

DEPTH AND HOMOLOGY DECOMPOSITIONS

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ABSTRACT. Homology decomposition techniques are a powerful tool used in the analysis of the homotopy theory of (classifying) spaces. The associated Bousfield-Kan spectral sequences involve higher derived limits of the inverse limit functor. We study the impact of depth conditions on the vanishing of these higher limits and apply our theory in several cases. We will that the depth of Stanley-Reisner algebras can be characterized in combinatorial terms of the underlying simplicial complexes, the depth of group cohomology in terms of depth of group cohomology of centralizers of elementary abelian subgroups, and the depth of polynomial invariants in terms of depth of polynomial invariants of point-wise stabilizer subgroups. The latter two applications follow from the analysis of an algebraic version of centralizer decompositions in terms of Lannes' T -functor.

1. INTRODUCTION

Homology decompositions are one of the most useful tools in the study of the homotopy theory of topological spaces. Given a cohomology theory h^* , a homology decomposition for a space X is, roughly speaking, a recipe to glue together spaces, desirably of a simpler homotopy type, such that the resulting space maps into X by an h^* -isomorphism.

Technically, this is described by a (covariant) functor $F : \mathcal{C} \rightarrow \text{TOP}$ from a (discrete) category \mathcal{C} into the category TOP of topological spaces together with a natural transformation $F \rightarrow 1_X$ from F to the constant functor 1_X which maps each object to X and each morphism to the identity. Passing to homotopy colimits, this natural transformation induces a map $f_X : \text{hocolim}_{\mathcal{C}} F \rightarrow X$.

In special circumstances we can decide by purely algebraic means, whether the map f_X is an h^* -equivalence. The cohomology of the homotopy colimit can be calculated with the help of the Bousfield-Kan spectral sequence [2]. The E_2 -page of this spectral sequence has the form

$$E_2^{i,j} \stackrel{\text{def}}{=} \lim_{\mathcal{C}^{op}}^i h^*(F)$$

and converges towards $h^{i+j}(X)$. If the derived limits $\lim_{\mathcal{C}^{op}}^i h^*(F)$ of the inverse limit $\lim_{\mathcal{C}^{op}} h^*(F)$ satisfy the equations

$$\lim_{\mathcal{C}^{op}}^i h^*(F) \cong \begin{cases} h^*(X) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1, \end{cases}$$

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then $f_X : \text{hocolim}_c \Phi \rightarrow X$ is an h^* -equivalence. The first isomorphism has to be induced by the natural transformation $1_{h^*(X)} \rightarrow h^*(F)$ respectively by the composition

$$h^*(X) \xrightarrow{h^*(f_X)} h^*(\text{hocolim}_c F) \rightarrow \lim_{c^{op}} h^*(F).$$

We would like to find conditions which ensure the vanishing of the higher derived limits. In particular, we are interested in the impact made by depth conditions.

Let R be a noetherian local ring with maximal ideal m_R and let M be a finitely generated R -module. A sequence $x_1, \dots, x_r \in m_R$ is called regular on M , if for all i , the element x_i is not a zero divisor for the quotient $M/(x_1, \dots, x_{i-1})M$. And the depth of M , denoted by $\text{depth } M$, is the maximal length of a regular sequence.

Let $\Phi : \mathcal{C} \rightarrow R_{f\bar{g}}\text{-mod}$ be a functor, defined on a discrete category \mathcal{C} and taking values in the category of finitely generated R -modules. Let $1_M \rightarrow \Phi$ be a natural transformation from the constant functor 1_M to Φ inducing a R -linear map $\rho : M \rightarrow \lim_c \Phi$. Our technical key result Lemma 2.1 relates depth conditions on M , depth conditions on the values of the functor, on the kernel and cokernel of ρ , and the vanishing of the higher limits of Φ .

We will apply Lemma 2.1 in several cases, namely in the context of Stanley-Reisner algebras, group cohomology, invariant theory and algebras over the Steenrod algebra. In all cases we will describe the algebraic object in question as the inverse limit of a functor defined on a finite category. The first two application have a topological background in terms of homology decompositions.

Example 1.1. (Stanley-Reisner algebras) The main invariant to study the combinatorics of an abstract simplicial complex K is the associated face ring respectively Stanley-Reisner algebra $\mathbb{F}(K)$ (\mathbb{F} a field), which is a quotient of a polynomial algebra generated by the vertices of the complex. Let $\text{CAT}(K^\times)$ denote the category given by the non-empty faces of K . Then, $\mathbb{F}(K)$ can be described as the inverse limit of a functor $\Phi_K : \text{CAT}(K^\times) \rightarrow \mathbb{F}\text{-Alg}$ from the category $\text{CAT}(K^\times)$ given by the non-empty faces of K into the category of \mathbb{F} -algebras. This algebraic setting has a topological realization. There exists a space $c(K)$ such that $H^*(c(K); \mathbb{F}) \cong \mathbb{F}(K)$ as well as a functor $F_K : \text{CAT}(K^\times)^{op} \rightarrow \text{TOP}$ and a natural transformation $F_K \rightarrow 1_{c(K)}$ such that $H^*(F_K; \mathbb{F}) \cong \Phi_K$ and such that $\text{hocolim}_{\text{CAT}(K^\times)^{op}} F_K \rightarrow c(K)$ is a homotopy equivalence [15](see also Section 7).

Properties of commutative algebra for $\mathbb{F}(K)$ are reflected in the geometry or combinatorics of the underlying simplicial complex. For example, Reisner proved such a theorem for the Cohen-Macaulay property [21]. With our methods we will generalize this result and characterize the depth of the face ring by the cohomological connectivity of the links of the faces of K (Theorem 7.1).

