MAPS BETWEEN CLASSIFYING SPACES AND APPLICATIONS

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Abstract. For connected Lie groups G and H the calculation of the mapping space $map(BG, BH)$ can be reduced to the case of simply connected Lie groups. This reduction method allows some applications. For example a homotopy classification of self maps $BG \longrightarrow BG$ which induce rational homotopy equivalences and a classification of fake Lie groups up to homotopy.

1. Introduction.

Let G and H be compact connected Lie groups with classifying spaces BG and BH. Recently the mapping space $map(BG, BH)$ has been studied intensively by many people. In this paper we will contribute the following theorem, which is more a 'philosophical' statement than an explicit result.

Theorem. *The calculation of the mapping space $map(BG, BH)$, where G and H are compact connected Lie groups, can be reduced to the case of simply connected Lie groups.*

Of course this statement needs some illustrations. We have to explain how this reduction can be carried out.

It is well known that for every compact connected Lie group G there exists a finite cover $\tilde{G}$ sitting in an exact sequence

$$1 \longrightarrow K_G \longrightarrow \widetilde{G} \xrightarrow{q_G} G \longrightarrow 1 ,$$

such that $\widetilde{G} = G_s \times T$ is a product of a simply connected Lie group and a torus. K_G is finite and central in $\widetilde{G}$. We call such coverings finite universal. The group $\widetilde{G}$ is unique up to isomorphism, but the map $q_G : \widetilde{G} \longrightarrow G$ is not unique. One can compose q_G with a finite self cover of T.

Now let

$$K_G \to \widetilde{G} \xrightarrow{q_G} G$$

and

$$K_H \to \widetilde{H} \xrightarrow{q_H} H$$

be finite universal coverings. We get induced maps

$$Bq_G{}^* : map(BG, BH) \to map(B\widetilde{G}, BH)$$

and

$$Bq_{H*} : map(BG, B\widetilde{H}) \to map(BG, BH) .$$

The analysis of these two maps allows the reduction to the case of universal Lie groups which are products of a simply connected Lie groups and a torus.

1.1 Proposition. *Let $K \to \widetilde{G} \xrightarrow{q} G$ be a finite covering (universal is not necessary).*

(1) *The map*

$$Bq^* : map(BG, BH) \to map(B\widetilde{G}, BH)$$

is a pseudo equivalence.

(2) *$\widetilde{f} : B\widetilde{G} \longrightarrow BH$ is in the image of $\pi_0(Bq^*)$ iff $\widetilde{f} \mid BK$ is homotopically trivial.*

Remark. A map $f : X \longrightarrow Y$ is called a pseudo equivalence if it induces an injection between the components and if $\pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for all $x \in X$ and all $n \geq 1$.

1.2 Proposition. *Let $K \to \widetilde{H} \xrightarrow{q} H$ be a finite covering (universal is not necessary).*

(1) *The map*

$$Bq_* : map(BG, B\widetilde{H}) \to map(BG, BH)$$

induces a π_0-injection.

(2) *For every $\widetilde{f} : BG \to B\widetilde{H}$ there exists a homotopy fibration*

$$BK \longrightarrow map(BG, B\widetilde{H})_{\widetilde{f}} \longrightarrow map(BG, BH)_{Bq \circ \widetilde{f}} \ .$$

Dwyer and Zabrodsky calculated $map(BP, BG)$ for a finite p-group P [D-Z]. In [N] this result was generalized to the case of p-toral groups, which are extensions of a torus by a finite p-group. To describe these mapping spaces, the following construction was introduced in [D-Z].

Let $\rho : P \longrightarrow H$ be a homorphism between groups. Denote by $C_H(\rho)$ the centralizer of ρ and by $Rep(P, H)$ the set of representations of P in H, i.e. the set of conjugation classes of homorphisms. The obvious homomorphism $C_H(\rho) \times P \longrightarrow H$ passes to a map between the classifying spaces, which has as adjoint the map

$$ad : BC_H(\rho) \longrightarrow map(BP, BH)_{B\rho} \ .$$

1.3 Theorem. *([D-Z], [N]) For a p-toral group P and a compact Lie group H the map*

$$ad : \coprod_{\rho \in Rep(P,H)} BC_H(\rho) \longrightarrow map(BP, BH)$$

is a mod-p equivalence.

Remark. A map $f : X \longrightarrow Y$ is called a mod-p equivalence, if it induces an isomorphism in $H^*(\ \ ; \mathbb{Z}/p)$.

This theorem allows us to carry out the next step in the program. We will pass from universal Lie groups to simply connected Lie groups.

1.4 Proposition. *Let T be a torus and let G and H be compact connected Lie groups. Then the map*

$$\coprod_{\rho \in Rep(T,H)} map(BG, BC_H(\rho)) \xrightarrow{ad_*} map(BG, map(BT, BH)) = map(BG \times BT, BH)$$

is a mod-p equivalence for every prime p.

For any homomorphism $\rho : T \longrightarrow H$ the centralizer $C_H(\rho)$ is generated by all maximal tori which contains the image of ρ [B–tD]. Thus $C_H(\rho)$ is connected. If $C_H(\rho)$ is not simply connected, we can use proposition 1.2 to pass to the finite universal cover $\widetilde{C}_H(\rho)$ which may contain a torus T as a factor. Splitting off the factor $map(BG, BT) \simeq \coprod_{H^2(BG;\pi_2(BT))} BT$ of $map(BG, B\widetilde{C}_H(\rho))$ leads to the situation, where the groups in the target and in the domain are simply connected. The equivalence follows because BT is an Eilenberg-MacLane space of degree 2.

Now we apply these results to self maps of BG. First we investigate the canonical map

$$K : [BG, BG] \to Hom_\lambda(K(BG), K(BG)) \ ,$$

where [,] denotes the set of homotopy classes of maps and $Hom_\lambda($, $)$ denotes λ-ring homomorphisms. A λ-ring homomorphism $\theta : K(BG) \longrightarrow K(BG)$ is called real if θ maps the image of $KO(BG) \to K(BG)$ into itself.

1.5 Theorem. *Any real λ-ring isomorphism*

$$\theta : K(BG) \to K(BG)$$

is contained in the image of K, and hence is realizable by a self map of BG.

This theorem has a localised version as well as a completed version (see theorem 5.3). Similiar results as the completed version are also proved by Wojtkowiak [Wo].

As a second application we will show that the map

$$K : [BG, BG] \longrightarrow Hom(K(BG), K(BG))$$

is injective on some part of $[BG, BG]$. This is a generalisation of a result of S. Jackowski, J.E. McClure and R. Oliver [J–M–O 1], which we recall now.

For a simple Lie group G, Hubbuck proved that every self map $f : BG \longrightarrow BG$ is the trivial map on rational cohomology or looks in rational cohomology like a composition $\psi^k \circ B\alpha^*$ of an Adams operation of degree k, $k \geq 1$, and a map induced by an outer automorphism α [H]. K. Ishiguro showed that only those k can occur, which are relative prime to the order of the Weyl group W_G of G [I]. This establishes a map

$$[BG, BG] \longrightarrow \{0\} \coprod Out(G) \times \{k > 0 : (k, |WG|) = 1\} \ .$$

The Dwyer-Zabrodsky construction induces a map $BC(G) \longrightarrow map(BG, BG)_{id}$, where $C(G)$ is the center of G. Denote by $Out(G)$ the set of outer homomorhism and $map(BG, BG)_{\alpha,k}$ the component of $\psi^k \circ B\alpha$.

1.6 Theorem. *([J–M–O 1]) Let G be a simple Lie group.*

(1) *The map*

$$[BG, BG] \to \{0\} \coprod Out(G) \times \{k > 0 : (k, |WG|) = 1\}$$

is a bijection.

(2) *The composition*

$$BC(G) \xrightarrow{ad} map(BG, BG)_{id} \xrightarrow{\psi^k B\alpha} map(BG, BG)_{\alpha,k}$$

is a mod-p equivalence for all primes p.

For any compact connected Lie group we denote by $HE_{\mathbb{Q}}(BG)$ the set of homotopy classes of self maps $BG \longrightarrow BG$ which are rational equivalences. If G is simple then $HE_{\mathbb{Q}}(BG)$ is the set of all nontrivial self maps.

1.7 Theorem. *Let G be a compact connected Lie group*

(1) *The map*
$$K : HE_{\mathbb{Q}}(BG) \to Hom_{\lambda}(K(BG), K(BG))$$
is an injection.

(2) *For every $f \in HE_{\mathbb{Q}}(BG)$ the composition*
$$BC(G) \longrightarrow map(BG, BG)_f$$
is a mod-p equivalence for every prime p.

