FIBRATIONS OF CLASSIFYING SPACES

BY
Kenshi Ishiguro and Dietrich Notbohm

Abstract. We investigate fibrations of the form $Z \longrightarrow Y \longrightarrow X$, where two of the three spaces are classifying spaces of compact connected Lie groups. We obtain certain finiteness conditions on the third space which make it also a classifying space. Our results allow to express some of the basic notions in group theory in terms of homotopy theory, i.e. in terms of classifying spaces. As an application we prove that every retract of the classifying space of a compact connected Lie group is again a classifying space.

1. Introduction.

In group theory one has the notions of a subgroup, i.e. a monomorphism, of a normal subgroup, of a kernel, and of an epimorphism. Following the idea of Rector [24] and Rector–Stasheff [26] to describe group theory or Lie group theory from the homotopy point of view, one may look for definitions of these notions in terms of classifying spaces. For a monomorphism, results of Quillen [23], Dwyer and Wilkerson [6] and the second author [20] give a strong hint to say that a map $f : BH \longrightarrow BG$ between the classifying spaces of compact Lie groups is a monomorphism if $H^*(BH; \mathbb{F}_p)$ is a finitely generated module over the ring $H^*(BG; \mathbb{F}_p)$ for any prime p. A similiar definition is already given in [24]. The notion of a kernel is discussed in [20]. In order to find homotopy theoretical definitions for a normal subgroup and an epimorphism, we are led to study fibrations of classifying spaces.

Suppose two of the three spaces of a fibration are homotopy equivalent to classifying spaces of compact connected Lie groups (at p). One might ask if the third space is homotopy equivalent to the classifying space of a compact Lie group (at p). There are 3 types of fibrations to consider:

(1) $BH \to BG \to X$

(2) $BH \to Y \to BK$

(3) $Z \to BG \to BK$

We see a negative answer to each of the three types. For type (1) and type (3), it is easy to find counter-examples; for instance, $BS^3 \to BSO(3) \to K(\mathbb{Z}/2, 2)$ and $S^2 \to BS^1 \to BS^3$, respectively. For type (2), J. Møller pointed out to us that there is a fibration $BS^3 \to Y \to BS^3$ such that $Y \in Genus(BS^3 \times BS^3)$, but $Y \not\simeq BS^3 \times BS^3$.

If the loop spaces ΩX and ΩZ satisfies certain finiteness conditions, however, Theorem 1 and Theorem 3 stated below give the affirmative answers and provide homotopy theoretical definitions for a normal subgroup and an epimorphism. The finiteness conditions are automatically satisfied if the map $BH \to BG$ is induced by a monomorphism, and if the map $BG \to BK$ is induced by an epimorphism. Theorem 2 will show that we can find a compact connected Lie group G such that $Y \in Genus(BG)$; i.e. the p–completed space $Y_p^\wedge$ is homotopy equivalent to $BG_p^\wedge$ for any prime p.

Theorem 1. *Suppose H and G are compact connected Lie groups.*

(1) *If $BH_p^\wedge \xrightarrow{f} BG_p^\wedge \to X$ is a fibration for a space X and if $H^*(\Omega X;\mathbb{F}_p)$ is finite, there is a compact connected Lie group K such that $X \simeq BK_p^\wedge$. In particuliar, if f is of form $(Bi)_p^\wedge$ for a monomorphism $i : H \to G$, then H is a normal subgroup of G, and hence the given fibration $BH_p^\wedge \xrightarrow{(Bi)_p^\wedge} BG_p^\wedge \to X$ is equivalent to $BH_p^\wedge \to BG_p^\wedge \to B(G/H)_p^\wedge$.*

(2) *If $BH \to BG \to X$ is a fibration for a space X whose loop space is homotopy equivalent to a finite CW–complex, there is a compact connected Lie group K such that $X \simeq BK$.*

Theorem 2. *Suppose H and K are compact connected Lie groups.*

(1) *If $BH_p^\wedge \to Y \to BK_p^\wedge$ is a fibration for some space Y, there is a compact connected Lie group G such that $Y \simeq BG_p^\wedge$.*

(2) *If $BH \to Y \to BK$ is a fibration for some space Y, there is a compact connected Lie group G such that $Y \in Genus(BG)$.*

To state a result in the third case we need to consider unstable Adams operations $\psi^{p^k} : BG_p^\wedge \longrightarrow BG_p^\wedge$, where G is a compact connected Lie group. We denote the homotopy fiber of ψ^{p^k} by $F(G,p^k)$ and call it an Adams fiber of p–type. If G is simply–connected and simple, we call $F(G;p^k)$ a special Adams fiber. As shown in Proposition 3.2, the Adams fiber is always homotopy equivalent to a CW–complex of finite type and p–complete. When G is a torus, it is clear that the Adams fiber is the classifying space of a finite abelian p–group. However, a special Adams fiber is never the completion of a classifying space of a compact Lie group (Proposition 3.2).