Example 1.2. (group cohomology) For a compact Lie group G and a fixed prime p , we denote by $\mathcal{F}_p(G)$ the Frobenius or Quillen category of G [19]. The objects are given by the non-trivial elementary abelian p -subgroups $E \subset G$ and the morphism are all monomorphism $E \rightarrow E'$ induced by conjugation by an element of G . For an object $E \in \mathcal{F}_p(G)$ we denote by $BC_G(E)$ the classifying space of the centralizer $C_G(E)$ of E . These assignments

fit together to establish a functor $F_G : \mathcal{F}_p(G)^{op} \rightarrow \text{TOP}$. The inclusions $C_G(E) \subset G$ establish a natural transformation $F_G \rightarrow 1_{BG}$ and the map $\text{hocolim}_{\mathcal{F}_p(G)^{op}} F_G \rightarrow BG$ induces an isomorphism in mod- p cohomology. Passing to mod- p cohomology gives rise to a functor $\Phi_G : \mathcal{F}_p(G) \rightarrow \mathbb{F}_p\text{-Alg}$ defined by $\Phi_G(E) \stackrel{\text{def}}{=} H^*(BC_G(E); \mathbb{F}_p)$ and to a natural transformation $1_{H^*(BG; \mathbb{F}_p)} \rightarrow \Phi_G$, which induces an isomorphism $H^*(BG; \mathbb{F}_p) \rightarrow \lim_{\mathcal{F}_p(G)} \Phi_G$. All these facts can be found in [10] and in [5]. We will show that $\text{depth } H^*(BG; \mathbb{F}_p)$ equals the minimum of the numbers $\text{depth } H^*(BC_G(E); \mathbb{F}_p)$ taken over all objects in $\mathcal{F}_p(G)$ (Theorem 5.1).

Example 1.3. (polynomial invariants) For a finite dimensional \mathbb{F}_p -vector space V and a representation $G \rightarrow \text{Gl}(V)$ of a finite group G , we denote by $\mathbb{F}_p[V]^G$ the ring of polynomial invariants of the G -action on the symmetric algebra $\mathbb{F}_p[V]$ of the dual of V . Let S denote the collection of all point wise stabilizer subgroups of non trivial subspaces of V whose order is divisible by p . Let $\mathcal{O}(G)$ denote the orbit category of G . That is objects are given by quotients G/H and the morphisms by G -equivariant maps. Let $\mathcal{O}_S(G) \subset \mathcal{O}(G)$ denote the full subcategory of the orbit category, whose objects are given by quotients G/H such that $H \in S$. There exists a functor $\Psi_G : \mathcal{O}_S(G)^{op} \rightarrow \mathbb{F}_p\text{-Alg}$ and natural transformation $1_{\mathbb{F}_p[V]^G} \rightarrow \Psi_G$. If p divides the order of G , then this natural transformation establishes an isomorphism $\mathbb{F}_p[V]^G \xrightarrow{\cong} \lim_{\mathcal{O}_S(G)} \Psi_G$ [16]. In this case If p divides the order of G , we will show that $\text{depth } \mathbb{F}_p[V]^G$ equals the minimum of the numbers $\text{depth } \mathbb{F}_p[V]^H$ taken over all subgroups $H \subset G$ contained in S (Theorem 6.1). If p does not divide the order of G , then $\mathbb{F}_p[V]^G$ is Cohen-Macaulay, i.e. $\text{depth } \mathbb{F}_p[V]^G$ equals the dimension of V [14].

The inequality $\text{depth } \mathbb{F}_p[V]^G \leq \text{depth } \mathbb{F}_p[V]^H$ for $H \in S$ was already proven by Kemper [11] as well as by Smith (see [14]).

Example 1.4. (algebraic centralizer decomposition) The second example is a specialization of a more general statement concerning algebras over the Steenrod algebra. We can think of this generalization as an algebraic centralizer decomposition for algebras over the Steenrod algebra.

Let p be a fixed prime. Let \mathcal{A} denote the mod- p Steenrod algebra and \mathcal{K} the category of unstable algebras over \mathcal{A} . Let A be a noetherian object of \mathcal{K} . i.e. A is noetherian just as algebra. Rector constructed a category $\mathcal{F}(A)$, whose objects (E, ϕ) consist of a non-trivial elementary abelian p -group E and a \mathcal{K} -morphism $\phi : A \rightarrow H^E \stackrel{\text{def}}{=} H^*(BE; \mathbb{F}_p)$ into the mod- p cohomology of the classifying space BE of E , such that ϕ makes H^E to a finitely generated A -module [20]. With the help of Lannes' T -functor [12], one can construct a functor $\Phi_A : \mathcal{F}(A) \rightarrow \mathcal{K}$ and a natural transformation $1_A \rightarrow \phi_A$ respectively a map $A \rightarrow \lim_{\mathcal{F}(A)} \Phi_A$ (see Section 4). We will establish a relation between the depth of A respectively of the functor values and the vanishing of the higher limits (Theorem 4.1).

Besides Lemma 2.1 there is one further input in all four examples. In all four cases it turns out that depth conditions on $\mathbb{F}(K)$, $H^*(BG; \mathbb{F}_p)$, $\mathbb{F}_p[V]^G$ or on A are inherited to the functor values of the associated functor. This is a consequence of Theorem 3.1, which states that an application of Lannes' T -functor may only increase the depth.

The paper is organized as follows. In the next section we establish our technical key result. In Section 3 we analyse the relation between and Lannes' T -functor. This passes the way to discuss the case of algebraic centralizer decompositions (Section 4), the case of group cohomology (Section 5), the case of polynomial invariants (Section 6) and, finally, the case of Stanley-Reisner algebras (Section 7).

2. THE KEY LEMMA

Let \mathbb{F} be a field. To simplify the discussion and since all applications are covered, we suppose that R is either a commutative local noetherian ring with maximal ideal m_R and residue field \mathbb{F} or a connected commutative graded noetherian \mathbb{F} -algebra. In the graded case, that is that $R^i = 0$ for $i < 0$ and $R^0 = \mathbb{F}$. In particular, R is also a graded local ring with residue field \mathbb{F} . The maximal ideal $m_R \stackrel{\text{def}}{=} R^+$ is the set of elements of positive degree. In the graded case, the depth of M is defined as the maximal length of a sequence of homogeneous elements in R which is regular on M . Let $R_{fg}\text{-mod}$ denote the category of finitely generated R -modules respectively the category of finitely generated non-negatively graded R -modules. Let $\Phi : \mathcal{C} \rightarrow R_{fg}\text{-mod}$ be a functor defined on a discrete category \mathcal{C} . Let M be an object of $R_{fg}\text{-mod}$ and $1_M \rightarrow \Phi$ a natural transformation from the constant functor 1_M to Φ . This establishes a R -linear map $\rho : M \rightarrow \lim_{\mathcal{C}} \Phi$. We measure the vanishing of the higher derived limits and the deviation of ρ being an isomorphism by the R -modules $L^i \stackrel{\text{def}}{=} L^i(M, \Phi)$, $i \geq -1$. That is L^{-1} is the kernel and L^0 the cokernel of ρ . And $L^i \stackrel{\text{def}}{=} \lim_{\mathcal{C}}^i \Phi_M$ for $i \geq 1$.