Remark. This result is also proved independently by Jackowski, McClure and Oliver [J–M–O 2]

As a last application we will classify up to homotopy the spaces in the adic genus $genus^{\wedge}_{\mathbb{Q}}(BG)$ of BG, where G is a compact connected Lie group. A nilpotent CW–complex of finite type Y is in the adic genus of BG if the p–completed spaces $Y^{\wedge}_p$ and $BG^{\wedge}_p$ for all primes p are homotopy equivalent, as well as $Y_{\mathbb{Q}}$ and $BG_{\mathbb{Q}}$.

1.8 Theorem. *Let Y_1 and Y_2 be in the adic genus of BG. $Y_1 \simeq Y_2$ if and only if there exist a real λ-ring isomorphism $K(Y_1) \cong K(Y_2)$.*

A finite loop space $X = \Omega BX$ is called an adic fake Lie group of type G, G a compact connected Lie group, if $BX \in genus^{\wedge}_{\mathbb{Q}}(BG)$.

Two finite loop spaces $X_1 = \Omega BX_1$ and $X_2 = \Omega BX_2$ are called homotopy equivalent if BX_1 and BX_2 are homotopy equivalent. Thus theorem 1.8 answers an open question about fake Lie groups of [N-S 1,2,3], namely to classify fake Lie groups up to homotopy.

The paper is organized as follows: Propositions 1.1, 1.2, and 1.4 are proved in the next section. The applications are discussed in same the order as they are stated, each in one section.

Whenever we deal with localizations $X_{(p)}$ or completions $X^{\wedge}_p$ we do this in the sense of [B-K], and define $X^{\wedge} := \prod_p X^{\wedge}_p$.

2. The reduction to simply connected Lie groups.

Let $K \longrightarrow G \longrightarrow \overline{G}$ be an exact sequence of compact Lie groups, with G and $\overline{G}$ connected and K finite, and let $BK \longrightarrow BG \longrightarrow B\overline{G}$ be the associated fibration between the classifying spaces. We think of the classifying spaces as given by the Milnor models. $\overline{G}$ acts on $\widetilde{BK} := EG/K$ from the right. $\widetilde{BK} \simeq BK$ and $\widetilde{BK}/\overline{G} \simeq BG$. The $\overline{G}$-action on $\widetilde{BK}$ is free. Hence we get, for any connected space X with trivial $\overline{G}$-action, a sequence of homotopy equivalences

$$\begin{aligned} map(BG, X) &\simeq map_{\overline{G}}(\widetilde{BK}, X) \\ &\simeq map_{\overline{G}}(E\overline{G} \times \widetilde{BK}, X) \\ &\simeq map_{\overline{G}}(E\overline{G}, map(\widetilde{BK}, X)) \\ &= map(\widetilde{BK}, X)^{h\overline{G}} \quad . \end{aligned}$$

Here, $map_{\overline{G}}(\ ,\ )$ is the space of equivariant maps, $\overline{G}$ acts on $map(\widetilde{BK}, X)$ via the action on the source, and we may take the last equation as the definition of homotopy fixed-point sets.

The component $map(\widetilde{BK}, X)_c$ of the constant map is fixed under the action. The canonical inclusion $X \longrightarrow map(\widetilde{BK}, X)_c$ is equivariant and induces a map

$$map(B\overline{G}, X) = X^{h\overline{G}} \longrightarrow map(\widetilde{BK}, X)_c^{h\overline{G}} \longrightarrow map(BG, X)$$

between homotopy fixed-point sets. The last map is a pseudo equivalence; it is the inclusion of some component into whole the mapping space. This proves the following lemma

2.1 Lemma. *If* $X \longrightarrow map(BK, X)_c$ *is a homotopy equivalence, then*

$$map(B\overline{G}, X) \longrightarrow map(BG, X)$$

is a pseudo equivalence.

Sufficient conditions on X for satisfying the assumptions may be found in [D–W].

Now we are prepared for

Proof of proposition 1.1. Let $K \longrightarrow G \longrightarrow \overline{G}$ be an exact sequence of compact Lie groups, where K finite. By lemma 2.1 we need only show that for a compact connected Lie group H,

$$BH \longrightarrow map(BK, BH)_c$$

is a homotopy equivalence. This follows from the Sullivan conjecture [M], which implies that the loop space of the pointed mapping space $map_*(BK, BH)_c$ is contractible. □

There is also a p-completed version of proposition 1.1, even for K beeing a compact Lie group. In [F–M] is shown that $map_*(BK, H_p^\wedge)$ is contractible, and hence, that $BK \longrightarrow map(BK, BH_p^\wedge)_c$ is homotopy equivalence. Now, the same proof works to show

2.2 Proposition. *Let* $K \to G \xrightarrow{q} \overline{G}$ *be an exact sequence of compact Lie groups and* H *a compact connected Lie group. Then,*

$$map(B\overline{G}, BH_p^\wedge) \longrightarrow map(BG, BH_p^\wedge)$$

is a pseudo equivalence.

Proof of proposition 1.2. Let $K \longrightarrow \widetilde{H} \xrightarrow{q} H$ be a finite cover of the compact connected Lie group H. Then K is central in $\widetilde{H}$ and finite. The fibration

$$BK \longrightarrow B\widetilde{H} \longrightarrow BH$$

is a principal fibration. By Thom–theory [T], as revisited in [N–S 4], every map $\widetilde{f}$: $BG \longrightarrow B\widetilde{H}$, G connected, establishes a principal fibration

$$Bq_* : map(BG, B\widetilde{H})_{\widetilde{f}} \longrightarrow map(BG, BH)_f$$

with structure group $map(BG, BK)$. f is defined by $f := Bq \circ \widetilde{f}$. Since BG is 1–connected, $map(BG, BK)$ consists only of the component of the constant map. The evaluation map $e : map(BG, BK) \longrightarrow BK$ is a homotopy equivalence and has as inverse the canonical section $s : BK \longrightarrow map(BG, BK)$. This proves part (ii) of proposition 1.2. To show that $Bq_* : map(BG, B\widetilde{H}) \longrightarrow map(BG, BH)$ is a π_0-injection we use obstruction theory. The obstruction groups $H^*(BG; \pi_*(BK))$ for lifting a homotopy vanish because BG is 1-connected and BK is an Eilenberg-McLane space of degree 1. This completes the proof. □

2.3 Corollary. *Let G and H be compact connected Lie groups and let $K \longrightarrow \widetilde{H} \longrightarrow H$ be a finite cover of H. Then there exists a finite universal cover $\widetilde{G} \longrightarrow G$ of G, such that every map $f : BG \longrightarrow BH$ can be lifted to a map $\widetilde{f} : B\widetilde{G} \longrightarrow B\widetilde{H}$. Moreover, two maps $f, g : BG \longrightarrow BH$ are homotopic if and only if the lifts $\widetilde{f}$ and $\widetilde{g}$ are homotopic.*

Proof. We choose a finite universal covering $q : \widetilde{G} \longrightarrow G$, where $\widetilde{G} = G_s \times T$ is a product of a simply connected Lie group and a torus. We can compose Bq with the map $id \times \psi^k : BG_s \times BT \longrightarrow BG_s \times BT$, where k is the order of K and ψ^k the Adams map inducing multiplikation with k in $H^2(BT)$. Then, for every map $f : BG \longrightarrow BH$ there is only one obstruction group $H^2(B\widetilde{G}; K) \cong H^2(BT; K)$ for a lift of $f \circ Bq$. If we try to lift the map $f \circ Bq \circ (id \times \psi^k)$, every obstruction must vanish. This proves the existence of lifts. The uniqueness of the lifts follows from the fact that in the two maps

$$map(B\widetilde{G}, B\widetilde{H}) \longrightarrow map(B\widetilde{G}, BH) \longleftarrow map(BG, BH)$$

are π_0-injections (proposition 1.1 and 1.2). $\square$

Finally, we proof proposition 1.4.

Proof of proposition 1.4. We only have to consider the case of a fixed homomorphism $\rho : T \longrightarrow H$. First, we show that ad_* is a π_0–injection. The centralizer $C_H(\rho)$ is connected, because it is generated by all maximal tori of H which contain the image of ρ [B–tD; IV,2.3]. Using completions we can build the commutative diagram

$$\begin{array}{ccc} [BG, BC_H(\rho)] & \longrightarrow & [BG, BC_H(\rho)^\wedge] \\ \downarrow & & \downarrow \\ [BG, map(BT, BH)_{B\rho}] & \longrightarrow & [BG, map(BT, BH)_{B\rho)}{}^\wedge]\ . \end{array}$$

The homotopy groups of the homotopy fiber F of $BC_H(\rho) \longrightarrow BC_H(\rho)^\wedge$ are all rational vector spaces, which are concentrated only in odd dimensions [F–M]. The rational cohomology of BG is concentrated in even degrees. Thus, all the obstruction groups $H^*(BG; \pi_*(F))$ for lifting homotopies vanish. That is that the top row is an injection. The right column is a bijection by theorem 1.3, and therefore ad_* is a π_0–injection.

