HOMOLOGY DECOMPOSITIONS FOR p-COMPACT GROUPS

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Abstract. We construct a homotopy theoretic setup for homology decompositions of classifying spaces of p-compact groups. This setup is then used to obtain a subgroup decomposition for p-compact groups which generalizes the subgroup decomposition with respect to p-stubborn subgroups for a compact Lie group constructed by Jackowski, McClure and Oliver.

Homology decompositions are among the most useful tools in the study of the homotopy theory of classifying spaces. Roughly speaking, a homology decomposition for a space X, with respect to some homology theory h_*, is a recipe for gluing together spaces, desirably of a simpler homotopy type, such that the resulting space maps into X by a map which induces an h_*-isomorphism.

When constructing a homology decomposition for a classifying space of a group G, it is natural to do so using classifying spaces of subgroups of G. For compact Lie groups two types of mod-p homology decompositions are known: the centralizer decomposition with respect to elementary abelian p-subgroups, due to Jackowski and McClure [JM], and the subgroup decomposition with respect to certain families of p-toral subgroups, due to Jackowski, McClure and Oliver [JMO].

A *p-compact group* is an $\mathbb{F}_p$-finite loop space X (i.e., a loop space whose mod-p homology is finite), whose classifying space BX is p-complete in the sense of [BK]. These objects, defined by Dwyer and Wilkerson [DW1], and extensively studied by them and others, are a far reaching homotopy theoretic generalization of compact Lie groups and their classifying spaces. Dwyer and Wilkerson also introduced in [DW2] a centralizer decomposition with respect to elementary abelian p-subgroups for p-compact groups, which generalizes the corresponding decomposition for compact Lie groups. The aim of this paper is to construct a subgroup decomposition for p-compact groups, analogous to the subgroup decomposition for compact Lie groups introduced by Jackowski, McClure and Oliver in [JMO]. We will in fact show that in the right setup, the Dwyer-Wilkerson theorem about existence of a centralizer decomposition for p-compact groups, with respect to their elementary abelian p-subgroups, implies the existence of subgroup decompositions with respect to certain other families of subgroups. Interestingly, as we will show, the opposite implication holds as well. More detail will be given shortly.

We start by explaining some of the concepts involved. A homomorphism between p-compact groups is a pointed map $\alpha : BY \longrightarrow BX$. A subgroup of a p-compact group X is a pair (Y, α) where Y is a p-compact group and $\alpha : BY \longrightarrow BX$ is a *monomorphism*, namely, a pointed map whose homotopy fibre is $\mathbb{F}_p$-finite. The phrase "(Y, α) is a subgroup of X" will frequently be abbreviated by $Y \leq_\alpha X$. [▶ 1] A p-compact torus

1991 *Mathematics Subject Classification.* Primary 55R35. Secondary 55R40, 20D20.

Key words and phrases. Classifying space, p-compact groups, Homology decompositions.

N.Castellana was partially supported by EPSRC grant GR/M7831 and MEC grant MTM2004-06686.

R. Levi is partially supported by EPSRC grants GR/M7831 and GR/N20218.

is a topological group of type $K(A, 1)$, where A is isomorphic to a finite product of copies of the p-adic integers, i.e. the p-completion of an ordinary torus. A p-compact toral group is a group containing a p-compact torus as a normal subgroup with p-power index. Every p-compact group admits a distinguished family of p-compact toral subgroups (S, ι), which are maximal in the sense that if (P, β) is any other p-compact toral subgroup of X, then there exists a map $f\colon BP \longrightarrow BS$, such that $\iota \circ f \simeq \beta$. Any such subgroup will be called a Sylow subgroup of X (see Definition A.7 and the following discussion).

For any p-compact group X, we consider two categories: the orbit category $\mathcal{O}(X)$ and the fusion category $\mathcal{F}(X)$. The objects in both categories are given by all subgroups (Y, α) of X. A morphism $(Y, \alpha) \longrightarrow (Y', \alpha')$ in $\mathcal{O}(X)$ is a homotopy class of [▶ 2] maps $h\colon BY \longrightarrow BY'$ such that $\alpha' \circ h \simeq \alpha$, whereas in $\mathcal{F}(X)$ such a morphism is a pointed homotopy class of a homomorphism $f\colon BY \longrightarrow BY'$ such that $\alpha' \circ f$ is freely homotopic to α.

For any p-compact group X, we consider certain full subcategories of $\mathcal{O}(X)$ and $\mathcal{F}(X)$, where the objects are restricted to particular collections of subgroups, defined by certain properties:

- A subgroup $Y \leq_\alpha X$ is said to be *centric* if the homotopy fibre of the natural map
$$\alpha_\#\colon \operatorname{Map}(BY, BY)_{id} \longrightarrow \operatorname{Map}(BY, BX)_\alpha$$
is weakly contractible.
- A p-compact toral subgroup $Y \leq_\alpha X$ of a p-compact group X is said to be *radical* if it is centric and if $\operatorname{Aut}_{\mathcal{O}(X)}(Y, \alpha)$ is finite and contains no normal non-trivial p-subgroup (i.e., it is finite and p-reduced).

This notion of a radical subgroup does not coincide with the classical one in the context of finite groups, where one does not require the subgroup to be centric. When $\pi_0(X)$ is a p-group, one can in fact replace the requirement that $P \leq_\alpha X$ is centric in the definition of a radical subgroup by the condition that the homotopy fibre of $\alpha_\#$ is homotopically discrete. However, with this weaker condition one can prove that every radical subgroup is in fact centric, and so one does not [▶ 3] lose any generality by making the centricity requirement. For a p-compact group X, we denote by $\mathcal{O}_p^c(X)$ and $\mathcal{O}_p^r(X)$, the full subcategories of $\mathcal{O}(X)$ whose objects are the centric and radical p-compact toral subgroups, respectively. Similar notation will be used for the fusion category. These categories are not generally small, but have small skeletal subcategories (see Proposition 2.6), so defining limits and colimits over them makes sense. Let $\mathcal{C}$ a familiy of subgroups of X closed under the conjugation and denote by $\mathcal{O}^{\mathcal{C}}(X)$ the full subcategory of $\mathcal{O}(X)$ whose objects are those subgroups in $\mathcal{C}$.

Let Sp denote the category of spaces, and hoSp its homotopy category. Also denote by $\underline{\mathrm{BX}}$ the constant functor with value BX on objects and with value the identity on all morphisms. We are now ready to state our main theorem.

Theorem A. *[▶ 4] Let X be a p-compact group and $\mathcal{C}$ a collection of centric p-compact toral subgroups which contains all radical p-compact toral subgroups. Then, there exists a functor $\Phi\colon \mathcal{O}^{\mathcal{C}}(X) \longrightarrow \mathsf{Sp}$ and a natural transformation $\eta\colon \Phi \longrightarrow \underline{BX}$ such that, for each object (P, α) we have $\Phi(P, \alpha) \simeq BP$, $\eta(P, \alpha) \simeq \alpha$ and $\Phi([h]) \simeq h$ for each morphism $[h]$ in $\mathcal{O}^{\mathcal{C}}(X)$. And, the map induced by $\underline{BX}$*

$$\operatorname*{hocolim}_{\mathcal{O}^{\mathcal{C}}(X)} \Phi \longrightarrow BX$$

is a mod-p homology equivalence.

The strategy for proving Theorem A is based on two auxiliary results, Theorems C and D, which might be of independent interest, are stated and shown to imply the main theorem.

Let $\pi\colon \mathsf{Sp} \longrightarrow \mathsf{hoSp}$ denote the obvious projection functor. For any p-compact group X, there are functors

$$\phi\colon \mathcal{O}(X) \longrightarrow \mathsf{hoSp} \quad \text{and} \quad \psi\colon \mathcal{F}(X)^{op} \longrightarrow \mathsf{hoSp},$$

defined as follows. The functor ϕ sends a subgroup (Y,α) to BY and any morphism to the respective homotopy class. The functor ψ takes a subgroup (Y,α) to the mapping space $\mathrm{Map}(BY,BX)_\alpha$ and a morphism to the homotopy class of the map induced by any representative. For a subgroup (Y,α) of X, we denote $\mathrm{Map}(BY,BX)_\alpha$ by $BC_X(Y,\alpha)$ or $BC_X(Y)$ for short, if no ambiguity can arise. The associated loop space $C_X(Y)$ is called the *centralizer* of (Y,α) in X. Dwyer and Wilkerson showed in [DW1, Propositions 5.1, 5.2] that if (P,α) is a p-compact toral subgroup of X, then $C_X(P)$ is a p-compact group and that the evaluation map $BC_X(P) \xrightarrow{ev} BX$ is a monomorphism. Thus, the pair $(C_X(P), ev)$ is a subgroup of X.

If $\mathcal{C}$ is a collection of p-compact toral subgroups of a p-compact group X, then we denote by $\mathcal{O}_\mathcal{C}(X)$ and $\mathcal{F}_\mathcal{C}(X)$ the full subcategories of $\mathcal{O}(X)$ and $\mathcal{F}(X)$, whose objects are the subgroups in $\mathcal{C}$. We denote by

$$\phi_\mathcal{C}\colon \mathcal{O}_\mathcal{C}(X) \longrightarrow \mathsf{hoSp} \quad \text{and} \quad \psi_\mathcal{C}\colon \mathcal{F}_\mathcal{C}(X)^{op} \longrightarrow \mathsf{hoSp},$$

the restriction of ϕ and ψ to the respective full subcategories.

Using the terminology introduced by Dwyer and Kan in [DK], a realization of the homotopy functor ϕ is a pair (Φ,γ), where $\Phi\colon \mathcal{O}(X) \longrightarrow \mathsf{Sp}$ is a functor and [▶ 5] γ is a natural isomorphism of functors $\pi\circ\Phi \xrightarrow{\gamma} \phi$. A realization of the homotopy functor ψ is defined analogously. Dwyer and Kan also define the notion of a weak equivalence between realizations (see Definition 1.1).

For any category $\mathcal{D}$ and a space Y, recall that $\underline{Y}\colon \mathcal{D} \longrightarrow \mathsf{Sp}$ denotes the constant functor with value Y. With this terminology, we can now define what we mean by subgroup and centralizer diagrams.

Definition 0.1. *Let $\mathcal{C}$ be a collection of subgroups of a p-compact group X.*

(i) *A subgroup diagram for X with respect to $\mathcal{C}$ is a triple $(\Phi_\mathcal{C},\gamma,\eta)$, where $(\Phi_\mathcal{C},\gamma)$ is a realization of $\phi_\mathcal{C}$, and $\eta\colon \Phi_\mathcal{C} \longrightarrow \underline{BX}$ is a natural transformation.*

(ii) *A centralizer diagram for X with respect to $\mathcal{C}$ is a triple $(\Psi_\mathcal{C},\delta,\zeta)$, where $(\Psi_\mathcal{C},\delta)$ is a realization of $\psi_\mathcal{C}$, and $\zeta\colon \Psi_\mathcal{C} \longrightarrow \underline{BX}$ is a natural transformation.*

Let $F\colon \mathcal{C} \longrightarrow \mathsf{Sp}$ be any functor, and assume a natural transformation $\alpha\colon F \longrightarrow \underline{Y}$ is given. Then one has a map $\operatorname{hocolim} F \longrightarrow Y$ given by the composite

$$\operatorname*{hocolim}_{\mathcal{C}} F \xrightarrow{\alpha_*} \operatorname*{hocolim}_{\mathcal{C}} \underline{Y} \simeq |\mathcal{C}| \times Y \xrightarrow{proj} Y.$$

In this way any subgroup diagram $(\Phi_\mathcal{C},\gamma,\eta)$ for X with respect to $\mathcal{C}$ gives rise to a map

$$\eta_*\colon \operatorname*{hocolim}_{\mathcal{O}_\mathcal{C}(X)} \Phi_\mathcal{C} \longrightarrow BX.$$

Similarly, a centralizer diagram $(\Psi_{\mathcal{C}}, \delta, \zeta)$ gives rise to a map

$$\zeta_*\colon \operatorname*{hocolim}_{\mathcal{F}_{\mathcal{C}}(X)^{op}} \Psi_{\mathcal{C}} \longrightarrow BX.$$

Generally the maps η_* and ζ_* are not guaranteed to have any good properties. Those diagrams for which these maps are well behaved are called decompositions. A precise definition is given next.

Definition 0.2. *We say that a subgroup diagram $(\Phi_{\mathcal{C}}, \gamma, \eta)$ (resp. centralizer diagram $(\Psi_{\mathcal{C}}, \delta, \zeta)$) is a subgroup (*resp. *centralizer) decomposition if the map η_* (resp. ζ_*) above induces a mod-p homology equivalence.*

A collection $\mathcal{C}$ of subgroups of a p-compact group X is called subgroup-ample if there exists a subgroup decomposition for X with respect to $\mathcal{C}$. Similarly $\mathcal{C}$ is said to be centralizer-ample if there exists a centralizer decomposition for X with respect to $\mathcal{C}$.

The claim of our main theorem thus amounts to saying that any p-compact group admits a subgroup decomposition with respect to the collection of its radical subgroups, or equivalently that the collection of the radical subgroups of a p-compact group X is subgroup-ample. Using this terminology, the Dwyer-Wilkerson theorem on homology decompositions for p-compact groups can be stated as claiming that for any p-compact group X, the collection of all its elementary abelian subgroups is centralizer-ample. The term "ample" is borrowed from [D], although there it refers only to a collection and not to the particular diagram. It is possible to show that if $\mathcal{C}$ is an arbitrary collection of subgroups of X, then $\mathcal{C}$ is centralizer-ample if and only if it is subgroup-ample. This would justify using the phrase "an ample collection" in the sense Dwyer does in [D], but we shall not discuss this terminology any further in this paper.

Next, we show that, for certain collections of subgroups, the existence and uniqueness of diagrams are guaranteed. Before we do so, notice that for discrete groups, pointed homotopy classes of maps between their classifying spaces correspond uniquely to homomorphisms. Thus, if $\mathcal{A}$ is a collection of discrete p-groups, then the functor $\psi_{\mathcal{A}}$ admits a canonical lift $\Psi_{\mathcal{A}}$ to the category of spaces.

Proposition B. *For any p-compact group X the following hold.*

(i) *If $\mathcal{C}$ is a collection of centric subgroups of X, then there exists a subgroup diagram $(\Phi_{\mathcal{C}}, \gamma, \eta)$ for X with respect to $\mathcal{C}$, which is unique up to a weak equivalence.*
(ii) *If $\mathcal{A}$ is a collection of finite abelian p-subgroups of X, then the triple $(\Psi_{\mathcal{A}}, Id, ev)$ is a centralizer diagram for X with respect to $\mathcal{A}$, which is unique up to a weak equivalence.*

Remark 0.3. In particular, notice that the uniqueness part of Proposition B implies that if a collection is subgroup ample, then any subgroup diagram with respect to this collection is a decomposition. A similar comment applies to centralizer diagrams.

The main theorem of this paper is the statement that for any p-compact group X, the collection of all its radical subgroups is subgroup ample. The first step in doing this is to show that this is in fact equivalent to the same statement where "radical" is replaced by "centric". More precisely:

Theorem C. *For any p-compact group X, the following statements are equivalent:*

(a) *The collection of all p-compact toral centric subgroups of X is subgroup ample.*
(b) *The collection of all p-compact toral radical subgroups of X is subgroup ample.*

Once this is proven, we proceed by proving the equivalence of another pair of statements of a rather different nature.

Theorem D. *The following statements are equivalent:*

(i) *For every p-compact group X the collection of all its non-trivial elementary abelian subgroups is centralizer ample.*
(ii) *For every p-compact group X the collection of all its centric p-compact toral subgroups is subgroup ample.*

Statement (i) of Theorem D is a theorem of Dwyer and Wilkerson [DW2, Theorem 8.1]. Thus, Theorems C and D imply Theorem A at once. Notice the difference between the two theorems: in Theorem C the conditions are stated for a given p-compact group, whereas in Theorem D they are stated for all p-compact groups. The reason for this difference is the different methods we employ in proving the two theorems. Theorem C is proved comparing homotopy colimits over the corresponding orbit categories, and the proof of Theorem D involves induction on the order of X (that is, number of components and cohomological dimension).

The rest of the paper is organized as follows. Section 1 contains the proof of Proposition B. Section 2 introduces discrete p-toral groups, and some basic properties. It also contains a discussion on the notions of the normalizer and Weyl spaces for subgroups of a p-compact group, as well as the Weyl group of a subgroup. The key properties of centric and radical subgroups are proven in Section 3. Section 4 is a study of the orbit category of radical subgroups. A slightly stronger form of Theorem C is shown in Section 5 (Proposition 5.1). The proof of Theorem D is contained in Section 6 (Proposition 6.4, again in a slightly stronger form). Background material needed along the paper is collected in Appendix A. In Appendix B we show that our decomposition theorem is indeed a generalization of the Jackowski-McClure-Oliver decomposition theorem.

We take the pleasure to thank Bob Oliver and Carles Broto for many useful conversations. We also thank the referee for doing a remarkably thorough job in refereeing this paper, proposing numerous simplifications, and correcting one of our statements. We would also like to thank the Max-Planck-Institut für Mathematik in Bonn, the Mittag-Leffler Institute, the Universities of Aberdeen and Leicester, and Universitat Autònoma de Barcelona for giving us several opportunities to meet. We also acknowledge support from EPSRC for grants obtained to partially fund this project.

1. Existence and uniqueness of subgroup and centralizer diagrams

This section is devoted to the proof of Proposition B. We first recall some terminology from [DK] that will be used in the proof.

Definition 1.1. *Let $\mathcal{D}$ be a category. If $\theta\colon \mathcal{D} \longrightarrow \mathsf{hoSp}$, then a realization of θ is a pair (Θ, γ), where $\Theta\colon \mathcal{D} \longrightarrow \mathsf{Sp}$ is a functor and $\pi \circ \Theta \xrightarrow{\gamma} \theta$ a natural isomorphism of functors. Two realizations (Θ, γ) and (Θ', γ') are weakly equivalent if there exists a natural transformation $\epsilon\colon \Theta \longrightarrow \Theta'$, which is a weak equivalence on each object $d \in \mathcal{D}$ and such that $\gamma' \circ \pi(\epsilon) = \gamma$.*

For a small category $\mathcal{D}$, a functor $\theta\colon \mathcal{D} \longrightarrow \mathsf{hoSp}$ is said to define a centric diagram over $\mathcal{D}$ if for every morphism $c \xrightarrow{f} d$ in $\mathcal{D}$, $\theta(f)$ is a the homotopy class of a centric map, namely, if for any representative f' for $\theta(f)$, the induced map

$$f'_{\#}\colon \operatorname{Map}(\theta(c), \theta(c))_{id} \longrightarrow \operatorname{Map}(\theta(c), \theta(d))_{\theta(f)}$$

is a weak equivalence.

If θ defines a centric diagram over $\mathcal{D}$, one has a sequence of functors $\theta_i\colon \mathcal{D}^{op} \longrightarrow \mathcal{A}b$ defined by

$$\theta_i(d) = \pi_i(\mathrm{Map}(\theta(d), \theta(d))_{id}),$$

on objects. [▶ 7] Note that $\theta_1(d)$ is always abelian since $\mathrm{Map}(\theta(d), \theta(d))_{id}$ is an H-space. Given a morphism $f : c \longrightarrow d$, the morphism $\theta_i(f)$ is defined as follows:

$$\theta_i(f)\colon \pi_i(\mathrm{Map}(\theta(d), \theta(d))_{id}) \longrightarrow \pi_i(\mathrm{Map}(\theta(c), \theta(d))_{\theta(f)}) \xleftarrow{\cong} \pi_i(\mathrm{Map}(\theta(c), \theta(c))_{id}).$$

By [DK, Theorem 1.1], if the groups $\varprojlim^j \theta_i$ vanish for all i and j, then there exists a realization Θ of θ which is unique up to weak equivalence.