Theorem 3. *Suppose G and K are compact connected Lie groups.*

(1) *Let $Z \to BG_p^\wedge \xrightarrow{f} BK_p^\wedge$ be a fibration. If $H^*(\Omega Z;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}$ is a finite dimensional $\mathbb{Q}_p^\wedge$–vector space, there is a compact Lie group Γ and special Adams fibers $F_1,\cdots,F_r$ of p–type such that $Z \simeq B\Gamma_p^\wedge \times \prod_{i=1}^r F_i$.*
If, in addition, $H^*(\Omega Z;\mathbb{F}_p)$ is finite, then $Z \simeq B\Gamma_p^\wedge$. In particuliar, if $f \simeq (B\rho)_p^\wedge$ for a homomorphism $\rho : G \to K$, then ρ is an epimorphism.

(2) *Let $Z \to BG \to BK$ be a fibration. If $H^*(\Omega Z;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–vector space, there exists a compact Lie group Γ and special Adams fibers F_i, $1 \le i \le s$, such that $Z \simeq B\Gamma \times \prod_{i=1}^s F_i$.*
If ΩZ is homotopy equivalent to a finite CW–complex, then $Z \simeq B\Gamma$.

According to these results we suggest the following definitions: A map $BH \to BG$ or $BH_p^\wedge \to BG_p^\wedge$ is called *normal* if there exists a fibration $BH \to BG \to X$ or $BH_p^\wedge \to BG_p^\wedge \to X$ such that X satisfies the finiteness conditions of Theorem 1. A map $BG \to BK$ or $BG_p^\wedge \to BK_p^\wedge$ is called a *weak epimorphism* if there exists a fibration $Z \to BG \to BK$ or $Z \to BG_p^\wedge \to BK_p^\wedge$ such that $H^*(\Omega Z;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–module or that $H^*(\Omega Z; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}_p^\wedge$–module, respectively. If ΩZ is equivalent to a finite CW–complex or if in addition $H^*(\Omega Z; \mathbb{F}_p)$ is finite then the map is called an *epimorphism.*

Our key result to prove the Theorems 1, 2, and 3 comes from observation of the following fibrations. Suppose π is a finite p–subgroup of a compact Lie group G such that $\pi_0(G)$ is a p–group. We consider the fibrations $B\pi \xrightarrow{(Bi)_p^\wedge} BG_p^\wedge \to X$ where $i : \pi \to G$ is the inclusion and X is a 1–connected space. The space X is 1–connected if and only if the quotient space G/π is connected. If π is a normal subgroup of G, then $B\pi \to BG \to B(G/\pi)$ is a fibration. Moreover, if this fibration is nilpotent, it is preserved by p–completion. The following shows that this is the only case.

Theorem 4. *Suppose π is a finite p–subgroup of a compact Lie group G such that $\pi_0(G)$ is a p–group. Let $i : \pi \to G$ be the inclusion. If $B\pi \xrightarrow{(Bi)_p^\wedge} BG_p^\wedge \to X$ is a fibration for a simply–connected space X, then π is a normal subgroup of G. In particular, if G is connected, then π is a central subgroup of G.*

The idea of the proofs of our main results is the following: We have to consider fibrations whose fibers are given by the classifying space BH of a compact connected Lie group H. Using Stasheff's result [27] fibrations of this type are classified by maps into $BHE(BH)$, which is the classifying space of the monoid of self equivalences of BH. The space $HE(BH_p^\wedge)$ is calculated by Jackowski, McClure and Oliver [11] [12]. There exists a complete description of the components, and after completion they prove that for any self–equivalence $f : BH \to BH$ the mapping space $map(BH, BH)_f$ is homotopy equivalent to $BZ(H)$, where $Z(H)$ denotes the center of H. This allows to get our hands on fibrations with fiber BH.

It is well known that for a compact connected Lie group G, there is a covering $\alpha_G \to \widetilde{G} \to G$ such that $\widetilde{G} = G_s \times T$ is a product of a simply-connected Lie group G_s and a torus T, and that α_G is a finite central subgroup of $\widetilde{G}$. We call $\alpha_G \to \widetilde{G} \to G$ a *universal finite covering* of G. If G is semi-simple, then it is the universal covering of G in the usual sense.

