HOMOLOGY DECOMPOSITIONS FOR CLASSIFYING SPACES OF FINITE GROUPS ASSOCIATED TO MODULAR REPRESENTATIONS

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Abstract. For a prime p, a homology decomposition of the classifying space BG of a finite group G consist of a functor $F : \mathbf{D} \longrightarrow$ **spaces** from a small category into the category of spaces and a map hocolim $F \longrightarrow BG$ from the homotopy colimit to BG which induces an isomorphism in mod-p homology. Associated to a modular representation $G \longrightarrow Gl(n; \mathbb{F}_p)$ we construct a family of subgroups closed under conjugation, which gives rise to three different homology decompositions, the so called subgroup, centralizer and normalizer decomposition. For an action of G on an $\mathbb{F}_p$-vector space V, this collection consists of the isotropy groups of all nontrivial (proper) subspaces of V with nontrivial p-Sylow subgroup. These decomposition formulas connect the modular representation theory of G with the homotopy theory of BG.

1.Introduction.

For a finite group G and a prime p, a homology decomposition or homology approximation of the classifying space BG is a a way to glue spaces together such that this new space allows a map into BG which induces an isomorphism in mod-p homology or cohomology. More explicit, it is given by a functor $F : \mathbf{D} \to$ **spaces** from a small, a discrete or, even better, a finite category $\mathbf{D}$ into the category of spaces and a mod-p homology isomorphism

$$\operatorname*{hocolim}_{\mathbf{D}} F \to BG$$

from the homotopy colimit of F to the classifying space BG of G. In practice, the category $\mathbf{D}$ is very often given by a full subcategory of the orbit category or of the category of subgroups of G. Homology decompositions of classifying spaces became an important tool in the analysis of the homotopy theory of these spaces. They can be used to make explicit calculations for G or to prove general theorems via an induction of the order of G. In the latter case one needs that the values of the functor F are homotopy equivalent to classifying spaces of smaller groups or of proper subgroups of G. And this turns out to be the case for the known decomposition results. A list of examples of applications of homology decompositions is given in [3].

In [3], Dwyer described many different homology decomposition formulas in terms of a single invariant: an associated poset of subgroups of G (see Section

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2). For each of these posets he constructed three different homology decompositions, the centralizer, the subgroup and the normalizer decomposition. Some of them were already known some of them gave new decompositions [3] [4].

The goal of this paper is to produce further homology decompositions of BG which are associated to modular representations. We hope to build a bridge between modular representation theory of a finite group and the homotopy theory of the classifying space at a fixed prime. This might give the chance to produce suitable decomposition formulas for questions about the homotopy theory of classifying spaces and/or invariant theory. The method of producing these new homology decompositions were already used with success in [9] and [10], where the author constructed spaces with polynomial mod-p cohomology and proved uniqueness results for them.

Before we formulate our results we first have to recall Dwyer's concepts of homology and sharp homology decompositions [3] [4].

Given a (covariant) functor $F : \mathbf{D} \to \mathbf{spaces}$, $\mathbf{D}$ a discrete category, the standard method to calculate the mod-p homology of $\operatorname{hocolim}_{\mathbf{D}} F$ is given by the Bousfield-Kan spectral sequence. It has the form

$$E_2^{i,j} \cong \varinjlim_{\mathbf{D}}{}_i \; H_j(F;\mathbb{F}_p) \quad \Longrightarrow \quad H_{i+j}(\operatorname{hocolim}_{\mathbf{D}} F;\mathbb{F}_p)$$

and converges to the mod-p homology of the homotopy colimit. Here, $\varinjlim_i$ denotes the i-th left derived functor of the functor $\varinjlim$ which maps diagrams of abelian groups to the direct limit of the diagram. And this also works for homology with twisted coefficients.

Let M be an $\mathbb{F}_p[G]$-module. If there exists an M-equivalence $\phi : \operatorname{hocolim}_{\mathbf{D}} F \to BG$, i.e. ϕ induces an isomorphism in $H_*(-;M)$-homology (twisted coefficients), then this map is called a *M-homology decomposition*. We call this decomposition *M-sharp*, if

$$\varinjlim_{\mathbf{D}}{}_i \; H_*(F;M) = \begin{cases} 0 & \text{for } \; i \geq 1 \\ H_*(BG;M) & \text{for } \; i = 0 \end{cases}$$

where, for $i = 0$, the isomorphism is induced by ϕ.

For a collection $\mathcal{C}$ of subgroups of G, i.e. a family which is closed under conjugation, Dwyer constructed three different decompositions, the subgroup, the centralizer and the normalizer decomposition. The first is based on full subcategory of the orbit category and has the form

$$b_{\mathcal{C}} : \operatorname{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \beta_{\mathcal{C}} \to BG \ ,$$

the second is based on a full subcategory of the category of subgroups and has the form

$$a_{\mathcal{C}} : \operatorname{hocolim}_{\mathbf{A}_{\mathcal{C}}} \alpha_{\mathcal{C}} \to BG \ ,$$

and the last one is based on the category of orbit simplices for the G-action on a complex $K_{\mathcal{C}}$ associated to $\mathcal{C}$ and has the form

$$d_{\mathcal{C}} : \operatorname{hocolim}_{\bar{s}\mathbf{S}_{\mathcal{C}}} \delta_{\mathcal{C}} \to BG \ .$$

For details see Section 2.

1.1 Definition. Let M be an $\mathbb{F}_p[G]$-module. We call a collection $\mathcal{C}$ *subgroup M-sharp* if the map $b_{\mathcal{C}}$ gives an M-sharp decomposition, *centralizer M-sharp* if $a_{\mathcal{C}}$ gives an M-sharp decomposition and *normalizer M-sharp* if $d_{\mathcal{C}}$ does so.

We will construct collections of subgroups associated to modular representations which are subgroup sharp. Let G be a finite group, V a finite dimensional $\mathbb{F}_p$-vector space and $\rho : G \to Gl(V)$ a modular representation. Then we define $\mathcal{I}(\rho)$ to be the collection of all subgroups K of G with order divisible by p which appear as isotropy subgroups of nontrivial proper subspaces of V. And $\mathcal{I}'(\rho)$ is the collection of all subgroups with the same properties but which appear as isotropy groups of nontrivial subspaces. In particular, $\mathcal{I}'(V)$ contains the kernel of ρ.

1.2 Theorem. *Let G be a finite group such that p divides the order of G. Let $\rho : G \to Gl(V)$ be a modular representation.*
(i) The collection $\mathcal{I}'(\rho)$ is subgroup M-sharp for any $\mathbb{F}_p[G]$-module M.
(ii) If ρ is faithful, then $\mathcal{I}(\rho)$ is subgroup $\mathbb{F}_p$-sharp.

For a G-space or a G-set X we define $\mathcal{I}'(X)$ and $\mathcal{I}(X)$ similarly as for a modular representation. $\mathcal{I}(X)$ is the collection of all subgroups of G with order divisible by p which appear as the isotropy group of nonempty proper subsets of X and $\mathcal{I}'(X)$ is the union of $\mathcal{I}(X)$ and the subgroup of all elements acting trivially on X. We say that G acts effectively, if this subgroup is the trivial group.

1.3 Theorem. *Let G be a finite group such that p divides the order of G. Let $P \subset G$ be a p-Sylow subgroup. Let X be a G-set such that the fixed point set X^P is nonempty.*
(i) The collection $\mathcal{I}'(X)$ is subgroup M-sharp for any $\mathbb{F}_p[G]$-module M.
(ii) If G acts effectively on X, then $\mathcal{I}(X)$ is subgroup $\mathbb{F}_p$-sharp.