Proof of Proposition B. Given a collection $\mathcal{C}$ of subgroups of a p-compact group X, define two enlarged collections: the collection $\mathcal{C}_1$ obtained by adding the subgroup $(X, 1_{BX})$ to $\mathcal{C}$, and the collection $\mathcal{C}_0$ obtained by adding the trivial subgroup $(\{1\}, *)$, where $*\colon B\{1\} \longrightarrow BX$ is the inclusion of the base point. Let $\iota_1\colon \mathcal{O}_{\mathcal{C}}(X) \longrightarrow \mathcal{O}_{\mathcal{C}_1}(X)$ and $\iota_0\colon \mathcal{F}_{\mathcal{C}}(X) \longrightarrow \mathcal{F}_{\mathcal{C}_0}(X)$ be the respective inclusion functors.

Remark 3.3 below implies that if $\mathcal{C}$ is a centric collection (i.e., a collection all of whose objects are centric) of p-compact toral subgroups of a p-compact group X, then $\phi_{\mathcal{C}}\colon \mathcal{O}_{\mathcal{C}}(X) \longrightarrow \mathsf{hoSp}$ defines a centric diagram. It is also immediate that the extended diagram defined by $\phi_{\mathcal{C}_1}$ is centric. The category $\mathcal{O}_{\mathcal{C}_1}(X)$ has a terminal object $(X, 1_{BX})$, and hence the higher limits of any contravariant functor from it to the category of abelian groups vanish. Thus by the Dwyer-Kan theorem stated above, a realization $(\Phi_{\mathcal{C}_1}, \gamma_1)$ of $\phi_{\mathcal{C}_1}$ exists and is unique up to weak equivalence.

Next, notice that since $(X, 1_{BX})$ is a terminal object in $\mathcal{O}_{\mathcal{C}_1}(X)$, the obvious map

$$\operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}_1}(X)} \Phi_{\mathcal{C}_1} \longrightarrow \Phi_{\mathcal{C}_1}(X, 1_{BX})$$

is a homotopy equivalence. Let $\rho_1\colon \pi \circ \Phi_{\mathcal{C}_1} \longrightarrow \phi_{\mathcal{C}_1}$ be a natural isomorphism. Then ρ_1 determines a homotopy class of a homotopy equivalence $\Phi_{\mathcal{C}_1}(X, 1_{BX}) \xrightarrow{\simeq} BX$. Fix a representative ι_X for this equivalence.

Let $\Phi_{\mathcal{C}}$ denote the composite

$$\mathcal{O}_{\mathcal{C}}(X) \xrightarrow{inc} \mathcal{O}_{\mathcal{C}_1}(X) \xrightarrow{\Phi_{\mathcal{C}_1}} \mathsf{Sp},$$

and let γ denote the restriction of γ_1 to $\Phi_{\mathcal{C}}$. Then $(\Phi_{\mathcal{C}}, \gamma)$ is clearly a realization of the homotopy functor $\phi_{\mathcal{C}}$ on $\mathcal{O}_{\mathcal{C}}(X)$. Let η denote the natural transformation defined by taking an object (P, α) of $\mathcal{O}_{\mathcal{C}}(X)$ to the composite

$$\Phi_{\mathcal{C}}(P, \alpha) = \Phi_{\mathcal{C}_1}(P, \alpha) \xrightarrow{\Phi_{\mathcal{C}_1}(\alpha)} \Phi_{\mathcal{C}_1}(X, 1_{BX}) \xrightarrow{\iota_X} BX.$$

Then the triple $(\Phi_{\mathcal{C}}, \gamma, \eta)$ is a subgroup diagram for X with respect to $\mathcal{C}$.

It is also clear that given a subgroup diagram, one can define a realization of ϕ_1. Therefore, we have shown that there a $1-1$ correspondence between equivalence classes of subgroup diagrams, and equivalence classes of realizations of ϕ_1 in the sense of Definition 1.1.

To prove (ii), notice that Remark 3.3, in conjunction with the fact that the centralizer in X of a p-compact toral subgroup is itself a p-compact group [DW1, Proposition 5.1], implies that if $\mathcal{A}$ is a collection of finite abelian p-subgroup of X, then the diagram defined by $\Psi_{\mathcal{A}}\colon \mathcal{F}_{\mathcal{A}}(X)^{op} \longrightarrow \mathsf{Sp}$ is centric [DW2, Lemma 11.15]. The evaluation map

$$\Psi_{\mathcal{A}}(A, \alpha) = \mathrm{Map}(BA, X)_{\alpha} \xrightarrow{ev} X$$

gives a natural transformation from $\Psi_{\mathcal{A}}$ to the constant functor $\underline{BX}$. Thus the triple $(\Psi_{\mathcal{A}}, Id, ev)$ is a centralizer diagram, which by an argument similar to the one given above, is unique up to weak equivalence. □

Proposition B can be stated in a more general context. Let $\mathcal{C}$ be a small category. Given a functor $F\colon \mathcal{C} \longrightarrow \mathsf{hoSp}$ and a natural transformation to the constant functor $\eta\colon F \longrightarrow \underline{Y}$ such that for each $f \in \mathrm{Mor}_{\mathcal{C}}(c,d)$ and $c \in \mathcal{C}$, the maps $F(f)\colon F(c) \longrightarrow F(d)$ and $\eta(c)\colon F(c) \longrightarrow Y$ are centric, then there exists a realization of F which is unique up to equivalence.

2. Discrete p-toral subgroups

One of the main objects of study in this paper is the orbit category of p-compact toral subgroups of a given p-compact group, and certain subcategories. We start by introducing some useful simplifications.

Two objects (Z,α) and (Z',α') of $\mathcal{O}(X)$ are said to be X-conjugate if they are isomorphic as objects of the orbit category. We shall use the same notion for isomorphic objects in any orbit category (e.g., of discrete p-toral subgroups, as defined below).

An essential ingredient in the study of p-compact groups is the concept of discrete p-toral groups, introduced by Dwyer-Wilkerson in [DW1] and used extensively in the literature.

Definition 2.1. *For a prime p, let $\mathbb{Z}/p^\infty$ denote the union $\cup_{n\geq 1}\mathbb{Z}/p^n$ under the obvious inclusions.*

(i) *A group T is said to be a discrete p-torus if T is isomorphic to $(\mathbb{Z}/p^\infty)^{\times n}$ for some n, which is called the* rank *of T.*
(ii) *A group P is called discrete p-toral if it contains a normal discrete p-torus of p-power index. The* dimension *of P is the rank of its maximal discrete p-torus.*
(iii) *By abuse of notation, if P is a discrete p-toral group with maximal torus T, then we denote by $\pi_0(P)$, the group P/T. The* order *of P is defined to be the pair* $(\dim(P), |\pi_0(P)|)$. *(Compare with Definition A.5)*

By [DW1, Proposition 6.9] every p-compact toral group P has a *discrete approximation*, namely, a discrete p-toral group $\check{P}$ together with a homomorphism $u\colon \check{P} \longrightarrow P$, which induces a mod-$p$ homology equivalence $Bu\colon B\check{P} \longrightarrow BP$. On the other hand, if Q is a p-discrete toral group, completion induces a mod-p equivalence $BQ \longrightarrow BQ^\wedge_p$ and $BQ^\wedge_p$ gives rise to p-compact toral group, which we denote by $\hat{Q}$. In particular, $B\hat{Q}$ and $BQ^\wedge_p$ denote the same classifying space.

For any two groups G,H, let $\mathrm{Rep}(G,H) \overset{\text{def}}{=} \mathrm{Hom}(G,H)/\mathrm{Inn}(H)$.

Lemma 2.2. *Let P and Q be p-compact toral groups and let $\check{P} \leq_{\iota_P} P$ and $\check{Q} \leq_{\iota_Q} Q$ be discrete approximations. Then one has induced maps*

$$\mathrm{Map}(B\check{P}, B\check{Q}) \xrightarrow{\iota_{Q\#}} \mathrm{Map}(B\check{P}, BQ) \xleftarrow{\iota_P^\#} \mathrm{Map}(BP, BQ),$$

and $\iota_{Q\#}$ is a mod p equivalence, while [▶ 9] $\iota_P^\#$ is a homotopy equivalence. In particular,

$$\mathrm{Rep}(\check{P},\check{Q}) \cong [B\check{P}, B\check{Q}] \cong [B\check{P}, BQ] \cong [BP, BQ].$$

Proof. That $\iota_P^\#$ is a homotopy equivalence follows since BQ is p-complete. The homotopy fibre of ι_Q is an Eilenberg-MacLane space of type $K(V,1)$, where V is a $\mathbb{Q}^\wedge_p$-vector space (see [DW1, Prop. 6.8]). For each homotopy class of maps $B\check{P} \xrightarrow{\alpha} BQ$, there is a fibration

$$BV^{h\check{P}} \longrightarrow \mathrm{Map}(B\check{P}, B\check{Q})_{\bar{\alpha}} \longrightarrow \mathrm{Map}(B\check{P}, BQ)_{\alpha}.$$

Since $\check{P}$ is discrete and V is a rational vector space, it follows that $BV^{h\check{P}}$ is again a space of type $K(U,1)$, where U is the invariant subspace in V under the $\check{P}$ action. In particular $BV^{h\check{P}}$ is connected and mod-p acyclic and so $\bar{\alpha}$ consists of a single component and

$$\mathrm{Map}(B\check{P}, B\check{Q})_{\bar{\alpha}} \xrightarrow{\iota_{Q\#}} \mathrm{Map}(B\check{P}, BQ)_{\alpha}$$

is a mod-p equivalence. In the second sequence of equivalences, the first is standard, and the others follow by taking components. □

The next definition gives a discrete analog for the concept of a Sylow subgroup.

Definition 2.3. *Let X be a p-compact group, let $\check{S}$ be a discrete p-toral group, and let $\check{\iota}\colon B\check{S} \longrightarrow BX$ be a map, which is a monomorphism of p-compact groups upon p-completion. Then $(\check{S}, \check{\iota})$ is said to be a discrete p-toral subgroup of X. The subgroup $(\check{S}, \check{\iota})$ is said to be a discrete Sylow subgroup of X if $\iota = \check{\iota}^{\wedge}_p\colon B\check{S}^{\wedge}_p \longrightarrow BX$ is a Sylow subgroup for X.*

Clearly, if $(\check{S}, \check{\iota})$ is a discrete Sylow subgroup for X, then any other discrete p-toral subgroup of X factors through $(\check{S}, \check{\iota})$ up to homotopy. Thus a discrete Sylow subgroup of X is unique up to X-conjugacy. It is also clear that if $\check{P} \xrightarrow{\phi} \check{S}$ is a monomorphism of discrete p-toral groups, then $(\check{P}, \check{\iota} \circ B\phi)$ is a discrete p-toral subgroup of X.

Proposition 2.4. *Let X be a p-compact group with a Sylow subgroup $S \leq_{\iota} X$, and let $P \leq S$ be a p-compact toral subgroup. Let $\check{P} \leq \check{S}$ be discrete approximations of $P \leq S$. Then $C_{\check{S}}(\check{P})$ is a discrete p-toral subgroup of $C_X(P)$. Furthermore, there exists a subgroup $\check{P}' \leq \check{S}$, X-conjugate to $\check{P}$, such that $C_{\check{S}}(\check{P}')$ is a discrete Sylow subgroup of $C_X(P)$.*

Proof. Let $\check{\iota}\colon B\check{S} \longrightarrow BX$ denote the map induced by inclusion followed by ι. Then $\check{\iota}$ induces a map

$$\mathrm{Map}(B\check{P}, B\check{S})_{inc} \xrightarrow{\check{\iota}_{\#}} \mathrm{Map}(B\check{P}, BX)_{\check{\iota}\circ inc} \simeq \mathrm{Map}(BP, BX)_{\iota_{|BP}},$$

where the last equivalence follows from [DW1, Prop 6.8], and the diagram

$$\begin{array}{ccc} \mathrm{Map}(B\check{P}, B\check{S})_{inc} & \xrightarrow{\check{\iota}_{\#}} & \mathrm{Map}(BP, BX)_{\iota_{|BP}} \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev} \\ B\check{S} & \xrightarrow{\quad\check{\iota}\quad} & BX \end{array}$$

commutes. Since both $\check{\iota}$ and the evaluation map on the left column are monomorphisms (the first is ι upon completion, and the second is induced by an inclusion of groups), their composite is a monomorphism, and hence $BC_{\check{S}}(\check{P}) = \mathrm{Map}(B\check{P}, B\check{S})_{inc}$ is a discrete p-toral subgroup of $BC_X(P) = \mathrm{Map}(BP, BX)_{\iota_{|BP}}$ by Lemma A.2(i).

To prove the second statement, let $Q \leq C_X(P)$ be a Sylow subgroup. Then, by adjointness, one has a map $BQ \times BP \longrightarrow BX$, and since (S, ι) is Sylow in X, this map factors up to homotopy through BS. Let $\check{Q} \times \check{P} \longrightarrow \check{S}$ be a discrete approximation of this factorization, and let $\check{P}' \leq \check{S}$ denote the image of $\{1\} \times \check{P}$ under this homomorphism. Then, by adjointness again, one has a group monomorphism

$\alpha\colon \check{Q} \longrightarrow C_{\check{S}}(\check{P}')$. Upon p-completion, this map becomes a monomorphism of p-compact groups by Lemma A.2(i), since its composition with the inclusion to $BC_X(P)$ is so. But, since Q is a Sylow subgroup of $C_X(P)$, α must be an isomorphism, and the proof is complete. □

Definition 2.5. *Let X be a p-compact group, and let $(\check{S}, \check{\iota})$ be a discrete Sylow subgroup. Define the orbit category $\mathcal{O}_{\check{S}}(X)$ to be the category whose objects are the subgroups of $\check{S}$, and whose morphisms are representations $[\rho] \in \operatorname{Rep}(\check{P}, \check{Q})$ such that $\check{\iota}_{|B\check{P}} \simeq \check{\iota}_{|B\check{Q}} \circ B\rho$.*

If we let (S, ι) denote the p-completion of $(\check{S}, \check{\iota})$, then one has an obvious functor $q_{\mathcal{O}}\colon \mathcal{O}_{\check{S}}(X) \longrightarrow \mathcal{O}_p(X)$, where the target category is the full subcategory of $\mathcal{O}(X)$, whose objects are all p-compact toral subgroups of X.

Proposition 2.6. *Let X be a p-compact group and let $S \leq_\iota X$ be a Sylow subgroup. Fix a discrete approximation $\check{S}$ of S, and let $\check{S} \leq_{\check{\iota}} X$ be the resulting discrete Sylow subgroup of X. Then the functor $q_{\mathcal{O}}$ defined above is an equivalence of categories.*

Proof. Let $\check{\mathcal{O}}_p(X)$ be the category whose objects are all discrete p-toral subgroups of X, and whose morphisms $(\check{P}, \alpha) \to (\check{Q}, \beta)$ are conjugacy classes of homomorphisms $\check{P} \to \check{Q}$ such that the obvious triangles commutes up to homotopy. Then p-completion induces a functor $\check{\mathcal{O}}_p(X) \to \mathcal{O}_p(X)$, which by Lemma 2.2 is an equivalence of categories. Since $(\check{S}, \check{\iota})$ is a discrete Sylow subgroup of X, the inclusion $\mathcal{O}_{\check{S}}(X) \to \check{\mathcal{O}}_p(X)$ is also an equivalence of categories. □

¿From this point on we will denote discrete p-toral groups by P, Q, S, etc. (i.e., omit the [▶ 10] decoration), as long as confusion cannot arise. But $\check{P}$ will always denote a p-discrete toral and $\hat{P}$ always a p-compact toral group. We end this section with a brief discussion of normalizer and Weyl spaces, adapted to the context of discrete p-toral subgroups.

In [DW3, Sec. 4], Dwyer and Wilkerson define the Weyl space $\mathcal{W}_X(Y) = \mathcal{W}_X(Y, \alpha)$, for a subgroup (Y, α) of a p-compact group X, to be the space of all selfmaps f of BY, such that $\alpha \simeq \alpha \circ f$. The Weyl space is a topological monoid under composition, and acts naturally on BY. By [DW3, Proposition 4.3], the set of components $\pi_0(\mathcal{W}_X(Y))$ is a group. Notice that this group is, by definition, the automorphism group of (Y, α) in $\mathcal{O}(X)$.

Similarly, if X is a p-compact group with a discrete Sylow subgroup S, then for each subgroup $P \leq S$, we define the Weyl space $\mathcal{W}_X(P)$ to be the space of all selfmaps f of BP, such that $\iota_P \simeq \iota_P \circ f$. This is again a topological monoid under composition, which acts naturally on BP.

The normalizer space of P in X, $\mathcal{N}_X(P)$, is the loop space whose classifying space is the Borel construction of the action on $\mathcal{W}_X(P)$ on BP,

$$BP \longrightarrow B\mathcal{N}_X(P) \longrightarrow B\mathcal{W}_X(P).$$

Since the action of $\mathcal{W}_X(P)$ on BP commutes with the inclusion $BP \longrightarrow BX$, there is natural map $B\mathcal{N}_X(P) \longrightarrow BX$ (see [DW3, Definition 4.4]).

The set of components $\pi_0(\mathcal{W}(P))$ is again a group. To see this note that every selfmap of BP is induced by an endomorphism of P up to homotopy, and this endomorphism is unique up to an inner automorphism of P (Lemma 2.2). Upon p-completion one obtains a selfmap of a p-compact toral group over BX, which by [DW3, Proposition

4.3] is an equivalence. Hence any endomorphism inducing this selfmap is in fact an automorphism.

Definition 2.7. *For any subgroup (Y, α) of X, define its Weyl group by*

$$W_X(Y, \alpha) \stackrel{\text{def}}{=} \pi_0(\mathcal{W}_X(Y, \alpha)).$$

If $S \leq_\iota X$ is a discrete Sylow subgroup, and $P \leq S$ is a subgroup, then the Weyl group of P, denoted by $W_X(P)$, is defined similarly and [▶ 11] $W_X(P) \leq \mathrm{Out}(P)$.

3. Centric subgroups and radical subgroups

This section is devoted to a discussion of centric and radical discrete p-toral subgroups of a p-compact group.

Recall that, in [DW1], Dwyer and Wilkerson define the centralizer of a p-compact toral subgroup (P, α) of a p-compact group X to be $BC_X(P) \stackrel{\text{def}}{=} \mathrm{Map}(BP, BX)_\alpha$. The center of a a p-compact toral group P can therefore be defined by $BZ(P) \stackrel{\text{def}}{=} \mathrm{Map}(BP, BP)_{id}$, which by Lemma 2.2 is approximated by $\mathrm{Map}(B\check{P}, B\check{P})_{id} \simeq BZ(\check{P})$. The last equivalence is of course well known for discrete groups. Thus one can define a centric p-compact toral subgroup of X to be a subgroup (P, α), such that the obvious map $BZ(P) \xrightarrow{\alpha_\#} BC_X(P)$ is a weak equivalence. We now specialize to discrete p-toral groups.

Definition 3.1. *Let $P \leq_\alpha X$ be a discrete p-toral subgroup of a p-compact group X. We say that P is centric in X if the natural map*

$$BZ(P) \simeq \mathrm{Map}(BP, BP)_{id} \xrightarrow{\alpha_\#} \mathrm{Map}(BP, BX)_\alpha$$

is a mod-p homology equivalence.