Next, we show that ad_* is also a surjection. This is based on the following lemma.

2.4 Lemma. *For two tori T_1 and T_2 the map*

$$ad_* : \coprod_{\rho \in Rep(T_2, H)} map(BT_1, BC(\rho)) \longrightarrow map(BT_1, map(BT_2, BH))$$

is a π_0-bijection.

Proof. Again, we only have to consider the case of a fixed homomorphism $\rho : T_2 \longrightarrow H$. For every map $f : BT_1 \times BT_2 \longrightarrow BH$ there exists a homomorphism $\alpha : T_1 \times T_2 \longrightarrow H$, such that $f \simeq B\alpha$ (theorem 1.3). Let $\tau := \alpha\,|_{T_1}$. Then we know that both, the maximal torus of H and the image $im(\tau)$ of τ, are contained in $C_H(\rho)$. That is that $B\tau$ lifts to a map $B\tau' : BT_1 \longrightarrow BC_H(\rho)$. Obviously, $ad_*(B\tau') = B\alpha \simeq f$, i.e ad_* is surjective. The injectivity was already proved above. $\square$

Let $f : BG \times BT \longrightarrow BH$ be a map such that $f|_{BT} \simeq B\rho$. By theorem 1.3, we can choose a homomorphism $\alpha : T_G \times T \longrightarrow H$ with the properties $f|_{BT_G} \times BT \simeq B\alpha$ and $\alpha|_T = \rho$. For the diagram

$$\begin{array}{ccc} BT_G \times BT & \xrightarrow{B\alpha} & BT_H \\ \downarrow & & \downarrow \\ BG \times BT & \xrightarrow{f} & BH \end{array}$$

there is an associated homomorphism $\beta : W_G \longrightarrow W_H$ between the Weyl groups such that $B\alpha$ is equivariant [A–M; 2.21]. $C(\rho)$ is a connected subgroup of H of maximal rank. This implies that $T_{C_H(\rho)} = T_H$ and $W_{C_H(\rho)}$ is a subgroup of W_H. Because W_G acts trivially on BT, β splits over $W_{C_H(\rho)}$.

Let $B\tau$ be the restriction of $B\alpha$ to BT_G. Then, $B\tau$ splits over $BC_H(\rho)$, is equivariant with respect to β, and $ad_*(B\tau) = B\alpha$. Passing to the Weyl group invariants in rational cohomology yields a map

$$\theta : H^*(BC(\rho); \mathbb{Q}) \longrightarrow H^*(BG; \mathbb{Q}) \ .$$

θ has a realization

$$f'_{\mathbb{Q}} : BG_{\mathbb{Q}} \longrightarrow BC(\rho)_{\mathbb{Q}} \ ,$$

which fits into

$$\begin{array}{ccc} (BT_G)_{\mathbb{Q}} & \xrightarrow{B\tau} & (BT_{C_H(\rho)})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ BG_{\mathbb{Q}} & \xrightarrow{f'_{\mathbb{Q}}} & BC_H(\rho)_{\mathbb{Q}} \end{array} \ .$$

Therefore, $ad_*(f'_{\mathbb{Q}})$ and $f_{\mathbb{Q}}$ induce the same map in rational cohomology and are homotopic.

From the homotopy equivalence $BC(\rho)^\wedge \simeq (map(BT, BH)_{B\rho})^\wedge$ we get a map $f'^\wedge : BG^\wedge \longrightarrow BC(\rho)^\wedge$ with $ad_*(f'^\wedge) \simeq f\wedge$.

$H^*(BG; \mathbb{Q}) \longrightarrow H^*(BT_G; \mathbb{Q})$ is injective. Restricting all the maps to BT_G lemma 2.4 implies that all maps agree over the adeles. Using the arithmetic square we glue $f'_{\mathbb{Q}}$ and $f'^\wedge$ together and get a map $f' : BG \longrightarrow BC(\rho)$ which has the property $ad_*(f') \simeq f$ as desired. This shows that ad_* is a surjection.

To finish the proof of proposition 1.4, we can now fix a map $f' : BG \longrightarrow BC_H(\rho)$ with adjoint $f : BG \times BT \longrightarrow BH$ respectively $f : BG \longrightarrow map(BT, BH)$. In [B–N] it is shown that, for any map $g : BG \longrightarrow BH$, G a compact Lie group and H compact connected Lie group, $map(BG, BH^\wedge_p)_g$ is p–complete and that $map(BG, BH)_g \longrightarrow map(BG, BH^\wedge_p)_g$ is a homotopy equivalent after completion. Hence, after completion, the Dwyer–Zabrodsky map induces an equivalence

$$BC_H(\rho)^\wedge_p \xrightarrow{\simeq} map(BT, BH^\wedge_p)_{B\rho} \ .$$

Now, in the commutative diagram

$$\begin{array}{ccccc} map(BG, BC_H(\rho))_f & \xrightarrow{ad_*} & map(BG, map(BT, BH)_{B\rho})_{f'} & \simeq & map(BG \times BT, BH)_{f'} \\ \downarrow & & \downarrow & & \downarrow \\ map(BG, BC_H(\rho)^\wedge_p)_f & \xrightarrow{ad_*} & map(BG, (map(BT, BH)_{B\rho})^\wedge_p)_{f'} & \simeq & map(BG \times BT, BH^\wedge_p)_{f'} \end{array}$$

the lower horizontal map is an equivalences. The second equivalence in the lower row also follows from [B–N]. The vertical arrows become equivalences after completion. This finishes the proof of proposition 1.4. □

3. λ–ring isomorphisms of $K(BG)$.

Let G be a compact connected Lie group. According to [W 2], any λ-ring homomorhism

$$\theta : K(BG) \otimes \mathbb{Z}_p^\wedge \longrightarrow K(BG) \otimes \mathbb{Z}_p^\wedge$$

induces a commutative diagram

$$(*) \qquad \begin{array}{ccc} K(BT_G) \otimes \mathbb{Z}_p^\wedge & \xrightarrow{\theta_T} & K(BT_G) \otimes \mathbb{Z}_p^\wedge \\ \uparrow & & \uparrow \\ K(BG) \otimes \mathbb{Z}_p^\wedge & \xrightarrow{\theta} & K(BG) \otimes \mathbb{Z}_p^\wedge \end{array} ,$$

where $T_G \hookrightarrow G$ is the maximal torus. In the case of product of a simply connected Lie group and a torus we describe all such automorphisms.

Let $\widetilde{G} = G_s \times T$ be a product of a simply connected Lie group and a torus, and let $G_s = G_1 \times ... \times G_r$ be th splitting of G_s into simple simply connected Lie groups. Passing to the rationalization of the classifying spaces we can rewrite this as $B\widetilde{G}_{\mathbb{Q}} = (BG_{1\mathbb{Q}})^{n_1} \times ... \times (BG_{s\mathbb{Q}})^{n_s} \times BT_{\mathbb{Q}}$, where we collect homotopy equivalent factors. Among the simple simply connected Lie groups, $Spin(2n+1)$ and $Sp(n)$, $n \geq 3$, are the only nonisomorphic Lie groups, which rationally have homotopy equivalent classifying spaces, and which have isomorphic complex K–theory [A–M: 2.7].

We call an automorphism of

$$K(B\widetilde{G}) \otimes \mathbb{Q} \cong (K(BG_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} ... \otimes_{\mathbb{Q}} (K(BG_r) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} (K(BT) \otimes_{\mathbb{Z}} \mathbb{Q})$$

simple if it is of the form

$$(\sigma \circ (\bigotimes_{i=1}^{r} \psi^{a_i} \circ B\alpha_i^* \circ \varepsilon_i^{k_i})) \otimes A_T \ .$$

Here, $\sigma \in \Sigma_{n_1} \times ... \times \Sigma_{n_s}$ permutes isomorphic factors of $K(BG_1) \otimes ... \otimes K(BG_r)$. ψ^{a_i} is an Adams operation of degree $a_i \in \mathbb{Q}$. α_i is an outer automorphism of G_i. ε_i is one of the exceptional maps described in [A–M; 2.15], which can only occur for $Sp(2)$, F_4, G_2. k_i is 1 or 0. $A_T : K(BT) \otimes \mathbb{Q}) \longrightarrow K(BT) \otimes \mathbb{Q}$ is an automorphism of $K(BT) \otimes \mathbb{Q}$.