For a collection $\mathcal{C}$ all the decompositions $b_{\mathcal{C}}$, $a_{\mathcal{C}}$ and $d_{\mathcal{C}}$ are M-homology decompositions if one of them is one [3] (see Theorem 2.5). Thus, all the collection of the above theorems also establish centralizer and normalizer homology decompositions for suitable $\mathbb{F}_p[G]$-modules. Actually, as shown in Section 4, our collections are normalizer M-sharp for suitable $\mathbb{F}_p[G]$-modules M. As the Example 5.3 shows, we cannot expect to have centralizer sharpness, too.

If a collection $\mathcal{C}$ contains the group G itself, we do not get an interesting decomposition formula. For example, in this case, the orbit category $\mathbf{O}_{\mathcal{C}}(G)$ contains G/G which is a terminal object. Therefore, the subgroup decomposition contains BG as a piece and is sharp by trivial reasons. To avoid this, one should look for modular representations which are fixed-point free, i.e. $V^G \neq \{0\}$ respectively for fixed-point free G-sets.

If the representation ρ is faithful or if the G-action is effective, then the collections $\mathcal{I}'(\rho)$ and $\mathcal{I}'(X)$ contain the trivial group $\{e\}$. In this case the group G also appears in the subgroup decomposition diagram, namely as the automorphism group of the object $G/\{e\}$. In contrast, for the collections $\mathcal{I}(\rho)$ and $\mathcal{I}(X)$, all automorphism groups of objects of the associated orbit category are quotients of proper subgroups of G. And this is the reason why we also looked at $\mathcal{I}(\rho)$ and $\mathcal{I}(X)$ and why we think that they give more interesting decomposition formulas than the other two.

Both statements are an easy consequences of the following theorem which states subgroup sharpness for collections satisfying the first and one of the last two of the following conditions:

(P) Every nontrivial p-subgroup is contained in an element of $\mathcal{C}$.
(S) The collection is closed under taking intersections
(wS) The intersection $H := H_1 \cap H_2$ of two elements of $\mathcal{C}$ is contained in $\mathcal{C}$ if H is nontrivial with order divisible by p.

1.4 Theorem. *Let G be a finite group whose order is divisible by p. Let $\mathcal{C}$ be a collection of subgroups of G.*
(i) If $\mathcal{C}$ satisfies the conditions (P) and (S), then it is subgroup M-sharp for any $\mathbb{F}_p[G]$-module M.
(ii) If $\mathcal{C}$ satisfies the conditions (P) and (wS), it is subgroup $\mathbb{F}_p$-sharp.

In Section 4, we also prove some normalizer sharpness properties for such collections.

Proof of Theorem 1.2 and Theorem 1.3. We only have to show that all the collections satisfy the correct conditions. For a p-subgroup $Q \subset G$, we have $Q \subset Iso_G(V^Q)$. For a nontrivial subgroup $Q \subset G$, we have $V^Q \neq V$ if ρ is faithful and $X^Q \neq X$ if G acts effectively. This shows that all the collections satisfy condition (P).

For two subspaces $V_1, V_2 \subset V$ or two subset $X_1, X_2 \subset X$, we have $Iso_G(V_1+V_2) = Iso_G(V_1) \cap Iso_G(V_2)$ and $Iso_G(X_1 \cup X_2) = Iso_G(X_1) \cap Iso_G(X_2)$. This shows that the collections $\mathcal{I}'(\rho)$ and $\mathcal{I}'(X)$ satisfy condition (S).

If ρ is faithful and $\{e\} \neq Q \subset Iso_G(V_1) \cap Iso_G(V_2) := H$, then $V_1 + V_2 \neq V$, since the sum is fixed by a nontrivial subgroup, and $H \in \mathcal{I}(\rho)$. Hence, $\mathcal{I}(\rho)$ satisfies (wS). A similar argument works for an effective G-action. □

The method of proof for Theorem 1.4 can also be used to reconstruct the Jackowski-McClure decomposition via centralizers of nontrivial elementary abelian p-subgroups [8]. Let $\mathcal{E}$ denote the collection of all nontrivial elementary abelian p-subgroups of G and $\mathcal{A}$ the collection of all nontrivial abelian p-subgroups. Let $\mathcal{C}en(\mathcal{E}) := \{C_G(E) | E \in \mathcal{E}\}$ and $\mathcal{C}en(\mathcal{A}) := \{C_G(A) | A \in \mathcal{A}\}$ denote the associated collection of centralizers.

1.5 Theorem. *The collections $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are subgroup and normalizer $\mathbb{F}_p$-sharp.*

Since the collection $\mathcal{E}$ is not subgroup sharp, one cannot expect that these collections are centralizer sharp.

The Jackowski-McClure decomposition via centralizers of elementary abelian p-subgroups is described via a functor defined on the Quillen-category of all nontrivial elementary abelian p-subgroups. This is a full subcategory of the category of subgroups. In fact, it is the centralizer decomposition associated to the collection $\mathcal{E}$. The subgroup decompositions associated to $\mathcal{C}en(\mathcal{E})$ comes from a functor defined on a subcategory of the orbit category, but one can show that both give the same decomposition formulas. That is, the collection $\mathcal{E}$ is centralizer sharp if and only $\mathcal{C}en(\mathcal{E})$ is subgroup sharp. We intend to analyze this relation in more detail in a future paper.

The proof of the theorems is based on the following idea. Let $\mathcal{C}$ be a collection of subgroups and denote by $\mathcal{P}$ the collection of all nontrivial p-subgroups and by $\mathcal{P}'$ the collection of all p-subgroups. Then we have inclusions

$$\mathcal{C} \to \mathcal{C} \cup \mathcal{P}' \leftarrow \mathcal{P}'$$

where $\mathcal{C} \cup \mathcal{P}'$ denotes the union of both collections. We will compare the decompositions associated to these three collections. For $\mathcal{I}(\rho)$ and $\mathcal{I}(X)$ we replace $\mathcal{P}'$ by $\mathcal{P}$ and for $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ by the collection $\mathcal{P}^c$ of all p-centric p-subgroups of G. A p-subgroup $P \subset G$ is called *p-centric* if the center $Z(P) \subset C_G(P)$ of P is a p-Sylow subgroup of the centralizer $C_G(P)$.

The paper is organized as follows: In the next section we recall material from [3] and [4]. In particular we explain the concept of subgroup, centralizer and normalizer decompositions. In Section 3 we compare different collection with respect to there sharpness properties. We set up the background for the proofs of the theorem. The proofs themselves are worked out in Section 4. And the last section is devoted to a discussion of some examples.

We notice that the first part of Theorem 1.2 is already proved in [6] but by different methods.

The paper is written with the convention that "spaces" means "simplicial sets". And a map $X \longrightarrow Y$ between spaces is a (weak) homotopy equivalence, if it induces a (weak) homotopy equivalence between the geometric realizations.

Throughout the paper p is a fixed prime, and $\mathbb{F}_p$ denotes the finite field of p-element. A map between spaces is called an M-equivalence, M an $\mathbb{F}_p[G]$-module, if it induces an isomorphism in $H_*(-;M)$-homology.

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2. The centralizer, the subgroup and the normalizer decomposition.

For each collection $\mathcal{C}$ of subgroups of a finite group G, Dwyer constructed three different decomposition, the centralizer , the subgroup and the normalizer decomposition. Moreover, Dwyer identified the Bousfield-Kan spectral sequence of the decompositions with the equivariant Bredon homology of certain G-spaces. We recall these constructions. All the material in this section is taken from [3] and [4].