Given a fibration, we can find another fibration involved with a universal finite covering. We will first discuss this kind of fibration and then deduce to the original fibration using Theorem 4. For example, in our proof of Theorem 1, we will consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} BH' & \longrightarrow & B\widetilde{G} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ BH & \longrightarrow & BG & \longrightarrow & X \end{array}$$

First we find $\widetilde{K}$ such that $\widetilde{X} \simeq B\widetilde{K}$. Theorem 4 then provides K with $X \simeq BK$.

We have another application of Theorem 4. Suppose X is a retract of the p–completed space $BG_p^\wedge$, where G is a compact connected Lie group. The first author showed [10] that there is a compact Lie group K such that $X \simeq BK_p^\wedge$. Our result is the un–completed version of this result.

Theorem 5. *Suppose G is a compact connected Lie group. A retract of BG is homotopy equivalent to a classifying space BK for some compact connected Lie group K.*

The group K in this theorem, however, need not be a subgroup of G. We will give such an example in §6.

This paper consists of 6 more sections. Each of the next five sections contains the proof of one of the above theorems. Section 2 is devoted to the discussion of normal p–subgroups of compact Lie groups; i.e. for a finite p–subgroup π of a connected Lie group G there exists a fibration $B\pi \to BG \to X$. Inparticuliar Theorem 4 is proved in this section. Theorem 2 is proved in §3 and Theorem 1 in §4. In §5 the study of Adams fibers leads to a proof of Theorem 3. Retracts of classifying spaces are discussed in §6. Some results about mapping spaces of classifying spaces of connected Lie groups are contained in the appendix §7. These results are used in the proofs of our main theorems.

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2. Normal p–subgroups of connected Lie groups.

In this section we will prove Theorem 4 and some other results about fibrations of type $B\pi \to BG \to X$. To prove Theorem 4 we need the following lemma.

Lemma 2.1. *Suppose Z is a subgroup of N and N is a subgroup of a group G. If Z is central in G and N/Z is a normal subgroup of the quotient group G/Z, then N is a normal subgroup of G.*

Proof. Suppose $x \in N$ and $xZ \in N/Z$. For any $g \in G$ we see that $g(xZ)g^{-1} = yZ$ for some $y \in N$, since N/Z is a normal subgroup of G/Z. Since Z is central in G, it follows that $g(xZ)g^{-1} = gxg^{-1}Z$. Consequently $gxg^{-1} \in yZ \subset N$. Therefore N is a normal subgroup of G. $\square$

Proof of Theorem 4. Let $Z(\pi)$ denote the center of the finite p–group π and let $j : Z(\pi) \to \pi$ be the inclusion. Since $\pi_0(G)$ is a p–group, $G/\pi_p^\wedge \to B\pi_p^\wedge \to BG_p^\wedge$ is, as the completion of a fibration, again a fibration [4]. The loop space ΩX is mod–p equivalent to the finite connected CW–complex G/π. Thus, by the Sullivan conjecture [14], the evaluation induces an equivalence $map(BZ(\pi), X)_0 \simeq X$ between the component of the constant map and X. We have the homotopy commutative diagrams of fibrations:

$$\begin{array}{ccccc}
map_*(BZ(\pi), B\pi)_{Bj} & \longrightarrow & map(BZ(\pi), B\pi)_{Bj} & \xrightarrow{ev} & B\pi \\
\downarrow & & \downarrow & & \downarrow \\
map_*(BZ(\pi), BG_p^\wedge)_{Bi} & \longrightarrow & map(BZ(\pi), BG_p^\wedge)_{Bi} & \xrightarrow{ev} & BG_p^\wedge \\
\downarrow & & \downarrow & & \downarrow \\
map_*(BZ(\pi), X)_0 & \longrightarrow & map(BZ(\pi), X)_0 & \xrightarrow{ev} & X
\end{array}$$

Here ev denotes an evaluation map and induces a homotopy equivalence in the top and bottom rows. Let $C_G(Z(\pi))$ denote the centralizer of $Z(\pi)$ in G. Since the map $BC_G(Z(\pi))_p^\wedge \to BG_p^\wedge$ is a homotopy equivalence and $\pi_0(G)$ is a p–group, a result of Mislin–Thevenaz [15] shows $Z(\pi)$ is central in G. Thus we obtain the homotopy commutative diagrams of fibrations:

$$\begin{array}{ccccc}
BZ(\pi) & \longrightarrow & BZ(\pi) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
B\pi & \longrightarrow & BG_p^\wedge & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
B(\pi/Z(\pi)) & \longrightarrow & B(G/Z(\pi))_p^\wedge & \longrightarrow & X
\end{array}$$