Clearly a subgroup $P \leq S$ is centric in X if and only if the corresponding p-compact toral subgroup of X is centric in the [▶ 12] sense described in the introduction since in that case both mapping spaces are p-complete (see [DW1, Proposition 5.8, and Proposition 6.1]). The following lemma gives an alternative condition, which avoids the need to pass to the p-completed classifying space.

Lemma 3.2. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_\iota X$, and let $P \leq S$ be a subgroup. Then P is centric in X if and only if P' is centric in S for each $P' \leq S$ which is X-conjugate to P.*

Proof. Assume $P' \leq S$ is centric in S for every P' which is X-conjugate to P. By Proposition 2.4, there is some P', X-conjugate to P, such that $C_S(P')$ is a discrete Sylow subgroup of $C_X(P)$. But $C_S(P') = Z(P') \cong Z(P)$ by assumption, and so by Remark A.11 the inclusion of $C_S(P')$ in $C_X(P)$ is a mod-p equivalence. Hence P is centric in X.

Conversely, if P is centric in X, then all $P' \leq S$ which are X-conjugate to P are centric in X. Hence the composite

$$BZ(P') \longrightarrow BC_S(P') \longrightarrow BC_X(P')$$

is a mod-p homology equivalence. Upon p-completion, the second map is both a monomorphism, and an epimorphism of p-compact groups (Lemma A.2), hence an isomorphism. It follows that the first map is a mod p homology isomorphism as well, and so P' is centric in S (by [DW1, Prop 6.8] and Lemma 2.2). □

Remark 3.3. Lemma 3.2 implies in particular that if $P \leq Q \leq X$ are discrete p-toral subgroups of X, and P is centric in X, then it is centric in Q.

The following is a useful property of centric subgroups.

Lemma 3.4. *Let S be a discrete p-toral group, and let $P \leq S$ be a centric subgroup. Then [▶ 13] $Z(P) \leq P$ intersects nontrivially with any nontrivial normal subgroup of S.*

Proof. Let $N \lhd S$ be a normal nontrivial subgroup. Assume first that N is finite. Then P acts on N, and the action factors through an action of a p-subgroup of $\mathrm{Aut}(N)$. Since a finite p-group acting on another finite p-group always has a fixed point, there is some $x \in N$ which is fixed by the conjugation action of P, and hence $x \in C_S(P) = Z(P)$. If N is infinite, it contains a non-trivial characteristic elementary abelian p-subgroup E. Hence the action of P on N induces an action on E. The argument given for the finite case can now be repeated to finish the proof. □

Proposition 3.5. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_\iota X$, and let $P \leq S$ be subgroup, which is centric in X. Then $W_X(P)$ is a finite group, and there are subgroups $P' \lhd Q' \leq S$ such that P' is X-conjugate to P, and such that [▶ 14] $Q'/P' \cong \mathrm{Out}_{Q'}(P') = \mathrm{Out}_S(P')$ is a Sylow p-subgroup of $W_X(P)$.*

Proof. The group $W_X(P)$ is finite by Corollary A.17. Let $\pi \in \mathrm{Syl}_p(W_X(P))$. Since $\mathcal{W}_X(P)$ is homotopically discrete, there is an obvious map $B\pi \to B\mathcal{W}_X(P)$, which induces the inclusion on fundamental groups. Let Q be the discrete p-toral group whose classifying space is the pull back space of the system $B\pi \longrightarrow B\mathcal{W}_X(P) \longleftarrow B\mathcal{N}_X(P)$. Thus we obtain an extension of discrete p-toral groups $P \xrightarrow{\nu} Q \to \pi$ and a map $BQ \longrightarrow BX$, which factors through a homomorphism $Q \xrightarrow{\alpha} S$. Then ν is clearly injective, and we claim that α is injective as well. Notice first that its restriction to P is injective. Furthermore, P is centric in X, hence by Lemma 3.2 it is centric in S, and thus also in Q. By Lemma 3.4, α is injective, since otherwise P intersects nontrivially with $\mathrm{Ker}(\alpha)$, which is a contradiction.

Now, let $Q' = \alpha(Q)$ and $P' = \alpha \circ \nu(P)$. Then $P' \lhd Q' \leq S$, P' is X-conjugate to P, and $Q'/P' \cong \pi$ is a Sylow p-subgroup of $W_X(P)$ by construction. □

We now specialize to centric collections of p-compact toral subgroups of a p-compact group X, i.e., collections all of whose objects are centric in X. We start by analyzing the automorphism group of a centric subgroup as an object in the orbit category $\mathcal{O}(X)$.

For a group G and a G-space Z, we denote by Z^{hG} the homotopy fixed point space of Z under the action of G, i.e., the space $\mathrm{Map}_G(EG, Z)$, where EG is a free contractible G-space, or equivalently the space of sections of the Borel construction

$$Z_{hG} \stackrel{\mathrm{def}}{=} Z \times_G EG \longrightarrow * \times_G EG = BG.$$

If X is a p-compact group and $Y \leq_\alpha X$ is a p-compact subgroup, we denote by X/Y the homotopy fibre of α. In particular, if $S \leq X$ is a discrete Sylow subgroup of X, and $P \leq S$, the space $X/\hat{P}$ is the homotopy fibre of $\iota_{\hat{P}}\colon B\hat{P} = BP^\wedge_p \longrightarrow BX$.

Proposition 3.6. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_\iota X$. Let $Y \leq_\beta X$ be a p-compact subgroup of X, and assume $P \leq S$ is centric in X. Then $(X/Y)^{hP}$ is homotopically discrete, and*

$$\mathrm{Mor}_{\mathcal{O}(X)}((P, \iota_P), (Y, \beta)) = \pi_0((X/Y)^{hP}).$$

In particular, $\mathcal{W}_X(P) = (X/\hat{P})^{hP}$*, and*

$$\mathrm{Mor}_{\mathcal{O}_S(X)}(P,Q) = \pi_0((X/\hat{Q})^{hP}).$$

Finally, if X *is p-compact toral and* P *is a proper subgroup of* S*, then* $W_X(P)$ *is a non-trivial finite p-group.*

Proof. [▶ 15] Let $\{\alpha\} \subseteq [BP, BY]$ be the set of components $[f]$ such that $\beta \circ f \simeq \alpha$. The homotopy fibre of the map

$$\beta_{\#}\colon \mathrm{Map}(BP, BY)_{\{\alpha\}} \longrightarrow \mathrm{Map}(BP, BX)_\alpha$$

is $(X/Y)^{hP}$ by [DW1, Lemma 10.4], and $\beta_{\#}$ is a homotopy equivalence on each component by Lemma 3.2 and Remark 3.3 since all components in $\mathrm{Map}(BP, BY)_{\{\alpha\}}$ are equivalent to $BZ(P)^\wedge_p$. Hence $(X/Y)^{hP}$ is homotopically discrete, and the morphism set $\mathrm{Mor}_{\mathcal{O}(X)}((P, \iota_P),(Y,\beta))$, given by the set of components $\{\alpha\}$, is in $1-1$ correspondence with the set of components $\pi_0((X/Y)^{hP})$. The identification with the automorphism group of P in $\mathcal{O}_S(X)$ is immediate.

If X is p-compact toral then $S \xrightarrow{\iota} X$ is a discrete approximation, and by Proposition 3.5 and Proposition 2.6 , $W_X(P)$ is a finite p-group isomorphic to $N_S(P)/P = (S/P)^P$, and the right hand side is nontrivial by Corollary A.17. □

The following lemma shows that the collection of subgroups $P \leq S$ which are centric in X, where S is a discrete Sylow subgroup in a p-compact group X is closed under overgroups.

Lemma 3.7. *Let* X *be a p-compact group with a discrete Sylow subgroup* $S \leq X$*, and let* $P \leq Q \leq S$ *be subgroups. Then if* P *is centric in* X*, then so is* Q*.*

Proof. By Lemma 3.2, P is centric in X if and only if all its X-conjugates $P' \leq S$ are centric in S. But, in a discrete group, centric subgroups are closed under overgroups. Hence Q and all its X-conjugates are centric in S, and so Q is centric in X. □

If $P \leq Y \leq X$, and P is centric in Y, then it is not generally the case that P is centric in X. The following lemma singles out a family of subgroups $Y \leq X$, which are a very useful exception to the rule.

Lemma 3.8. *Let* X *be a p-compact group with a discrete Sylow subgroup* $S \leq X$*. Let* $E \leq S$ *be an elementary abelian p-subgroup, and let* $P \leq C_X(E)$ *be a discrete p-toral subgroup. Then* P *is centric in* $C_X(E)$ *if and only if it is centric in* X*.*

Proof. By Proposition 2.4, there is some $E' \leq S$, X-conjugate to E, such that $C_S(E')$ is a discrete Sylow subgroup of $C_X(E)$. Thus P is $C_X(E)$-conjugate to a subgroup of $C_S(E')$, and, replacing E' by E, it suffices to prove the statement for subgroups $P \leq C_S(E)$. [▶ 16] Note that in this situation $C_S(P) = C_{C_S(E)}(P)$.

If $P \leq C_S(E)$ is centric in X, then, by Lemma 3.2, P and all its X-conjugates $P' \leq S$ are centric in S. In particular those X-conjugates of P, which are contained in $C_S(E)$ and are conjugate in $C_X(E)$, are also centric in S, and therefore in $C_S(E)$. Thus P is centric in $C_X(E)$ by Lemma 3.2 again.

Conversely, if $P \leq C_S(E)$ is centric in $C_X(E)$, then it is centric in $C_S(E)$. For any $P' \leq S$, X-conjugate to P, we must show that P' is centric in S. Let $\mathsf{E}(P)$ denote the maximal central elementary abelian subgroup of P. Since $P \leq C_S(E)$, and since E is central in $C_S(E)$, it follows that $E \leq \mathsf{E}(P)$. Clearly $P \leq C_S(\mathsf{E}(P))$, and is centric there. An X-conjugation $P \to P'$ is in particular a group isomorphism, and hence takes $\mathsf{E}(P)$ isomorphically to $\mathsf{E}(P')$, and induces an isomorphism of p-compact groups $C_X(\mathsf{E}(P')) \to C_X(\mathsf{E}(P))$. Since P is centric in $C_X(\mathsf{E}(P))$, P' is centric in $C_X(\mathsf{E}(P'))$, and thus in $C_S(\mathsf{E}(P'))$. Let C' denote $C_S(\mathsf{E}(P'))$. It is now easy to verify that $C_{C'}(P') = C_{C_S(P')}(\mathsf{E}(P'))$. The right hand side is equal to $C_S(P')$ since $\mathsf{E}(P') \leq P'$ is central, while the left hand is $Z(P')$, since P' is centric in C'. This shows that P' is centric in S, and thus completes the proof. □

Next we discuss some basic properties of the collection of radical p-compact toral subgroups of a p-compact group X. In direct analogy to p-toral subgroups of a p-compact group, we define radical discrete p-toral subgroups of a p-compact group X.

Definition 3.9. *[▶ 17] We say that a discrete p-toral subgroup (P, α) of a p-compact group X is radical if it is centric and if $\mathrm{Aut}_{\mathcal{O}(X)}(P, \alpha)$ is finite and p-reduced (that is, it has no nontrivial normal p-subgroups).*

We start by showing that any discrete Sylow subgroup of a p-compact group is radical.

Lemma 3.10. *Let $S \leq X$ be a discrete Sylow subgroup of a p-compact group X. Then S is radical in X.*

Proof. By Lemma 3.2, S is centric since $C_S(S) = Z(S)$. Furthermore, by Proposition 3.5, $\mathrm{Out}_S(S) = 1$ is a Sylow p-subgroup of $W_X(S)$, therefore $|W_X(P)|$ is of order prime to p and is thus p-reduced. □

The following lemma is a one-sided analogue of 3.8 for radical subgroups. Notice that if $P \leq Y \leq X$ and P is radical in X, it is not necessarily radical in Y.

Lemma 3.11. *Let X be a p-compact group with a discrete Sylow subgroup S. Let $P \leq S$ be a subgroup, and let $E \stackrel{\text{def}}{=} \mathsf{E}(P) \leq_\epsilon X$ be a maximal central elementary abelian p-subgroup of P. Then $P \leq C_X(E)$ and if P is radical in X, then it is radical in $C_X(E)$.*

Proof. The first statement is obvious (see Remark A.15). Write $K \stackrel{\text{def}}{=} C_X(E)$ for short. If P is radical in X, then it is centric there by definition, and hence $\mathcal{W}_X(P)$ is homotopically discrete by Proposition 3.6. By the same Proposition, since $P \leq K$, P is also centric in K, and $\mathcal{W}_K(P)$ as well as $(X/K)^{hP}$ are homotopically discrete. The Weyl group $W_K(P)$ is finite by Proposition 3.6, and it remains to show that it is p-reduced.

To show that, we construct a homomorphism

$$\rho\colon W_X(P) \longrightarrow \mathrm{Aut}(E),$$

with kernel $W_K(P)$. Having done that, the claim follows from the assumption that $W_X(P)$ is p-reduced and Lemma 3.12 below.

Let $\iota_P\colon P \longrightarrow X$ denote the inclusion. We can identify $W_X(P)$ with those $\beta \in \mathrm{Out}(P)$ such that $B\iota_P \circ B\beta \simeq B\iota_P$. Since P is centric in S, β induces an automorphism $\beta^\sharp \in \mathrm{Aut}(E)$. Define $\rho(\beta) = \beta^\sharp$.

Now, since $\beta^\sharp$ is the restriction of β to E, β also induces a selfmap of $K = C_X(E)$, and if $\beta^\sharp = Id$, then the induced map on K is also homotopic to the identity. Therefore, if we let $j_P\colon P \longrightarrow K$ be the inclusion, then $Bj_P \circ B\beta \simeq Bj_P$, and thus $\beta \in W_K(P)$. Conversely, let $\beta \in W_K(P)$. Consider the morphism $m\colon E \times P \longrightarrow X$ induced by multiplication with kernel E. In this situation $Id \times \beta \in \mathrm{Aut}(E \times P)$ induces an automorphism over X, that is, $m \circ (Id \times \beta) \simeq m$ since E is central in P and $\beta \in W_K(P) \leq W_X(P)$. The restriction of m to the kernel is the automorphism of E given by $\beta^\sharp = Id$. □

Lemma 3.12. *Let $H \lhd G$ be a normal subgroup of the finite group G. If G is p-reduced, then so is H.*

Proof. Let $Q \leq H$ be the intersection of all Sylow p-subgroups of H. Then, $Q \lhd H$ is a characteristic subgroup of H. Since $H \lhd G$, it follows that $Q \lhd G$. Moreover, H is p-reduced if and only if Q is the trivial group. Thus, if H is not p-reduced then G is also not p-reduced, which proves the statement. □

4. The orbit category of radical subgroups

In this section we study the orbit category of radical p-compact subgroups of a p-compact group X. In particular we study the behavior of the orbit category of radical subgroups and the respective subgroup diagrams under extension of a p-compact group by a p-compact toral group. We also show that the orbit category of radical subgroups of a p-compact group has a finite skeletal subcategory. This last observation is crucial in carrying out inductive procedures later on.

Definition 4.1. *Let X and Z be p-compact groups. An extension of Z by X is a fibration*

$$BX \longrightarrow BY \longrightarrow BZ,$$

where both the projection and the fibre inclusion are homomorphisms (i.e. pointed maps).

Notice that an extension of p-compact groups as above gives rise to a p-compact group, whose classifying space BY is the total space in the defining fibration. To see this, notice that BY is p-complete, since $\pi_1(BZ)$ is a p-group and hence acts nilpotently on the mod p homology of BX. Hence by [BK], p-completion preserves the fibration, while not changing its base and fibre spaces, implying that BY is p-complete as well. Furthermore, $Y = \Omega BY$ is the total space in a fibration with $\mathbb{F}_p$-finite base and fibre, and is thus itself $\mathbb{F}_p$-finite by inspection of the associated Serre spectral sequence.

We are now ready to state and prove our main claim about extensions: if Y is an extension of a p-compact group X by a p-compact toral group K, then X and Y have equivalent orbit categories of p-compact toral radical subgroups. More precisely we have the following.

Proposition 4.2. *Let*

$$BK \xrightarrow{\iota} BX \xrightarrow{\pi} BY$$

be an extension of p-compact groups where K is p-compact toral. Then there is an equivalence of categories

$$\Psi \colon \mathcal{O}^r(Y) \longrightarrow \mathcal{O}^r(X).$$

Proof. Let (S, ι_S) be a discrete Sylow subgroup for X, and let (R, ι_R) be a discrete Sylow subgroup of Y. The map π induces an epimorphism $\breve{\pi} \colon S \longrightarrow R$, which is well defined up to conjugation in R. This follows since the pull back of the extension along ι_R gives a discrete p-toral subgroup of X, which is clearly Sylow by a standard Euler characteristic argument. Let $\breve{K}$ denote $\mathrm{Ker}(\breve{\pi})$. Since $K \leq_\iota X$ is a p-compact toral subgroup of X such that $\check{\pi} \circ \iota \simeq *$, there is an induced map $B\check{K} \longrightarrow BK$, which is a discrete approximation. To see that it is, notice that in the fibration

$$B\check{K} \longrightarrow BS \longrightarrow BR$$

the action of R on $H_*(B\check{K}, \mathbb{F}_p)$ factors through a finite p-group quotient, and is therefore nilpotent. Hence the fibration is preserved under p-completion, and $B\check{K}^\wedge_p \simeq BK$. In particular, there is a [▶ 18] 1–1 correspondence between subgroups of R and subgroups of S containing $\check{K}$.

By Proposition 2.6, to prove our statement, it suffices to show that there is an isomorphism of categories $\Psi_S \colon \mathcal{O}^r_R(Y) \longrightarrow \mathcal{O}^r_S(X)$, where $\mathcal{O}^r_S(X)$ is the full subcategory of $\mathcal{O}_S(X)$ whose objects are radical in X, and similarly $\mathcal{O}^r_R(Y)$. The functor Ψ_S we use is in fact defined on $\mathcal{O}_R(Y)$, but will only be shown to be an isomorphism of categories when restricted to $\mathcal{O}^r_R(Y)$.

Definition of Ψ_S: For an object $P \le R$ of $\mathcal{O}_R(Y)$, define $\Psi_S(P) = \check{\pi}^{-1}(P) \le S$. Notice that the square

$$\begin{array}{ccc} B(P') & \xrightarrow{\iota_{P'}} & BX \\ \downarrow & & \downarrow \pi \\ BP & \xrightarrow{\iota_P} & BY \end{array}$$

is a homotopy pullback square upon p-completion, where $P' = \Psi_S(P)$. Unless ambiguity may arise, we will denote throughout this proof $\Psi_S(P)$, $\Psi_S(Q)$ by P', Q' etc., for short.

We next define Ψ_S on morphisms. A morphism $P \xrightarrow{[f]} Q$ in $\mathcal{O}_R(Y)$ is a Q-conjugacy class of a homomorphism $f \colon P \longrightarrow Q$, such that $\iota_Q \circ Bf \simeq \iota_P$. Thus given such a morphism, applying the classifying space functor and p-completion, and using the universal property of a pull-back, one has an induced map

$$f' \colon BP' \longrightarrow BQ'$$

such that $[\iota_{Q'}] \circ f' = [\iota_{P'}]$. Let $\Psi_S([f]) \colon P' \longrightarrow Q'$ denote any homomorphism, which induces f' up to homotopy.