3.1 Proposition. *If $\widetilde{\theta} : K(B\widetilde{G}) \otimes \mathbb{Q} \longrightarrow K(B\widetilde{G}) \otimes \mathbb{Q}$ is an isomorphism, then $\widetilde{\theta}$ fits into a commutative diagram*

$$\begin{array}{ccc} K(BT_{\widetilde{G}}) \otimes \mathbb{Q} & \xrightarrow{\widetilde{\theta}_T} & K(BT_{\widetilde{G}}) \otimes \mathbb{Q} \\ \uparrow & & \uparrow \\ K(B\widetilde{G}) \otimes \mathbb{Q} & \xrightarrow{\widetilde{\theta}} & K(B\widetilde{G}) \otimes \mathbb{Q} \end{array} ,$$

if and only if $\widetilde{\theta}$ is simple.

Proof. For a simple isomorphism $\widetilde{\theta} = \sigma\circ(\bigotimes_{i=1}^{r} \psi^{a_i}\circ B\alpha_i^*\circ\varepsilon_i^{k_i})\otimes A_T$, every factor $\psi^{a_i}\circ B\alpha_i^*\circ \varepsilon_i^{k_i}$ lifts to an isomorphism of $K(BT_{G_i})\otimes\mathbb{Q}$, and σ to an isomorphism of $K(BT_{G_s})\otimes\mathbb{Q}$. This proves one part of the statement.

Now let $\widetilde{\theta}$ be an isomorphism, which has a lift $\widetilde{\theta}_T : K(BT_{\widetilde{G}})\otimes\mathbb{Q}\longrightarrow K(BT_{\widetilde{G}})\otimes\mathbb{Q}$. For the classifying space of a compact connected Lie group, the Chern character induces an isomorphism $K(\ )\otimes\mathbb{Q}\cong H^*(\ ;\mathbb{Q})$. This allows to pass to rational cohomology, where we have analogous diagrams and maps. We use the same notation for the maps.

In 2–dimensional rational cohomology, $\widetilde{\theta}_T$ can be described by a matrix

$$M = \begin{pmatrix} \theta_{s,s} & \theta_{T,s} \\ \theta_{s,T} & \theta_{T,T} \end{pmatrix} ,$$

where

$$\begin{aligned} \theta_{s,T} : H^2(BT_{G_s};\mathbb{Q}) &\rightarrow H^2(BT_{G_s};\mathbb{Q})\oplus H^2(BT;\mathbb{Q}) \\ &\xrightarrow{\widetilde{\theta}_T} H^2(BT_{G_s};\mathbb{Q})\oplus H^2(BT;\mathbb{Q}) \rightarrow H^2(BT;\mathbb{Q}) \ . \end{aligned}$$

All the other entries are given by analogous compositions, and all entries are equivariant with respect to a suitable isomorphism of W_G. $\widetilde{\theta}_{s,T}$ is a lift of $H^*(BG_s;\mathbb{Q})\longrightarrow H^*(BT;\mathbb{Q})$. All maps can be realized over the rationals. Therefore, $\widetilde{\theta}_{s,T}$ is trivial.

It is well known that the isomorphism

$$H^2(BT_{G_s};\mathbb{Q}) \cong \bigoplus_i H^2(BT_{G_i};\mathbb{Q})$$

is a splitting into irreducible W_{G_s}–modules. Thus, $H^2(BT_{G_s};\mathbb{Q})$ contains no submodul with trivial W_{G_s} action. This implies that $\theta_{T,s}$, as an W_{G_s} equivariant map, is also trivial, that $\widetilde{\theta}_T = \theta_{s,s}\oplus\theta_{T,T}$ on $H^2(BT_{\widetilde{G}};\mathbb{Q})$ and $\widetilde{\theta}_T = \theta_{s,s}\otimes\theta_{T,T}$ on $H^*(BT_{\widetilde{G}};\mathbb{Q})$, and that, by passing to the Weyl group invariats, $\widetilde{\theta} = \theta_{G_s}\otimes\theta_{T,T}$, where θ_{G_s} is an automorphism of $H^*(BG_s;\mathbb{Q})$.

The elements of positive degree in $H^*(BG_s;\mathbb{Q})$ generate an ideal $J\subset H^*(BT_{G_s};\mathbb{Q})$. Since $H^*(G_s/T_{G_s};\mathbb{Q})\cong H^*(BT_{G_s},\mathbb{Q})/J$, the above considerations establish an automorphism $\overline{\theta}$ of $H^*(G_s/T_{G_s};\mathbb{Q})$. In [P], these automorphisms are calculated. They look like being induced by simple automorphisms of $H^*(BG_s;\mathbb{Q})$. The map $G_s/T_{G_s}\longrightarrow BT_{G_s}$ induces an isomorphism in two dimensional rational cohomology. Therefore $\overline{\theta}$ determines $\widetilde{\theta}_{T_s}$ as well as θ_s. Hence, θ_{G_s} and $\widetilde{\theta}$ are simple isomorphisms, which finishes the proof. □

3.2 Remark. In the case of a $\mathbb{Q}_p^\wedge$–isomorphism $K(B\widetilde{G})\otimes\mathbb{Q}_p^\wedge\longrightarrow K(B\widetilde{G})\otimes\mathbb{Q}_p^\wedge$, all the arguments also work. Hence, the same statement is true over $\mathbb{Q}_p^\wedge$. In this case the degree of the Adams operations are given by p–adic rationals.

For any compact connected Lie group G, we choose a finite universal covering $K\longrightarrow(\widetilde{G} = G_s\times T)\longrightarrow G$ of G. This induces rational equivalences $T_{\widetilde{G}}\hookrightarrow T_G$ and $\widetilde{G}\hookrightarrow G$. for any λ–ring homomorphism $\theta : K(BG)\otimes\mathbb{Z}_p^\wedge\longrightarrow K(BG)\otimes\mathbb{Z}_p^\wedge$, we can tensor the diagram $(*)$ with $\mathbb{Q}_p^\wedge$ yielding

$$\begin{array}{ccc} K(BT_{\widetilde{G}})\otimes\mathbb{Q}_p^\wedge & \xrightarrow{\widetilde{\theta}_T} & K(BT_{\widetilde{G}})\otimes\mathbb{Q}_p^\wedge \\ \uparrow & & \uparrow \\ K(B\widetilde{G})\otimes\mathbb{Q}_p^\wedge & \xrightarrow{\widetilde{\theta}} & K(B\widetilde{G})\otimes\mathbb{Q}_p^\wedge \end{array} .$$

We assume that, after tensoring with $\mathbb{Q}_p^\wedge$, θ and $\widetilde{\theta}$ are isomorphisms. Proposition 3.1 tells us that $\widetilde{\theta}$ is of the form

$$\widetilde{\theta} = (\sigma \circ (\bigotimes_{i=1}^{r} \psi^{a_i} \circ B\alpha_i^* \circ \varepsilon_i^{k_i})) \otimes A_T \ .$$

We would like to deduce more information about the exponenents of the Adams operations .

3.3 Lemma. *$a_i \in \mathbb{Z}_p^\wedge$ for all i.*

Proof. Let us start with some remarks. The image of

$$G_1 \times ... \times G_{i-1} \times G_{i+1} \times ... \times G_r \times T \longrightarrow G$$

is normal in G and the quotient is denoted by $\overline{G}_i$. We introduce the diagram

$$\begin{array}{ccccc}
K(B\overline{G}_i) \otimes \mathbb{Z}_p^\wedge & \longrightarrow & K(B\overline{G}_i) \otimes \mathbb{Q}_p^\wedge & \xrightarrow{\phi_i} & K(BG_i) \otimes \mathbb{Q}_p^\wedge \\
\downarrow & & & & \downarrow {\scriptstyle \psi^{a_i} \circ B\alpha_i^* \circ \varepsilon_i^{k_i}} \\
K(BG) \otimes \mathbb{Z}_p^\wedge \xrightarrow{\theta} K(BG) \otimes \mathbb{Z}_p^\wedge & & & & K(BG_i) \otimes \mathbb{Q}_p^\wedge \\
\qquad\qquad\qquad\qquad \downarrow & & & & \downarrow {\scriptstyle \beta_\sigma} \\
\qquad\qquad K(B\widetilde{G}) \otimes \mathbb{Z}_p^\wedge & \longrightarrow & K(BG_{\sigma(i)}) \otimes \mathbb{Z}_p^\wedge & \longrightarrow & K(BG_{\sigma(i)}) \otimes \mathbb{Q}_p^\wedge
\end{array} ,$$

where ϕ_i is the map induced by the homomorphism $G_i \longrightarrow \overline{G}_i$, and β_σ the map induced by the permutation σ. In the case $G_i \neq G_{\sigma(i)}$, i.e. one of the groups is $Sp(n)$ and the other $Spin(2n+1)$, the inclusions

$$Sp(n) \hookleftarrow U(n) \hookrightarrow SO(2n) \hookrightarrow SO(2n+1)$$

correspond to the identity on the maximal tori, which is equivariant with respect to the Weyl groups of $Sp(n)$ and $SO(2n+1)$. For $G_i = Sp(n)$ and $G_{\sigma(i)} = Spin(2n+1)$, β_σ is given by $K(BSp(n)) \cong K(BSO(2n+1)) \longrightarrow K(BSpin(2n+1))$ and vice versa by the rational inverse. In the case of $G_i = G_{\sigma(i)}$, β_σ is the identity.