Because the composition $BZ(\pi) \to BG \to X$ is nullhomotopic, the right arrow in the bottom row exists [9]. Notice that the bottom fibration $B(\pi/Z(\pi)) \to (B(G/Z(\pi))_p^\wedge \to X$ satisfies the assumption of this theorem. Repeating the same procedure, we obtain a sequence of fibrations:

$$\begin{array}{ccccc}
B\pi & \longrightarrow & BG_p^\wedge & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
B(\pi/Z(\pi)) & \longrightarrow & B(G/Z(\pi))_p^\wedge & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & B\overline{G}_p^\wedge & \longrightarrow & X
\end{array}$$

where $\overline{G}$ is a suitable quotient group. Since a finite p–group has a non-trivial center, the fibre of the bottom fibration is contractible. Using Lemma 2.1, an inductive argument shows that π is a normal subgroup of G. $\square$

Corollary 2.2. *Suppose π is a finite subgroup of a compact connected Lie group G. If $B\pi \xrightarrow{Bi} BG \to X$ is a fibration, then π is central in G.*

Proof. The exact sequence of homotopy groups implies that π is abelian and X is 1–connected. Since the fibration is nilpotent, it is preserved by p–completion. Notice that $(B\pi)_p^\wedge = B\pi_p$, where π_p is a p–Sylow subgroup of π, and that X is p–equivalent to the finite complex G/π. Theorem 4 shows that for any p the p–Sylow subgroup of π is normal in G. Since G is connected, π_p is central in G. Consequently, π is central in G. $\square$

Remark. If there is a fibration $B\pi \xrightarrow{Bi} BG \to X$, then the loop space ΩX is homotopy equivalent to the quotient space G/π. This means that G/π admits a loop space structure that makes the quotient map $G \to G/\pi$ deloopable. However we point out that G/π can have a loop structure even if π is not central. In fact, there is a compact Lie group which is homotopy equivalent to G/π such that π is not central in G. Such an example is given by the following. Let $i : \mathbb{Z}/k \to U(n)$ be the monomorphism which sends $x \in \mathbb{Z}/k$ to the diagonal unitary $n \times n$ matrix

$diag(x,1,\dots,1)$. Then $U(n)/\mathbb{Z}/k$ is homeomorphic to $SU(n)\times S^1$, and $\mathbb{Z}/k$ is not a central subgroup of $U(n)$ for $n\geq 2$.

We will discuss the mod p version of Corollary 2.2. In this case, the finite subgroup π need not be abelian. Our result will show, however, that π must be p–nilpotent. Casacuberta shows [5] that π is p–nilpotent if and only if $(B\pi)^\wedge_p \simeq B\pi_p$, where π_p is a p–Sylow subgroup of π.

In [9] the kernel $ker(f)$ of a map $f: BG \to X$ is defined. This is the set of all elements $g\in N_pT_G$ in the p–toral Sylow subgroup N_pT_G of G, which generates a finite p–group $\langle g\rangle$, such that $f|_{B\langle g\rangle}$ is nullhomotopic. Under certain finiteness conditions on the space X, $ker(f)$ is a normal subgroup of N_pT_G [20]. In particuliar, if $(p,|W_G|)=1$, $ker(f)$ is an abelian group.

The following observation, formulated as a lemma, is needed for later purpose. The proof is obvious.

Lemma 2.3. *Suppose the following diagram of groups is commutative:*

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{\widetilde{\psi}} & \widetilde{G} \\ {\scriptstyle q_H}\downarrow & & \downarrow{\scriptstyle q_G} \\ H & \xrightarrow{\psi} & G \end{array}$$

where the vertical homomorphisms are surjective. If the image $\widetilde{\psi}(\widetilde{H})$ is normal in $\widetilde{G}$, then $\psi(H)$ is normal in G. (This result holds when "normal" is replaced by "central".)

Theorem 2.4. *Suppose π is a finite subgroup of a compact connected Lie group G. If $(B\pi)^\wedge_p \xrightarrow{(Bi)^\wedge_p} (BG)^\wedge_p \xrightarrow{f} X$ is a fibration, the following hold:*

(1) *The finite group π is p–nilpotent.*

(2) *Its p–Sylow subgroup π_p is central in G.*

Proof. Notice that if π_p is a p–Sylow subgroup of π, then π_p is a subgroup of $Kerf$ [20]. We need to consider four cases.