Higher derived limits of the functor Φ can be defined as the cohomology groups of a certain cochain complex $(C^*(\mathcal{C}, \Phi), \delta)$ over R (for details see [18]). For $n \geq 0$, the R -module $C^n \stackrel{\text{def}}{=} C^n(\mathcal{C}, \Phi)$ is defined as

$$C^n \stackrel{\text{def}}{=} \prod_{c_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_n} F(c_n).$$

The differential $\delta : C^n \rightarrow C^{n+1}$ is given by the alternating sum $\sum_{k=0}^n (-1)^k \delta^k$ where δ^k is defined on $u \in C^n$ by

$$\delta^k(u)(c_0 \rightarrow \dots \rightarrow c_{n+1}) \stackrel{\text{def}}{=} \begin{cases} u(c_0 \rightarrow \dots \rightarrow \widehat{c}_k \rightarrow \dots \rightarrow c_{n+1}) & \text{for } k \neq n+1 \\ \phi(c_n \rightarrow c_{n+1})u(c_0 \rightarrow \dots \rightarrow c_n) & \text{for } k = n+1. \end{cases}$$

In particular, δ^k as well as δ are R -linear. And all the groups L^i are in a natural way finitely generated R -modules.

We fix all the above notation through out this section. We say that the sequence L^i , $i \geq -1$, is almost trivial, if, for all $i \geq -1$, either $L^i = 0$ or $\text{depth } L^i = 0$. The following result is the key lemma for proving our main results.

Lemma 2.1. *Let $\Phi : \mathcal{C} \rightarrow R_{fg}\text{-mod}$ be a functor. Suppose that $\text{depth } \Phi(c) \geq r$ for all objects $c \in \mathcal{C}$ and that the sequence L^i , $i \geq -1$, is almost trivial. Then, $\text{depth } M \geq r$ if and only if $L^i = 0$ for $i \leq r - 2$.*

The rest of this section is devoted to a proof of this lemma. The proof is based on spectral sequences associated to double complexes. To fix notation we recall the basic concept. Details may be found in [24].

A double complex over R or a differential bigraded R -module $(D^{*,*}, d_h, d_v)$ is a bigraded R -module $D^{*,*}$ with two R -linear maps $d_h : D^{*,*} \rightarrow D^{*+1,*}$ and $d_v : D^{*,*} \rightarrow D^{*,*+1}$ of bidegree $(1, 0)$ and $(0, 1)$ such that $d_h d_h = 0 = d_v d_v$ and $d_h d_v + d_v d_h = 0$. We think of d_h as the horizontal and of d_v as the vertical differential. To each double complex $D^{*,*}$ we associate a total complex $Tot^*(D)$ which is a differential graded R -module defined by $Tot^n(D) \stackrel{\text{def}}{=} \bigoplus_{i+j=n} D^{i,j}$ with differential $d \stackrel{\text{def}}{=} d_h + d_v$.

If B_* is a chain complex and C^* a cochain complex over R then $\text{Hom}_R(B_*, C^*)$ can be made into a bigraded R -module. We define $D^{i,j} \stackrel{\text{def}}{=} \text{Hom}_R(B_j, C^i)$, $d_v \stackrel{\text{def}}{=} \text{Hom}_R(d_B, \text{id})$ and $d_h \stackrel{\text{def}}{=} (-1)^j \text{Hom}_R(\text{id}, d_C)$ as the map $D^{i,*} \rightarrow D^{i+1,*}$.

For a double complex $(D^{*,*}, d_h, d_v)$, we can take horizontal or vertical cohomology groups denoted by $H_h^*(D^{*,*})$ and $H_v^*(D^{*,*})$. The boundary maps d_v and d_h induce again boundary maps on these cohomology groups. We can consider cohomology groups of the form $H_h^*(H_v^*(D^{*,*}))$ and $H_v^*(H_h^*(D^{*,*}))$.

If $D^{i,j} = 0$ for $i < 0$ or $j < 0$, there exist two spectral sequences converging towards $H^*(Tot(D), d)$. In one case, we have ${}_I E_2^{i,j} = H_h^i(H_v^j(D))$ and in the other case ${}_{II} E_2^{i,j} = H_v^j(H_h^i(D))$. In the first case the differentials on the E_r -page have degree $(1 - r, r)$ and in the second case degree $(r, 1 - r)$.

Proof of Lemma 2.1: Since R is local, the depth of a finitely generated R -module N is characterized by the smallest number j , such that $\text{Ext}_R^j(\mathbb{F}, N) \neq 0$ [24, Theorem 4.4.8]. Hence, we have to relate certain Ext-groups.

If $r = 0$ there is nothing to show since each R -module has depth ≥ 0 . If $r = 1$ we have an exact sequence

$$0 \longrightarrow \text{Ext}_R^0(\mathbb{F}, L^{-1}) \longrightarrow \text{Ext}_R^0(\mathbb{F}, M) \longrightarrow \text{Ext}_R^0(\mathbb{F}, \bigoplus_{c \in C} M(c)) \cong \bigoplus_{c \in C} \text{Ext}_R^0(\mathbb{F}, M(c)) = 0.$$

Hence, since $L^{-1} = 0$ or $\text{depth } L^{-1} = 0$, we have $\text{depth } M \geq 1$ if and only if $\text{depth } L^{-1} \geq 1$ if and only if $L^{-1} = 0$. This proves the statement for $r = 1$.

Now we assume that $\text{depth } M \geq r \geq 2$. In particular, $L^{-1} = 0$. We want to show that $L^i = 0$ for $i \leq r - 2$. Let Q_* be a R -projective resolution of \mathbb{F} as R -module, let $C^* \stackrel{\text{def}}{=} C^*(\mathcal{C}; \Phi)$ and let $D^{*,*}$ denote the differential bigraded R -module $\text{Hom}_R(Q_*, C^*)$. Since $D^{*,*}$ is concentrated in the first quadrant, both spectral sequences converge towards $H^*(Tot(D))$. We have

$${}_I E_2^{i,j} = H_h^i(H_v^j(D)) \cong \lim^i \text{Ext}_R^j(\mathbb{F}, \Phi),$$

which vanishes for $j \leq r - 1$. On the other hand, we have

$${}_{II} E_2^{i,j} = H_v^j(H_h^i(D)) \cong \begin{cases} \text{Ext}_R^j(\mathbb{F}, L^i) & \text{for } i \geq 1 \\ \text{Ext}_R^j(\mathbb{F}, \lim_c^0 \Phi) & \text{for } i = 0. \end{cases}$$

The long exact sequence of Ext-groups for the short exact sequence

$$0 \longrightarrow M \longrightarrow \lim_c \Phi \longrightarrow L^0 \longrightarrow 0$$

shows that $\text{Ext}_R^i(\mathbb{F}, \lim_c \Phi) \cong \text{Ext}_R^i(\mathbb{F}, L^0)$ for $i \leq r - 2$.