We have similiar diagrams in terms of the maximal tori and K-theory as well as in terms of the maximal tori and cohomology with $\mathbb{Z}_p^\wedge$ or $\mathbb{Q}_p^\wedge$ as coefficients. In particular, if we look at the 2-dimensional cohomology of the maximal tori, all maps are described by matrices.

The above diagram implies that $\beta_\sigma \circ \psi^{a_i} \circ B\alpha_i^* \circ \varepsilon_i^{k_i} \circ \phi_i$ is a $\mathbb{Z}_p^\wedge$-matrix. Because $B\alpha_i^*$ is invertible over $\mathbb{Z}$, we can drop it and the rest is still a $\mathbb{Z}_p^\wedge$-matrix.

We have to consider several different cases. Let us start with those where the exceptional map ε can occur. For G_2, $Sp(2)$ and F_4, ε looks like $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, M or $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$, where M is given by $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ [A–M; 2.5, 2.6]. For G_2 and F_4 the center is trivial, i.e. $\phi_i = id$. We can assume that $\psi^{a_i} \circ \varepsilon_i^{k_i}$ is a $\mathbb{Z}_p^\wedge$-matrix, which implies that $a_i \in \mathbb{Z}_p^\wedge$.

The center of $Sp(2)$ is $\mathbb{Z}/2$. Thus, for $G_i = Sp(2)$, ϕ_i is either the identity or is represented by $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$. In both cases the product matrix $\varepsilon_i^{k_i} \circ \phi_i$ has a 1 as an entry, and, hence, $a_i \in \mathbb{Z}_p^\wedge$.

For $G_i = G_{\sigma(i)} = Spin(2n+1)$ or $Sp(n)$, β_σ is the identity, and we can assume that $\psi^{a_i} \circ \phi_i$ is a $\mathbb{Z}_p^\wedge$-matrix. If the denominator of a_i contains a power of p, then every entry in the matrix ϕ_i must be divisible by p and the group $\mathbb{Z}/p \times ... \times \mathbb{Z}/p$, with $rank\ G$ factors, is in the kernel of the homomorphism $G_i \longrightarrow \overline{G}_i$. This is a contradiction in view of the center of G_i (check all the cases) apart from the case $G_i = SU(2)$ and $p = 2$. In this situation a_i may be of the form $b/2$, b a unit in $\mathbb{Z}_2^\wedge$. Bigger powers of 2 cannot occur. But this implies that $K(BSO(3)) \otimes \mathbb{Z}_2^\wedge$ and $K(BSU(2)) \otimes \mathbb{Z}_2^\wedge$ are isomorphic as λ-rings, which is also a contradiction.

There is only one case left, namely $G_i = Spin(2n+1)$ and $G_{\sigma(i)} = Sp(n)$ or vice versa, with $n \geq 3$. We get $det(\phi_i) = 1$ or 2, $det(\beta_\sigma) = 2$ or $1/2$ and $det(\psi^{a_i}) = a_i^n$; in particuliar $4a_i^n \in \mathbb{Z}_p^\wedge$. Since $n \geq 3$, $a_i \in \mathbb{Z}_p^\wedge$. $\square$

3.4 Proposition. *If $\theta : K(BG) \otimes \mathbb{Z}_p^\wedge \longrightarrow K(BG) \otimes \mathbb{Z}_p^\wedge$ is a real λ-ring isomorphism, θ is realizable as a map $BG_p^\wedge \longrightarrow BG_p^\wedge$.*

Proof. First we choose a finite universal cover $q_2 : \widetilde{G} = G_s \times T \longrightarrow G$ of G. By the proof of lemma 3.3, θ induces a map $\widetilde{\theta}_T : H^2(BT_{\widetilde{G}}; \mathbb{Z}_p^\wedge) \longrightarrow H^2(BT_{\widetilde{G}}; \mathbb{Z}_p^\wedge)$, which has the form $(\sigma \circ (\bigoplus \psi^{a_i} \circ B\alpha_i \circ \varepsilon_i^{k_i})) \oplus A_T$ (notation as above, but we can write the map as a direct sum because $H^2(BT_{\widetilde{G}}; \mathbb{Z}_p^\wedge)$ splits into a direct sum). We will show that this map can be realized as a topological map $B\widetilde{G} \longrightarrow B\widetilde{G}$.

σ splits in a product of cycles, which, after renumbering the factors of $G_s = G_1 \times \ldots \times G_r$, can be represented by the symbol $(l_0, \ldots, l_1)(l_1+1, \ldots, l_2) \ldots (l_j - 1, \ldots, l_j)$ with $l_0 = 1$ and $l_j = r$. We compute the characteristic polynomial $\chi(\widetilde{\theta}_T)$. To do this, we define $d_i := rank\ G_i$ and $e_i := det(\varepsilon_i)$. Then,

$$\chi(\widetilde{\theta}_T) = \prod_{n=1}^{j} \left(t^{(l_n - l_{n-1}) d_{l_n}} \pm (\prod_{m=l_{n-1}+1}^{l_n} a_n^{d_m} e_n^{k_m}) \right) \cdot \chi(A_T) \ .$$

$det(\beta_\sigma)$ does not occur, because the number of maps $Spin \longrightarrow Sp$ is equal the number of maps $Sp \longrightarrow Spin$. The determinant of $B\alpha_i$ is ± 1.

Since $\widetilde{\theta}_T$ and θ_T are conjugate they must have the same characteristic polynomial. Thus $\chi(\widetilde{\theta}_T)$ has $\mathbb{Z}_p^\wedge$-coefficients. $\mathbb{Z}_p^\wedge$ is integrally closed in $\mathbb{Q}_p^\wedge$. All polynomials have leading coefficient 1. a_i and e_i are p-adic integers. All together, this implies that $\chi(A_T)$ is also a p-adic integral polynomial.

From the assumptions follows that

$$det(\theta_T) = det(\widetilde{\theta}_T) = (\prod_i a_i^{d_i} e_i^{k_i}) det(A)$$

is a p-adic unit. So all the factors must be p-adic units; in particuliar a_i and $e_i^{k_i}$. This shows that all Adams operations can be realized on $B\widetilde{G}$ [W 1] and that the exeptional maps ε_i cannot occur for $Sp(2)$ and F_4 for $p = 2$, as well as for G_2 for $p = 3$. They can be realized in all other cases [F].

Can we find a realisation of σ? For p odd there is no problem, because $BSpin(2n+1)_p^\wedge$ and $BSp(n)_p^\wedge$ are equivalent.

For $p = 2$, we have to consider permutation of $Spin$ and Sp. If there are any permutations of this type, we map $Spin$ to Sp at least once. Let $G_i = Spin(n+1)$ and $G_{\sigma(i)} = Sp(n)$. Because the exponents of all the Adams operations which occur in the splitting of $\widetilde{\theta}$ are p-adic integers, the big diagram of the proof of lemma 3.2 leads to

$$\begin{array}{ccccc} K(B\overline{G}_i)\otimes\mathbb{Z}_2^\wedge & \xrightarrow{\phi_i} & K(BSpin(2n+1))\otimes\mathbb{Z}_2^\wedge & \xrightarrow{\beta_\sigma} & K(Sp(n))\otimes\mathbb{Z}_2^\wedge \\ \downarrow & & & & \uparrow \\ K(BG)\otimes\mathbb{Z}_2^\wedge & \xrightarrow{\theta} & K(BG)\otimes\mathbb{Z}_2^\wedge & \longrightarrow & K(B\widetilde{G})\otimes\mathbb{Z}_2^\wedge \end{array} .$$

Outer automorphism don't occur. For $\overline{G}_i = SO(2n+1)$, $\beta_\sigma \circ \phi_i$ is a λ-ring isomorphism but not a real λ-ring map. This is a contradiction, because θ is a real λ-ring map by assumption. For $\overline{G}_i = Spin(2n+1)$, $\phi_i = id$ and $det(\beta_\sigma) = det(\beta_\sigma \circ \phi_i) = 1/2 \notin \mathbb{Z}_2^\wedge$, which is also a contradiction. Thus σ permutes only factors of the same 2-adic type and is induced by a topological map.

