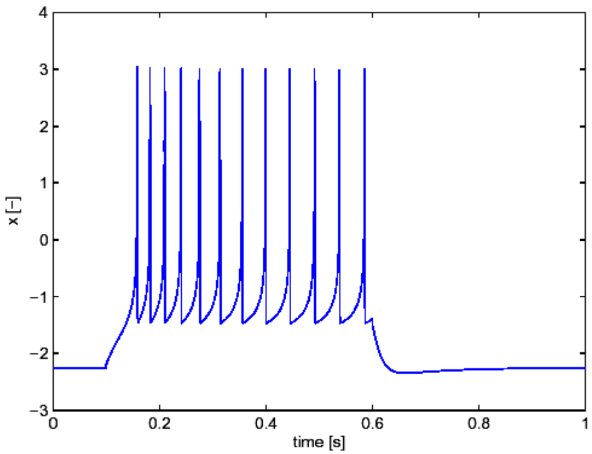
minimal mathematical model

$$\dot{x} = -ax^3 + bx^2 + y - z + \alpha u$$
$$\dot{y} = c - \beta y - dx^2$$
$$\dot{z} = \varepsilon x - \tau z + g$$

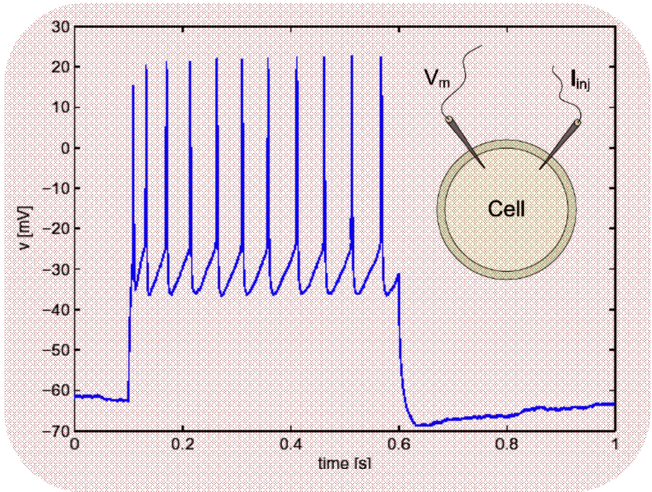
is not in adaptive observer canonic form

$$\dot{x} = \boldsymbol{\theta}_1^T \boldsymbol{\phi}_1(x,u) + x - y$$
$$\dot{y} = \boldsymbol{\theta}_2^T \boldsymbol{\phi}_2(x,y)$$
$$\dot{z} = \boldsymbol{\theta}_3^T \boldsymbol{\phi}_3(x,z)$$

(typical trajectory) model

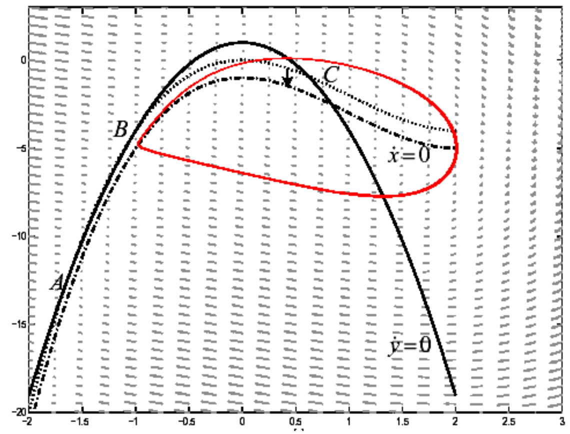


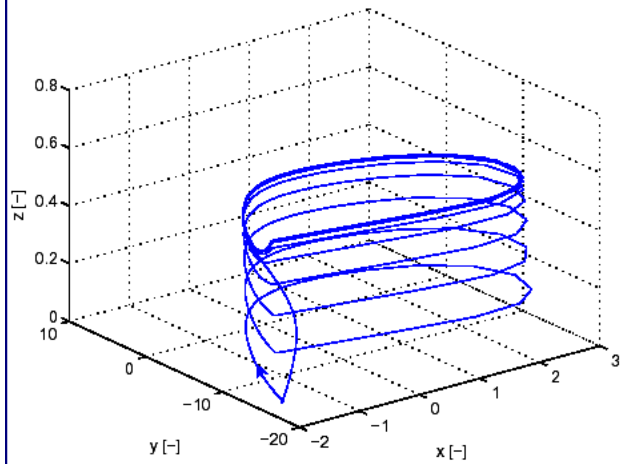
real measurements



in total: 11 unknown parameters (including initial conditions with respect to y, z) and only 1 accessible variable