By degree reasons and since no differential starts or ends at ${}_{II}E_2^{0,0} \cong \text{Ext}_R^0(\mathbb{F}, L^0)$, this group has to vanish. Since $L^0 = 0$ or $\text{depth } L^0 = 0$, this shows that $L^0 = 0$, that $M \cong \lim_c^0 \Phi$ and that ${}_{II}E_2^{0,j} = 0$ for $j < r$. We can repeat the argument successively for $i = 1, \dots, r - 2$, which implies that $L^i = 0$ for $i \leq r - 2$. Since there might be a non-trivial differential

$${}_{II}E_2^{r-1,0} \cong \text{Ext}^0(\mathbb{F}, L^{r-1}) \longrightarrow {}_{II}E_2^{0,r} \cong \text{Ext}_R^r(\mathbb{F}, M),$$

we cannot repeat the argument further.

If $L^i = 0$ for $i \leq r - 2$, the E_2 -page of first spectral sequence is exactly the same, but may fail the vanishing statement. Since $M \cong \lim_c^0 \Phi$, the E_2 -page of the second spectral sequence satisfies the equations

$${}_{II}E_2^{i,j} \cong \begin{cases} \text{Ext}_R^j(\mathbb{F}, M) & \text{for } i = 0 \\ 0 & \text{for } 1 \leq i \leq r - 2. \end{cases}$$

A similar degree argument shows that $\text{Ext}_R^j(\mathbb{F}, M) = 0$ for $j \leq r - 1$ and that $\text{depth } M \geq r$. This finishes the proof. \square

We also will apply Lemma 2.1 in the form of the following corollary.

Corollary 2.2. *If $L^i = 0$ for all $i \geq -1$ then $\text{depth } M \geq \min\{\text{depth } \Phi_M(c) : c \in \mathcal{C}\}$.*

3. DEPTH AND THE T -FUNCTOR

To fix notation we recall some basic concepts from the T -functor technology. For more details see [22] and [12]

Let p be a fixed prime. As already mentioned in the introduction, \mathcal{A} denotes the mod- p Steenrod algebra, \mathcal{U} the category of unstable modules and \mathcal{K} the category of unstable algebras over \mathcal{A} .

For an elementary abelian p -group E we denote by $H^E \stackrel{\text{def}}{=} H^*(BE; \mathbb{F}_p)$ the mod- p cohomology of the classifying space BE . The functor $\otimes H^E : \mathcal{U} \longrightarrow \mathcal{U}$ has a left adjoint in \mathcal{U} denoted by $T_E : \mathcal{U} \longrightarrow \mathcal{U}$. This functor is exact, commutes with tensor products and restricts therefore to a functor $T_E : \mathcal{K} \longrightarrow \mathcal{K}$ also denoted by T_E . The adjoint of $\text{id} : T_E(M) \longrightarrow T_E(M)$ gives a map $M \longrightarrow T_E(M) \otimes H^E$ and projecting to the first factor via the augmentation $H^E \longrightarrow \mathbb{F}_p$ a map $e : M \longrightarrow T_E(M)$.

We can specialize further. For an algebra $A \in \mathcal{K}$ we denote by $A\text{-}\mathcal{U}$ the category of $A \odot \mathcal{A}$ -modules. An $A \odot \mathcal{A}$ -module is both, an object $M \in \mathcal{U}$ and an A -module such that the structure map $A \otimes M \longrightarrow M$ is \mathcal{A} -linear. Here, \mathcal{A} acts on the tensor product via the Cartan formula. We also denote by $A_{fg}\text{-}\mathcal{U} \subset A\text{-}\mathcal{U}$ the full subcategory of all objects which are finitely generated as A -modules.

If $M \in A\text{-}\mathcal{U}$, then, since T commutes with tensor products, the \mathcal{U} -object $T_E(M)$ is actually an object of $T_E(A)\text{-}\mathcal{U}$. And if $\phi : A \rightarrow H^E$ is a \mathcal{K} -map, we define $M(\phi) \stackrel{\text{def}}{=} T_E(M; \phi) \stackrel{\text{def}}{=} T_E(M) \otimes_{T_E^0(A)} \mathbb{F}_p(\phi)$, where $\mathbb{F}_p(\phi)$ denotes the $T_E(A)$ -module whose structure map is induced by the adjoint of the \mathcal{K} -map $\phi : A \rightarrow H^E$.

Let $F(n) \in \mathcal{U}$ denote the free object in \mathcal{U} generated by one element in degree n . The module $F(n)$ is defined by the equation $\text{Hom}_{\mathcal{U}}(F(n), M) \stackrel{\text{def}}{=} M^n$ for any object $M \in \mathcal{U}$. Here, M^n denotes the set of elements of M of degree n (for details see [22]). For any object $M \in A\text{-}\mathcal{U}$, we have $\text{Hom}_{A\mathcal{U}}(A \otimes_{\mathbb{F}_p} F(n), M) = \text{Hom}_{\mathcal{U}}(F(n), M) = M^n$. Therefore, there exists an $A\mathcal{U}$ -epimorphism $P \rightarrow M$ such that P is free as A -module as well as a resolution

$$\rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

of $A\text{-}\mathcal{U}$ -modules, such that each P_i is free as A -module.