Now we choose a finite cover $\rho : T \longrightarrow T$ of T such that $A_T \circ B\rho^*$ has only p-adic integers as matrix entries, and is realizable. Using the finite universal covering $q_1 := (id \times \rho) \circ q_2$, we can construct a map $\widetilde{f} : B\widetilde{G}_p^\wedge \longrightarrow B\widetilde{G}_p^\wedge$ which fits into a commutative diagram

$$\begin{array}{ccc} K(B\widetilde{G})\otimes\mathbb{Z}_p^\wedge & \xrightarrow{\widetilde{f}^*} & K(B\widetilde{G})\otimes\mathbb{Z}_p^\wedge \\ {\scriptstyle Bq_1^*}\uparrow & & {\scriptstyle Bq_2^*}\uparrow \\ K(BG)\otimes\mathbb{Z}_p^\wedge & \xrightarrow{\theta} & K(BG)\otimes\mathbb{Z}_p^\wedge \end{array} .$$

The map $Bq_2 \circ \widetilde{f}|_{BK}$, where K is the kernel of q_1, induces the trivial map in K-theory and therefore, is homotopically trivial [Z]. By proposition 1.1, $Bq_1 \circ \widetilde{f}$ factors over Bq_2 and induces a map $f : BG_p^\wedge \longrightarrow BG_p^\wedge$, such that $f^* = \theta$. □

Theorem 1.5 is included in the following statement:

3.5 Corollary. *Let R be $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$. Every real λ-ring isomorphism*

$$\theta : K(BG)\otimes R \longrightarrow K(BG)\otimes R$$

is induced from a topological map $f : BG_R \longrightarrow BG_R$.

BG_R means BG localized or completed, and $BG_{\mathbb{Z}} = BG$.

Proof. We desribe the proof only for $R = \mathbb{Z}$. Tensoring with $\mathbb{Z}_p^\wedge$ or $\mathbb{Q}$ yields λ-ring isomorphisms

$$\theta_p^\wedge : K(BG)\otimes\mathbb{Z}_p^\wedge \longrightarrow K(BG)\otimes\mathbb{Z}_p^\wedge$$

for all primes p and

$$\theta_{\mathbb{Q}} : K(BG)\otimes\mathbb{Q} \longrightarrow K(BG)\otimes\mathbb{Q} .$$

$\theta_p^\wedge$ can be realized by propositiobn 5.2 and $\theta_{\mathbb{Q}}$ can be realized because $BG_{\mathbb{Q}}$ is a product of Eilenberg-MacLane spaces. Glueing them together with the arithmetic square, we get a map $f : BG \longrightarrow BG$ as desired. □

3.6 Remark. If the finite universal cover $\widetilde{G}\longrightarrow G$ of G does not contain both, $Sp(n)$ and $Spin(2n+1)$, for any $n \geq 3$, then there are no problems with the permutation in 3.2 and 3.3. In this case or if we know that Sp and $Spin$ are not permuted, we do not have to assume anything about KO-theory.

4. Homotopy classification of rational self equivalences of BG.

Let $f, g : BG\longrightarrow BG$ be two rational self equivalences of BG, where G is a compact connected Lie group, i.e. f and g induce isomorphisms in rational cohomology. In view of corollary 2.3, we can choose two finite universal covers $q_1, q_2 : \widetilde{G}\longrightarrow G$ independently of f and g, such that f and g lift to rational equivalences $\widetilde{f}, \widetilde{g} : B\widetilde{G}\longrightarrow B\widetilde{G}$. Moreover, f and g are homotopic if and only if $\widetilde{f}$ and $\widetilde{g}$ are homotopic.

$\widetilde{G} = G_1 \times \ldots \times G_r \times T$ is a product of simple simply connected Lie groups G_i and a torus T. In section 3, we described how $\widetilde{f}$ and $\widetilde{g}$ look like in terms of rational cohomology. If f and g induce the same map in K-theory, then they also do it in rational cohomology, as do $\widetilde{f}$ and $\widetilde{g}$. This means that in rational cohomology or K–theory both, $\widetilde{f}$ and $\widetilde{g}$, look like

$$(\sigma \circ \bigotimes_i (\psi^{a_i} \circ B\alpha_i \circ \varepsilon^{k_i})) \otimes A_T \ .$$

The notation is as in section 3.

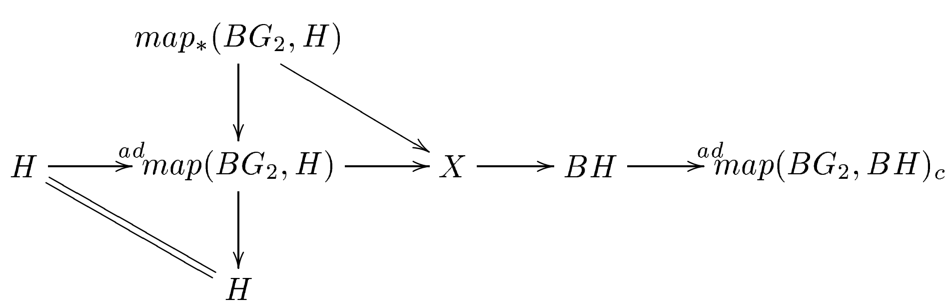
Because $\widetilde{f}$ is a map over the integers, σ can only permute factors of the same integral type. In particuliar σ is realizable by a map, which we also call σ. To show that $\widetilde{f}$ and $\widetilde{g}$ are homotopic it is sufficient to consider $(\sigma^{-1} \times id) \circ \widetilde{f}$ and $(\sigma^{-1} \times id) \circ \widetilde{g}$. Both of these maps look in rational cohomology like a product of maps. The main step in the proof of theorem 1.7 is to show that both these maps split topologically into a product of maps.

4.1 Lemma. *If G_1, G_2 and H are compact connected Lie groups and $c : BG_2\longrightarrow BH$ is the constant map, the map*

$$ad_* : map(BG_1, BH)\longrightarrow map(BG_1, map(BG_2, BH)_c)$$

is a mod-p equivalence for every prime p.

Proof. We introduce the following fiber sequences



where X is the homotopy fiber of ad, $map_*(\ , \ )$ denotes the mapping space of pointed maps and e is the evaluation map. In this situation ad is nothing but the right inverse of the evaluation map. Therefore $map_*(BG_2, H)\longrightarrow X$ is an equivalence.

The homotopy groups of $map_*(BG_2, H)$ are computed in [F–M]. The result is that X

is a product of Eilenberg-McLane spaces and that all homotopy groups are rational vector spaces, concentrated only in even dimensions. Hence $BH \longrightarrow map(BG_2, BH)_c$ is a mod-p equivalence for all primes p.

Let us look at the lifting problem

$$\begin{array}{ccc} & & BH \\ & \nearrow & \downarrow \\ BG_1 & \longrightarrow & map(BG_2, BH) \end{array} \quad .$$

Since the obstruction groups $H^{*+1}(BG_1; \pi_*(X))$ vanish, ad_* is a π_0-surjection.

To prove that ad_* is a π_0-injection, we use the diagram

$$\begin{array}{ccc} [BG_1, BH] & \longrightarrow & [BG_1, BH^\wedge] \\ \downarrow & & \downarrow \\ [BG_1, map(BG_2, BH)_c] & \longrightarrow & [BG_1, map(BG_2, BH)_c^\wedge] \end{array} \ .$$

By obstruction theory the upper horizontal arrow is injective, and the right column is a bijection.