Let EG denote the universal cover of BG. For a G-space X we denote by $X_{hG} := (X \times EG)/G$ the Borel construction of G.

Let $\tilde{\gamma}$ be a functor from a small category $\mathbf{D}$ into transitive G-sets. Every G-set can be considered as a discrete G-space. Then, the homotopy colimit $X := \operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ is a G-space. It can be identified with the nerve of the category whose objects are pairs (x, d) such that $x \in \tilde{\gamma}(d)$ and whose morphisms are maps $d \longrightarrow d'$ such that $\tilde{\gamma}(d \rightarrow d')(x) = x'$.

Let $\gamma := \tilde{\gamma}_{hG}$. The natural maps $\tilde{\gamma}(d) \longrightarrow *$ are G-equivariant and induce maps $\gamma(d) \longrightarrow BG$ which are compatible with the morphisms in $\mathbf{D}$, and therefore a map $\operatorname{hocolim}_{\mathbf{D}} \gamma \longrightarrow BG$. If this map is a mod-$p$ equivalence, then γ provides a homology decomposition of BG.

A coefficient functor for G is a functor Φ from the category of $\mathbb{F}_p[G]$-modules to the category of $\mathbb{F}_p$-vector spaces which preserves arbitrary direct sums. If K is a subgroup of G, $\Phi|_K$ denotes the coefficient functor for K defined by $\Phi|_K(A) := \Phi(\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[K]} A)$. Examples are given by twisted homology groups of G, namely $H_*(G; M \otimes -)$ where M is an $\mathbb{F}_p[G]$-module. In this case , by Shapiro's lemma, the restriction to K is isomorphic to $H_*(K; M \otimes -)$.

If X is a G-space and Φ a coefficient functor for G, then we can construct a chain complex $C_n^G(X; \Phi) := \Phi(\mathbb{F}_p[X_n])$ where X_n denotes the set of n-simplices of X. The boundary map is given by the alternating sum of the face maps in X. The Bredon homology groups $H_*^G(X;\Phi)$ are defined to be the homology groups of this chain complex. By construction, Bredon homology is a functor from the category of G-spaces to the category of abelian groups. A G-map $f: X \to Y$ is called a Φ-equivalence if it induces an isomorphism $H_*^G(f;\Phi)$ for Bredon homology. And a G-space X is called acyclic with respect to Φ if the map $X \to *$ induces a Φ-equivalence.

Let M be an $\mathbb{F}_p[G]$-module. The E^2-term of the Bousfield-Kan spectral sequence for $H_*(-;M)$-homology of $\operatorname{hocolim}_{\mathbf{D}} \gamma$ can be identified with the Bredon homology groups of $\operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ with coefficient functor $H_*(G; M \otimes -)$ [4; §3, 4.4 and 6.2]. And $\operatorname{hocolim}_{\mathbf{D}} \gamma \to BG$ is an M-sharp decomposition if and only if $\operatorname{hocolim}_{\mathbf{D}} \tilde{\gamma}$ is acyclic with respect to $H_*(G; M \otimes -)$.

In the following $\mathcal{C}$ is a collection of subgroups. We can associate to $\mathcal{C}$ a simplicial complex $K_{\mathcal{C}}$ [1; 6.2]. The n-simplices of $K_{\mathcal{C}}$ are the subsets $\{H_i\} \subset \mathcal{C}$ of cardinality $n+1$ which are totally ordered by proper inclusions. The group G acts on $\mathcal{C}$ and, since this action respects inclusions, it also acts on $K_{\mathcal{C}}$.

Before we explain the different decompositions associated to the collection $\mathcal{C}$, we notice that, for every transitive G-set X considered as a discrete G-space, the Borel construction X_{hG} produces a space homotopy equivalent to BG_x for some $x \in X$, where G_x is the stabilizer group of x.

2.1 The subgroup decomposition. We denote by $\mathbf{O}_{\mathcal{C}}(G) \subset \mathbf{O}(G)$ the full subcategory of the orbit category whose objects are given by the homogeneous spaces G/H where $H \in \mathcal{C}$. As usual the morphisms are given by G-equivariant maps. Considering transitive G-sets as discrete G-spaces defines a functor $\tilde{\beta}_{\mathcal{C}}$: $\mathbf{O}_{\mathcal{C}}(G) \to G - \mathbf{spaces}$. Composing $\tilde{\beta}_{\mathcal{C}}$ with the Borel construction $(-)_{hG}$ gives a functor

$$\beta_{\mathcal{C}} : \mathbf{O}_{\mathcal{C}}(G) \to \mathbf{spaces}$$

whose value at G/H has the homotopy type of BH. The natural maps $\beta_{\mathcal{C}}(G/H) = (G/H)_{hG} \to BG$ are compatible with the morphisms of the orbit category and induce a map

$$b_{\mathcal{C}} : \operatorname*{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \beta_{\mathcal{C}} \to BG.$$

This is an M-homology decomposition if $b_{\mathcal{C}}$ is an M-equivalence.

The space $X_{\mathcal{C}}^{\beta} := \operatorname{hocolim}_{\mathbf{O}_{\mathcal{C}}(G)} \tilde{\beta}_{\mathcal{C}}$ carries a G-action and is the nerve of the category $\mathbf{X}_{\mathcal{C}}^{\beta}$ whose objects are pairs $(gH, G/H)$ and whose morphism are G-maps $G/H \to G/H'$ which map gH on $g'H'$. The decomposition is M-sharp if and only if $(X_{\mathcal{C}}^{\beta})$ is acyclic with respect to $H_*(G; M \otimes -)$.

2.2 The centralizer decomposition. The $\mathcal{C}$-conjugacy category $\mathbf{A}_{\mathcal{C}}$ is the category in which the objects are pairs (H, Θ) where H is a group and Θ a conjugacy class of monomorphisms $i: H \to G$ with $i(H) \in \mathcal{C}$. A morphism $(H,\Theta) \to (H', \Theta')$ is a group homomorphism $j: H \to H'$ (actually a monomorphism) which under composition carries Θ' into Θ. Restricting H, for instance by requiring that H is a subgroup of G or that $|H| \leq |G|$, makes $\mathbf{A}_{\mathcal{C}}$ into a finite category.

We define a (covariant) functor $\tilde{\alpha}_{\mathcal{C}} : \mathbf{A}_{\mathcal{C}}^{op} \to G - \mathbf{spaces}$ by assigning to each object (H,Θ) the transitive G-set Θ. Composing $\tilde{\alpha}_{\mathcal{C}}$ with the Borel construction

yields a functor
$$\alpha_{\mathcal{C}} : \mathbf{A}_{\mathcal{C}}^{op} \to \mathbf{spaces} \ ,$$
whose value at (H,Θ) is homotopy equivalent to $BC_G(i(H))$ for some $i \in \Theta$. The natural transformation from $\tilde{\alpha}_{\mathcal{C}}$ to the constant functor with value a point, establishes a natural transformation from $\alpha_{\mathcal{C}}$ to the constant functor whose value is BG. Hence, we get a map
$$a_{\mathcal{C}} : \operatorname*{hocolim}_{\mathbf{A}_{\mathcal{C}}^{op}} \alpha_{\mathcal{C}} \to BG \ ,$$
If $a_{\mathcal{C}}$ is an M-equivalence we get a decomposition formula.