<u>*Case 1.*</u> G is simple, $|W(G)|\equiv 0 \bmod p$ except that $G\neq G_2$ and $p=3$

In this case, according to results of [9, 10], we see π_p is a central subgroup of G. Suppose $N(\pi_p)$ denotes the normalizer in π. Using Swan's theorem [28] we can show that

$$H^*(B\pi;\mathbb{F}_p)\cong H^*(BN(\pi_p);\mathbb{F}_p)\cong H^*(B\pi_p;\mathbb{F}_p)^{N(\pi_p)/\pi_p}\cong H^*(B\pi_p;\mathbb{F}_p)\ .$$

Therefore $(B\pi)^\wedge_p\simeq B\pi_p$.

<u>*Case 2.*</u> $|W(G)|\not\equiv 0 \bmod p$

Notice that π_p is abelian. We claim that π_p is trivial. If π_p was non-trivial, we would find a nontrivial subgroup $Z/p\subset\pi_p$. We consider the diagram associated with $map(B\mathbb{Z}/p,-)$.

$$\begin{array}{ccc} map(B\mathbb{Z}/p,(B\pi)^\wedge_p)_{Bj} & \longrightarrow & (B\pi)^\wedge_p \\ \downarrow & & \downarrow \\ map(B\mathbb{Z}/p, BG^\wedge_p)_{Bi} & \longrightarrow & BG^\wedge_p \\ \downarrow & & \downarrow \\ map(B\mathbb{Z}/p, X)_0 & \longrightarrow & X \end{array}$$

By Proposition 7.4, $map(B\mathbb{Z}/p, (B\pi)^\wedge_p)_{Bj} \simeq BC_\pi(\mathbb{Z}/p)^\wedge_p$, therefore the top arrow is rationally homotopy equivalent. The bottom arrow is a homotopy equivalence. Because $(p, |W_G|) = 1$, the subgroup $\mathbb{Z}/p \xrightarrow{i} G$ is not central, and $BC_G(i) \to BG$ is rationally not a homotopy equivalence. This is a contradiction and implies the desired result.

<u>*Case 3.*</u> $G = G_2$ and $p = 3$

From [9], we see that π_3 is a subgroup of $\mathbb{Z}/3$. Note that $map(B\mathbb{Z}/3, (BG_2)^\wedge_3)_{Bi} \simeq BSU(3)^\wedge_3$. If $\pi_3 = \mathbb{Z}/3$, a diagram analoguous to the above produces a contradiction. Consequently, π_3 is trivial.

<u>*Case 4.*</u> G is any compact connected Lie group.

Let $\alpha_G \to \widetilde{G} \to G$ be a universal finite covering of G. The group extension $\alpha_G \to \widetilde{\pi} \to \pi$ induced by the monomorphism $i : \pi \to G$ is central. Consequently the fibration $B\alpha_G \to B\widetilde{\pi} \to B\pi$ is nilpotent, and is preserved by p–completion. We have the homotopy commutative diagram:

$$\begin{array}{ccccc}
(B\alpha_G)^\wedge_p & \longrightarrow & (B\alpha_G)^\wedge_p & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
(B\widetilde{\pi})^\wedge_p & \longrightarrow & (B\widetilde{G})^\wedge_p & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
(B\pi)^\wedge_p & \longrightarrow & (BG)^\wedge_p & \longrightarrow & X
\end{array}$$

Suppose $\widetilde{G} = \prod_i \widetilde{G}_i$ where each $\widetilde{G}_i$ is either a simply-connected simple Lie group or a torus. Let $\widetilde{G}_a = \prod_j \widetilde{G}_j$ with $|W(\widetilde{G}_j)| \not\equiv 0 \bmod p$ or $\widetilde{G}_j = G_2$ and $p = 3$. If $q : \widetilde{G} \to \widetilde{G}_a$ is the projection and $\widetilde{\pi}_p$ is a p–Sylow subgroup of $\widetilde{\pi}$, an argument analogous to the ones used in Case 2 and Case 3 shows that $q(\widetilde{\pi}_p) = 1$. Hence $\widetilde{\pi}_p$ is central. This implies that $(B\widetilde{\pi})^\wedge_p \simeq B\widetilde{\pi}_p$. Since α_G is abelian, we see $(B\alpha_G)^\wedge_p \simeq B(\alpha_G)_p$. Therefore $(B\pi)^\wedge_p \simeq B\pi_p$. Finally we see π_p is central by lemma 2.3. □

Theorem 2.4 is also true if the total space of the fibration is $BO(n)$. We note that the orthogonal group $O(n)$ is disconnected and that the center of $O(n)$ is isomorphic to the finite group $\mathbb{Z}/2$.