Now, we can argue anologously as in the proof of proposition 1.4 and get

$$map(BG_1, BH)_{f}{}^\wedge_p \simeq map(BG_1, map(BG_2, BH)_c)_{ad^*(f)}{}^\wedge_p \ .$$

This completes the proof. □

For a map $f : BG_1 \times BG_2 \longrightarrow BH_1 \times BH_2$ we define the maps f_{kl} to be the canonical compositions

$$f_{kl} : BG_k \xrightarrow{i_k} BG_1 \times BG_2 \longrightarrow BH_1 \times BH_2 \xrightarrow{pr_l} BH_l \ .$$

4.2 Lemma. *If for $f : BG_1 \times BG_2 \longrightarrow BH_1 \times BH_2$ the maps f_{12} and f_{21} are homotopically trivial, then $f \simeq f_{11} \times f_{22}$, and*

$$map(BG_1 \times BG_2, BH_1 \times BH_2)_f \longrightarrow map(BG_1, BH_1)_{f_{11}} \times map(BG_2, BH_2)_{f_{22}}$$

is a mod-p equivalence for every prime p.

Proof. For the first statement we have to check the equivalence of $pr_k \circ f$ and $pr_k \circ (f_{11} \times f_{22})$, $k = 1, 2$. $pr_1 \circ f \circ i_2$ is homotopically trivial. Thus, by lemma 4.1, we get $pr_1 \circ f \simeq ad_*(f_{11}) \simeq pr_1 \circ (f_{11} \times f_{22})$, and the analogous for $pr_2 \circ f$.

Also, by lemma 4.1, $map(BG_1, BH_1)_{f_{11}} \longrightarrow map(BG_1, map(BG_2, BH_1)_{pr_1 \circ f}$ is a mod-p equivalence. This establishes the second statement. □

We can now complete the proof of theorem 1.7.

Proof of theorem 1.7. Let $f, g \in HE_{\mathbb{Q}}(BG)$ be two rational self equivalences with $K(f) = K(g)$. At the beginning of this section we chose finite universal coverings $q_1, q_2 : \widetilde{G} = \prod G_i \times T \longrightarrow G$ of G and found maps $\widetilde{f}, \widetilde{g} \in HE_{\mathbb{Q}}(B\widetilde{G})$ which are lifts of f and g. $\widetilde{f}$ and $\widetilde{g}$ contain the same permutation σ. This implies that $f \simeq g$ iff $\widetilde{f}_\sigma := (\sigma^{-1} \times id) \circ \widetilde{f} \simeq (\sigma^{-1} \times id) \circ \widetilde{g} =: \widetilde{g}_\sigma$. In K-theory both maps look like a tensor product of maps. A map between classifying spaces is homotopically trivial if and only if the induced map in K-theory is trivial [Z]. Hence, by corollary 4.2, $\widetilde{f}_\sigma \simeq \prod_i f \times f_T$ and $\widetilde{g}_\sigma \simeq \prod g_i \times g_T$ split into a product of maps $f_i, g_i : BG_i \longrightarrow BG_i$ and $f_T, g_T : BT \longrightarrow BT$. All factors look equally in K-theory. Because G_i is simple, the maps must be equivalent, by theorem 1.6. This shows that $f \simeq g$ and finishes the proof of the first part.

Using theorem 1.6 and corollary 4.2 again we get a sequence of equivalences or mod-p equivalences

$$
\begin{gathered}
BC(\widetilde{G}) \longrightarrow \prod BC(G_i) \times BT \longrightarrow \prod map(BG_i, BG_i)_{id} \times map(BT, BT)_{id} \longrightarrow \\
\prod map(BG_i, BG_i)_{f_i} \times map(BT, BT)_{f_T} \longrightarrow map(B\widetilde{G}, B\widetilde{G})_{\widetilde{f}_\sigma} \longrightarrow map(B\widetilde{G}, B\widetilde{G})_{\widetilde{f}}\ ,
\end{gathered}
$$

which fit into a commutative diagram

$$
\begin{array}{ccccc}
BC(\widetilde{G}) & \longrightarrow & map(B\widetilde{G}, B\widetilde{G})_{id} & \longrightarrow & map(B\widetilde{G}, B\widetilde{G})_{\widetilde{f}} \\
 & & & & \big\downarrow {\scriptstyle Bq_{2*}} \\
{\scriptstyle Bq_2|_{C(\widetilde{G})}} \big\downarrow & & & & map(B\widetilde{G}, BG)_{Bq_2 \circ \widetilde{f}} \\
 & & & & \big\uparrow {\scriptstyle Bq_1^*} \\
BC(G) & \longrightarrow & map(BG, BG)_{id} & \longrightarrow & map(BG, BG)_f
\end{array}
\quad .
$$

Let K be the kernel of q_2. By proposition 1.1 Bq_1^* is an equivalence. BK is the fiber of Bq_2 and, by proposition 1.2, the fiber of Bq_{2*}. This shows that the bottom arrow is a mod-p equivalence for all primes p. □

5. Fake Lie Groups and the Genus of BG.

First we proof theorem 1.8.

Proof of theorem 1.8. Let $\theta : K(Y_2) \longrightarrow K(Y_1)$ be a real λ–ring isomorphism, where Y_1 and Y_2 are in the adic genus of BG. Because Y_1 and Y_2 are Eilenberg-McLane spaces over the rationals, $\theta \otimes \mathbb{Q}$ can be realized by a map $f_{\mathbb{Q}} : Y_{1\mathbb{Q}} \longrightarrow Y_{2\mathbb{Q}}$. For every p we choose homotopy equivalences $g_1, g_2 : BG_p^\wedge \longrightarrow Y_{1,2}$. Then

$$K(g_1) \circ \theta \otimes \mathbb{Z}_p^\wedge \circ K({g_2}^{-1}) : K(BG) \otimes \mathbb{Z}_p^\wedge \longrightarrow K(BG) \otimes \mathbb{Z}_p^\wedge$$

is a real λ-ring isomorphism and can be realized by a topological map. This gives a realization $f_p^\wedge : {Y_1}_p^\wedge \longrightarrow {Y_2}_p^\wedge$ of $\theta \otimes \mathbb{Z}_p^\wedge$. The rational and the p–adic realization, $f_{\mathbb{Q}}$ and $f_p^\wedge$, must agree over $\mathbb{Q}_p^\wedge$. So the arithmetic square produces a map $f : Y_1 \longrightarrow Y_2$ with $K(f) = \theta$.

The next lemma will show that the maps $f_p^\wedge$ are homotopy equivalences, and hence that f is a homotopy equivalence, which completes the proof. □

5.1 Lemma. *A map $g : BG_p^\wedge \longrightarrow BG_p^\wedge$ is a homotopy equivalence if and only if*

$$K(g) : K(BG) \otimes \mathbb{Z}_p^\wedge \longrightarrow K(BG) \otimes \mathbb{Z}_p^\wedge$$

is an isomorphism.

Proof. Let $g : BG_p^\wedge \longrightarrow BG_p^\wedge$ be a map which induces an isomorphism in K-theory. Analogously to the second proof of [N-S; 2.1], there exists a self homotopy equivalence $g_T : BT_{G_p^\wedge} \longrightarrow BT_{G_p^\wedge}$ of the maximal torus of G and a commutative diagram

$$\begin{array}{ccccc} G/T_{G_p^\wedge} & \longrightarrow & BT_{G_p^\wedge} & \longrightarrow & BG_p^\wedge \\ {\scriptstyle g_{G/T}}\big\downarrow & & {\scriptstyle g_T}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ G/T_{G_p^\wedge} & \longrightarrow & BT_{G_p^\wedge} & \longrightarrow & BG_p^\wedge \ . \end{array}$$

In [N-S 1; 2.1] it is assumed that g is a homotopy equivalence, but the proof works also in the present case; it is based on depends only on K-theory information.

$H^*(g_{G/T}; \mathbb{Q}_p^\wedge)$ is an isomorhism, because g is an equivalence over $\mathbb{Q}_p^\wedge$. $H^*(G/T_G; \mathbb{Z}_p^\wedge)$ is torsion free. Therefore the kernel of $H^*(g_{G/T}; \mathbb{Z}_p^\wedge)$ is trivial. Now let us look at the diagram

$$\begin{array}{ccccccc} H^*(BT_{G_p^\wedge}; \mathbb{Z}_p^\wedge) & \longrightarrow & H^*(G/T_G; \mathbb{Z}_p^\wedge) & \longrightarrow & coker & \longrightarrow & 1 \\ {\scriptstyle g_T^*}\big\downarrow & & {\scriptstyle g_{G/T}^*}\big\downarrow & & {\scriptstyle \overline{g}^*}\big\downarrow & & \\ H^*(BT_G; \mathbb{Z}_p^\wedge) & \longrightarrow & H^*(G/T_G; \mathbb{Z}_p^\wedge) & \longrightarrow & coker & \longrightarrow & 1 \ . \end{array}$$

coker is finite, because $G/T_{G_p^\wedge} \longrightarrow BT_{G_p^\wedge}$ induces a surjection in rational cohomology. Since $g_{G/T}^*$ is one to one, $\overline{g}^*$ an injection . This implies that $\overline{g}^*$ is an isomorphism as well as $g_{G/T}^*$. Thus, $g_{G/T}$ and g are homotopy equivalences. $\square$

To study fake Lie groups and to distinguish them from the genuine article, the concept of a maximal torus is used in [N-S 1,2,3]. We will recall the definitions and some of the results. For $R = \mathbb{Z}_{(p)}, (\mathbb{Z}_p^\wedge)$ we denote by X_R the p-localisation or p-completion of X, for $R = \mathbb{Z}$, $X_R := X$.