The space $X_{\mathcal{C}}^{\alpha} := \operatorname*{hocolim}_{\mathbf{A}_{\mathcal{C}}^{op}} \tilde{\alpha}_{\mathcal{C}}$ carries a G-action and is the nerve of the category $\mathbf{X}_{\mathcal{C}}^{\alpha}$ whose objects are pairs (i,H) where $i \in \Theta$ and whose morphisms are maps $\rho : H \to H'$ such that $i'\rho = i$. The centralizer decomposition is M-sharp if and only if $X_{\mathcal{C}}^{\alpha}$ is acyclic with respect to $H_*(G; M \otimes -)$.

2.3 The normalizer decomposition. Let $\overline{s}\mathbf{S}_{\mathcal{C}}$ denote the category of the orbit simplices for the G-action on $K_{\mathcal{C}}$. That is the objects are the G-orbits $\overline{\sigma}$ of the simplices σ of $K_{\mathcal{C}}$. There is exactly one morphism $\overline{\sigma} \to \overline{\sigma}'$ if there exist $\sigma \in \overline{\sigma}$ and $\sigma' \in \overline{\sigma}'$ such that $\sigma \subset \sigma'$, i.e. σ is a face of σ'. There is a functor $\tilde{\delta}_{\mathcal{C}} : \overline{s}\mathbf{S}_{\mathcal{C}} \to G - \mathbf{spaces}$ which assigns to each object $\overline{\sigma}$ the transitive G-set $\overline{\sigma}$. Passing to the Borel construction yields a functor
$$\delta_{\mathcal{C}} : \overline{s}\mathbf{S}_{\mathcal{C}} \to \mathbf{spaces} \ ,$$
whose value at $\overline{\sigma}$ is homotopy equivalent to $B(\bigcap_i N_G(H_i))$ where $(H_0 \subset ... \subset H_n) \in \overline{\sigma}$. The natural transformation from $\tilde{\delta}_{\mathcal{C}}$ to the constant functor with value a point establishes a natural transformation from $\delta_{\mathcal{C}}$ to the constant functor with value BG and therefore a map
$$d_{\mathcal{C}} : \operatorname*{hocolim}_{\overline{s}\mathbf{S}_{\mathcal{C}}} \delta_{\mathcal{C}} \to BG \ .$$
If this is an M-equivalence then we get an M-homology decomposition of BG.

Let $\mathrm{sd}X_{\mathcal{C}}^{\delta} := \operatorname*{hocolim}_{\overline{s}\mathbf{S}_{\mathcal{C}}} \tilde{\delta}_{\mathcal{C}}$. and let $X_{\mathcal{C}}^{\delta}$ be the nerve of the category $\mathbf{X}_{\mathcal{C}}^{\delta}$ whose objects are the subgroups $H \in \mathcal{C}$ and where there exists a unique morphism $H \to H'$ if $H \subseteq H'$. Both spaces carry an G action which is induced from the G-action on $\mathcal{C}$. The space $\mathrm{sd}X_{\mathcal{C}}^{\delta}$ is the barycentric subdivision of $X_{\mathcal{C}}^{\delta}$. And $d_{\mathcal{C}}$ is an M-sharp decomposition if and only if $\mathbf{X}_{\mathcal{C}}^{\delta}$ is acyclic with respect to $H_*(G; M \otimes -)$.

By construction, $K_{\mathcal{C}}$ consists of the non degenerate simplices of $X_{\mathcal{C}}^{\delta}$, and both are G-equivariant homotopy equivalent.

All this three decomposition are closely connected as the next theorem shows. Let $*$ denote the one-point space with trivial G-action.

2.4 Definition. The collection $\mathcal{C}$ is called M-ample if the G-equivariant map $K_{\mathcal{C}} \to *$ induces an M-equivalence $(K_{\mathcal{C}})_{hG} \to BG$.

2.5 Theorem. *(Dwyer [3]) Let G be a finite group and M an $\mathbb{F}_p[G]$-module. If, for a collection $\mathcal{C}$, the complex $K_{\mathcal{C}}$ is M-ample then all three maps $a_{\mathcal{C}}$, $b_{\mathcal{C}}$ and $d_{\mathcal{C}}$ are M-homology decompositions. And if one of this three maps is a M-homology decomposition then $K_{\mathcal{C}}$ is M-ample.*

3. Comparison of collections.

A collection $\mathcal{C}$ of subgroups of a finite group G is a partially ordered set and can be treated as a category denoted by $\mathbf{S}_{\mathcal{C}}$. This follows our convention to denote categories by boldface characters. Let $S_{\mathcal{C}}$ denote the nerve of $B S_{\mathcal{C}}$. Then $K_{\mathcal{C}} \simeq S_{\mathcal{C}}$, since the simplices of $K_{\mathcal{C}}$ are just the non degenerate simplices of $S_{\mathcal{C}}$.

An inclusion of collections $\mathcal{H} \subset \mathcal{L}$ can be interpreted as a functor $\iota : \mathbf{S}_{\mathcal{H}} \to \mathbf{S}_{\mathcal{L}}$. For an element $L \in \mathcal{L}$ there is a standard over category $\iota{\to}L$ and a standard under category $L{\to}\iota$, here denoted by $\mathbf{S}_{\mathcal{H}}{\to}L$ and $L{\to}\mathbf{S}_{\mathcal{H}}$. Both of these categories come from posets. The first is the category associated to the set of elements of $\mathcal{H}$ containing L and the second is given by the poset $\mathcal{H}_L$ of all elements of $\mathcal{H}$ which are contained in L. The first is a set of subgroups of G, while $\mathcal{H}_L$ is actually a collection of subgroups of L. If all over categories or all under categories are contractible, Quillen's Theorem A [11] implies that the functor $\iota : \mathbf{S}_{\mathcal{H}} \to \mathbf{S}_{\mathcal{L}}$ induces equivalences $S_{\mathcal{H}} \simeq S_{\mathcal{L}}$ and $K_{\mathcal{H}} \simeq K_{\mathcal{L}}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two collections of subgroups of a finite group G. We want to show results of the following form. If $\mathcal{K}$ satisfies some ampleness or sharpness properties, then so does $\mathcal{H}$. Of course we have to put some conditions on $\mathcal{H}$. For a subgroup $H \subset G$ we denote by $\mathcal{K}_H := \{K \in \mathcal{K} | K \subset H\}$ the set of all elements in $\mathcal{K}$ which are contained in H. This is a collection of subgroups of H. And, considered as a category, this is the under category $\mathbf{S}_{\mathcal{K}}{\to}H$. Actually, we will compare $\mathcal{H}$ with $\mathcal{H}\cup\mathcal{K}$ and $\mathcal{K}$ with $\mathcal{H}\cup\mathcal{K}$. In the latter case we like to think of $\mathcal{H}\cup\mathcal{K}$ as an extension of $\mathcal{K}$ and in the first case of $\mathcal{H}$ as a reduction of $\mathcal{H} \cup \mathcal{K}$.

3.1 Definition. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be three collections of subgroups of G. And let M be an $\mathbb{F}_p[G]$-module.
(i) The collection $\mathcal{H}$ is called a *sharp reduction* of $\mathcal{L}$ if $\mathcal{H} \subset \mathcal{L}$ and if for every $L \in \mathcal{L} \setminus \mathcal{H}$ the over category $L{\to}\mathbf{S}_{\mathcal{H}}$ has an initial element.
(ii) We say that $\mathcal{L}$ is an *M-ample extension* of $\mathcal{K}$, if $\mathcal{K} \subset \mathcal{L}$ and if $\mathcal{K}_L$ is an M-ample collection for all $L \in \mathcal{L} \setminus \mathcal{K}$. The collection is called a *subgroup M-sharp extension* if, in addition, $\mathcal{K}_L$ is a subgroup M-sharp collection for L.