Corollary 2.5. *Suppose π is a finite subgroup of an orthogonal group $O(n)$. If $B\pi \xrightarrow{Bi} BO(n) \to X$ is a nilpotent fibration, then π is central in $O(n)$.*

Proof. Notice that $(B\pi)^\wedge_p \to BO(n)^\wedge_p \to X^\wedge_p$ is a fibration for any p. We recall that if $n = 2m$ or $2m+1$ and p is odd, then $BO(n)^\wedge_p = BSO(2m+1)^\wedge_p$ [3]. Hence the p–Sylow subgroup π_p is trivial by Theorem 2.4. According to Theorem 4, we see that $\pi = \pi_2$ is normal in $O(n)$, and hence central in $O(n)$. □

3. Fibrations of type $BH \to Y \to BK$.

Proof of Theorem 2. (1) Let $\alpha_K \to \widetilde{K} \to K$ be a universal finite covering of the compact connected Lie group K. We consider the diagram:

$$\begin{array}{ccccc} * & \longrightarrow & B\alpha_{K}{}^{\wedge}_{p} & \longrightarrow & B\alpha_{K}{}^{\wedge}_{p} \\ \downarrow & & \downarrow & & \downarrow \\ BH^{\wedge}_{p} & \longrightarrow & \widetilde{Y} & \longrightarrow & B\widetilde{K}^{\wedge}_{p} \\ \downarrow & & \downarrow & & \downarrow \\ BH^{\wedge}_{p} & \longrightarrow & Y & \longrightarrow & BK^{\wedge}_{p} \end{array}$$

where $\widetilde{Y}$ is the pullback. The middle horizontal fibration is classified by a map $B\widetilde{K}^{\wedge}_{p} \to B(HE(BH^{\wedge}_{p}))$ [27]. This map lifts to $B(SHE(BH^{\wedge}_{p}))$. Since H is connected, we see $B(SHE(BH^{\wedge}_{p})) \simeq B^2(Z(H))^{\wedge}_{p}$, [12]. Taking a self covering of $\widetilde{K}$, if necessary, we may assume the classifying map of the fibration is trivial. Consequently $\widetilde{Y} \simeq B(H \times \widetilde{K})^{\wedge}_{p}$ and $Y \simeq (B(H \times \widetilde{K})/\alpha_K)^{\wedge}_{p}$ by Theorem 4.

(2) We have the following diagram analogous to (1):

$$\begin{array}{ccccc} * & \longrightarrow & B\alpha_{K} & \longrightarrow & B\alpha_{K} \\ \downarrow & & \downarrow & & \downarrow \\ BH & \longrightarrow & \widetilde{Y} & \longrightarrow & B\widetilde{K} \\ \downarrow & & \downarrow & & \downarrow \\ BH & \longrightarrow & Y & \longrightarrow & BK \end{array}$$

Since $(B^2Z(H))^{\wedge}_{p}$ is contractible for large prime p, taking a suitable finite self-covering of $\widetilde{K}$, we can assume that the p–completion of the middle horizontal fibration $BH^{\wedge}_{p} \to \widetilde{Y}^{\wedge}_{p} \to B\widetilde{K}^{\wedge}_{p}$ is trivial at any prime p. Consequently $\widetilde{Y}^{\wedge}_{p} \simeq B(H \times \widetilde{K})^{\wedge}_{p}$ at any p. Thus $Y^{\wedge}_{p} \simeq BG^{\wedge}_{p}$ for any p if $G = (H \times \widetilde{K})/\alpha_K$. $\square$

4. Fibrations of the type $BH \to BG \to X$.

This chapter is devoted to the proof of theorem 1:

Proof of Theorem 1. (1) First we assume the compact connected Lie group G is semi–simple. We claim that H is semi–simple. If not, the Serre spectral sequence of the fibration $G^{\wedge}_{p} \to \Omega X \to BH^{\wedge}_{p}$ with coefficients $\mathbb{Q}$ would tell us that $H^1(\Omega X; \mathbb{Q}) = 0$ and $H^2(\Omega X; \mathbb{Q}) \neq 0$. This is a contradiction, since ΩX is a mod–p finite H–space and $H^*(\Omega X; \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}^{\wedge}_{p}}(x_{2k_1+1}, ..., x_{2k_n+1})$ an exterior algebra generated by odd degree elements.