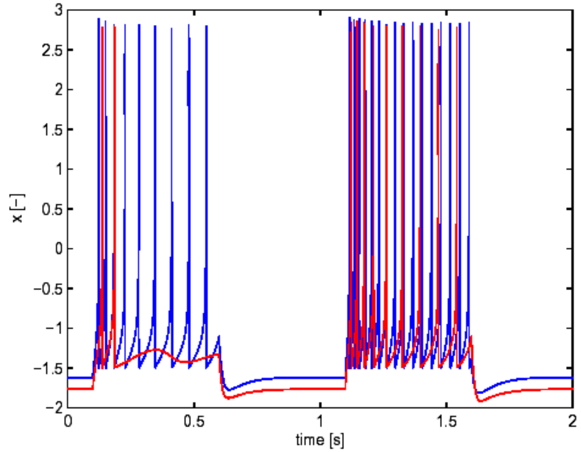
Analysis of the model





5.2. Identification of the cell dynamics (membrane potential)

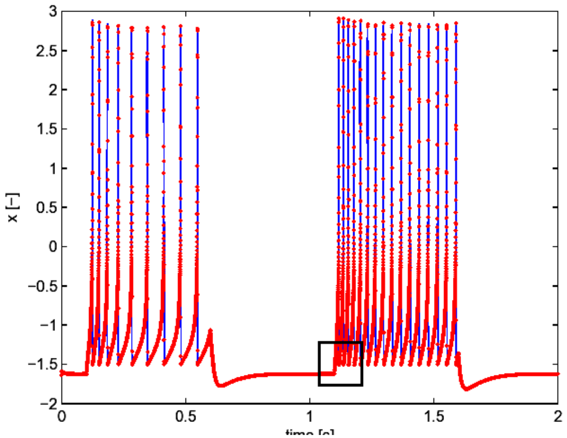
$$\Sigma = \begin{cases} \dot{\hat{x}} = \boxed{-\hat{a}\,x^3 + \hat{b}\,x^2 + \rho} \boxed{+\,\hat{y}} \boxed{-\,s\,w + \hat{\alpha}\,I + \lambda_0 (x - \hat{x})} \\ \dot{\hat{y}} = \boxed{-\,\hat{\beta}\,\hat{y} - \hat{d}\,x^2} \end{cases}$$

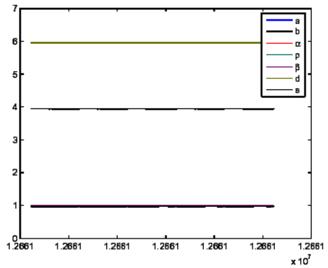


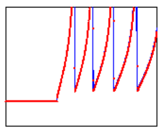
$$\hat{\beta} = \beta + \varepsilon$$
$$|\varepsilon| < 0.05$$

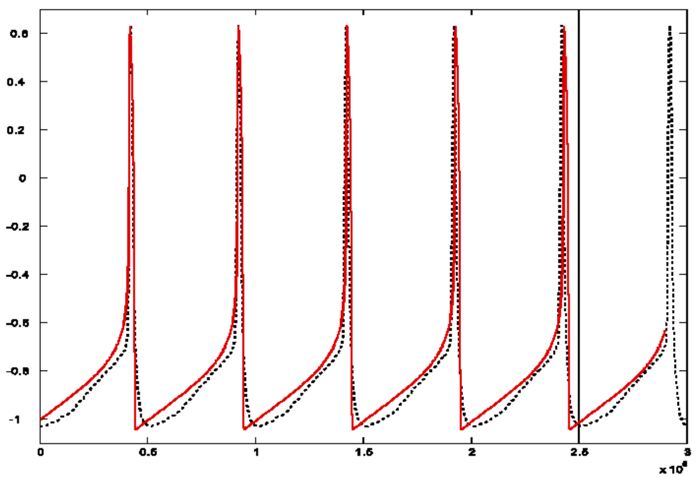
- 8 unknown parameters
- exhaustive search for a grid consisting of 20 points in each dimension will take about 296 days (1 test in 1 ms)
- 100 points > 100 000 000 days

with our method the problem is solved within hours









model

actual data

Concluding remarks

- We proposed tools for the analysis of asymptotic behavior of a class of dynamical systems with unstable invariant sets.

- Our results allow to address a variety of problems in which convergence may not be unform with respect to initial conditions (behavior of uncertain systems at the point of bifurcation, cascades of integral ISS systems, problems of optimization).

- The method does not require complete knowledge of the dynamical systems in question. Only qualitative information is necessary.

- The method offers the model “reduction” with respect to the prior information about the elements of the system

- When slow and fast motions are present (small-gain condition), the results are constructive, e.g. they automatically provide the bounds for the time constants.

- The result can be applied immediately to the problems of nonlinear regulation and parameter identification of nonlinear parameterized systems.

- Other domains of applications are being currently explored (**neuroscience, pattern recognition** etc.)