3.2 Proposition. *Let $\mathcal{H} \subset \mathcal{L}$ be two collections of subgroups of a finite group G. Let M be an $\mathbb{F}_p[G]$-module. If $\mathcal{H}$ is a sharp reduction of $\mathcal{L}$, then the following holds:*
(i) The map $K_{\mathcal{H}} \to K_{\mathcal{L}}$ is a G-equivariant homotopy equivalence and $\mathcal{H}$ is M-ample if and only if $\mathcal{L}$ is M-ample.
(ii) The maps $X^{\beta}_{\mathcal{H}} \to X^{\beta}_{\mathcal{L}}$ and $X^{\delta}_{\mathcal{H}} \to X^{\delta}_{\mathcal{L}}$ are G-equivariant homotopy equivalences.

In fact, Proposition 3.2 says that $\mathcal{H}$ has the same M-ampleness and the same subgroup and normalizer M-sharpness properties as $\mathcal{L}$.

3.3 Proposition. *Let $\mathcal{K} \subset \mathcal{L}$ be two collections of subgroups of a finite group G. and let M be an $\mathbb{F}_p[G]$-module.*
(i) If $\mathcal{L}$ is an M-ample extension of $\mathcal{K}$, then the map $K_{\mathcal{K}} \to K_{\mathcal{L}}$ is an M-equivalence.
(ii) If $\mathcal{L}$ is a subgroup M-sharp extension of $\mathcal{K}$, then $\mathcal{L}$ is subgroup M-sharp if and only if $\mathcal{K}$ is.

Proof of Proposition 3.2. If $\mathcal{H} \to \mathcal{L}$ is a sharp reduction we can define a map $F : \mathcal{L} \to \mathcal{L}$ by $F(L) := H_L$ where $L \to H_L$ is the initial element of $L{\to}\mathbf{S}_{\mathcal{H}}$. This map is idempotent, respects inclusion, is G-equivariant and maps onto $\mathcal{H}$. The G-

equivariance follows, since $gH_Lg^{-1} = H_{gLg^{-1}}$. Actually, $\mathcal{H} \subset \mathcal{L}$ is a sharp reduction if and only if such a map exists.

The map allows us to construct G-equivariant left inverses s_β and s_δ for the inclusion functors $i_\beta : \mathbf{X}^\beta_\mathcal{H} \subset \mathbf{X}^\beta_\mathcal{L}$ and $i_\delta : \mathbf{X}^\delta \subset \mathbf{X}^\delta_\mathcal{L}$ and G-equivariant natural transformations $id \to i_\beta s_\beta$ and $id \to i_\delta s_\delta$. As Lemma 3.4 below will show this is sufficient to prove part (ii).

For the subgroup decomposition, s_β is given by $(G/L, gL) \mapsto (G/F(L), gF(L))$. The values on the morphisms are the obvious ones. By the properties of F, this is well defined. And the functor s_β is G-equivariant. The inclusion $L \subset F(L)$ induces a morphisms $(G/L, gL) \to (G/F(L), gF(L))$. The family of these morphisms establishes a natural transformation $id \to i_\beta s_\beta$ which, in addition, is G-equivariant.

For the normalizer decomposition, the left inverse is given by $L \mapsto F(L)$ and by the obvious definition for morphisms. This is well defined and G-equivariant. Again, the inclusion $L \subset F(L)$ establishes a morphism $L \to F(L)$ and the family of all these morphisms a G-equivariant natural transformation $id \to i_\delta s_\delta$.

The first part follows from the second since, for any collection $\mathcal{C}$, $K_\mathcal{C}$ and $X^\delta_\mathcal{C}$ are G-equivariant homotopy equivalent (see 2.3). □

Since, in general, the centralizer $C_G(L)$ is not contained in $C_G(F(L))$, we are not able to construct a G-equivariant left inverse for the functor $i_\alpha : \mathbf{X}^\alpha \subset \mathbf{X}^\alpha_\mathcal{L}$.

3.4 Lemma. *Let $i : \mathbf{C} \to \mathbf{D}$ be a G-equivariant functor between small categories. which has a G-equivariant left inverse $s : \mathbf{D} \to \mathbf{C}$. If there exists a G-equivariant natural transformation $id \to i \circ s$, the map i induces a G-equivariant homotopy equivalence between the classifying spaces of $\mathbf{C}$ and $\mathbf{D}$.*

Proof. It is sufficient to show that the map i induces a homotopy equivalence between the fixed-point sets of the classifying spaces $B\mathbf{C}$ and $B\mathbf{D}$ of the categories for all subgroups of G. Since $(B\mathbf{C})^H \simeq B(\mathbf{C}^H)$, passing to fixed-points maintains the situation described in the statement but where we forget the equivariant setting. We only have to prove the non equivariant version of the statement. But that is obvious since a natural transformation between functors induces a homotopy between the induced maps on the classifying spaces.

Proof of Proposition 3.3. Let $\mathcal{K} \to \mathcal{L}$ be an M-ample extension. We want to show that $K_\mathcal{K}$ is M-ample if and only $K_\mathcal{L}$ is or, equivalently, that $b_\mathcal{K} : \operatorname{hocolim}_{\mathbf{O}_\mathcal{K}(G)} \beta_\mathcal{K} \to BG$ is an M-homology decomposition if and only if $b_\mathcal{L}$ is one.

For the inclusion $\mathbf{O}_\mathcal{K}(G) \to \mathbf{O}_\mathcal{L}(G)$, the left Kan-extension of $\beta_\mathcal{K}$ is given by $L(\beta_\mathcal{K})(G/L) := \operatorname{hocolim}_{\mathbf{O}_\mathcal{K} \to G/L} \beta_\mathcal{K}$ and $\operatorname{hocolim}_{\mathbf{O}_\mathcal{K}(G)} \beta_\mathcal{K} \simeq \operatorname{hocolim}_{\mathbf{O}_\mathcal{L}(G)} L(\beta_\mathcal{K})$ (e.g. see [5; 9.8]). In particular, this equivalence is an M-equivalence for any $\mathbb{F}_p[G]$-module M.

Since all objects as well as all morphisms in the under category $\mathbf{O}_\mathcal{K} \to G/L$ are given by G–maps, there exists a natural transformation $T : L(\beta_\mathcal{K}) \to \beta_\mathcal{L}$. There is a functor $\mathbf{O}_{\mathcal{K}_L}(L) \to (\mathbf{O}_\mathcal{K} \to G/L)$ which is an equivalence of categories. The inverse is given by mapping $G/K \to G/L$ to the counter image of the orbit $1 \cdot L \in G/L$. Since the collection $\mathcal{K}_L$ is M-ample the natural transformation T consists of M-equivalences. This establishes an isomorphism between the E_2-pages of the Bousfield-Kan spectral sequences for $H_*(-; M)$-homology. Hence, $\operatorname{hocolim}_{\mathbf{O}_\mathcal{L}(G)} L(\beta_\mathcal{K}) \to \operatorname{hocolim}_{\mathbf{O}_\mathcal{L}(G)} \beta_\mathcal{L}$ is an M-equivalence as well as $\operatorname{hocolim}_{\mathbf{O}_\mathcal{K}(G)} \beta_\mathcal{K} \to \operatorname{hocolim}_{\mathbf{O}_\mathcal{L}(G)} \beta_\mathcal{L}$. This proves part (i).