In the following we assume that A is noetherian as an algebra, which simplifies the discussion. In particular, $\text{Hom}_{\mathcal{K}}(A, H^E)$ is a finite set and $T_E^0(A) \cong \mathbb{F}_p^{\text{Hom}_{\mathcal{K}}(A, H^E)}$ is a finite dimensional vector space isomorphic to the dual of $\text{Hom}_{\mathcal{U}}(A, H^E)$. And, if $M \in A\text{-}\mathcal{U}$, then

$$T_E(M) \cong T_E(M) \otimes_{T_E^0(A)} T^0(A) \cong \bigoplus_{\phi \in \text{Hom}_{\mathcal{K}}(A, H^E)} M(\phi).$$

The above map $e : M \rightarrow T_E(M)$ respects all these additional structures and, restricting to a particular summand of $T_E(M)$, establishes a map $e_\phi : M \rightarrow M(\phi)$. This is a map of A -modules, where A acts on $M(\phi)$ via the \mathcal{K} -map $A \rightarrow A(\phi)$, and therefore a $A\text{-}\mathcal{U}$ -morphism. In fact, if $A \in \mathcal{K}$ is noetherian and $A \rightarrow H^E$ a \mathcal{K} -map, then the functor $T_E(-, \phi)$ restricts to a functor $T_E(-, \phi) : A_{fg}\text{-}\mathcal{U} \rightarrow A_{fg}\text{-}\mathcal{U}$ [9]. In particular, if A is noetherian, so is $A(\phi) \stackrel{\text{def}}{=} T_E(A, \phi)$.

In this section we want to prove the following theorem.

Theorem 3.1. *Let $A \in \mathcal{K}$ be a noetherian connected algebra and $M \in A_{fg}\text{-}\mathcal{U}$. Let E be an elementary abelian group and let $\phi : A \rightarrow H^E$ be a \mathcal{K} -map. Then, $\text{depth } M \leq \text{depth } M(\phi)$.*

Using the Auslander Buchsbaum equation (e.g see [24]) we will reduce this statement to a claim about homological dimension.

Let A be a connected commutative graded algebra and M a graded A -module. We say that M has finite homological dimension if there exists a finite resolution

$$0 \rightarrow P_r \rightarrow P_{r-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

of graded A -modules, such that P_i is projective for $0 \leq i \leq r$. We say that $\text{hom-dim}_A M = t$ if the shortest projective resolution has length t .

Proposition 3.2. *Let A be a noetherian object of \mathcal{K} , M and object of $A\text{-}\mathcal{U}$, and $\phi : A \rightarrow H^E$ a \mathcal{K} -map. If M has finite homological dimension over A , then $M(\phi)$ has finite homological dimension over $A(\phi)$ and $\text{hom-dim}_A M \geq \text{hom-dim}_{A(\phi)} M(\phi)$.*

Proof. Let $\text{hom-dim}_A M = t$. We can choose a resolution

$$0 \rightarrow P_t \rightarrow P_{t-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

of $A\mathcal{U}$ -modules, such that P_i is free as A -module for $0 \leq i \leq t-1$. Since $\text{hom-dim}_A M = t$, this implies that P_t is a projective A -module. And since A is connected and noetherian, in particular a graded local ring ring, P_t is a free A -module.

Applying the exact functor $T_E(-, \phi)$ establishes an exact $A(\phi)\mathcal{U}$ -sequence

$$0 \longrightarrow P_t(\phi) \longrightarrow P_{t-1}(\phi) \longrightarrow \dots \longrightarrow P_0(\phi) \longrightarrow M(\phi) \longrightarrow 0.$$

Lannes T -functor maintains freeness properties [8]. Hence, for all i , the module $P_i(\phi)$ is a free $A(\phi)$ -module, which proves the claim. \square

The proof of Theorem 3.1 needs some further preparation. Let $\alpha : A \longrightarrow B$ and $\phi : A \longrightarrow H^E$ be \mathcal{K} -maps, A and B noetherian. We denote by $e(\phi, \alpha)$ the set of all \mathcal{K} maps $\psi : B \longrightarrow H^E$ such that $\psi\alpha = \phi$. This is a finite set.

Lemma 3.3. *Let $\alpha : A \longrightarrow B$ and $\phi : A \longrightarrow H^E$ as above. For each object M of $B\mathcal{U}$, the T -functor induces an $A\mathcal{U}$ -isomorphism $M(\phi) \cong \bigoplus_{\psi \in e(\phi, \alpha)} M(\psi)$.*

Proof. The algebra A acts on $B(\psi)$ via the composition $A \longrightarrow B \longrightarrow B(\psi)$ of \mathcal{K} -maps. Since $T^0(B) \otimes_{T^0(A)} \mathbb{F}_p(\phi) \cong \bigoplus_{\psi \in e(\phi, \alpha)} \mathbb{F}_p(\psi)$ as algebras, we have

$$\begin{aligned} M(\phi) \cong T(M) \otimes_{T^0(A)} \mathbb{F}_p(\phi) &\cong T(M) \otimes_{T^0(B)} T^0(B) \otimes_{T^0(A)} \mathbb{F}_p(\phi) \\ &\cong \bigoplus_{\psi \in e(\phi, \alpha)} T(M) \otimes_{T^0(B)} \mathbb{F}_p(\psi) \cong \bigoplus_{\psi \in e(\phi, \alpha)} M(\psi). \end{aligned}$$

And this composition is obviously a map of A -modules. \square

The Dickson algebra $D_r \cong \mathbb{F}_p[d_1, \dots, d_r] \cong \mathbb{F}_p[x_1, \dots, x_r]^{\text{Gl}(r, \mathbb{F}_p)}$ is a polynomial algebra isomorphic to the invariants of the general linear group $\text{Gl}(r, \mathbb{F}_p)$ acting on the polynomial algebra $\mathbb{F}_p[x_1, \dots, x_r]$ with r -generators, which we give the degree 2. Then, this algebra carries an action of \mathcal{A} which inherits an action to D_r . More details can be found in [14].

For a positive integer l we denote by $D_r^l \subset D_r$ the subalgebra of all p^l -powers of elements of D_r . Then, $D_r^l \cong \mathbb{F}_p[d_1^{p^l}, \dots, d_r^{p^l}]$ is again a polynomial algebra, in particular a noetherian object of \mathcal{K} [22].

Proof of Theorem 3.1: Since A is noetherian, there exist integers $n, l \geq 1$ and a \mathcal{K} map $\alpha : D \stackrel{\text{def}}{=} D_n^l \longrightarrow A$ making A into a finitely generated D -module [1, Appendix]. Hence, M is a finitely generated D -module as well, and for the calculation of $\text{depth } M$ we can consider M as an D -module [23]. On the other hand, for each \mathcal{K} $\phi : A \longrightarrow H^E$, we have $M(\phi\alpha) \cong \bigoplus_{\psi \in e(\phi\alpha, \alpha)} M(\psi)$ as D -modules and, since $\phi \in e(\phi\alpha, \alpha)$, we have $\text{depth } M(\phi) \geq \text{depth } M(\phi\alpha)$. Hence, we can and will assume that $A = D$.