Next, we check that $\Psi_S([f])$ is well defined, and that Ψ_S induces a bijection on morphism sets. Let $P, Q \in \mathcal{O}_R(Y)$ be any two objects, and let $P', Q' \le X$ denote $\Psi_S(P)$ and $\Psi_S(Q)$ respectively, as before. Then one has the following sequence of homotopy equivalences

$$(X/\hat{Q}')^{hP'} \simeq (Y/\hat{Q})^{hP'} \simeq ((Y/\hat{Q})^{h\check{K}})^{hP} \simeq (Y/\hat{Q})^{hP}. \tag{1}$$

The first equivalence holds since $X/\hat{Q}' \simeq Y/\hat{Q}$. The second follows from [DW1, Lemma 10.5]. For the last equivalence, note that, since the composite

$$BK \longrightarrow BX \longrightarrow BY$$

is nullhomotopic, the pullback of the fibration $B\hat{Q} = BQ^\wedge_p \longrightarrow BY$ along this composite is the trivial fibration $BK \times Y/\hat{Q} \longrightarrow BK$. Hence $(Y/\hat{Q})^{hK} \simeq \mathrm{Map}(BK, Y/\hat{Q}) \simeq Y/\hat{Q}$, where the second equivalence follows since $Y/\hat{Q}$ is p-complete and $\mathbb{F}_p$-finite, and so the evaluation map

$$\mathrm{Map}(BK, Y/\hat{Q}) \xrightarrow[ev]{\simeq} Y/\hat{Q}$$

is an equivalence, by the Sullivan conjecture for p-compact groups [DW2, Theorem 9.3]. Notice that $(Y/\hat{Q})^{hK} \simeq (Y/\hat{Q})^{h\check{K}}$. Taking components, and using Proposition 3.6, one has [▶ 19]

$$\mathrm{Mor}_{\mathcal{O}(X)}(P', Q') = \pi_0((X/\hat{Q}')^{hP'}) \cong \pi_0((Y/\hat{Q})^{hP}) = \mathrm{Mor}_{\mathcal{O}(Y)}(P, Q).$$

This shows that Ψ_S is well defined and is a bijection on morphism sets.

Ψ_S is injective on objects: Since objects in $\mathcal{O}_R(Y)$ are in 1–1 correspondence via Ψ_S with objects of $\mathcal{O}_S(X)$ which contain $\check{K}$, it follows that Ψ_S, and hence its restriction $\mathcal{O}^r_R(Y)$, is injective on objects.

[▶ 20] **The functor Ψ_S sends radical p-compact toral subgroups to radical p-compact toral subgroups:** Set $P = Q$ in Equation (1) above. Taking components, one obtains a group isomorphism $W_X(P') \cong W_Y(P)$. Hence, $P' \leq_{\iota_{P'}} X$ is radical if and only if $P \leq_{\iota_P} Y$ is radical. This shows that Ψ_S restricted to $\mathcal{O}^r_R(Y)$ takes values in $\mathcal{O}^r_S(X)$.

Ψ_S is surjective on objects: To show that Ψ_S is surjective on objects, it suffices to show that any subgroup $Q \leq S$ such that Q is radical in X must contain $\check{K}$, and thus is in the image of Ψ_S. This part of the proof is the only place where radicality is used.

We first consider the case when $\check{K}$ is abelian. Assume that Q is radical and does not contain $\check{K}$. Set $U = Q \cap \check{K}$. We first verify that $Q \lneq N_{Q\check{K}}(Q)$. To see this, notice that the action of Q on $\check{K}/U$ by conjugation has a nontrivial fixed subgroup (this is clear if $\check{K}/U$ is finite, and if $\check{K}/U$ is infinite, then the action of Q on characteristic subgroup of p-torsion elements in $\check{K}/U$ must have a nontrivial fixed subgroup. There is an element $g \in \check{K} \setminus Q$ such that $gU \in (\check{K}/U)^Q$, and therefore g normalizes Q and is hence in $N_{Q\check{K}}(Q)$.

Let $g \in N_{\check{K}Q}(Q) \setminus Q$. Since $\check{K}$ is abelian, so is U, and since $g \in \check{K}$, the conjugation action of g on Q, restricts to the identity on U and also on Q/U. Let $A \leq \mathrm{Aut}_X(Q)$ denote the subgroup of all X-automorphisms of Q with this property. For any $\alpha \in A$, and $q \in Q$, $\alpha(q) = qu_q$ for some $u_q \in U$. Since every X-automorphism of Q preserves $\check{K}$ and hence U, this implies that A is a normal subgroup of $\mathrm{Aut}_X(Q)$. Moreover, since for every $u \in U$, $|u| = p^m$ for some $m \geq 1$, it follows that for every $q \in Q$, there is some m such that $\alpha^{p^m}(q) = q$. Since the group of automorphisms $\mathrm{Aut}_{\mathcal{O}_S(X)}(Q)$ is finite (Proposition 3.6), the observations above imply that the group $A/(\mathrm{Inn(Q)} \cap \mathrm{A}) \lhd \mathrm{Aut}_{\mathcal{O}_S(X)}(\mathrm{Q})$ is a normal p-subgroup, and since Q is radical, A must be contained in $\mathrm{Inn(Q)}$. Thus $c_g = c_q$ for some $q \in Q$, and hence gq^{-1} centralizes Q. But since $g \notin Q$, and since Q is centric, this contradicts Lemma 3.2. This shows that Q must contain $\check{K}$. In particular, this finishes the proof of the proposition in the case $\check{K}$ is abelian.

Let $\check{K}$ be arbitrary, let T be a maximal torus of $\check{K}$ and set $\pi = \check{K}/T$. Notice that X/T is a p-compact group. Since T is abelian, the particular case just handled implies that there is an equivalence of categories $\Psi : \mathcal{O}^r(X) \longrightarrow \mathcal{O}^r(X/T)$. Since there is an extension of p-compact groups

$$\pi \longrightarrow X/T \longrightarrow Y,$$

it is enough to consider the case where K is a finite p-group. But in this case we can proceed similarly by induction on the order of π using the fact that a nontrivial finite p-group has a nontrivial center. This completes the proof in the general case. □

Proposition 4.2 allows to compare subgroup ampleness of radical subgroups of a p-compact group Y, with that of radical subgroups of any extension of Y [▶ 21] by a p-compact toral group.

Proposition 4.3. *Let X be an extension of a p-compact group Y by a p-compact toral group K. Then, the collection of all radical subgroups of X is subgroup-ample if and only if the collection of all radical subgroups of Y is subgroup-ample.*

Proof. Let $S \leq X$ and $R \leq Y$ be discrete Sylow subgroups, such that $\check{K} \lhd S$ is a discrete approximation for K, and $S/\check{K} \cong R$. Let $S \xrightarrow{\pi} R$ denote the projection. It suffices to prove the claim for the collections of all subgroups of S and R which are radical in X and Y respectively.

By Proposition B, there exists a subgroup diagram (Definition 0.1)

$$\bar{\phi}\colon \mathcal{O}^r_R(Y) \longrightarrow \mathsf{Sp}.$$

Let

$$\phi\colon \mathcal{O}^r_R(Y) \longrightarrow \mathsf{Sp}$$

be the functor which takes a subgroup $P \leq R$ to the pullback space in the diagram

$$\begin{array}{ccc} \phi(P) & \longrightarrow & \bar{\phi}(P) \\ \eta\big\downarrow & & \big\downarrow\bar{\eta} \\ BX & \xrightarrow{\ \pi\ } & BY = 1_{BY}(P) \end{array}.$$

Here 1_{BY} is the constant functor on $\mathcal{O}^r_R(Y)$ with value BY, and $\bar{\eta}$ is the natural transformation associated to $\bar{\phi}$. Notice that since π is a fibration, ϕ is well defined, and comes equipped with an obvious natural transformation $\eta\colon \phi \longrightarrow \underline{BX}$. Furthermore, since ϕ is defined using the pullback construction, the conditions of Puppe's theorem [Dr, pp.179] are automatically satisfied, and the commutative square above gives rise to a commutative diagram of fibrations

$$\begin{array}{ccccc} BK & \longrightarrow & \operatorname{hocolim}_{\mathcal{O}^r_R(Y)} \phi & \longrightarrow & \operatorname{hocolim}_{\mathcal{O}^r_R(Y)} \bar{\phi} \\ =\big\downarrow & & |\eta|\big\downarrow & & |\bar{\eta}|\big\downarrow \\ BK & \longrightarrow & BX & \longrightarrow & BY. \end{array}$$

Since Y and X are p-compact groups, their fundamental groups are finite p-groups. Hence $|\bar{\eta}|$ (resp. $|\eta|$) is a mod-p equivalence if and only if its homotopy fibre is mod-p acyclic. Since the fibres of η and $\bar{\eta}$ are homotopy equivalent, it follows that $|\eta|$ is a mod-p equivalence if and only if $|\bar{\eta}|$ is a mod-p equivalence.

Finally, by Lemma 4.2 the categories $\mathcal{O}^r_S(X)$ and $\mathcal{O}^r_R(Y)$ are equivalent categories, and the composite

$$\mathcal{O}^r_S(X) \xrightarrow{\Psi_S^{-1}} \mathcal{O}^r_R(Y) \xrightarrow{\ \phi\ } \mathsf{Sp}$$

is obviously a subgroup diagram for X. The proposition follows at once. □

We end this section by showing the for every p-compact group X the category $\mathcal{O}^r(X)$ is equivalent to a finite category.

Proposition 4.4. *For any p-compact group X, the orbit category $\mathcal{O}^r(X)$ has a finite skeletal subcategory [▶ 22](that is, $\mathcal{O}^r(X)$ has finitely many isomorphism classes of objects and finite morphism sets).*

Proof. Since all radical subgroups in X are centric in X by definition, each morphism set $\mathrm{Mor}_{\mathcal{O}^r(X)}(P,Q)$ is given by the set of components of the respective homotopy orbit space $(X/Q)^{hP}$, which is homotopically discrete and finite by Corollary A.17. Hence, it suffices to show that $\mathcal{O}^r(X)$ has a finite number of isomorphism classes of objects.

If X is a p-compact toral group and $Q \leq X$ is a proper subgroup, then the Weyl group $W_X(Q)$ is always a nontrivial finite p-group by Corollary 3.6. Hence, the only radical subgroup of X is X itself. This proves the claim in this case.

Let X be an arbitrary p-compact group, which is not p-compact toral. Let Y be the centerfree quotient of X, which exists by Lemma A.12. Since $\mathcal{O}^r(X) \simeq \mathcal{O}^r(Y)$ by Proposition 4.2, we are reduced to showing the statement for centerfree p-compact groups.

We proceed by downward induction on the order of X. The order of X is the pair (d_X, o_X), where d_X is the mod-p cohomological dimension of X and o_X is the order of its group of components (see Definition A.5). Thus assume the claim holds for all p-compact groups of order strictly less than that of X.

[▶ 23] By [DW2, Proposition 8.3], there exist only finitely many conjugacy classes of elementary abelian subgroups of X. If $P \leq X$ is a radical subgroup, then $P \leq C_X(\mathsf{E}(P))$, where $\mathsf{E}(P)$ is the maximal central elementary abelian subgroup of P, and P is radical there by Lemma 3.11. Since X is centerfree $|C_X(\mathsf{E}(P))| < |X|$, and by induction hypothesis $C_X(\mathsf{E}(P))$ has only finitely many conjugacy classes of radical p-compact toral subgroups. Hence the conjugacy class of P can only be one of a finite list of conjugacy classes of p-compact toral subgroups of X, each of which is radical in $C_X(E)$ for some elementary abelian p-subgroup $E \leq X$. This completes the proof. □

5. Subgroup ampleness of centric and radical collections

Let $\mathcal{C}$ be a family of centric subgroups of a p-compact group X, which contains at least one representative from the conjugacy class of each radical subgroup of X. The objective of this section is to prove the following proposition. Notice that this is a slightly more general form of the equivalence of statements (a) and (b) in Theorem C.

Proposition 5.1. *Let X be a p-compact group, and let $\mathcal{C}$ be a collection of subgroups all of which are centric in X, and such that $\mathcal{C}$ contains all subgroups which are radical in X. Then the collection of all radical subgroups in X is subgroup ample if and only if $\mathcal{C}$ is subgroup ample.*

Without lost of generality, we can restrict attention to collections which are contained in a fixed discrete Sylow subgroup (see Proposition 2.6). Thus, let X be a p-compact group with a discrete Sylow subgroup $S \leq_{\iota_S} X$, and let $\mathcal{C}$ be a collection of subgroups of S, all of which are centric in X, containing all subgroups of S which are radical in X. Recall that we denote by $\mathcal{O}^r_S(X)$ (resp. $\mathcal{O}^c_S(X)$) the orbit category of subgroups of a fixed discrete Sylow $S \leq_{\iota_S} X$ which are radical in X (resp. centric in X). Let $\mathcal{O}_{\mathcal{C}}(X)$ be the orbit category of subgroups of S which belong to the family $\mathcal{C}$.

¿From this point onwards, our discussion becomes quite categorical in nature. The required material is collected in the Appendix for the convenience of the reader, and will be referred to in due course.

Let

$$\tau_{\mathcal{C}} \colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}_{\mathcal{C}}(X)$$

denote the inclusion functor. For each object $P \in \mathcal{C}$, we denote by $P{\downarrow}\tau_{\mathcal{C}}$ the undercategory of P with respect to $\tau_{\mathcal{C}}$. Objects in the undercategory are pairs $(Q,[u])$, where $Q \leq_{\iota_Q} X$ is radical, and $P \xrightarrow{[u]} Q$ is a morphism in $\mathcal{O}_{\mathcal{C}}(X)$. A morphism in $P{\downarrow}\tau_{\mathcal{C}}$

$$[g]\colon (Q,[u]) \longrightarrow (Q',[u'])$$

is determined by a morphism $Q \xrightarrow{[g]} Q'$ in $\mathcal{O}^r_S(X)$, such that $[g]\circ[u] = [u']$. Note that, by Lemma A.18, there is at most one morphism between two objects in $P{\downarrow}\tau_{\mathcal{C}}$.

The functor $\tau_{\mathcal{C}}$ is said to be *right cofinal* if the nerve $|P{\downarrow}\tau_{\mathcal{C}}|$ is contractible for each $P \in \mathcal{C}$. The claim that this is indeed the case (Proposition 5.2 below) is the key ingredient in the proof of Proposition 5.1.

Proposition 5.2. *For any p-compact group X, the inclusion functor*

$$\mathcal{O}^r_S(X) \xrightarrow{\tau} \mathcal{O}^c_S(X)$$

is right cofinal.

Assuming Proposition 5.2, we now prove Proposition 5.1.

Proof of Proposition 5.1. If $\mathcal{C}$ is a collection of subgroups of S which are centric in X, and $\mathcal{C}$ contains all subgroups of S which are radical in X, then by Proposition 5.2 the inclusion functor

$$\tau_{\mathcal{C}}\colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}_{\mathcal{C}}(X)$$

is clearly right cofinal. If $(\Phi_{\mathcal{C}},\gamma,\eta)$ is a subgroup diagram (where $(\Phi_{\mathcal{C}},\gamma)$ is a realization of $\phi_{\mathcal{C}}$, and $\eta\colon \Phi_{\mathcal{C}} \longrightarrow \underline{BX}$ is a natural transformation), then its restriction to $\mathcal{O}^r_S(X)$ via $\tau_{\mathcal{C}}$ is also a subgroup diagram, and one has a commutative square

$$\begin{array}{ccc}
\operatorname*{hocolim}_{\mathcal{O}^r_S(X)} \Phi_{\mathcal{C}}\circ\tau_{\mathcal{C}} & \longrightarrow & \operatorname*{hocolim}_{\mathcal{O}_{\mathcal{C}}(X)} \Phi_{\mathcal{C}} \\
\downarrow & & \downarrow \\
BX & = & BX,
\end{array}$$

where the top row is a mod p equivalence by Theorem A.19.

[▶ 24] Since the top and the bottom rows of the commutative diagram are mod p equivalences, the left arrow is a mod p equivalence if and only if the right arrow is.

□

The proof of Proposition 5.2 will occupy the rest of the section. The following technical lemma is our main tool in an inductive proof of Proposition 5.2.

Lemma 5.3. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_{\iota_S} X$. Let $P \leq S$ be a subgroup which is centric in X, and let*

$$P = P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_j \leq P_{j+1} \leq \cdots Q = \operatorname*{colim}_j P_j \leq S$$

be a sequence of discrete p-toral subgroups, such that for each $j \geq 0$, P_j is a normal subgroup of finite index in P_{j+1}. Then there exists a positive integer j_0 such that for all $j \geq j_0$ the functor

$$Q{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau,$$

induced by the inclusion $P_j \leq Q$, is an equivalence of categories. Here, as before, $\tau\colon \mathcal{O}^r_S(X) \longrightarrow \mathcal{O}^c_S(X)$ denotes the inclusion.

Proof. Let $R \leq S$ be a subgroup which is radical in X. Then one has a sequence of maps between the homotopy fixed point spaces (recall that $X/\hat{R}$ is the homotopy fiber of the induced map $BR^\wedge_p \to BX$),

$$(2) \qquad (X/\hat{R})^{hQ} \longrightarrow \ldots \longrightarrow (X/\hat{R})^{hP_{j+1}} \longrightarrow (X/\hat{R})^{hP_j} \longrightarrow \ldots \longrightarrow (X/\hat{R})^{hP}.$$

Since R is radical, it is also centric by definition, and so all homotopy fixed point sets in the sequence are either empty or homotopically discrete by Proposition 3.6. Since $P_j \lhd P_{j+1}$ for all j, one has $(X/\hat{R})^{hP_{j+1}} = ((X/\hat{R})^{hP_j})^{h(P_{j+1}/P_j)}$ by [DW1, Propositions 6.8, 6.9 and Lemma 10.5]. Furthermore, since $(X/\hat{R})^{hP_j}$ is either empty or homotopically discrete, and the projection from any G-space to the G-set of its connected components is G-equivariant, we have the following equivalences

$$((X/\hat{R})^{hP_j})^{h(P_{j+1}/P_j)} \simeq (\pi_0((X/\hat{R})^{hP_j}))^{h(P_{j+1}/P_j)} = (\pi_0((X/\hat{R})^{hP_j}))^{P_{j+1}/P_j},$$

where the equality follows from the fact that for discrete G-sets homotopy fixed points and fixed points coincide by definition (in the empty case there is nothing to prove). This shows that all maps in the sequence (2) above induce monomorphisms on sets of path components.

Since $\pi_0((X/\hat{R})^{hP})$ is finite by Proposition 3.6, the induced sequence on sets of path components has to stabilize above some sufficiently large $j(R)$, depending only on the isomorphism class of R in $\mathcal{O}^r_S(X)$. By Proposition 4.4, there are only finitely many isomorphism classes of objects in this category. Hence one can define j_0' to be the maximum of all $j(R)$, where R runs over a set of representatives of isomorphism classes of objects in $\mathcal{O}^r_S(X)$. Thus, for all $j \geq j_0'$ and all $R \in \mathcal{O}^r_S(X)$, one has equivalences

$$(X/\hat{R})^{hP_{j_0}} \simeq (X/\hat{R})^{hP_j} \simeq (X/\hat{R})^{hQ},$$

[▶ 25] where the last equivalence follows from tha fact that $(X/\hat{R})^{hQ} \simeq \operatorname{holim}(X/\hat{R})^{hP_j}$.

By definition of the undercategory, for all $j \geq 0$

$$Obj(P_j{\downarrow}\tau) = \bigcup_{R \in \mathcal{O}^r_S(X)} \operatorname{Mor}_{\mathcal{O}^c_S(X)}(P_j, R) = \bigcup_{R \in \mathcal{O}^r_S(X)} \pi_0((X/\hat{R})^{hP_j}).$$

[▶ 26] What we just showed implies that the sequence of functors

$$\cdots \longrightarrow P_{j+1}{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau \longrightarrow \cdots$$

stabilizes on objects for all $j > j_0'$.