Now we consider the case of subgroup sharp extensions. We use the same idea. Let $L(H_*(\beta_{\mathcal{K}};M) := \varinjlim_{\mathbf{O}_{\mathcal{K}}\to G/L} H_*(\beta_{\mathcal{K}};M)$ be the left Kan extension of $H_*(\beta_{\mathcal{K}};M)$ for the inclusion $\mathbf{O}_{\mathcal{K}}(G) \to \mathbf{O}_{\mathcal{L}}(G)$ and denote by L^i the left derived functors of $\varinjlim_{\mathbf{O}_{\mathcal{K}}\to G/L} H_*(\beta_{\mathcal{K}};M)$. Since we can identify the under category $\mathbf{O}_{\mathcal{K}} \to G/L$ with the orbit category $\mathbf{O}_{\mathcal{K}_L}(L)$ and since , by assumption, $\mathcal{K}_L$ is subgroup M-sharp, L^i vanishes for $i > 0$ and $L^0 \cong H_*(\beta_{\mathcal{L}};M)$. An application of the composition of functor spectral sequence (cf. [7; Appendix II, 3.6]) proves the claim. □

4. Proof of Theorem 1.4 and Theorem 1.5.

The proof of the subgroup sharpness in both theorems is a consequence of the following two lemmas. For two collections $\mathcal{H}$ and CK we denote by $\mathcal{HK} := \mathcal{H} \cup \mathcal{K}$ the union of both.

4.1 Lemma. *Let G be a finite group and let M be an $\mathbb{F}_p[G]$-module. Let $\mathcal{C}$ be a collection of subgroups.*
(i) If $\mathcal{C}$ satisfies the conditions (P) and (S), then $\mathcal{C} \subset \mathcal{CP}'$ is a sharp reduction and $\mathcal{P}' \subset \mathcal{CP}'$ is a subgroup M-sharp extension.
(ii) If $\mathcal{C}$ satisfies the conditions (P) and (wS), then $\mathcal{C} \subset \mathcal{CP}$ is a sharp reduction and $\mathcal{P} \subset \mathcal{CP}$ is a subgroup $\mathbb{F}_p$-sharp extension.

4.2 Lemma. *Let G be a finite group such that p divides the order of G. For $\mathcal{C} = \mathcal{C}en(\mathcal{E})$ or $\mathcal{C}en(\mathcal{A})$, the inclusion $\mathcal{C} \subset \mathcal{C} \cup \mathcal{P}^c$ is a sharp reduction and $\mathcal{P}^c \subset \mathcal{C} \cup \mathcal{P}^c$ is a subgroup $\mathbb{F}_p$-sharp extension.*

Proof of Lemma 4.1. For each (nontrivial) p-subgroup $Q \subset G$ we denote by H_Q the intersection of all elements $H \in \mathcal{C}$ which contain Q. Because of condition (P), the subgroup H_Q is nonempty, and, because of condition (S) respectively (wS), $H_Q \in \mathcal{C}$. Moreover, $Q \subset H_Q$ is an initial element of $Q{\to}\mathbf{S}\mathcal{C}$, which shows that $\mathcal{C} \subset \mathcal{CP}'$ respectively $\mathcal{C} \subset \mathcal{CP}$ is a sharp reduction.

For any $H \in \mathcal{C}$, the collection $\mathcal{P}'_H$ is the collection of all p-subgroups and $\mathcal{P}_H$ the collection of all nontrivial p-subgroups of H. The collection $\mathcal{P}'_H$ is subgroup M-sharp for any $\mathbb{F}_p[G]$-module M and the $\mathcal{P}_H$ is subgroup $\mathbb{F}_p$-sharp [4]. Hence, the inclusion $\mathcal{P} \subset \mathcal{CP}'$ is a subgroup M-sharp extension and the inclusion $\mathcal{P} \subset \mathcal{CP}$ a subgroup $\mathbb{F}_p$-sharp extension. □

Proof of Lemma 4.2. For a subgroup $K \subset G$ we denote by $E(K)$ the maximal elementary abelian subgroup of the center $Z(K)$ of K. For $P \in \mathcal{P}$ we set $H_P := C_G(E(P))$ and for $K \in \mathcal{C}en(\mathcal{E}) \setminus \mathcal{P}$, we set $H_K = K$.

If $H := C_G(E) \in \mathcal{C}en(\mathcal{E})$, then $E \subset E(H)$ and $H \subset C_G(E(H)) \subset C_G(E) = H$. Hence, in the latter sequence, all groups are equal. If $P \in \mathcal{P}^c$ and $H := C_G(E) \in \mathcal{C}en(\mathcal{E})$ such that $P \subset H$, then $Z(P) \subset C_G(P) =: C$ is a p-Sylow subgroup and, in particular, $E(H) \subset E(P)$. Thus $H_P \subset C_G(E(H)) = H$. This shows that $P \subset H_P$ is an initial element of $P{\to}\mathbf{S}_{\mathcal{C}en(\mathcal{E})}$ and that $\mathcal{C}en(\mathcal{E}) \subset \mathcal{C}en(\mathcal{E})\mathcal{P}^c$ is a sharp reduction.

Moreover, if $H := C_G(E)$ and $P \subset H$ is p-centric in G or H, then $E(H) \subset Z(P)$ and $C_H(P) = C_G(P)$. That is to say that P is a p-centric subgroup of H if and only if it is a p-centric subgroup of G. Hence, $\mathcal{P}^c_H$ is the collection of all p-centric subgroups of H and is therefore subgroup $\mathbb{F}_p$-sharp [4; 10.3]. This proves the second part of statement.

For $\mathcal{C}en(\mathcal{A})$, the proof is analogously. We only have to replace $E(H)$ by the center $Z(H)$ of H. $\square$

4.3 Remark. Let $\rho : G \to Gl(V)$ be a modular representation. Let $\mathcal{I}_{\mathcal{P}}(\rho)$ $(\mathcal{I}_{\mathcal{P}'})$ denote the collection of all isotropy groups of the fixed-point set V^Q where $Q \in \mathcal{P}$. $(Q \in CP')$. These are collection which in general are smaller than $\mathcal{I}(\rho)$ respectively $\mathcal{I}'(\rho)$. For any $Q \in \mathcal{P}'$ the over categories $Q \to \mathbf{S}_{\mathcal{I}_{\mathcal{P}'}(\rho)}$ have an initial element given by $Q \to Iso_G(V^Q)$. Therefore, $\mathcal{I}_{\mathcal{P}}(\rho) \to \mathcal{I}_{\mathcal{P}}(\rho) \cup \mathcal{P}$ is a sharp reduction. If ρ is faithful, the same holds $\mathcal{I}_{\mathcal{P}}(\rho) \to \mathcal{I}_{\mathcal{P}}(\rho) \cup \mathcal{P}$. And as the proofs show, Theorem 1.2 is also true for these collections.

Obviously, the analogue results for an (effective) G-action on a space X are also true, if $X^P \neq \emptyset$ for a p-Sylow subgroup $P \subset G$.

In the rest of this section we prove normalizer sharpness properties for our collections.

4.4 Theorem. *Let G be a finite group such that p divides the order of G. Let $\mathcal{C}$ be a collection of subgroups of G.*
(i) If $\mathcal{C}$ satisfies the conditions (P) and (S) then $\mathcal{C}$ is normalizer M-sharp for any $\mathbb{F}_p[G]$-module M.
(ii) If $\mathcal{C}$ satisfies (P) and (wS), then $\mathcal{C}$ is normalizer $\mathbb{F}_p$-sharp.