Since D is a polynomial algebra, every finitely generated D -module has finite homological dimension (see [24]). By Proposition 3.2 we have $\text{hom-dim}_D M \geq \text{hom-dim}_{D(\phi)} M(\phi)$. Since D is a polynomial algebra, the same holds for $D(\phi)$ and $D(\phi)$ is a free finitely generated D -module [8]. In particular, every projective $D(\phi)$ -module is a projective D -module and $\text{hom-dim}_{D(\phi)} M(\phi) \geq \text{hom-dim}_D M(\phi)$. By the Auslander-Buchsbaum equation (see [24]), we have $\text{depth } N = \text{depth } D - \text{hom-dim}_D N$ for any finitely generated D -module. Proposition 3.2 shows that $\text{depth } M \leq \text{depth } M(\phi)$. \square

4. DEPTH AND THE ALGEBRAIC CENTRALIZER DECOMPOSITION

We use the same notation as in the last section. For a noetherian object $A \in \mathcal{K}$, Rector defined a category $\mathcal{F}(A)$ as follows [20]. The objects are given by pairs (E, ϕ) , where E is a non-trivial elementary abelian p -group and where $\phi : A \rightarrow H^E$ is a \mathcal{K} -map such that H^E becomes a finitely generated A -module. And a morphism $\alpha : (E, \phi) \rightarrow (E', \phi')$ is a monomorphism $i_\alpha : E \rightarrow E'$ such that $\phi H^*(Bi_\alpha) = \phi'$. Since A is noetherian, this category is finite.

Let M be an object of $A\text{-}\mathcal{U}$. Since the T -functor is natural with respect to homomorphism $E \rightarrow E'$, the maps $e_\phi : M \rightarrow M(\phi)$ are compatible with all morphisms in the category $\mathcal{F}(A)$. This defines a covariant functor

$$\Phi_M : \mathcal{F}(A) \rightarrow A\text{-}\mathcal{U} : M \mapsto M(\phi)$$

as well as a natural transformation $1_M \rightarrow \Phi_M$. In particular, there exists an $A\text{-}\mathcal{U}$ -morphism

$$\rho_M : M \rightarrow \lim_{\mathcal{F}(A)} M(\phi).$$

For a topological interpretation of these algebraic data and their relation to centralizer decompositions of compact Lie groups see [5] (see also Section 5).

As in Section 2, we define

$$L^i(M) \stackrel{\text{def}}{=} \begin{cases} \lim_{\mathcal{F}(A)}^i \Phi_M & \text{for } i \geq 1 \\ \text{coker}(\rho_M) & \text{for } i = 0 \\ \text{ker}(\rho_M) & \text{for } i = -1 \end{cases}$$

Theorem 4.1. *Let $A \in \mathcal{K}$ be a connected noetherian algebra and let M be an object of $A_{fg}\text{-}\mathcal{U}$. Then, $\text{depth } M \geq r$ if and only if the following two conditions hold:*

- (i) *For all objects $(E, \phi) \in \mathcal{F}(A)$, we have $\text{depth } M(\phi) \geq r$.*
- (ii) *$L^i(M) = 0$ for $i \leq r - 2$*

Proof. In [9], Henn has shown that the A -modules L^i are finite dimensional \mathbb{F}_p -vector spaces. In particular, either $L^i(M) = 0$ or $\text{depth } L^i = 0$ for all $i \geq -1$. That is, the sequence $\{L^i : i \geq -1\}$ is almost trivial. Moreover, by Theorem 3.1, we have $\text{depth } M(\phi) \geq \text{depth } M$. An application of Lemma 2.1 finishes the proof. \square

5. DEPTH AND GROUP COHOMOLOGY

Let G be a compact Lie group with classifying space BG . And let p be a fixed prime. In this section, cohomology is always taken with \mathbb{F}_p -coefficients and $H^*(-) \stackrel{\text{def}}{=} H^*(-; \mathbb{F}_p)$.

As in the introduction we denote by $\mathcal{F}_p(G)$ the Quillen category of G and by $F_G : \mathcal{F}_p(G)^{op} \rightarrow \text{TOP}$ the functor mapping E to $BC_G(E)$. The natural transformation $F_G \rightarrow 1_{BG}$ establishes a map $\text{hocolim}_{\mathcal{F}_p(G)^{op}} F_G \rightarrow BG$. Jackowski and McClure showed that this map induces an isomorphism in mod- p cohomology. In fact they showed that $H^*(BG) \cong \lim_{\mathcal{F}_p(G)} H^*(F_G)$, that $\lim_{\mathcal{F}_p(G)}^i H^*(F_G) = 0$ for $i \geq 1$ [10].

Passing to classifying spaces and mod- p cohomology induces a functor

$$\mathcal{F}_p(G)^{op} \longrightarrow \mathcal{F}(H^*(BG)),$$

which turns out to be an equivalence of categories [5]. Moreover, for a subgroup $i_e : E \subset G$, we have $H^*(BC_G(E)) \cong T_E(H^*(BG), H^*(i_E))$ [12]. Hence, identifying both categories, there exists a natural equivalence $H^*(F_G) \xrightarrow{\cong} \Phi_G$, where $\Phi_G \stackrel{\text{def}}{=} \Phi_{H^*(BG)} : \mathcal{F}(H^*(BG)) \longrightarrow H^*(BG)\text{-}\mathcal{U}$ denotes the functor constructed in the previous section in the case $M = A = H^*(BG)$. In particular, this implies that $L^i(\mathcal{F}(H^*(BG))) = 0$ for all $i \geq -1$. and that $\text{depth } H^*(BG) \leq \text{depth } H^*(BC_G(E))$ (Theorem 3.1).

The inclusion $C_G(E) \subset G$ induces a \mathcal{K} -map $H^*(BG) \longrightarrow H^*(BC_G(E))$ and makes the target into a finitely generated $H^*(BG)$ -module [19]. Hence, $\text{depth}_{H^*(BG)} H^*(BC_G(E)) = \text{depth}_{H^*(BC_G(E))} H^*(BC_G(E))$ [23].

Applying Corollary 2.2 and fitting all the above arguments together, this proves the following statement.