[▶ 27] It remains to show that the sequence stabilizes on morphism sets. The morphism set in each of the categories $P_j{\downarrow}\tau$ and in $Q{\downarrow}\tau$, between objects $(R, [u])$ and $(R', [u'])$, is a subset of $\operatorname{Mor}_{\mathcal{O}^r_S(X)}(R, R')$, which is finite for all $R, R' \in \mathcal{O}^r_S(X)$ by Proposition 4.4, and its cardinality depends only on the isomorphism classes of R and R' in $\mathcal{O}^r_S(X)$. Moreover, by Lemma A.18, there is at most one morphism between two objects in $P{\downarrow}\tau$. That is, for every pair of objects $(R, [u])$ and $(R', [u'])$ there exists $j_{R,R'}$ such that $\operatorname{Mor}_{Q\downarrow\tau}((R, [u]), (R', [u'])) = \operatorname{Mor}_{P_i\downarrow\tau}((R, [u]), (R', [u']))$ for all $i > j_{R,R'}$.

By Proposition 4.4 again, there are only finitely many isomorphism classes of objects in $\mathcal{O}^r_S(X)$, and so the sequence of functors above must stabilize on morphism sets for all $j > j_0''$ for some sufficiently large j_0''. Let $j_0 = \max\{j_0', j_0''\}$. Then for all $j > j_0$ the functor $Q{\downarrow}\tau \longrightarrow P_j{\downarrow}\tau$ is an equivalence of categories, as claimed. □

Our next aim is to show that the nerve of the undercategories $P{\downarrow}\tau$ is mod p-acyclic, thus proving Proposition 5.2. To achieve this one more step is required.

Let X be a p-compact group, and let $P \leq_{\iota_P} X$ be a centric discrete p-toral subgroup. By Proposition 3.5, we can choose P in its isomorphism class such that $W_P \stackrel{\text{def}}{=} \operatorname{Out}_S(P)$ is a Sylow subgroup of [▶ 28] $W \stackrel{\text{def}}{=} W_X(P) \leq \operatorname{Out}(P)$. Let $N = \mathcal{N}_X(P)$ be the normalizer space for $(B\iota_P)^\wedge_p \colon BP^\wedge_p \to BX$ and $\eta_P \colon BN \longrightarrow BX$ be the natural map [DW3, Definition 4.4]. Then there is a diagram

$$\begin{array}{ccccc} BP^\wedge_p & \xrightarrow{\;inc\;} & BN & \longrightarrow & BW \\ & {\scriptstyle (B\iota_P)^\wedge_p}\searrow & & \swarrow{\scriptstyle \eta_P} & \\ & & BX & & \end{array},$$

where the row is a fibration, up to homotopy, since the Weyl space $\mathcal{W}_X(P)$ is homotopically discrete. Moreover, since P is a centric subgroup of S, there is an extension

$$P \longrightarrow N_S(P) \xrightarrow{\;q\;} W_P.$$

For every p-subgroup $\pi \leq W_P$, define a discrete p-toral subgroup $P_\pi \leq S$ of X by $P_\pi = q^{-1}(\pi)$. Let $\alpha_\pi \colon (BP_\pi)^\wedge_p \longrightarrow BX$ be given by the composition of the map induced by inclusion $P_\pi \leq N_S(P) \leq S$ followed by $\iota_S \colon BS \to BX$. Notice that $(BP_\pi)^\wedge_p$ is given as the pull-back space of the system

$$BN \longrightarrow BW \xleftarrow{\;inc\;} B\pi, \tag{3}$$

and α_π is the composition of η_P with the obvious map $(BP_\pi)^\wedge_p \longrightarrow BN$. Also, by Lemma 3.7, P_π is centric in X since it contains P as a subgroup. Thus the class of the inclusion $P \leq P_\pi$ is a morphism in $\mathcal{O}^c(X)$.

Let $\mathcal{S}_p(W)$ denote the poset category of all nontrivial p-subgroups of W. Let $\pi \leq \pi' \leq W_p$ be p-subgroups of W and $i \in \operatorname{Mor}_{\mathcal{S}_p(W)}(\pi, \pi')$. By the universal property of a pull back (Diagram (3) above), one has an induced map $\hat{\delta}(i) \colon BP_\pi \longrightarrow BP_{\pi'}$, well defined up to homotopy (and hence a corresponding representation $[\delta(i)] \colon P_\pi \longrightarrow P_{\pi'}$), such that $\hat{\delta}(i) \circ \alpha_\pi \simeq \alpha_{\pi'}$. In other words, $[\delta(i)]$ is a morphism in $\mathcal{O}^c(X)$. Moreover, any representative of $[\delta(i)]$ restricts to the homotopy class of the identity on P. By naturality of this construction, for every discrete p-toral subgroup $P \leq S$ which is centric in X one gets a functor

$$\delta = \delta_P \colon \mathcal{S}_p(W) \longrightarrow \mathcal{O}^c(X)$$

which takes $\pi \leq W$ to P_π, and $i \in \operatorname{Mor}_{\mathcal{S}_p(W)}(\pi, \pi')$ to $[\delta(i)]$.

Let $P \leq S$ be a non-radical discrete p-toral subgroup. We define a functor

$$\rho \colon P{\downarrow}\tau \longrightarrow \mathcal{S}_p(W)$$

as follows. First notice that if $(Q, [h])$ is an object in $P{\downarrow}\tau$, we can fix a representative $h \colon P \longrightarrow Q$ and define $\rho(Q, [h]) = h^{-1} \operatorname{Out}_Q(h(P))h \leq W$. Note that this definition does not depend on the choice of representatives for $[h]$.

Given a morphism $[g] : (Q, [h]) \longrightarrow (Q', [h'])$, and a fixed choice of representatives h and h' as above, there is an inclusion $\rho([g]) \colon h^{-1} \operatorname{Out}_Q(h(P))h \longrightarrow (h')^{-1} \operatorname{Out}_{Q'}(h'(P))h'$ of subgroups of W. Moreover, if $P \leq Q$ and $\pi = \operatorname{Out}_Q(P)$, then $P_\pi = N_Q(P) \leq Q$, and the composite of $P \leq P_\pi$ followed by the inclusion in Q is an extension of h up to conjugacy in Q.

Remark 5.4. Given a morphism $[h] \in \operatorname{Mor}_{\mathcal{O}_S(X)}(P, Q)$, with a fixed representative h, and a subgroup $\pi \leq_i h^{-1} \operatorname{Out}_Q(h(P))h \leq W$, there is a factorization $[h] = [\iota^Q] \circ [\delta(i)] \circ [\gamma_\pi]$ where ι^Q is the inclusion $N_Q(P) \leq Q$, and $P \leq_{\gamma_\pi} P_\pi$.

Notice that ρ cannot be defined at all if P is radical in X, since in that case $(P,[Id_P])$ is an object in the over category, and $\rho(P,[Id_P])$, as defined above, is the trivial subgroup, which is not an object of $\mathcal{S}_p(W)$.

Lemma 5.5. *Given a non-trivial p-subgroup $\pi \leq W$, and a minimal $(P_\pi,[h_\pi]) \in P{\downarrow}\tau$ such that $\rho(P_\pi,h_\pi)=\pi$, there is an isomorphism of categories $\pi{\downarrow}\rho \cong P_\pi{\downarrow}\tau$.*

Proof. We will construct functors $R\colon P_\pi{\downarrow}\tau \longrightarrow \pi{\downarrow}\rho$ and $L\colon \pi{\downarrow}\rho \longrightarrow P_\pi{\downarrow}\tau$ such that $L\circ R = Id_{P_\pi{\downarrow}\tau}$ and $R\circ L = Id_{\pi{\downarrow}\rho}$.

The objects in $\pi{\downarrow}\rho$ are given by $(Q,[h],f)$ where $(Q,[h])$ is an object in $P{\downarrow}\tau$, and $\pi \leq_f h^{-1}\operatorname{Out}_Q(h(P))h$. Given two objects $(Q,[h],f)$ and $(Q',[h'],f')$, a morphism from the first to the second is a morphism $[g]\colon (Q,[h]) \longrightarrow (Q',[h'])$ in $P{\downarrow}\tau$ such that $\rho([g])\colon h^{-1}\operatorname{Out}_Q(h(P))h \longrightarrow (h')^{-1}\operatorname{Out}_{Q'}(h'(P))h'$ satisfies $\rho([g])\circ f = f'$.

Define R on objects as follows. For each object $(Q,[h])$ in $P{\downarrow}\tau$ fix a representative h. Given $(Q,[g])$ in $P_\pi{\downarrow}\tau$, one has two objects $(P_\pi,[\gamma_\pi])$ and $(Q,[g]\circ[\gamma_\pi])$ in $P{\downarrow}\tau$, where $P \leq_{\gamma_\pi} P_\pi$ is the inclusion, and $[g]$ is a morphism from the first to the second. Applying the functor ρ defined above, one has a morphism in $\mathcal{S}_p(W)$ corresponding to the inclusion

$$\rho([g])\colon \rho(P_\pi,[\gamma_\pi]) = \operatorname{Out}_{P_\pi}(P) = \pi \longrightarrow g^{-1}\operatorname{Out}_Q(g(P))g = \rho(Q,[\gamma_\pi]\circ[g]).$$

Thus let $R(Q,[g]) = (Q,[g]\circ[\gamma_\pi],\rho([g]))$.

If $[h]\colon (Q,[g]) \longrightarrow (Q',[g'])$ is a morphism in $P_\pi{\downarrow}\tau$, then it is also a morphism in $P{\downarrow}\tau$ from $(Q,[g]\circ[\gamma_\pi])$ to $(Q',[g']\circ[\gamma_\pi])$. Since $[h]\circ[g]=[g']$, where all morphisms are considered in $P{\downarrow}\tau$, one has $\rho([h])\circ\rho([g]) = \rho([g'])$, and so $[h]$ is also a morphism $R(Q,[g]) \longrightarrow R(Q',[g'])$.

The functor L takes $(Q,[h],f)$ in $\pi{\downarrow}\rho$ to $(Q,[\iota^Q]\circ[\delta(f)])$ in $P_\pi{\downarrow}\tau$, where ι^Q is the inclusion $N_Q(P) \leq Q$, and a morphism $[g]\colon (Q,[h],f) \longrightarrow (Q',[h'],f')$ to itself. Notice that the definition of L on morphisms makes sense because of the relation pointed out in Remark 5.4 and Lemma A.18.

¿From the construction, $L\circ R = Id_{P_\pi{\downarrow}\tau}$. Finally,

$$R\circ L(Q,[h],[f]) = (Q,[\iota^Q]\circ[\delta(f)]\circ[\gamma_\pi],\rho([\iota^Q]\circ[\delta(f)])) = (Q,[h],[f]),$$

since $[\iota^Q]\circ[\delta(f)]\circ[\gamma_\pi]$ is a factorization of $[h]$ by Remark 5.4, and $\rho([\iota^Q]\circ[\delta(f)]) = f$. Therefore, $R\circ L = Id_{\pi{\downarrow}\rho}$. □

We are now ready to conclude the section with a proof of Proposition 5.2, and thus complete the proof of Proposition 5.1.

Proof of Proposition 5.2. [▶ 30] We have to show that for all discrete centric p-toral subgroups $P \leq S$ the undercategories $P{\downarrow}\tau$ are contractible. We do this by a descending induction on the dimension of objects in $\mathcal{O}_S^c(X)$ (see Definition 2.1).

For every discrete p-toral radical subgroup $P \leq S$ of X, the undercategory $P{\downarrow}\tau$ has an initial object, and is therefore contractible. Thus, the claim holds for all discrete p-toral radical subgroups of X, and in particular for any discrete Sylow subgroup (which is radical by Lemma 3.10).

Let P be an object in $\mathcal{C}$ which is not radical. Assume first that the claim holds for every discrete centric p-toral subgroup Q of the same dimension as that of P, and such that $|\pi_0(Q)| > |\pi_0(P)|$. In particular we may assume that for every p-subgroup $\{1\} \neq \pi \leq W_X(P)$, $|P_\pi{\downarrow}\tau|$ is contractible. By Lemma 5.5 $|P_\pi{\downarrow}\tau| \simeq |\pi{\downarrow}\rho|$, and so the functor ρ is right cofinal. Hence $|P{\downarrow}\tau| \simeq |\mathcal{S}_p(W)|$, and the right hand side is contractible since W is not p-reduced (see [Q]).

Next, assume that the claim holds for all subgroups whose dimension is strictly larger than that of P, and that it does not hold for P. By the previous paragraph, there must exist a non-trivial p-subgroup $\pi \leq W_X(P)$, such that $|P_\pi{\downarrow}\tau|$ is not mod p acyclic. Let

$$P_1 \overset{\text{def}}{=} P_\pi,$$

and notice that $P \leq_{\gamma_\pi} P_1$ is a proper subgroup since π is nontrivial. Repeating this argument produces a chain of infinite length

$$P \lneqq P_1 \lneqq \cdots \lneqq P_n \lneqq \cdots$$

of centric discrete p-toral subgroups. Furthermore, since all P_n have the same cohomological dimension, and since all homomorphisms $P_n \longrightarrow P_{n+1}$ are proper monomorphisms, the order of $\pi_0(P_n)$ must increase strictly with n. Let $Q \overset{\text{def}}{=} \operatorname{colim}_n P_n$ and let Q be the resulting subgroup of X. Then, $\dim(Q) \gneqq \dim(P)$ and, by Lemma 5.3, $|Q{\downarrow}\tau| \simeq |P_n{\downarrow}\tau|$ for all sufficiently large values of n . Therefore, the claim can not hold for Q, which contradicts the induction hypothesis and we have shown that $|P{\downarrow}\tau|$ is contractible. □

6. Subgroup ampleness of centric subgroups and centralizer ampleness of elementary abelian subgroups

This section contains the proof of Theorem C. The core of the proof is a comparison result between the homotopy type of homotopy colimits over $\mathcal{O}^c(X)$ and $\mathcal{F}^e(X)^{op}$.

Let X be a p-compact group, and fix a discrete Sylow subgroup $S \leq_{\iota_S} X$. We start by constructing a functor $\mathsf{E}\colon \mathcal{O}^c_S(X) \longrightarrow \mathcal{F}^e_S(X)^{op}$. Let $P \leq S$ be a subgroup which is centric in X. Define $\mathsf{E}(P) \leq Z(P)$ to be the maximal elementary abelian subgroup of $Z(P)$ (see Lemma A.13). Then $\mathsf{E}(P)$ is an elementary abelian subgroup of X via the inclusion to P followed by $P \leq_{\iota_P} X$. The following lemma shows how to define E on morphisms, and will be useful throughout the section.

Lemma 6.1. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_{\iota_S} X$. Let $P, Q \leq S$ be subgroups which are centric in X. Then for every morphism $[h]\colon P \longrightarrow Q$ in $\mathcal{O}^c_S(X)$, there is a unique homomorphism $\mathsf{E}[h]\colon \mathsf{E}(Q) \longrightarrow \mathsf{E}(P)$ such that the diagram*

$$\begin{array}{ccc} B\mathsf{E}(P) & \xrightarrow{inc} & BP \\ {\scriptstyle B\mathsf{E}[h]}\uparrow & & \downarrow{\scriptstyle h} \\ B\mathsf{E}(Q) & \xrightarrow{inc} & BQ \end{array}$$

commutes up to homotopy.

Proof. The morphism $[h]$ can be represented by a homomorphism $h\colon P \longrightarrow Q$, which is unique up to conjugation in Q. Since $h(P)$ is centric in Q by Lemma 3.2, $Z(h(P)) = C_Q(h(P)) \geq Z(Q)$. Define $\mathsf{E}[h]$ to be the composite

$$\mathsf{E}(Q) \leq Z(Q) \leq Z(h(P)) \xrightarrow{h^{-1}|_{Z(h(P))}} Z(P).$$

Since $\mathsf{E}(P)$ and $\mathsf{E}(Q)$ are fully characteristic, the image of this composite is in $\mathsf{E}(P)$. Since conjugation in Q leaves $Z(Q)$ fixed, the definition does [▶ 31] not depend on the choice of h. Commutativity of the square above is clear, and implies uniqueness. □

[▶ 32] Lemma 6.1 provides the definition of E on morphisms in $\mathcal{O}^c_S(X)$, and shows that $\mathsf{E}\colon \mathcal{O}^c_S(X) \longrightarrow \mathcal{F}^e_S(X)^{op}$ is well defined as a functor (that E respects composition follows from the uniqueness statement in the lemma). The next proposition provides a useful identification of overcategories of E.

Proposition 6.2. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_{\iota_S} X$. Let $E_0 \leq S$ be an elementary abelian subgroup and let $S_0 \leq C_X(E_0)$ be a discrete Sylow subgroup. Then there is an equivalence of categories*

$$\Theta_{E_0}\colon \mathcal{O}^c_{S_0}(C_X(E_0)) \longrightarrow \mathsf{E} \downarrow E_0.$$

Moreover, this equivalence is natural with respect to morphisms in $\mathcal{F}^e_S(X)$.

Proof. Fix an elementary abelian subgroup $E_0 \leq S$ and a discrete Sylow subgroup $S_0 \leq C_X(E_0)$. By Proposition 2.4, we can assume $S_0 = C_S(E_0)$. Objects in the overcategory $\mathsf{E} \downarrow E_0$ are pairs (P, f), where $P \leq S$ is centric in X, and $f\colon E_0 \longrightarrow \mathsf{E}(P)$ is a morphism in $\mathcal{F}^e_S(X)$. A morphism $[h]\colon (P, f) \longrightarrow (P', f')$ is determined by a morphism $[h]\colon P \longrightarrow P'$ in $\mathcal{O}^c_S(X)$, such that $\mathsf{E}[h] \circ f' = f$ in $\mathcal{F}^e_S(X)$.

If $P \leq S_0$ is an object in $\mathcal{O}^c_{S_0}(C_X(E_0))$, then $E_0 \leq_{j_P} \mathsf{E}(P)$. Define

$$\Theta_{E_0}(P) = (P, j_P).$$

Notice that this is well defined, since $P \leq S_0 \leq S$ is centric in X by Lemma 3.8. For a morphism $[h]\colon P \longrightarrow P'$ in $\mathcal{O}^c_{S_0}(C_X(E_0))$, let $\Theta_{E_0}([h]) = [h]$. Since $[h]$ also represents a morphism in $\mathcal{O}^c_S(X)$, and since $\mathsf{E}[h]\colon \mathsf{E}(P') \longrightarrow E(P)$ restricted to E_0 is the identity by construction, this is well defined.

Define a functor $T_{E_0}\colon \mathsf{E}{\downarrow}E_0 \longrightarrow \mathcal{O}^c_{S_0}(C_X(E_0))$ as follows. Given an object (P, f) in $\mathsf{E}{\downarrow}E_0$, consider the composite

$$BP \xrightarrow{\iota_P} BC_X(\mathsf{E}(P)) \xrightarrow{Bf^{\#}} BC_X(E_0).$$

Since S_0 is a discrete Sylow subgroup in $C_X(E_0)$, there exists a subgroup $P_f \leq S_0$, unique up to $C_X(E_0)$-conjugacy, such that $(BP_f, B\iota_{P_f})$ and $(BP, Bf^{\#} \circ \iota_P)$ are isomorphic as objects over $BC_X(E_0)$. For each object (P, f), fix a choice of an isomorphism (as objects in $\mathcal{O}(C_X(E_0))$).

$$[\alpha_{P_f}]\colon (P_f, B\iota_{P_f}) \longrightarrow (P, Bf^{\#} \circ \iota_P).$$

Define $T_{E_0}(P, f) = P_f$, and for a morphism $[h]\colon (P, f) \longrightarrow (Q, g)$ in $\mathsf{E}{\downarrow}E_0$, define $T_{E_0}([h]) = [\alpha^{-1}_{Q_g}] \circ [h] \circ [\alpha_{P_f}]$. In particular we may require that if $P \leq S_0$, then $T_{E_0}(P, j_P) = P$, and that in that case $[\alpha_P] = [1_P]$. To see that T_{E_0} is well defined is suffices to show that the diagram

$$\begin{array}{ccc}
BP & \xrightarrow{\quad h \quad} & BQ \\
\downarrow & & \downarrow \\
BC_X(\mathsf{E}(P)) & \xrightarrow{\mathsf{E}[h]^{\#}} & BC_X(\mathsf{E}(Q)) \\
& {\scriptstyle f^{\#}}\searrow \quad \swarrow{\scriptstyle g^{\#}} & \\
& BC_X(E_0) &
\end{array}$$

commutes. Indeed, the triangle commutes since $\mathsf{E}[h] \circ g = f$, and the rectangle by construction of $\mathsf{E}[h]$, and so the diagram is homotopy commutative. Clearly T_{E_0} depends on the choices made only up to a natural isomorphism.