4.5 Corollary. *Let G be a finite group with order divisible by p. Let $\rho : G \to Gl(V)$ be a modular representation. Then, the collection $\mathcal{I}'(\rho)$ is normalizer M-sharp for any $\mathbb{F}_p[G]$-module. And, if ρ is faithful, $\mathcal{I}(\rho)$ is normalizer $\mathbb{F}_p$-sharp.*

The analogue results holds obviously for a G-action on a space X if $X^P \neq \emptyset$ for a p-Sylow subgroup $P \subset G$.

Proof of Theorem 4.4. We use the techniques of [4; §7]. We first assume that $\mathcal{C}$ satisfies (wS). For a nontrivial p-subgroup $Q \subset G$ we again denote by H_Q the intersection of all elements of $\mathcal{C}$ which contain Q.

For any nontrivial p-subgroup $Q \subset G$, the fixed-point set $(X^{\delta}_{\mathcal{C}})^Q$ is the nerve of the category $(\mathbf{X}^{\delta}_{\mathcal{C}})^Q$ [4; 7.2]. And $\mathcal{C}^Q$ is the full subcategory of all $H \in \mathcal{C}$ such that $Q \subset N_G(H)$. Obviously, the group H_Q is fixed, thus $\mathbf{X}^{\delta}_{\mathcal{I}(\rho)^Q}$ is nonempty. The natural transformations induced by the inclusions $K \subset H_{K \cdot Q} \supset H_Q$ show that the identity is homotopic to the constant functor. Therefore, $(\mathbf{X}^{\delta}_{\mathcal{C}})^Q$ is contractible.

Moreover, for each simplex $\sigma := (H_0 \subset H_1 \subset ... \subset H_n)$ of $X^{\delta}_{\mathcal{C}}$, the isotropy group $G_\sigma = \bigcap_i N_G(H_i)$ contains H_0 and therefore an element of order p which acts trivially on $\mathbb{F}_p$. This is sufficient to conclude that $X^{\delta}_{\mathcal{C}}$ is acyclic for the functors $H_*(G; -)$ [4; 6.10] and that $\mathcal{C}$ is normalizer $\mathbb{F}_p$-sharp which proves part (ii).

If $\mathcal{C}$ satisfies (S), then the category $\mathbf{X}^{\delta}_{\mathcal{C}}$ has an initial element, namely the isotropy group of V and $X^{\delta}_{\mathcal{C}}$ is contractible. In particular, this and the above argument show that $X^{\delta}_{\mathcal{C}}$ is acyclic for the functors $H_*(P; M \otimes -)$, where $P \subset G$ is a p-Sylow subgroup [4; 4.8]. By [4; 6.4], this implies that $X^{\delta}_{\mathcal{C}}$ is acyclic for $H_*(G; M \otimes -)$ and that $\mathcal{C}$ is normalizer M-sharp. $\square$

4.6 Remark. A long the same lines, one can proof that $\mathcal{C}$ is subgroup $\mathbb{F}_p$-sharp. We use the same notation as above. The fixed-point set $(X^{\beta}_{\mathcal{C}})^Q$ is the nerve of the full subcategory of all objects $(gH, G/H)$ such that $Q \subset gHg^{-1}$. Hence, it is nonempty, and, since there is a unique map $(eH_Q, G/H_Q) \to (gH, G/H)$ for each object, it has an initial element and is therefore contractible.

A typical simplex of $X^{\beta}_{\mathcal{C}}$ has as its isotropy group an element of $\mathcal{C}$ and contains therefore an element of order p. Analogously as above, this implies that $\mathcal{I}(\rho)$ is subgroup sharp.

For the collection $\mathcal{I}_{\mathcal{P}}(\rho)$ (see Remark 4.3) we can improve Corollary 4.5.

4.7 Theorem. *Let G be a finite group such that p divides the order of G, let $\rho : G \to Gl(V)$ be a modular representation and let M be an $\mathbb{F}_p[G]$-module such that there exists a nontrivial element of order p in G acting trivially on M. Then, the collection $\mathcal{I}_{\mathcal{P}}(\rho)$ is normalizer M-sharp.*

Proof. For any nontrivial p-subgroup $Q \subset G$, we set $H_Q := Iso_G(V^Q)$. Analogously as above follows that $(X^{\delta}_{\mathcal{C}})^Q$ is contractible. For each simplex $\sigma := (H_0 \subset H_1 \subset \ldots \subset H_n)$ of $X^{\delta}_{\mathcal{C}}$, let $Q_i \subset H_i$ be a p-Sylow subgroup, such that $Q_i \subset Q_{i+1}$. Then, $H_i = Iso_G(V^{Q_i})$, and the centralizer $C_G(Q_n)$ centralizes all Q_i's, normalizes all H_i's and is contained in the isotropy group $G_\sigma = \bigcap_i N_G(H_i)$ of σ. By [4; 7.1] there a nontrivial element of order p in G_σ acting trivially on M. And by [4; 6.10], this implies that $\mathcal{I}_{\mathcal{P}}(\rho)$ is normalizer M-sharp. □

Finally we want to show that the collection $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are normalizer sharp.

4.8 Theorem. *Let G be a finite group whose order is divisible by p. Let M be an $\mathbb{F}_p[G]$-module such that there exists a nontrivial element of order p acting trivially on M. Then, the collections $\mathcal{C}en(\mathcal{E})$ and $\mathcal{C}en(\mathcal{A})$ are normalizer M-sharp.*

Proof. We only prove this statement for $\mathcal{C} := \mathcal{C}en(\mathcal{E})$. A slight modification of the argument will work for $\mathcal{C}en(\mathcal{A})$.

Let Q be a nontrivial p-subgroup. Then, $(X^{\delta}_{\mathcal{C}})^Q$ is the nerve of the full subcategory of $\mathbf{X}^{\delta}_{\mathcal{C}}$ whose objects satisfy the condition $Q \subset N_G(C_G(E)$. This category is nonempty since $Q \subset C_G(E(Q)$. Moreover, Q acts on $E' := E(C_G(E))$. The zig-zag

$$C_G(E) \subset C_G((E')^Q) \supset C_G((E')^Q \cdot E(Q)) \subset C_G(E(Q))$$

shows that this fixed point set is contractible. Moreover, by [4; 7.1], $N_G(C_G(E))$, actually $C_G(E)$, contains an element of order p acting trivially on M. Therefore, $(X^{\delta}_{\mathcal{C}})$ is acyclic for the coefficient functor $H_*(G; M \otimes -)$ [4; 6.10] and $\mathcal{C}$ is normalizer M-sharp. □

5. Examples.

In this section we discuss some examples. We do not claim originality for the decomposition described in the following examples. Some of them might already be in the literature or known to experts.

5.1 Polynomial invariants of finite groups.

This example was used in [9] and [10] to construct spaces with polynomial mod-p cohomology and to prove uniqueness results for these spaces. Actually it was the motivating example for this note. It is also discussed in [6].

For a finite group G, the collection $\mathcal{P}'$ is subgroup M-sharp for any $\mathbb{F}_p[G]$-module M [4; 10.1 and the proof], even if we replace homology by cohomology. For cohomology, this is proved in [8].