Definition. Let X be a fake Lie group of type G. A maximal torus over R is a map

$$f : (BT_X)_R \longrightarrow BX_R \ ,$$

where T_X is a torus, such that

(1) $rank(T_X) \ = \ rank(G)$

(2) the homology $H^*(hofib(f); R)$ is a finitely generated R-module.

The set of homotopy classes of maps $W_X := \big\{[w : BT_R \to BT_R] \mid f \circ w \simeq f\big\}$ is called the Weyl group of X over R.

Remark. This definition is slightly different from the one in [N–S 1]. Instead of (2), a sharper condition is used. For $R = \mathbb{Z}$ it is supposed that $hofib(f)$ has the homotopy type of a finite CW-complex. Also it is assumed that BX is in the genus of BG, not in the adic genus. The genus is defined using localisations instead of completions. But it turns out that even under these weaker conditions, all the statements of [N-S 1] remain true, as the proofs show. Using [N–S; 2.1,3.9], one can show that $hofib(f)$ is simply connected. Hence $hofib(f)$ has the homotopy type of a finite CW-complex for $R = \mathbb{Z}$.

Now we will compare the adic genus and the normal genus of BG. Examples, due to A.K.Bousfield and G.Mislin (see [N–S 2,appendix]), show that these two genera are different. But we can offer

5.2 Proposition. *Let BX be in the adic genus of BG. The following statements are equivalent:*

(1) *For every prime p, BX has p–locally a maximal torus $(BT_X)_{(p)} \longrightarrow BX_{(p)}$.*
(2) *BX is in the genus of BG.*

Proof. Let BX be in the adic genus of BG, and let $(BT_X)_{(p)} \longrightarrow BX_{(p)}$ be a maximal torus of X over $\mathbb{Z}_{(p)}$ with Weyl group W_X. By [N–S; 2.1] these maps fit into a commutative diagram

$$\begin{array}{ccc} BT_{X}{}_p^\wedge & \longrightarrow & BT_{G}{}_p^\wedge \\ \downarrow & & \downarrow \\ BX_p^\wedge & \xrightarrow{\overline{g}} & BG_p^\wedge \end{array} ,$$

where the right column is given by a classical maximal torus $T_G \longrightarrow G$. The rows are equivalences. The upper row induces a conjugation of the two representations

$$W_X, W_G \longrightarrow Gl(n, \mathbb{Z}_p^\wedge) ,$$

$n = rank(G)$, given by the action of W_X and W_G on the 2–dimesional (p)-local cohomology of the maximal tori. This conjugation establishes a homomorphism $\alpha : W_X \longrightarrow W_G$. By [C–R; 30.17] α can be represented by a conjugation in $GL(n, \mathbb{Z}_{(p)})$. That is to say that there exists a p-local equivalence $g' : BT_{(p)} \longrightarrow BT_{(p)}$ which is equivariant with respect to α. Because all the spaces are rational Eilenberg-MacLane spaces, this produces an diagram

$$\begin{array}{ccc} BT_{\mathbb{Q}} & \xrightarrow{g'} & BT_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ BX_{\mathbb{Q}} & \xrightarrow{\overline{g}'} & BG_{\mathbb{Q}} \end{array} ,$$

where the rows are equivalences. The left column is a maximal torus of X over $\mathbb{Q}$, whose Weyl group is isomorphic to WX [N–S; 3.6].

Of course, we will use an arithmetic square argument to construct a p-local equivalence. But the maps $\overline{g}$ and $\overline{g}'$ may not be homotopic over $\mathbb{Q}_p^\wedge$. We have to change the map $\overline{g}$. To do this we take the map $h := g' \circ g^{-1} : BT_{G}{}_p^\wedge \longrightarrow BT_{G}{}_p^\wedge$, which is equivariant with respect to the identity on WG. Restricting this map to the invariants of the WG-action on $K(BT_G) \otimes \mathbb{Z}_p^\wedge$, we get a λ-ring isomorphism $\theta : K(BG) \otimes \mathbb{Z}_p^\wedge \longrightarrow K(BG) \otimes \mathbb{Z}_p^\wedge$.

If we pass to to a finite universal cover of G, $\widetilde{\theta}$ still commutes with the Weyl group (notation as in section 5) and hence cannot contain any permutation. By remark 3.6 and theorem 5.3, θ is realizable by a map $\overline{h} : BG_p^\wedge \longrightarrow BG_p^\wedge$. The composition $\overline{h} \circ \overline{g}$ agrees with $\overline{g}'$ over $\mathbb{Q}_p^\wedge$. The arithmetic square argument is now applicable and establishes an equivalence $BX_{(p)} \simeq BG_{(p)}$.

If BX is in the genus of BG, then there is nothing to prove. $\square$

References.

[A-M] J.F. Adams and Z. Mahmud. Maps between classifying spaces, Inv. Math. 35 (1976), 1-41.

[B-K] A.K. Bousfield and D.M. Kan. Homotopy limits,completions and localizations, LNMS 304, Springer 1972.

[B-tD] T. Bröcker and T. tom Dieck. Representations of compact Lie groups, Springer 1985.

[C-R] C.W. Curtis and I. Reiner. Methods of representation theory I, II, Wiley, New York 1984,1986.

[D-W] W.G.Dwyer and C.W.Wilkerson, Spaces of null homotopic maps Astérisque 66, (1991), 97-108.

[D-Z] W.G. Dwyer and A. Zabrodsky. Maps between classifying spaces, Proc. Barcelona Conf., SLNM 1298, 106-119.

[F] E.M. Friedlander. Exeptional isogeniesand the classifying space of simple Lie groups, Ann. Math. 101 (1975), 510-520.

[F-M] E.Friedlander and G.Mislin, Locally finite approximation of Lie groups I, Inv. Math. 83 (1986), 425-436.

[H] J. Hubbuck. Mapping degrees for classifying spaces I, Quart. J. Math. Oxford Ser. (2) 25 (1974),113-133.

[I] K. Ishiguro. Unstable Adams operations on classifying spaces, Math. Proc. Camb. Phil. Soc. 102 (1987), 71-75.

[J-M-O 1] S. Jackowski, J.McClure, and R.Oliver. Homotopy classification of self maps of BG via G actions, Part I and II, Ann. Math. 135 (1992), 183-226, 227-270.

[J-M-O 2] S. Jackowski, J.McClure, and R.Oliver. Self homotopy equivalences of BG, Preprint.

[M] H.R.Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. 120 (1984), 39-87.

[N] D. Notbohm. Maps between classifying spaces, Math. Z. 207 (1991) 153-168.

[N-S 1,2,3] D. Notbohm and L. Smith. Fake Lie groups and maximal tori I, II, III, Math. Ann. 288 (1990), 637-661, 663-673, Math. Ann. 290 (1991), 629-641.

[N-S 4] D. Notbohm and L. Smith. Rational homotopy of the space of homotopy equivalences of a flag manifold, Proccedings of the Barcelona Conference, SLNM 1509, 301-312.

[P] S. Papadima. Rigidity properties of compact Lie groups modulo maximal tori, Math. Ann. 275 (1986), 637-652.

[S] D. Sullivan. Genetics of homotopy theory and the Adams conjecture, Ann. Math. 100 (1974),1-79.

[T] R. Thom. L'homologie des espaces fonctionelle, Colloque de Topologie algébrique, Louvain 1957.

[W 1] C.W. Wilkerson. Self maps of classifying spaces, localisations in group theory and homotopy theory, LNMS 118, Springer 1974, 150-157.

[W 2] C.W. Wilkerson. Lambda rings, binomial domains and vector bundles over $CP(\infty)$, Comm. Alg. 10 (1982), 311-328.

[Wo] Z. Wojtkowiak. Maps between p-completed classifying spaces II,III Proc. Royal Soc. Edin. 118A (1991), 133-137, Preprint.

[Z] A. Zabrodsky. Maps between classifying spaces, Algebraic topology and algebraic K-theory, Annals of Math. Studies 113, Princeton 1987, 228-246.

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