Next we will show that there is a compact connected semi–simple Lie group K such that $X \simeq BK^{\wedge}_{p}$. Let α be the kernel of the homomorphism $\pi_2(BG^{\wedge}_{p}) \to \pi_2(X)$. If $\beta := \pi_2(BG^{\wedge}_{p})$ and $\gamma := \pi_2(X)$, we obtain the homotopy commutative diagram:

$$\begin{array}{ccccc} B\alpha & \longrightarrow & B\beta & \longrightarrow & B\gamma \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & B\widetilde{G}^{\wedge}_{p} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ BH^{\wedge}_{p} & \longrightarrow & BG^{\wedge}_{p} & \longrightarrow & X \end{array}$$

where $\widetilde{G}$ is a universal covering of G, where $\widetilde{X}$ is the homotopy fibre of the map $X \to K(\pi_2 X, 2)$ whose induced homomorphism on the second homotopy group is isomorphic, and where F is the homotopy fibre of the induced map $B\widetilde{G}_p^\wedge \to \widetilde{X}$. We claim here that $F \simeq (BH')_p^\wedge$ for some semi–simple Lie group H'. If $\delta = \pi_2(BH_p^\wedge)$, then we have the homotopy commutative diagram:

$$\begin{array}{ccccc}
* & \longrightarrow & B\delta & \longrightarrow & B\delta \\
\downarrow & & \downarrow & & \downarrow \\
B\alpha & \longrightarrow & \widetilde{F} & \longrightarrow & B\widetilde{H}_p^\wedge \\
\downarrow & & \downarrow & & \downarrow \\
B\alpha & \longrightarrow & F & \longrightarrow & BH_p^\wedge
\end{array}$$

where $\widetilde{H}$ is a universal covering of H, and $\widetilde{F}$ is the pullback. The middle horizontal fibration $B\alpha \to \widetilde{F} \to B\widetilde{H}_p^\wedge$ is characterized, according to Stasheff's theorem [27], by a map $B\widetilde{H}_p^\wedge \to B(HE(B\alpha))$. Since H is semi–simple, the space $B\widetilde{H}_p^\wedge$ is 3–connected. Hence the characteristic map is trivial, and we see $\widetilde{F} \simeq B(\alpha \times \widetilde{H})_p^\wedge$. Thus $F \simeq B((\alpha \times \widetilde{H})/\delta)_p^\wedge$. Let $H' = (\alpha \times \widetilde{H})/\delta$. Since F is 1–connected, the Lie group H' is connected. We next claim that the fibration $F \simeq (BH')_p^\wedge \to B\widetilde{G}_p^\wedge \to \widetilde{X}$ is trivial. The fibration is characterized by a map $\widetilde{X} \to BHE(BH'^\wedge_p)$ [27], which lifts to $BSHE(BH'^\wedge_p) \simeq B^2 Z(H')_p^\wedge$ [12] and is therefore nullhomotopic, since $\widetilde{X}$ is 2–connected. Here $SHE(\ )$ denotes the component of the identity of $HE(\ )$. It follows that $\widetilde{X}$ is a retract of $B\widetilde{G}_p^\wedge$. A result of [10] implies $\widetilde{X} \simeq B\widetilde{K}_p^\wedge$ for some subgroup $\widetilde{K}$ of $\widetilde{G}$, where $\widetilde{K} = \prod_i \widetilde{K}_i$, a product group of 1–connected simple Lie groups. Since $H^*(\Omega X; \mathbb{F}_p)$ is finite, using the Sullivan conjecture we can show that the map $B\gamma \to B\widetilde{K}_p^\wedge$ is induced by a monomorphism $\gamma \to \widetilde{K}$. If K is the quotient group $\widetilde{K}/\gamma$, Theorem 4 implies $X \simeq BK_p^\wedge$.

Suppose $f = (Bi)_p^\wedge$ for a monomorphism $i : H \to G$. We will show $i(H) \lhd G$ using some results in §7. Notice that we now have the following homotopy commutative diagram:

$$\begin{array}{ccccc}
B\widetilde{H}_p^\wedge & \xrightarrow{(B\widetilde{i})_p^\wedge} & B\widetilde{G}_p^\wedge & \xrightarrow{r} & B\widetilde{K}_p^\wedge \\
\downarrow & & \downarrow & & \downarrow \\
BH_p^\wedge & \xrightarrow{(Bi)_p^\wedge} & BG_p^\wedge & \longrightarrow & BK_p^\wedge
\end{array}$$