Clearly, $T_{E_0} \circ \Theta_{E_0} = 1_{\mathcal{O}^c_{S_0}(C_X(E_0))}$. On the other hand, $\Theta_{E_0} \circ T_{E_0}(P, f) = (P_f, j_{P_f})$, and so to complete the proof we must produce an isomorphism in $\mathsf{E}{\downarrow}E_0$ between (P, f) and (P_f, j_{P_f}).

But, by Lemma 6.1 one has a commutative diagram of discrete abelian p-toral groups:

$$\begin{array}{ccccc} & & \mathsf{E}(P) & \xrightarrow{inc} & \mathcal{Z}(P) \\ & \overset{[f]}{\nearrow} & \uparrow & & \uparrow \\ E_0 & & \cong \Big| \mathsf{E}[\alpha_{P_f}] & & \cong \Big| \\ & \underset{[j_{P_f}]}{\searrow} & & & \\ & & \mathsf{E}(P') & \xrightarrow{inc} & \mathcal{Z}(P') \end{array} .$$

Thus $[\alpha_{P_f}]$ gives the necessary isomorphism, and Θ_{E_0} is an equivalence of categories as claimed.

Finally, it follows from the construction that if $\alpha \in \operatorname{Hom}_{\mathcal{F}^e_S(X)}(E_1, E_0)$, the square

$$\begin{array}{ccc} \mathcal{O}^c_{S_0}(C_X(E_0)) & \xrightarrow{\alpha_*} & \mathcal{O}^c_{S_1}(C_X(E_1)) \\ \Theta_{E_0} \downarrow & & \downarrow \Theta_{E_1} \\ \mathsf{E}{\downarrow}E_0 & \xrightarrow{\mathsf{E}{\downarrow}\alpha} & \mathsf{E}{\downarrow}E_1 \end{array}$$

commutes. Notice that the existence of the functor on centric orbit categories is guaranteed by Lemma 3.8. This proves the naturality statement. □

The next proposition is a key ingredient in our analysis, as it sets the ground for an inductive proof of the equivalence of statements (i) and (ii) in Theorem C. Recall that $\Psi \colon \mathcal{F}^e_S(X)^{op} \longrightarrow \mathsf{Sp}$ is the functor that takes a subgroup $E \leq S$ to the mapping space $\operatorname{Map}(BE, BX)_\iota$, where ι is the composite $BE \longrightarrow BS \longrightarrow X$, and a morphism to the induced map.

Proposition 6.3. *Let X be a p-compact group with a discrete Sylow subgroup $S \leq_{\iota_S} X$. Assume that for every [▶ 33] nontrivial elementary abelian p-subgroup $E_0 \leq S$, and any discrete Sylow subgroup S_0 of $C_X(E_0)$, the collection of all subgroups of S_0 which are centric in $C_X(E_0)$ is subgroup-ample for $C_X(E_0)$. Let (Φ, γ, η) be a subgroup diagram for X, where*

$$\Phi \colon \mathcal{O}^c_S(X) \longrightarrow \mathsf{Sp},$$

and let $\mathsf{E} \colon \mathcal{O}^c_S(X) \longrightarrow \mathcal{F}^e_S(X)^{op}$ be the functor constructed above, and let $L_{\mathsf{E}}(\Phi)$ denote the left homotopy Kan extension of Φ along E. Then, there exists a natural transformation $\rho \colon L_{\mathsf{E}}(\Phi) \longrightarrow \Psi$ such that the triple

$$(L_{\mathsf{E}}(\Phi), \rho, L_{\mathsf{E}}(\eta))$$

is a centralizer diagram for X naturally equivalent to the standard centralizer diagram $(\Psi, 1, ev)$.

Proof. The left Kan extension is defined as follows

$$L_{\mathsf{E}}(\Phi)(E_0) = \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi.$$

Given an object (P, f) in $\mathsf{E}{\downarrow}E_0$, since $f(E_0)$ is a central subgroup of P, there is a map

$$BE_0 \times \Phi(P) \xrightarrow{\mu(Bf \times Id)} \Phi(P) \xrightarrow{\eta(P)} BX,$$

where μ is induced by group multiplication, and this map is natural with respect to morphisms in $\mathsf{E}{\downarrow}E_0$. By adjointness, we get a map

$$\rho(E_0)\colon L_{\mathsf{E}}(\Phi)(E_0) \longrightarrow \mathrm{Map}(BE_0, BX)_\iota$$

where ι is the composite $BE_0 \longrightarrow BS_0 \longrightarrow BS \to X$. Moreover, this construction in natural in $\mathcal{F}^e_S$. That is, given $\alpha \in \mathrm{Hom}_{\mathcal{F}^e_S(X)}(E_0, E_1)$, the functor $\alpha^\sharp\colon \mathsf{E}{\downarrow}E_1 \longrightarrow \mathsf{E}{\downarrow}E_0$, induces a commutative diagram

$$\begin{array}{ccccc} BE_0 \times \Phi(P) & \xrightarrow{\mu(B(f\circ\alpha)\times Id)} & \Phi(P) & \xrightarrow{\eta(P)} & BX \\ {\scriptstyle B\alpha\times id}\downarrow & & =\downarrow & & =\downarrow \\ BE_1 \times \Phi(P) & \xrightarrow{\mu(Bf\times Id)} & \Phi(P) & \xrightarrow{\eta(P)} & BX. \end{array}$$

Summarizing, we have defined a natural transformation $\rho : L_{\mathsf{E}}(\Phi) \longrightarrow BC_X(-)$.

Finally, the morphism $BE_0 \times \Phi(P) \xrightarrow{\mu(Bf\times Id)} \Phi(P) \xrightarrow{\eta(P)} BX$ induces a commutative diagram

$$\begin{array}{ccc} BE_0 \times L_{\mathsf{E}}(\Phi)(E_0) & \xrightarrow{\mu(Bf\times Id)} & L_{\mathsf{E}}(\Phi)(E_0) \\ {\scriptstyle \mu(Bf\times L_{\mathsf{E}}(\eta)(E_0))}\downarrow & & \downarrow{\scriptstyle L_{\mathsf{E}}(\eta)(E_0)} \\ BX & \xrightarrow{Id} & BX. \end{array}$$

which, by adjointness, shows the compatibility of both diagrams over X.

$$\begin{array}{ccc} L_{\mathsf{E}}(\Phi)(E_0) & \xrightarrow{id} & L_{\mathsf{E}}(\Phi)(E_0) \\ {\scriptstyle \rho(E_0)}\downarrow & & \downarrow{\scriptstyle L_{\mathsf{E}}(\eta)} \\ BC_X(E_0) & \xrightarrow{ev} & BX. \end{array}$$

It remains to check that $L_{\mathsf{E}}(\Phi)$ is a centralizer diagram. By Proposition 6.2, the functor Θ_{E_0} induces a homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{O}^c_{S_0}(C_X(E_0))} \Theta^*_{E_0}(\Phi) \simeq \operatorname*{hocolim}_{\mathsf{E}\downarrow E_0} \Phi.$$

Notice that $\Theta^*_{E_0}(\rho)$ defines a natural transformation between $\Theta^*_{E_0}(\Phi)$ and $1_{BC_X(E_0)}$. Therefore, $\Theta^*_{E_0}(\Phi)$ is a subgroup diagram for $BC_X(E_0)$ which, by hypothesis, is a decomposition. This shows that ρ is an equivalence between $L_{\mathsf{E}}(\Phi)$ and Ψ, and so that $L_{\mathsf{E}}(\Phi)$ is a centralizer diagram for BX. □

We are now ready to prove the equivalence of statements (i) and (ii) in Theorem C. The following proposition claims the equivalence of two more general statements, which implies the equivalence claimed in the theorem. The proposition makes use of the concept of the order $|X|$ of a p-compact group X, i.e., the pair (d_X, o_X), where d_X is the cohomological dimension of X, and $o_X = |\pi_0(X)|$. (See Definition A.5). These pairs are ordered lexicographically.

Proposition 6.4. *Fix an ordered pair of nonnegative integers (d, o). Then the following statements are equivalent.*

(i) *For every p-compact group X with a discrete Sylow subgroup S, such that $|X| \leq (d, o)$, the collection of all subgroups of S which are centric in X is subgroup ample.*

(ii) *For every p-compact group X with a discrete Sylow subgroup S, such that $|X| \le (d,o)$, the collection of all elementary abelian subgroups $1 \ne E \le S$ is centralizer ample.*

Proof. (i)⇒(ii): Fix a p-compact group X of order $|X| \le (d,o)$ with a discrete Sylow subgroup $S \le X$. For any elementary abelian subgroup $E \le S$, $|C_X(E)| \le (d,o)$ (see Lemma A.6). Hence, assumption (i) applied to $C_X(E)$ for any elementary abelian $E \le S$ is that for any discrete Sylow subgroup $S' \le C_X(E)$, the collection of all subgroups $Q \le S'$ which are centric in $C_X(E)$ is subgroup ample. Thus the hypotheses of Lemma 6.3 are satisfied, and it follows that if Φ is a subgroup diagram for X with respect to the collection of all $P \le S$ which are centric in X, then the left Kan extension $L_{\mathsf{E}}(\Phi)$ is a centralizer diagram for X with respect to all elementary abelian subgroups $E \le S$. Furthermore, one has a commutative diagram

$$\tag{4} \begin{array}{ccccc} \operatorname*{hocolim}_{\mathcal{O}_S^c(X)} \Phi & \simeq & \operatorname*{hocolim}_{\mathcal{F}_S^e(X)^{op}} L_{\mathsf{E}}(\Phi) & \xrightarrow[\simeq]{\rho} & \operatorname*{hocolim}_{\mathcal{F}_S^e(X)^{op}} \operatorname{Map}(-,BX)_- \\ \eta\downarrow & & L_{\mathsf{E}}(\eta)\downarrow & & ev\downarrow \\ BX & = & BX & = & BX, \end{array}$$

where the first equivalence is given by the property of the left homotopy Kan extension with respect to homotopy colimits (see [HV, §4]). Hence if Φ is a subgroup decomposition of X, then $L_{\mathsf{E}}(\Phi)$ is a centralizer decomposition. In particular, assuming (i) for X, every subgroup diagram for X with respect to the collection of all subgroups $P \le S$ which are centric in X, is a subgroup decomposition. Hence (ii) holds for X.

(ii)⇒(i): Notice first that for finite p-groups P (i.e., p-compact groups of cohomological dimension 0), (i) and (ii) hold independently of each other ([▶ 34] since in this situation the fusion category has an initial object $E \le Z(P)$, and the orbit category has a terminal object P).

Assume by induction that (ii)⇒(i) for all p-compact groups Y such that $|Y| < (d,o)$. Let X be a p-compact group of order (d,o), and assume (ii) holds for X. We must show that (i) hold for X as well.

Consider first the special case where X is centerfree. With this assumption $|C_X(E)| < |X|$ by Lemma A.6, for any nontrivial elementary abelian subgroup $E \le S$. For a fixed elementary abelian subgroup $E \le S$, assumption (ii), applied to $C_X(E)$ is that for any discrete Sylow subgroup $S' \le C_X(E)$ the collection of all nontrivial elementary abelian subgroups $F \le S'$ is centralizer ample for $C_X(E)$. By induction hypothesis, the collection of all subgroups $Q \le S'$ which are centric in $C_X(E)$ is subgroup ample. Let $\Phi\colon \mathcal{O}_S^c(X) \longrightarrow \mathsf{Sp}$ be a subgroup diagram, which exists by Proposition B, and let $L_{\mathsf{E}}(\Phi)$ denote the left Kan extension of Φ along E. The hypotheses of Proposition 6.3 are satisfied, and it follows that $L_{\mathsf{E}}(\Phi)$ is a centralizer diagram for X with respect to the collection of all elementary abelian subgroups $F \le S$. Assumption (ii) applied to X is that this collection is centralizer ample, and so there are homotopy equivalences over BX (see diagram 4)

$$\operatorname*{hocolim}_{\mathcal{O}_S^c(X)} \Phi \simeq \operatorname*{hocolim}_{\mathcal{F}_S^e(X)^{op}} L_{\mathsf{E}}(\Phi) \simeq BX.$$

[▶ 35] The first equivalence follows again by the property of the left homotopy Kan extension [HV, §4] with respect to homotopy colimits, and the second by ampleness of the collection of elementary abelian p-subgroups. This shows that the collection of all subgroups $P \le S$ which are centric in X is subgroup ample and completes the proof in this case.

Let X be an arbitrary p-compact group of order (d, o). By Proposition 5.1, the collection of all subgroups $P \leq S$ which are centric in X is subgroup ample if and only if the collection of all subgroups of S which are radical in X is subgroup ample. By Proposition A.12, X is an extension of a centerfree p-compact group X' by a p-compact toral group K. Let S' be a discrete Sylow subgroup for X'. By Proposition 4.3, it suffices to show that the collection of all subgroups $P' \leq S'$ which are radical in X' is subgroup ample, which is equivalent to the statement that the collection of all subgroups of S' which are centric in X' is subgroup ample, by Proposition 5.1 again. But $|X'| \leq |X|$ and X' is centerfree, in which case we have already proven the claim. This completes the proof in the general case. □

Appendix A. p-compact groups

A p-compact group is a triple (X, BX, e) such that X is $\mathbb{F}_p$-finite, BX is pointed and p-complete, and $e\colon X \longrightarrow \Omega BX$ is a homotopy equivalence. The space BX is called the classifying space of X.

Homomorphisms. A homomorphism $f\colon X \to Y$ of p-compact groups $f\colon X \to Y$ is a pointed map $Bf\colon BX \to BY$ between their classifying spaces. The homotopy fibre of Bf is denoted by $Y/f(X)$.

Definition A.1. *A homomorphism of p-compact groups $f\colon X \longrightarrow Y$ is said to be*

- *a monomorphism if $Y/f(X)$ is $\mathbb{F}_p$-finite,*
- *an epimorphism if $Y/f(X)$ is the classifying space of a p-compact group, and*
- *an isomorphism if Bf is an homotopy equivalence.*

A short exact sequence of p-compact groups is a sequence of homomorphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration, up to homotopy. Such a short exact sequence is also called an extension of Z by X.

The behavior of monomorphisms and epimorphisms between p-compact groups with respect to composition was considered in several papers (e.g. [DW1, DW3, MN]). The following Lemma gives a summary for future reference.

Lemma A.2. *Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be homomorphisms of p-compact groups.*

(i) *If $f \circ g$ is a monomorphism, then g is a monomorphism.*
(ii) *If g is a monomorphism and $f \circ g$ is an epimorphism, then f is an epimorphism.*
(iii) *If f is a monomorphism and an epimorphism then it is an isomorphism.*

Proof. (i) is implicit in [DW1, Theorem 7.3] and [MN, Theorem 2.17]. For (iii), see [DW1, Remark 3.3]. To prove (ii), consider the following diagram of fibrations

$$\begin{array}{ccccc}
\Omega(Z/f(Y)) & \longrightarrow & Y/g(X) & \longrightarrow & BK \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & BX & \xrightarrow{=} & BX \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f\circ g} \\
Z/f(Y) & \longrightarrow & BY & \xrightarrow{f} & BZ
\end{array}$$

where $Y/g(X)$ is $\mathbb{F}_p$ finite since g is a monomorphism and K is a p-compact group since the composite $f \circ g$ is an epimorphism. The space $Z/f(Y)$ is p-complete, being the homotopy fibre of a map between p-complete spaces, and $\Omega(Z/f(Y))$ is $\mathbb{F}_p$ finite by inspection of the Serre spectral sequence of the fibration $K \to \Omega(Z/f(Y)) \to Y/g(X)$. Hence, $Z/f(Y)$ is the classifying space of a p-compact group, which means that f is an epimorphism. □

Subgroups, Maximal Tori, and Sylow Subgroups. One of the most important concepts in this paper is that of a subgroup.

Definition A.3. *A subgroup of a p-compact group X, is a pair (Y, α), where Y is a p-compact group, and $\alpha\colon BY \longrightarrow BX$ is a monomorphism.*

A p-compact torus of rank n is a p-compact group T, such that

$$BT \simeq K((\mathbb{Z}_p^\wedge)^n, 2) \simeq (B(\mathbb{Z}/p^\infty\mathbb{Z})^n)_p^\wedge,$$

where $\mathbb{Z}/p^\infty$ is the direct limit of all cyclic groups $\mathbb{Z}/p^r$ under inclusion. The group $(\mathbb{Z}/p^\infty\mathbb{Z})^n$ is called a p-discrete torus or a discrete approximation for T (where the prime p is understood). A maximal torus of a p-compact group X is a subgroup (T_X, α), where T_X is a p-compact torus, which is maximal in the sense that if (T, β) is any other subgroup with T a p-compact torus, then there exists a homomorphism $k\colon T \to T_X$ such that $\beta \simeq \alpha \circ k$.

Theorem A.4. [DW1, Theorem. 8.13] *Any p-compact group admits a maximal torus unique up to conjugacy.*

The next useful concepts we introduce are those of the *order* of a p-compact group, and the index of a subgroup.

Definition A.5. *Let X be a p-compact group. Define the order of X to be the pair (d_X, o_X), where d_X is the mod-p cohomological dimension of X and o_X is the order of its group of components. The order of X is denoted by $|X|$. If Y is another p-compact group, then we say $|Y| \le |X|$ if [▶ 36] $d_Y < d_X$ or if $d_Y = d_X$ and $o_Y \le o_X$. More generally, if $Y \le X$ is a subgroup, define the index of Y in X, denoted as usual $|X : Y|$, to be the pair $(d_{X/Y}, o_{X/Y})$, where $d_{X/Y}$ is the mod-p cohomological dimension of X/Y, and $o_{X/Y}$ the order of the set of components of X/Y. Thus $|X| = |X : 1|$. We say that Y is a subgroup of X of finite index, or of index n, if $|X : Y| = (0, n)$ for some n (essentially finite).*

Similarly, for a compact Lie group G, define the order $|G|$ of G to be the pair (d_G, o_G), where d_G is the dimension of G, and $o_G \overset{\text{def}}{=} |\pi_0(G)|$. Lexicographical ordering, as above, endows the class of all compact Lie groups with a linear order.

The next lemma shows that the order behaves as one would expect under taking subgroups.