Theorem 5.1. *Let G be a compact Lie group. Then,*

$$\text{depth } H^*(BG) = \min\{\text{depth } H^*(BC_G(E)) : E \in \mathcal{F}_p(G)\}.$$

If $E \cong E' \times E''$ then we have $E \subset C_G(E')$ and $C_G(E) = C_{C_G(E')}(E)$. Hence, in the above theorem, we only have to take the minimum over all 1-dimensional elementary abelian p -subgroups.

Remark 5.2. Results of the Jackowski-McClure type do exist for homotopy theoretic versions of groups. In particular, Theorem 5.1 also holds for p -compact groups and p -local finite groups. For definitions and concepts of these notions see [6] and [7] respectively [3].

6. DEPTH AND POLYNOMIAL INVARIANTS

We use the same notation as in the introduction, V denotes a finite dimensional \mathbb{F}_p -vector space, $G \longrightarrow \text{Gl}(V)$ is a (faithful) representation, $\mathbb{F}_p[V]$ denotes the ring of polynomial functions on V respectively the symmetric algebra on the dual of V , $\mathbb{F}_p[V]^G \subset \mathbb{F}_p[V]$ the subalgebra of polynomial invariants, S the collection of all pointwise stabilizer subgroups of non-trivial subspaces of V whose order is divisible by p , $\mathcal{O}_S(G) \subset \mathcal{O}(G)$ the full subcategory of the orbit category associated to the collection S and $\Psi_G : \mathcal{O}_S \longrightarrow \mathbb{F}_p[V]^G$ – mod the functor given by $G/H \mapsto \mathbb{F}_p[V]^H$. For a linear subspace $U \subset V$, we denote by G_U the point wise stabilizer of U .

Theorem 6.1. *If p divides the order of G and $G \longrightarrow \text{Gl}(V)$ is a faithful representation, then*

$$\text{depth } \mathbb{F}_p[V]^G = \min\{\text{depth } \mathbb{F}_p[V]^{G_U} : 0 \neq U \subset V\}.$$

In particular, this says that $\text{depth } \mathbb{F}_p[V]^G \leq \text{depth } \mathbb{F}_p[V]^{G_U}$ for all non-trivial subspaces $U \subset V$. Since, for $U \subset W \subset V$, we have $G_U = (G_W)_U$, we get away in the above statement by taking the minimum over all 1-dimensional subspaces.

Proof. The proof is based on Lemma 2.1 and Theorem 3.1. Before we can apply these results, we have to recall the necessary setting. We make $\mathbb{F}_p[V]$ into a graded \mathbb{F}_p -algebra by giving all polynomial generators the degree 2. Then, there exists a unique \mathcal{A} -action on $\mathbb{F}_p[V]$ and $\mathbb{F}_p[V]$ becomes an object of \mathcal{K} . Moreover, the \mathcal{A} -action is inherited to the ring of polynomial invariants $\mathbb{F}_p[V]^G$. If $U \subset V$, the composition $\phi : \mathbb{F}_p[V]^G \longrightarrow \mathbb{F}_p[V] \longrightarrow \mathbb{F}_p[U] \longrightarrow H^U$ is a \mathcal{K} map and $T_U(\mathbb{F}_p[V]^G, \phi) \cong \mathbb{F}_p[V]^{G_U}$. For example, all these constructions and claims can be found in [14]. In particular, this shows that $\text{depth } \mathbb{F}_p[V]^G \leq \text{depth } \mathbb{F}_p[V]^{G_U}$ for all non-trivial subspaces $U \subset V$ (Theorem 3.1).

By [16] we have

$$\lim^i_{\mathcal{O}_S(G)} \Phi_G \cong \begin{cases} \mathbb{F}_p[V]^G & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}$$

Since $\text{depth } \mathbb{F}_p[V]^G \leq \text{depth } \Phi_G(G/H) = \mathbb{F}_p[V]^H$ for all objects $G/H \in \mathcal{O}_S(G)$, an application of Corollary 2.2 finishes the proof. \square

7. STANLEY-REISNER ALGEBRAS

An abstract simplicial complex K with m -vertices given by the set $V \stackrel{\text{def}}{=} \{1, \dots, m\}$ consists of a finite set $K \stackrel{\text{def}}{=} \{\sigma_1, \dots, \sigma_r\}$ of subsets of V , which is closed with respect to formation of subsets. The subsets $\sigma_i \subset V$ are called the faces of K . The dimension of K is denoted by $\dim K$ and $\dim K = n - 1$, if every face σ of K has order $\#\sigma \leq n$ and there exists a face μ of order $\#\mu = n$. We consider the empty set \emptyset as a face of K .

For a field \mathbb{F} we denote by $\mathbb{F}(K)$ the associated face ring or Stanley-Reisner algebra of K over \mathbb{F} . It is the quotient $\mathbb{F}[V]/(v_\sigma : \sigma \notin K)$, where $\mathbb{F}[V] \stackrel{\text{def}}{=} \mathbb{F}[v_1, \dots, v_m]$ is a polynomial algebra on m -generators and $v_\sigma \stackrel{\text{def}}{=} \prod_{j \in \sigma} v_j$. We can think of $\mathbb{F}(K)$ as a graded object. It will be convenient to choose the topological grading and give the generators of $\mathbb{F}(K)$ and $\mathbb{F}[V]$ the degree 2.

Each abstract simplicial complex K has a geometric realization, denoted by $|K|$. Let $\Delta(V)$ denote the full simplicial complex whose faces are given by all subsets of V . One realization of $\Delta(V)$ is given by the standard $m-1$ -dimensional simplex $|\Delta(V)| \stackrel{\text{def}}{=} \{\sum_{i=1}^m t_i e_i : 0 \leq t_i \leq 1, \sum_i t_i = 1\} \subset \mathbb{R}^m$, where e_1, \dots, e_m denotes the standard basis. And a topological realization of K is given by the subset $|K| \subset |\Delta(V)|$ defined by the subset relation $K \subset \Delta(V)$. We define the cohomology groups $H^*(K)$ of K as the cohomology groups $H^*(|K|)$ of the topological realization. And $\tilde{H}^*(K)$ will denote the reduced cohomology.

For a face $\sigma \in K$, the star $\text{st}_K(\sigma)$ and the link $\text{link}_K(\sigma)$ of σ are defined as the simplicial subcomplexes $\text{st}_K(\sigma) \stackrel{\text{def}}{=} \{\tau \in K : \sigma \cup \tau \in K\}$ and $\text{link}_K(\sigma) \stackrel{\text{def}}{=} \{\tau \in K : \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\}$.