Let $\rho : G \to Gl(V)$ be a modular representation. The collections $\mathcal{I}'(\rho)$ and $\mathcal{I}_{\mathcal{P}'}(\rho)$ are both subgroup M-sharp for every $\mathbb{F}_p[G]$-module M. For homology this follows from Theorem 1.2, and for cohomology one has to dualize all the arguments in the proof of Theorem 1.2.

Let $\phi : G \to Gl(W)$ be another representation of G. The the polynomial algebra $A := \mathbb{F}_p[W]$ of formal polynomial function on W carries a G-action. For $\mathcal{J} = \mathcal{I}'(\rho)$ or $\mathcal{I}_{\mathcal{P}'}(\rho)$, the functor $H^*_A : \mathbf{O}^{op}_{\mathcal{J}}(G) \to \mathcal{A}b$ defined by $H^*_A(G/H) := H^*(H; A)$ (group cohomology) is isomorphic to $H^*(\beta_{\mathcal{J}}; A)$ (twisted coefficients). We get the algebraic decomposition formula

$$\lim_{\overleftarrow{\mathcal{J}}}{}^i H^*_A \cong \begin{cases} H^*(G; A) & \text{for } \ i = 0 \\ 0 & \text{for } \ i \geq 1 \end{cases}$$

And, specializing to H^0_A, this reads as

$$\lim_{\overleftarrow{\mathcal{J}}}{}^i \mathbb{F}_p[W]^H \cong \begin{cases} \mathbb{F}_p[W]^G & \text{for } \ i = 0 \\ 0 & \text{for } \ i \geq 1 \end{cases}$$

5.2 Decompositions for the symmetric group Σ_n.
The set $X := \{1, ..., n\}$ given by the first n positive integers carries a Σ_n-action. For a p-subgroup $P \subset \Sigma_n$ a P-orbit has length 1 or a power of p. Hence, the fixed-point set X^P is a subset of X of order r such that p divides $n - r$. And the isotropy subgroup of X^P is conjugate to Σ_{n-r}.

Since every subset of X of order r such that $p|(n - r)$ appears as the fixed-point set of a p-subgroup, the collection of all subgroups conjugate to some $\Sigma_m \subset \Sigma_n$, where $p|m$, is subgroup and normalizer $\mathbb{F}_p$-sharp.

Let $\Sigma_n \to Gl(n, \mathbb{F}_p)$ denote the permutation representation. And let $B := \{e_1, ..., e_n\}$ denote the standard basis of $\mathbb{F}_p^n =: V$. For a subset $B' \subset B$ we denote by $V(B') \subset V$ the subspace generated by the sum of all elements in B'. The isotropy subgroup of $V(B')$ is conjugate to $\Sigma_r \times \Sigma_{n-r}$ where r is the order of B'. For a nontrivial p-subgroup $P \subset \Sigma_n$ the fixed-point set V^P is the direct sum of subspaces $V(B')$ where B' runs through the P-orbits of B. Since each of these orbits has length a power of p, the isotropy subgroup of V^P is conjugate to a subgroup of the form $\Sigma_{p^{n_1}} \times ... \times \Sigma_{p^{n_k}}$ where $p^{n_1} + ... + p^{n_k} = n$ and where $n_1 \geq 1$. Let $\mathcal{C}$ denote the collection of all subgroups conjugate to a product of symmetric groups as described above. Each of these groups appears as the isotropy subgroup of the fixed-point set of some p-subgroup. Therefore, the collection $\mathcal{C}$ is subgroup and normalizer sharp.

5.3 A decomposition for $Gl(n, \mathbb{F}_p)$.
The general linear group $G := Gl(n; \mathbb{F}_p)$ acts on $V := \mathbb{F}_p^n$ by matrix multiplication. Again, we denote by $B := \{e_1, ..., e_n\}$ the standard basis of V. Let V_r denote the subspace generated by the first r basis vectors. The group G acts transitively on the set of r-dimensional subspaces. Therefore, the isotropy subgroup of a proper nontrivial subspace $V' \subset V$ is conjugate to the isotropy subgroup of V_r, where $1 \leq r = dim_{\mathbb{F}_p} V'$. And this is the subgroup $G(r)$ of all matrices of the form $\begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix}$, where $D \in Gl(n - r, \mathbb{F}_p)$ and where B is a $r \times (n - r)$ matrix over $\mathbb{F}_p$. All these subgroups contain elements of order p. Therefore, the collection $\mathcal{C}$ of all subgroups conjugate to some $G(r)$, $0 < r < n$ is subgroup and normalizer sharp.

The centralizer of $G(r)$ in G is the center of G which is a cyclic group of order $p - 1$. Therefore, the values of the functor $\alpha_{\mathcal{C}}$ are mod-p acyclic and the collection $\mathcal{C}$ is not centralizer $\mathbb{F}_p$-sharp.

5.4 Decompositions for finite groups associated to permutation representation.
Let $H \subset G$ be a subgroup of the finite group G. The G-action of G on the homogeneous space G/H induces an action of G on the vector space $V := \mathbb{F}_p[G/H]$ whose standard basis is given by the elements $gH \in G/H$. As in 5.2, we associate for each subset of $B \subset G/H$ the 1-dimensional subspace $V(B) \subset V$ generated by the sum of all elements in B. For a p-subgroup $P \subset G$ the fixed-point set V^P is the direct sum of all subspaces $V(B)$ where B runs through the P-orbits in G/H. And a subgroup $K \subset G$ with $P \subset K$ is contained in the isotropy subgroup of V^P if the obvious map $P\backslash G/H \to K\backslash G/H$ is a bijection. That is to say that, for all $gH \in G/H$ and for all $k \in K$, $kgH \subset PgH$ or , equivalently, such that $k \in P \cdot gHg^{-1}$. The product is understood as the set of elements ph, $p \in P$ and $h \in gHg^{-1}$, not as the subgroup generated by P and gHg^{-1}. The isotropy group of V^P is therefore the subgroup $H_P := \bigcap_{gH \in G/H} P \cdot gHg^{-1}$. The collection of all subgroups H_P, $P \in \mathcal{P}$ is subgroup and normalizer $\mathbb{F}_p$-sharp.

5.5 Decomposition associated to collections of subgroups.
Let $X := \mathcal{C}$ be a collection of subgroups of G, such that there exists an element $C \in \mathcal{C}$ whose normalizer contains a p-Sylow subgroup of G. For a nontrivial p-subgroup $P \subset G$, the isotropy subgroup of X^P is the intersection $\bigcap N_G(C)$ taken over all subgroups C such that $P \subset N_G(C)$. The collection of all nontrivial subgroups of this form is nonempty and subgroup and normalizer $\mathbb{F}_p$-sharp.

If we choose $\mathcal{C}$ to be $\mathcal{P}$ itself then the collection of all intersection $N_G(P_1) \cap ... \cap N_G(P_k)$ with nontrivial p-Sylow subgroup is subgroup and normalizer $\mathbb{F}_p$-sharp.

If we choose $\mathcal{C}$ to be the collection of all p-Sylow subgroups, then the collection of all intersections $N_G(P_1) \cap ... \cap N_G(P_k)$ such that $P_1 \cap ... \cap P_k$ is a nontrivial intersection of p-Sylow subgroups is subgroup and normalizer $\mathbb{F}_p$-sharp.

If we choose X to be the set of all objects of the category $\mathbf{X}_{\mathcal{C}}^{\beta}$ where $\mathcal{C}$ is again the collection of all p-Sylow subgroups, then the collection of all nontrivial intersections $P_1 \cap ... \cap P_k$, P_i a p-Sylow subgroup for all i, is subgroup and normalizer $\mathbb{F}_p$-sharp.

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