Lemma A.6. *Let X be a p-compact group and let $Y \le_\alpha X$ be a subgroup. Then $|Y| \le |X|$ with equality holding if and only if α is an isomorphism.*

Proof. By [DW1, Proposition 6.15], $d_X = d_Y + d_{X/Y}$. Hence $d_Y \le d_X$ with equality if and only if $d_{X/Y} = 0$, i.e., if and only if X/Y is homotopically discrete. By [DW1, Remark. 6.16] this is the case if and only if α induces a homotopy equivalence between Y and a union of components of X, or equivalently if and only if $o_Y \le o_X$. Hence

$|Y| \leq |X|$ and equality holds if and only if α is a homotopy equivalence, i.e., an isomorphism of p-compact groups. □

A p-compact toral group is a p-compact group P, which is an extension of a finite p-group π by a p-compact torus. An important family of p-compact toral subgroups of any p-compact group is the collection of its maximal p-compact toral subgroups, which behaves in many ways like Sylow p-subgroups do in a finite group.

Definition A.7. *[▶ 37] A Sylow subgroup of a p-compact group X is a p-compact toral subgroup $S \leq_\iota X$, which is maximal in the sense that every other p-compact toral subgroup $P \leq_\alpha X$ factors through it. In other words, there exists a homomorphism $BP \xrightarrow{f} BS$, such that $\iota \circ f \simeq \alpha$.*

Notice that a Sylow subgroup of X is unique up to conjugacy. Since the prime p is fixed and since a p-compact toral group is in general not a p-group, we omit the prime p from the terminology and use "Sylow subgroup", rather than "Sylow p-subgroup".

By [DW2, Proposition 2.10] the p-normalizer [▶ 38] $\mathcal{N}_p(T)$ of the maximal torus in a p-compact group X is a subgroup such that $\chi(X/\mathcal{N}_p(T))$ is relatively prime to p, and by Proposition 2.14 in the same paper $\mathcal{N}_p(T)$ is a Sylow subgroup of X in the sense defined here. With the existence of at least one Sylow subgroup granted, the following lemma demonstrates the analogy of our concept with Sylow p-subgroups in the usual sense.

Lemma A.8. *Let X be a p-compact group, and let $P \leq_\alpha X$ be a p-compact toral subgroup. Then the following conditions are equivalent.*

- (i) *(P, α) is a Sylow subgroup in X.*
- (ii) *The Euler characteristic $\chi(X/P)$ is not divisible by p.*
- (iii) *(P, α) is a p-compact toral subgroup of maximal order.*

Proof. The implication (ii)⇒(i) is [DW2, Proposition 2.14]. Conversely, if (P, α) is a Sylow subgroup for X, then (P, α) is conjugate to $\mathcal{N}_p(T)$, and hence $\chi(X/P) = \chi(X/\mathcal{N}_p(T)$, which is relatively prime to p, by [M2, Theorem 1.2].

Next we prove (i)⇒(iii). Let (P, α) be a Sylow subgroup of X, and let $Q \leq_\beta X$ be any other p-compact toral subgroup. Then, by definition, there is a monomorphism $f\colon BQ \longrightarrow BP$, and by Lemma A.5 $|Q| \leq |P|$, so P is a subgroup of maximal order.

Finally we show (iii)⇒(i). Let (P, α) be a p-compact toral subgroup of X of maximal order, and let (Q, β) be a Sylow subgroup. Then there is a monomorphism $f\colon P \longrightarrow Q$, such that $\beta \circ f \simeq \alpha$. But, by the previous argument, $|P| \leq |Q|$, and so by maximality $|P| = |Q|$, and f is an isomorphism. This shows that (P, α) is also a Sylow subgroup. □

Centralizers and Centers. Let X be a p-compact group and let $Y \leq_\alpha X$ be a subgroup. The centralizer of (Y, α) in X is defined to be the loop space of the space

$$BC_X(Y, \alpha) \overset{\text{def}}{=} \mathrm{Map}(BY, BX)_\alpha.$$

Proposition A.9. *Let $P \leq_\alpha X$ be a p-compact toral subgroup of a p-compact group X. Then $C_X(P, \alpha)$ is a p-compact group.*

Proof. [DW1, Proposition 5.1, and Proposition 6.1] □

Next we define the center of a p-compact group.

Definition A.10. *A subgroup $Z \leq_\alpha X$ is central if $C_X(Z) \cong X$, or in other words if $ev\colon \operatorname{Map}(BZ, BX)_\alpha \to BX$ is a homotopy equivalence. A central subgroup of X is said to be the center of X, if every other central subgroup factors through it. A p-compact group X is said to be centerfree if it has no nontrivial central subgroup.*

For a p-compact group X, Dwyer and Wilkerson showed that

$$\mathcal{Z}(X) = \Omega \operatorname{Map}(BX, BX)_{id}$$

is an abelian p-compact toral group and has the property that the evaluation map $B\mathcal{Z}(X) \longrightarrow BX$ is, up to homotopy, a final object among all central monomorphisms into X [DW2, Theorems 1.2, 1.3]. Thus, whenever we say "the center of X", we mean the subgroup $(\mathcal{Z}(X), ev)$. Notice that X is centerfree if and only if $\mathcal{Z}(X)$ is weakly contractible.

Remark A.11. If the Sylow subgroup S of a p-compact group X is central, then they are equivalent since the p-compact group quotient $X/Z(X)$ is trivial (there are no non-trivial maps $B\mathbb{Z}/p \to B(X/Z(X))$).

The following lemma is useful in reducing certain claims to the centerfree case. [▶ 39] The notion of an extension of p-compact groups was defined in Definition 4.1.

Lemma A.12. *Every p-compact group X is an extension of a centerfree p-compact group by a p-compact toral group.*

Proof. For a p-compact group X we denote by X_0 the component of the identity element and by $\pi = \pi_0(X)$ the group of components. The canonical fibration

$$BX_0 \longrightarrow BX \longrightarrow B\pi$$

is classified by a map $B\pi \xrightarrow{\alpha} B\operatorname{Aut}(BX_0)$. Note that BX_0 is the 1-connected cover of BX where the fundamental group π of BX acts freely.

Let $Z_0 = \mathcal{Z}(X_0)$ be the center, and let $X_0' = X_0/Z_0$ be the centerfree quotient of X_0 (see [DW2, Theorem 6.3]). The classifying space of BX_0' is described by the Borel construction of the action of the topological group BZ_0 on $\operatorname{Map}(BZ_0, BX_0)_\iota$,

$$BX_0' \simeq (\operatorname{Map}(BZ_0, BX_0)_\iota)_{hBZ_0}.$$

The fundamental group acts freely on $\operatorname{Map}(BZ_0, BX_0)_\iota$, so $\operatorname{Map}(BZ_0, BX_0)_\iota/\pi \simeq BX$. Consider BX' to be the p-compact group $B(X/Z_0) = (\operatorname{Map}(BZ_0, BX_0)_\iota)_{hBZ_0}/\pi$. There is a commutative diagram of extensions of p-compact groups

$$\begin{array}{ccccc} BX_0 & \longrightarrow & BX & \longrightarrow & B\pi \\ \downarrow & & \downarrow & & \downarrow{=} \\ BX_0' & \longrightarrow & BX' & \longrightarrow & B\pi \end{array} .$$

If X' is centerfree, we are done, as we have presented X as an extension of a centerfree p-compact group by Z_0, and abelian p-compact group (in particular p-compact toral). Otherwise, let $Z' = \mathcal{Z}(X')$ be the center of X'. Since X_0' is centerfree, the composite $BZ' \longrightarrow BX' \longrightarrow B\pi$ is a monomorphism, and Z' is a central subgroup of π. Taking the quotients of X' and π by Z' gives a commutative diagram of extensions of p-compact groups

$$\begin{array}{ccccc} BX_0' & \longrightarrow & BX' & \longrightarrow & B\pi \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ BX_0' & \longrightarrow & BX_1 & \longrightarrow & B\pi_1 \end{array},$$

where $|\pi_1| \lneq |\pi|$. Notice that the homotopy fibre of the homomorphism $BX \longrightarrow BX_1$ is an extension of Z', which is a finite abelian p-group, by Z_0 which is an abelian p-compact group. Thus the homotopy fibre is the classifying space of a p-compact toral group.

If X_1 is centerfree, the proof is complete. Otherwise, divide X_1 by its center, which by the same argument as above is also a central subgroup of π_1, to obtain an extension X_2 of a finite p-group π_2, with $|\pi_2| \lneq |\pi_1|$, by X_0', and such that the homotopy fibre of the projection $BX \longrightarrow BX_2$ is the classifying space of a p-compact toral group. Applying this process repeatedly yields in finitely many steps (since π is finite) a p-compact group quotient Y of X, such that Y is centerfree, and the homotopy fibre of the projection $BX \longrightarrow BY$ is the classifying space of a p-compact toral group. □

Next, we consider the maximal central elementary abelian subgroup of a p-compact group.

Lemma A.13. *Any p-compact group X admits a maximal central elementary abelian subgroup $\mathsf{E}(X)$.*

Proof. Since $\mathcal{Z}(X)$ is an abelian p-compact toral group, it admits a maximal elementary abelian p-subgroup of $\mathsf{E}(X) \leq_\iota \mathcal{Z}(X)$.

The subgroup $(\mathsf{E}(X), \iota)$ is clearly maximal in the sense that if $F \leq_\beta X$ is any other central elementary abelian p-subgroup of X, then β factors up to homotopy through $B\mathsf{E}(X)$. This follows from the fact that $(\mathcal{Z}(X), ev)$ is a final object among all central subgroups ([DW2, Proposition 1.2]) □

Remark A.14. Notice that the symbol $\mathsf{E}(X)$ is used elsewhere to denote a functor. The reader should not be confused by this abuse of notation. If X is a p-compact toral group and $\check{X}$ is a discrete approximation, then the algebraic center of $\check{X}$ is a discrete approximation for the center of X. With this setup, it is possible to define $\mathsf{E}(X)$ canonically. We choose to use the same symbol here to emphasize that it is practically the same construction we discuss here, but without specifying discrete approximations.

Remark A.15. Let $Y \leq_\alpha X$ be a subgroup, let $\mathsf{E}(Y) \leq \mathcal{Z}(Y)$ denote a maximal central elementary abelian subgroup and consider the maps δ and ϵ given by the composites

$$\delta \stackrel{\text{def}}{=} \big(BY \times B\mathsf{E}(Y) \longrightarrow BY \times B\mathcal{Z}(Y) \xrightarrow{mult} BY \xrightarrow{\alpha} BX\big)$$

and

$$\epsilon \stackrel{\text{def}}{=} \big(\mathsf{E}(Y) \longrightarrow \mathcal{Z}(Y) \longrightarrow Y \xrightarrow{\alpha} X\big).$$

Then the map $ad(\delta)\colon BY \longrightarrow \mathrm{Map}(B\mathsf{E}(Y), BX)_\epsilon$ makes the following diagram homotopy commutative

$$\begin{array}{ccc} BY & \xrightarrow{ad(\delta)} & \mathrm{Map}(B\mathsf{E}(Y), BX)_\epsilon \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle ev} & \\ & BX & \end{array}$$

Thus $Y \leq C_X(\mathsf{E}(Y)) \leq X$ is a factorization of $Y \leq_\alpha X$.

Homotopy fixed points. For a group G and a G-space Z, we denote by Z^{hG} the homotopy fixed point space of Z under the action of G, i.e., the space $\mathrm{Map}_G(EG, Z)$, where EG is a free contractible G-space, or equivalently the space of sections of the Borel construction

$$Z_{hG} \stackrel{\text{def}}{=} Z \times_G EG \longrightarrow * \times_G EG = BG.$$

A proxy action of G on Z is a space W homotopy equivalent to Z together with an action of G on W (see [DW1, Section 10]). Standard constructions on G-spaces can be defined for proxy actions. In [DW1, Lemma 10.4], Dwyer and Wilkerson show that given a fibration $p\colon E \longrightarrow BG$, there is a proxy action of G on the homotopy fiber F of p such that F^{hG} is homotopy equivalent to the space of sections of p. [▶ 40] The notions of proxy actions and homotopy fixed points are defined analogously when G is a p-compact group.

In this context, given a subgroup $Y \leq_\alpha X$, one can replace α by a fibration and consider a proxy action of X on the homogeneous space X/Y. If $Z \leq_\beta X$ is another subgroup, considering the pullback fibration, one can easily identify the space of homotopy fixed points $(X/Y)^{hZ}$ with the homotopy fiber of

$$\mathrm{Map}(BZ, BY)_{\{\gamma\}} \longrightarrow \mathrm{Map}(BZ, BX)_\beta$$

where $\{\gamma\}$ is the set of components which are homotopic to β when composed with α. In particular, there is an epimorphism

$$\pi_0((X/Y)^{hZ}) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}(X)}((Z,\beta),(Y,\alpha)).$$

Proposition A.16. *[▶ 41] Let X be a p-compact group, and $Q \leq X$ be p-compact toral subgroup of X. Given an increasing sequence $P_0 \leq P_1 \leq \cdots \leq P = \mathrm{colim}_n P_n$ of discrete p-toral groups, there exists an $N > 0$ such that $(X/Q)^{hP} \simeq (X/Q)^{hP_i}$ for any $i > N$.*

Proof. Let T_n be the kernel of the composite $P_n \longrightarrow \check{P} \longrightarrow \pi$, where $\pi = \pi_0(BP^\wedge_p)$. Then $\check{T} \stackrel{\text{def}}{=} \mathrm{colim}_n T_n$ is a maximal discrete p-torus of P, and for a sufficiently large n, the map $P_n \longrightarrow \pi$ is an epimorphism. For each n one has a map

$$(X/Q)^{hP} \simeq ((X/Q)^{hT})^{h\pi} \longrightarrow ((X/Q)^{hT_n})^{h\pi} \simeq (X/Q)^{hP_n}.$$

Hence, if we show that for n sufficiently large the map $(X/Q)^{hT_n} \longrightarrow (X/Q)^{hT}$ is an equivalence, then the lemma holds for all p-compact toral groups.

Thus assume $P = T$, a p-compact torus, and notice that it is enough to consider the case in which $\check{T} = \mathrm{colim}\, T_n$ is a p-discrete approximation. Let ι_n be the restriction of $\iota_T\colon BT \longrightarrow BX$ to BT_n. Then, there is a homotopy commutative diagram of fibrations [▶ 41]

$$\begin{array}{ccc}
(X/Q)^{hT_{n+1}} & \longrightarrow & (X/Q)^{hT_n} \\
\downarrow & & \downarrow \\
\mathrm{Map}(BT_{n+1}, BQ)_{\{\iota_{n+1}\}} & \xrightarrow{inc^\sharp} & \mathrm{Map}(BT_n, BQ)_{\{\iota_n\}} \\
{\scriptstyle \iota_{Q\sharp}}\downarrow & & {\scriptstyle \iota_{Q\sharp}}\downarrow \\
\mathrm{Map}(BT_{n+1}, BX)_{\iota_{n+1}} & \xrightarrow{inc^\sharp} & \mathrm{Map}(BT_n, BX)_{\iota_n}
\end{array} \tag{5}$$

where $\{\iota_k\}$ is the set of homotopy classes of maps $BT_k \xrightarrow{f} BQ$, such that $\iota_Q \circ f \simeq \iota_k$. Notice that the sets $\{\iota_k\}$ are finite sets, since by [DW1, Theorem 4.6, and Theorem 5.8], $(X/Q)^{hT_k}$ is $\mathbb{F}_p$-finite for all k.

Next we show that, for n sufficiently large, the map between the total spaces in the diagram above induces a monomorphism on the sets of components. This is equivalent to the claim that, if $f, g\colon BT_{n+1} \longrightarrow BQ$ are two maps whose homotopy classes are contained in $\{\iota_{n+1}\}$, and such that their restriction to BT_n are homotopic, then $f \simeq g$. In other words, if f and g are as above and both render the diagram

$$\begin{array}{ccc} BT_n & \xrightarrow{\iota_n} & BQ \\ {\scriptstyle j}\downarrow & \overset{f}{\underset{g}{\nearrow}} & \downarrow{\scriptstyle \iota_Q} \\ BT_{n+1} & \xrightarrow[\iota_{n+1}]{} & BX \end{array}$$

homotopy commutative, then $f \simeq g$.

Apply the functor $\mathrm{Map}(T_n, -)$ to the diagram above. Since ,T_n and T_{n+1} are abelian, this gives a homotopy commutative diagram

$$\begin{array}{ccc} BT_n & \xrightarrow{\iota_{n\sharp}} & BC_Q(T_n) \\ {\scriptstyle j_\sharp}\downarrow & \overset{f_\#}{\underset{g_\#}{\nearrow}} & \downarrow{\scriptstyle \iota_{Q\sharp}} \\ BT_{n+1} & \xrightarrow[h_{n+1\sharp}]{} & BC_X(T_n) \end{array}.$$

For n sufficiently large, the map $BC_X(T_{n+1}) \longrightarrow BC_X(T_n)$ is an equivalence by [DW1, Propsition 6.18], and hence the composite

$$BT_{n+1} \longrightarrow BC_X(T_{n+1}) \longrightarrow BC_X(T_n)$$

is central. By [DW2, Lemma 6.5] it now follows that $f_\sharp \simeq g_\sharp$ and hence that $f \simeq g$ [▶ 43] by evaluation.

We have thus shown that, for n sufficiently large, the map between total spaces in Diagram (5) above induces a monomorphism on path components. Hence the sequence of path components stabilizes, and is equal to the set $\{\iota_T\}$ which is finite. By [DW1, Proposition 6.18] again, the map between [▶ 44] components of total spaces in the diagram is a homotopy equivalence for n sufficiently large. Hence for such an n, the induced map on homotopy fibres is a homotopy equivalence, and the result follows. □

Corollary A.17. *[▶ 45] For any p-compact toral subgroups $P \leq_\alpha X$ and $Q \leq_\beta X$, $(X/Q)^{hP}$ is $\mathbb{F}_p$-finite and $\chi((X/Q)^{hP}) \equiv \chi(X/Q)$ mod p. In particular, $\mathrm{Mor}_{\mathcal{O}(X)}(P, Q)$ is finite.*

If P is centric, then $(X/Q)^{hP}$ is either empty or it is homotopically discrete. If X is p-compact toral and P is a proper subgroup of X, then the group $\mathrm{Aut}_{\mathcal{O}(X)}(P)$ is a non-trivial finite p-group.

Proof. [▶ 46] By Proposition A.16, applied to a discrete approximation $\check{P} = \operatorname{colim} P_n$ of P, $(X/Q)^{hP} \simeq (X/P)^{hK}$, for some finite p-group K. By [DW1, Proposition 5.8], for every subgroup $L \leq K$, each component of $(X/Q)^{hL}$ is p-complete. Hence by [DW1, Theorem 4.6] $(X/Q)^{hP} \simeq (X/Q)^{hK}$ is $\mathbb{F}_p$-finite and $\chi((X/Q)^{hP}) \equiv \chi(X/Q)$ mod p. The map

$$\pi_0((X/Q)^{hP}) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}(X)}((P, \alpha), (Q, \beta))$$

induced by the fibration

$$(X/Q)^{hP} \to \mathrm{Map}(BP, BQ)_{\{\alpha\}} \to \mathrm{Map}(BP, BX)_\alpha,$$

where $\{\alpha\} = \mathrm{Hom}_{\mathcal{O}(X)}((P,\alpha),(Q,\beta))$, is exhaustive. This shows that $\mathrm{Hom}_{\mathcal{O}(X)}((P,\alpha),(Q,\beta))$ is finite since $\pi_0((X/Q)^{hP})$ is so.