Theorem 7.1. *Let K be an abstract finite simplicial complex. Let $r \leq \dim K + 1$. Then the following conditions are equivalent:*

(i) $\text{depth } \mathbb{F}(K) \geq r$.

- (ii) For all faces $\sigma \in K$ we have $\tilde{H}^i(\text{link}_K(\sigma); \mathbb{F}) = 0$ for $i \leq r - \#\sigma - 2$.
(iii) $\tilde{H}^i(K; \mathbb{F}) = 0 = H^i(|K|, |K| - x; \mathbb{F})$ for $i \leq r - 2$ and for all points $x \in |K|$.

The equivalence of the last two conditions was shown by Munkres [13]. In fact, for each face $\sigma \in K$ and each inner point $x \in \sigma$, he showed that $\tilde{H}^{i-\#\sigma}(\text{link}_K(\sigma)) \cong H^i(|K|, |K| - x)$. This shows that the depth condition only depends on the topological structure of $|K|$, and not on the simplicial structure of K .

Since the Krull dimension of $\mathbb{F}(K)$ equals $\dim K + 1 = n$, the face ring $\mathbb{F}(K)$ is Cohen-Macaulay if and only if $\text{depth } \mathbb{F}(K) = n$. This shows that the above theorem generalizes Reisner's characterization of Cohen-Macaulay face rings [21].

The proof needs some preparation. The poset structure of K , given by the subset relation between the faces, gives rise to a category denoted by $\text{CAT}(K)$, and the non-empty faces of K to the full subcategory $\text{CAT}(K^\times) \subset \text{CAT}(K)$. There exists a functor $\Phi_K : \text{CAT}(K^\times) \rightarrow \text{Alg}_{\mathbb{F}}$ taking values in the category of \mathbb{F} -algebras. For a face $\sigma \in K$ we define $\Phi_K(\sigma) \stackrel{\text{def}}{=} \mathbb{F}(\text{st}(\sigma))$. For an inclusion $\sigma \subset \tau$, we have $\text{st}_K(\tau) \subset \text{st}_K(\sigma)$, which defines the map $\mathbb{F}(\text{st}_K(\sigma)) \rightarrow \mathbb{F}(\text{st}_K(\tau))$.

The following theorem is proved in [15]:

Theorem 7.2. *Let K be an abstract finite simplicial complex. Then,*

$$\lim_{\text{CAT}(K^\times)}^i \Phi_K \cong \begin{cases} \mathbb{F}(K) \oplus \tilde{H}^0(K; \mathbb{F}) & \text{for } i = 0 \\ H^i(K; \mathbb{F}) & \text{for } i \geq 1. \end{cases}$$

Proposition 7.3. *Let K be an abstract finite simplicial complex. For all faces $\sigma \in K$ we have $\text{depth } \mathbb{F}(\text{link}_K(\sigma)) + \#\sigma = \text{depth } \mathbb{F}(\text{st}_K(\sigma)) \geq \text{depth } \mathbb{F}(K)$.*

Proof. Since $\mathbb{F}(\text{st}_K(\sigma)) \cong \mathbb{F}[\sigma] \otimes \mathbb{F}(\text{link}_K(\sigma))$, the first equation always holds. Here, $\mathbb{F}[\sigma] \subset \mathbb{F}[V]$ denotes the polynomial subalgebra generated by all v_i such that $i \in \sigma$.

For the proof of the second equation we first assume that $\mathbb{F} = \mathbb{F}_p$. Then, $\mathbb{F}_p(K)$ is an unstable algebra over the Steenrod algebra. For $\sigma \subset V$ we denote by E^σ the $\#\sigma$ -fold product of the cyclic group C_p of order p . Then, $\mathbb{F}_p[\sigma] \subset H^\sigma \stackrel{\text{def}}{=} H^{E^\sigma}$ is a subalgebra in the category \mathcal{K} . In fact, it is the polynomial part of H^σ . Let ϕ_σ denote the composition $\phi_\sigma : \mathbb{F}_p(K) \rightarrow \mathbb{F}_p[\sigma] \rightarrow H^\sigma$. By [17], there exists an isomorphism $T_{E^\sigma}(\mathbb{F}_p(K), \phi_\sigma) \cong \mathbb{F}_p(\text{st}_K(\sigma))$. Now, the second equation follows from Theorem 3.1.

If \mathbb{F} is a general field of characteristic $p > 0$, the claim follows from the observation that $\text{depth } \mathbb{F}_p(K) = \text{depth } \mathbb{F}_p(K) \otimes_{\mathbb{F}_p} \mathbb{F} = \text{depth } \mathbb{F}(K)$.

If $\mathbb{F} = \mathbb{Q}$, then we notice that $\text{depth } \mathbb{Q}(K)$ is the maximum of the set $\{\text{depth } \mathbb{F}_p(K) : p \text{ a prime}\}$. And if \mathbb{F} is a general field of characteristic 0, then we can argue as in the case of positive characteristic. \square

Proof of Theorem 7.1: By Theorem 7.2, the groups $L^i \stackrel{\text{def}}{=} L^i(\mathbb{F}(K), \Phi_K)$ are all finite and vanish for $i = -1$, and $i \geq \dim K + 1$. We can apply Lemma 2.1.

If $\text{depth } \mathbb{F}(K) \geq r$, then $\text{depth } \mathbb{F}(\text{st}_K(\sigma)) = \text{depth } \mathbb{F}(\Phi(\sigma)) \geq r$ for all $\sigma \in K$ (Proposition 7.3), and $\tilde{H}^i(K; \mathbb{F}) = L^i(\text{CAT}(K^\times); \Phi_K) = 0$ for $i \leq r - 2$. (Lemma 2.1).

If $\text{depth } \mathbb{F}(\text{st}_K(\sigma)) \geq r$ for all $\emptyset \neq \sigma \in K$ and if $\tilde{H}^i(K; \mathbb{F}) = 0$ for $i \leq r - 2$, Lemma 2.1 shows that $\text{depth } \mathbb{F}(K) \geq r$.

Since $\text{depth } \mathbb{F}(\text{link}_K(\sigma)) = \text{depth } \mathbb{F}(\text{st}_K(\sigma)) - \#\sigma$ and since $\dim \text{link}_K(\sigma) < \dim K$, an induction over the dimension of the simplicial complexes proves the claim. \square

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