Assume now that P is centric. The homotopy fibre of the map

$$\beta_\#\colon \mathrm{Map}(BP, BQ)_{\{\alpha\}} \longrightarrow \mathrm{Map}(BP, BX)_\alpha$$

is $(X/Q)^{hP}$ by [DW1, Lemma 10.4], and $\beta_\#$ is a homotopy equivalence on each component by Lemma 3.2. Hence $(X/Q)^{hP}$ is homotopically discrete, and the morphism set $\mathrm{Mor}_{\mathcal{O}(X)}((P,\alpha),(Q,\beta))$, given by the set of components $\{\alpha\}$, is in $1-1$ correspondence with the set of components $\pi_0((X/Q)^{hP})$. The identification with the automorphism group of P in $\mathcal{O}_S(X)$ is immediate.

If X is p-compact toral and $S \xrightarrow{\iota} X$ is a discrete approximation, by Proposition 3.5 and Proposition 2.6 , $W_X(P)$ is a finite p-group isomorphic to $N_S(P)/P = (S/P)^P$, which is non-trivial since $\chi((X/Q)^{hP}) \equiv \chi(X/Q) \bmod p$. □

Finally, an interesting observation on morphisms in the orbit category of a p-compact group.

Lemma A.18. *Let X be a p-compact group X, with a discrete Sylow subgroup $S \leq X$. [▶ 47] Then every morphism in $\mathcal{O}^c_S(X)$ is an epimorphism in the categorical sense, i.e., if $P, Q, R \leq S$ are objects, and $[f]\colon P \longrightarrow Q$, and $[g],[h]\colon Q \longrightarrow R$ are morphisms such that $[g]\circ[f] = [h]\circ[f]$, then $[g] = [h]$.*

Proof. By Proposition 3.6, since all subgroups in question are centric in X, it suffices to show that for P, Q, R and f as above, the induced map

$$(X/\hat{R})^{hQ} \xrightarrow{f^*} (X/\hat{R})^{hP} \tag{6}$$

is injective on components. From now on, we identify P as a subgroup of Q. Let $T \leq Q$ denote the maximal torus in Q. Let $T_r \leq T$ denote the subgroup of all elements of order at most p^r. Then

$$P \leq P\cdot T_1 \leq \ldots P\cdot T_r \leq \ldots P\cdot T \leq Q$$

all inclusions are of (finite) p-power index, and all subgroups are centric in X. Hence the sequence can be refined to a sequence of inclusions

$$P = P_0 \leq P_1 \leq \ldots \leq P\cdot T = Q_0 \leq Q_1 \leq \ldots Q_s = Q, \tag{7}$$

where each subgroup is normal of p-power index in the following one (and is centric in X). Hence the map (6) can be refined into a sequence

$$(X/\hat{R})^{hP_0} \longleftarrow (X/\hat{R})^{hP_1} \longleftarrow \cdots (X/\hat{R})^{hQ_0} \longleftarrow \cdots (X/\hat{R})^{hQ}.$$

By Proposition A.16 there exists $n > 0$ big enough such that $(X/\hat{R})^{hP_i} \xleftarrow{\simeq} (X/\hat{R})^{hQ_0}$ for all $i > N$. Now, let $U_i \lhd U_{i+1}$ be any two consecutive groups in the sequence (7). Then

$$(X/\hat{R})^{hU_{i+1}} \simeq ((X/\hat{R})^{hU_i})^{h(U_{i+1}/U_i)} \simeq \pi_0((X/\hat{R})^{hU_i})^{U_{i+1}/U_i},$$

where the last equivalence follows since $(X/\hat{R})^{hU_i}$ is homotopically discrete, and the projection to its set of component is equivariant. This shows that each map in the refinement of (6) is a monomorphism, and hence that the map (6) itself is a monomorphism. □

Some categorical constructions. A standard reference for this material is [HV]. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between small categories. For an object $d \in \mathcal{D}$, the *undercategory* $d{\downarrow}F$ is the category with objects given by pairs (c, α), where c is an object in $\mathcal{C}$, and $\alpha\colon F(c) \longrightarrow d$ is a morphism in $\mathcal{D}$. A morphism $(c, \alpha) \longrightarrow (c', \alpha')$ in $d{\downarrow}F$ is a morphism $\varphi\colon c \longrightarrow c'$ in $\mathcal{C}$, such that $\alpha' \circ F(\varphi) = \alpha$. The functor F is said to be *right cofinal* (resp. *mod p right cofinal*) if the nerve of the undercategory $d{\downarrow}F$ is contractible (resp. *mod p acyclic*) for every object d in $\mathcal{D}$. The *overcategory* $F{\downarrow}d$ is defined by analogy, and F is said to be *left cofinal* (resp. *mod p left cofinal*) if $F{\downarrow}d$ is contractible (resp. *mod p acyclic*) for every $d \in \mathcal{D}$.

Theorem A.19. *If $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is right cofinal (resp.* mod p right cofinal*) and $\phi\colon \mathcal{D} \longrightarrow \mathsf{Sp}$ is any functor then the induced map [*▶ 49*]*

$$\operatorname*{hocolim}_{\mathcal{C}} \phi \circ F \longrightarrow \operatorname*{hocolim}_{\mathcal{D}} \phi$$

is a homotopy equivalence (resp. mod p homology equivalence).

The mod p right cofinality statement in Theorem A.19 can be easily obtained by adapting the proof in [HV].

Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor between small categories, and let $\mathcal{C} \xrightarrow{\phi} \mathsf{Sp}$ be any functor. The *left homotopy Kan extension* of ϕ along F is the functor $L_F(\phi)\colon \mathcal{D} \longrightarrow \mathsf{Sp}$ defined on objects by

$$L_F(\phi)(d) \stackrel{\text{def}}{=} \operatorname*{hocolim}_{F\downarrow d} \phi \circ \iota,$$

where $\iota\colon F{\downarrow}d \longrightarrow \mathcal{C}$ is the obvious functor taking (c, α) to c. The left homotopy Kan extension has the property that there is a natural homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{D}} L_F(\phi) \simeq \operatorname*{hocolim}_{\mathcal{C}} F.$$

Appendix B. Subgroup decompositions for classifying spaces of compact Lie groups

The main theorem of this paper is the existence of a subgroup homology decomposition for p-compact groups with respect to the collection of all their radical subgroups. The first such decomposition was constructed for classifying space of compact Lie groups by Jackowski, McClure and Oliver [JMO]. In this appendix we show that our main theorem is in fact a generalization of the Jackowski-McClure-Oliver result. By this we mean that the orbit category of radical subgroups, as defined in [JMO] is equivalent to the orbit category of radical subgroups constructed in this paper from the homotopy type of the respective p-completed classifying space. Furthermore, the decomposition functor constructed in this paper, and the one used in [JMO] also coincide up to homotopy, as we explain below. We start by recalling the basic construction from [JMO].

For a compact Lie group G, the orbit category $\mathcal{O}_p(G)$ is a category whose objects are G-orbits G/P, where $P \le G$ is a p-toral subgroup, and whose morphisms are G-maps $G/P \longrightarrow G/Q$. The morphism set $\mathrm{Mor}_{\mathcal{O}_p(G)}(G/P, G/Q)$ can be identified with the fixed point set $(G/Q)^P$. We call this category "*the group theoretic orbit category of all p-toral subgroups of G*". Let $\mathcal{O}^r(G) \subset \mathcal{O}_p(G)$ denote the full subcategory whose objects are orbits G/P, where $P \le G$ is a p-toral p-radical subgroup of G, i.e. those subgroups P whose Weyl group $W_G(P) \overset{\text{def}}{=} N_G(P)/P$ is finite and p-reduced. There is a functor

$$\Phi_{\mathfrak{g}} \colon \mathcal{O}^r(G) \longrightarrow \mathsf{Sp},$$

which takes an orbit G/P to the homotopy orbit space $(G/P)_{hG} \overset{\text{def}}{=} G/P \times_G EG$, and a G-map $G/P \longrightarrow G/Q$ to the induced map. Furthermore, the obvious natural transformation from the forgetful functor $\mathcal{O}^r(G) \longrightarrow G\text{-}\mathsf{Sp}$ to the constant functor with value a point induces a natural transformation $\xi \colon \Phi_{\mathfrak{g}} \longrightarrow 1_{BG}$. Thus one gets a map

$$\xi_* \colon \operatorname*{hocolim}_{\mathcal{O}^r(G)} \Phi_{\mathfrak{g}} \longrightarrow BG,$$

which by [JMO] is a mod-p equivalence.

If G is a compact Lie group and $\pi_0(G)$ is a finite p-group, then $G^\wedge_p$ is a p-compact group with classifying space $B(G^\wedge_p) \simeq (BG)^\wedge_p$. The orbit category, as defined in this paper, is called "*the homotopy theoretic orbit category of all p-toral subgroups of G*". Our aim is to show that the group theoretic orbit category of p-radical subgroups of G is equivalent to the orbit category of radical subgroups of $G^\wedge_p$. We will also observe that this claim fails if one does not restrict to p-radical subgroups.

Let

$$\varphi_G \colon \mathcal{O}^r(G) \longrightarrow \mathcal{O}_p(G^\wedge_p)$$

be the functor taking an object G/Q to the p-compact toral subgroup $(Q^\wedge_p, \iota_Q)$, where $\iota_Q \colon BQ^\wedge_p \to BG^\wedge_p$ is the p-completion of the map

$$BQ \simeq (G/Q)_{hG} \longrightarrow *_{hG} = BG.$$

For a morphism $G/Q \xrightarrow{a} G/Q'$ in $\mathcal{O}^r(G)$, $\varphi_G(a)$ is defined to be the homotopy class of the induced map.

The proof of the following proposition is given later at the end of the appendix.

Proposition B.1. *Let G be a compact Lie group such that $\pi_0(G)$ is a finite p-group. Then, the functor φ_G takes values in $\mathcal{O}^r(G^\wedge_p)$ and*

$$\varphi_G \colon \mathcal{O}^r(G) \longrightarrow \mathcal{O}^r(G^\wedge_p)$$

is an equivalence of categories.

Let $\Phi \colon \mathcal{O}^r(G^\wedge_p) \longrightarrow \mathsf{Sp}$ be any subgroup decomposition functor, and consider the composite functor $\Phi \circ \varphi_G$ on $\mathcal{O}^r(G)$. By Proposition B.1, $\mathcal{O}^r(G)$ can be identified with $\mathcal{O}^r(G^\wedge_p)$, and the functors $(\Phi_{\mathfrak{g}})^\wedge_p$ and $\Phi \circ \varphi_G$ are clearly subgroup diagrams on it. By Proposition B of the Introduction, these two functors are naturally homotopy equivalent. Hence, one obtains a homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{O}^r(G)} (\Phi_{\mathfrak{g}})^\wedge_p \simeq \operatorname*{hocolim}_{\mathcal{O}^r(G)} \Phi \circ \varphi_G.$$

This shows that our decomposition, restricted to the class of p-compact groups which arise as the p-completed classifying spaces of appropriate Lie groups, coincides with the Jackowski-McClure-Oliver decomposition, up to p-completion.

The following two lemmas are needed for the proof of Proposition B.1. Recall that a p-toral group is an extension of a finite p-group π by a torus $T = (S^1)^n$ for some $n \geq 0$. A space X is said to be p-good if the completion map $X \longrightarrow X^\wedge_p$ is a mod-p equivalence.

Lemma B.2. *Let Q be a p-toral compact Lie group and K a p-good finite Q-complex. Then, $(K^Q)^\wedge_p \simeq (K^\wedge_p)^{h(Q^\wedge_p)}$.*

Proof. By the generalized Sullivan conjecture for p-toral compact Lie groups [N1], the map $K^Q \longrightarrow (K^\wedge_p)^{hQ}$ is a mod-p equivalence. Hence, we have to show that $(K^\wedge_p)^{hQ} \simeq (K^\wedge_p)^{h(Q^\wedge_p)}$. As homotopy fixed point sets, these spaces are given as the fibre of the left and right vertical arrows in the diagram

$$\begin{array}{ccccc} \mathrm{Map}(BQ, (K^\wedge_p)_{hQ})_{\{id\}} & \xrightarrow{l_*} & \mathrm{Map}(BQ, (K^\wedge_p)_{h(Q^\wedge_p)})_{\{l\}} & \xleftarrow[\simeq]{l^*} & \mathrm{Map}(BQ^\wedge_p, (K^\wedge_p)_{h(Q^\wedge_p)})_{\{id\}} \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Map}(BQ, BQ)_{id} & \xrightarrow{l_*} & \mathrm{Map}(BQ, BQ^\wedge_p)_l & \xleftarrow[\simeq]{l^*} & \mathrm{Map}(BQ^\wedge_p, BQ^\wedge_p)_{id}, \end{array}$$

where l denotes the completion map $BQ \longrightarrow BQ^\wedge_p$, and the map π denotes, in each case, the map induced by the projection from the homotopy orbit space to the respective classifying space. Since $(K^\wedge_p)_{h(Q^\wedge_p)}$ and $BQ^\wedge_p$ are p-complete, both arrows marked l^* are homotopy equivalences, and so the homotopy fibres of the right and middle vertical arrows are equivalent. The left square arises by applying the functor $\mathrm{Map}(BQ, -)$ to a pull-back diagram and is therefore itself a pull-back diagram. This shows that homotopy fibres of all vertical arrows homotopy equivalent and finishes the proof. □

Lemma B.3. *Let G be a compact Lie group such that $\pi_0(G)$ is a finite p-group. Let (Q, β) be a p-compact toral subgroup of $G^\wedge_p$, which is either finite or radical in $BG^\wedge_p$. Then there exists a p-toral subgroup $P \leq G$ and a mod-p equivalence $h\colon BP \longrightarrow BQ$ such that the diagram*

$$\begin{array}{ccc} BP & \xrightarrow{B\iota_P} & BG \\ {\scriptstyle h}\downarrow & & \downarrow \\ BQ & \xrightarrow{\beta} & BG^\wedge_p \end{array}$$

commutes up to homotopy, where $P \xrightarrow{\iota_P} G$ is the inclusion.

Proof. If Q is a finite p-group, then $\mathrm{Map}(BQ, BG^\wedge_p) \simeq \mathrm{Map}(BQ, BG)^\wedge_p$ (see for instance [BL, Proposition 2.1]. In particular both sides have the same path components, and the components in the right hand side are given by $\mathrm{Rep}(Q, G) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(Q, G)/\sim$, where the equivalence relation is given by conjugation in G. Thus let $Q \xrightarrow{\varphi} G$ be a homomorphism, such that $B\varphi \simeq \beta$, let $P \stackrel{\mathrm{def}}{=} \mathrm{Im}\,\varphi$, and let $BP \xrightarrow{h} BQ$ the map induced by the inverse of φ (considered as a map $Q \to P$). Then the statement holds for P and h.

Let G be a compact Lie group with $\pi_0(G)$ a finite p-group. Let $O_p(G)$ denote the maximal normal p-toral subgroup of G, and let $\bar{G}$ denote the quotient group $G/O_p(G)$. (Notice that $O_p(G)$ exists, since it can be taken to be the intersection of all Sylow subgroups of G) Then $\bar{G}$ contains no normal p-toral subgroup, and in particular its center contains no such subgroup. On the other hand, since $Z(\bar{G})$ is an abelian compact Lie group, it is isomorphic to a product of a torus and a finite abelian group. Hence, $Z(\bar{G})$ is a finite group of order prime to p. Notice also that $\pi_0(\bar{G})$ is a finite p-group, since $\bar{G}$ is a quotient group of G. Thus $\bar{G}^\wedge_p$ is a p-compact group, and since $Z(\bar{G})$ is a finite group of order prime to p, $\bar{G}^\wedge_p$ is centerfree. By Proposition 4.2, there is a 1–1 correspondence between isomorphism classes of p-compact toral radical subgroups of

$G^\wedge_p$ and those of $\bar{G}^\wedge_p$ (considered as objects of the orbit category in both cases). Hence, it suffices to prove the claim for compact Lie groups whose center is a finite group of order prime to p.

The claim is obvious if G is p-toral, since the only radical subgroup of $G^\wedge_p$ in that case is $G^\wedge_p$ itself. The proof now proceeds by induction on the order. Let G be an arbitrary compact Lie group, such that $\pi_0(G)$ is a finite p-group. Assume the lemma holds for all compact Lie groups H, satisfying the same condition, and such that $|H| \lneqq |G|$. We must show that it holds for G.

By the discussion above, we may assume that $Z(G)$ is finite of order prime to p. Thus $G^\wedge_p$ is a centerfree p-compact group. Let $Q \leq_\beta G^\wedge_p$ be a radical p-compact toral subgroup, and let $\mathsf{E}(Q)$ be the maximal central elementary abelian subgroup in Q. Since $\mathsf{E}(Q)$ is a finite p-group, we may assume by the discussion above that $\mathsf{E}(Q)$ is a p-subgroup of G. Since

$$C_G(\mathsf{E}(Q))^\wedge_p \simeq \Omega(\mathrm{Map}(B\mathsf{E}(Q), BG)_{inc})^\wedge_p \simeq \Omega(\mathrm{Map}(B\mathsf{E}(Q), BG^\wedge_p)_{inc}) = C_{G^\wedge_p}(\mathsf{E}(Q)),$$

and since $Q \leq G^\wedge_p(\mathsf{E}(Q))$ is radical there by Lemma 3.11, it suffices to prove the claim for $C_G(\mathsf{E}(Q))$. But since $Z(G)$ is finite of order prime to p, $|C_G(\mathsf{E}(Q))| \lneqq |G|$ and the claim follows from the induction hypothesis. □

Proof of Proposition B.1. Fix a compact Lie group G with $\pi_0(G)$ a finite p-group. For radical p-toral subgroups $Q, Q' \leq G$, the fixed point set $(G/Q')^Q$ is finite or empty. Hence there are homotopy equivalences

$$(G/Q')^Q \simeq (G/Q')^{Q^\wedge_p} \simeq ((G/Q')^\wedge_p)^{h(Q^\wedge_p)},$$

where the second equivalence follows from Lemma B.2. This shows that $Q \leq G$ is radical if and only if $Q^\wedge_p \leq G^\wedge_p$ is radical, and so the functor φ_G takes values in the category $\mathcal{O}^r(G^\wedge_p)$. Furthermore, since the morphisms in the respective categories are the path components of the left and right hand sides of the spaces in the equation above, φ_G induces an isomorphism on morphism sets. It is also clear that φ_G is an injection on isomorphism classes of objects, and by Lemma B.3, it is also an epimorphism on the isomorphism classes of objects. Thus φ_G is an equivalence of categories as stated. □

We end this appendix with the observation that Lemma B.3 (and hence our argument in the proof of Lemma B.1) fails if one does not require that the subgroup $Q \leq_\beta G^\wedge_p$ is either finite or radical.

Remark B.4. Let $G \stackrel{\text{def}}{=} S^1 \times S^1$, let $\alpha, \beta \in \mathbb{Z}^\wedge_p$ be units, and let $f\colon (BS^1)^\wedge_p \longrightarrow BG^\wedge_p$ be a map induced by the monomorphism $\mathbb{Z}^\wedge_p \xrightarrow{(\alpha,\beta)} \mathbb{Z}^\wedge_p \times \mathbb{Z}^\wedge_p$, sending 1 to (α, β). If Lemma B.3 held with respect to this setup, it would mean that there is a map $g\colon BS^1 \longrightarrow BG$ and a mod-p equivalence $h\colon BS^1 \longrightarrow (BS^1)^\wedge_p$, such that $g^\wedge_p \simeq f \circ h^\wedge_p$. But g must be induced by a monomorphism $\mathbb{Z} \xrightarrow{(a,b)} \mathbb{Z} \times \mathbb{Z}$, sending 1 to the pair (a, b) for some integers a, b, whereas h is induced by multiplication by some p-adic unit u. An easy calculation now shows that $\frac{\alpha}{\beta} = \frac{a}{b}$, and since the right hand side is a rational number, there are clearly choices of α and β, where this equation cannot hold. Thus the lemma fails in this case.

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