where $\widetilde{i}$ is a lifted homomorphsim of Lie groups. We need a general result about maps between classifying spaces of simple Lie groups. Suppose G' and K' are 1–connected compact simple Lie groups. If both homotopy sets $[(BG')_p^\wedge, B(K')_p^\wedge]$ and $[(BK')_p^\wedge, (BG')_p^\wedge]$ are non-trivial, then $(BG')_p^\wedge$ is homotopy equivalent to $(BK')_p^\wedge$. This is a consequence of the following two results: (1) $[(BG')_p^\wedge, B(K')_p^\wedge] \neq 0$ implies that $rank(G') \leq rank(K')$ and that the Weyl group $W(G')$ is a subgroup of $W(K')$, [1]. (2) $[BSp(n)_2^\wedge, BSpin(2n+1)_p^\wedge] = 0$ for $n \geq 3$, [9]. Since $B\widetilde{K}_p^\wedge$ is a retract of $B\widetilde{G}_p^\wedge$, each $(B\widetilde{K}_i)_p^\wedge$ is also a retract of $B\widetilde{G}_p^\wedge$. By Proposition 6.1, for each i ,we can find a factor subgroup $\widetilde{G}_{j_i}$ of $\widetilde{G}$ such that $r|_{(B\widetilde{G}_{j_i})_p^\wedge} : (B\widetilde{G}_{j_i})_p^\wedge \to (B\widetilde{K}_i)_p^\wedge$ is a homotopy equivalence. Let $\widetilde{G}_K = \prod_i \widetilde{G}_{j_i}$, the product of such subgroups of $\widetilde{G}$, and let $\widetilde{G}_H$ be the product of the rest of factor subgroups of $\widetilde{G}$ so that $\widetilde{G} = \widetilde{G}_H \times \widetilde{G}_K$. Since the restricted map $r|_{(B\widetilde{G}_K)^\wedge_p}$ is a homotopy equivalence, we see $\widetilde{i}(\widetilde{H}) = \widetilde{G}_H$. Consequently $\widetilde{i}(\widetilde{H})$ is a normal subgroup of $\widetilde{G}$. Lemma 2.3 implies that $H \triangleleft G$ is a normal subgroup. This finishes the proof of part (1) for G semi simple.

Now we prove (1) for the general case. For a compact connected Lie group G, we can find a group extension $\overline{G}_s \to G \to \overline{T}_g$, where $\overline{G}_s$ is semi-simple and $\overline{T}_g$ is a torus (for details see for example [19]). Note that $H^2(B\overline{T}_g;\mathbb{Z}) = H^2(BG;\mathbb{Z})$ and hence $\overline{T}_g = K(H^2(BG;\mathbb{Z}),1)$. If H' is the kernel of the composite homomorphsim $H \to G \to \overline{T}_g$, we obtain the homotopy commutative diagram:

$$\begin{array}{ccccc}
(BH')^\wedge_p & \longrightarrow & (B\overline{G}_s)^\wedge_p & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
BH^\wedge_p & \longrightarrow & BG^\wedge_p & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
B(H/H')^\wedge_p & \longrightarrow & K(H^2(BG;\mathbb{Z}),2)^\wedge_p & \longrightarrow & K(H^2(X;\mathbb{Z}),2)
\end{array}$$

Since $\overline{G}_s$ is semi-simple, we see $\widetilde{X} \simeq (BK')^\wedge_p$ for some compact connected Lie group K'. Since H/H' is a torus, we see that $K(H^2(X;\mathbb{Z}),2)$ is the p–completion of the classifying space of some torus. Theorem 2 shows $X \simeq BK^\wedge_p$ for some compact connected Lie group K.

Suppose $f = (Bi)^\wedge_p$. Let $\alpha_G \to \widetilde{G} \to G$ be a universal finite covering, where $\widetilde{G} = G_s \times T_g$ is a product of a simply–connected Lie group G_s and a torus T_g. (Warning: G_s and T_g can be different from $\overline{G}_g$ and $\overline{T}_g$ respectively.) For suitable universal finite coverings $H_s \times T_h \to H$ and $G_s \times T_g \to G$, Lie theory provides a homomorphism $\widetilde{i} : H_s \times T_h \to G_s \times T_g$ such that the following diagram of Lie groups commutes:

$$\begin{array}{ccc}
H_s \times T_h & \xrightarrow{\widetilde{i}} & G_s \times T_g \\
\downarrow & & \downarrow \\
H & \xrightarrow{i} & G
\end{array}$$

Let $\widetilde{H} = H_s \times T_h$. The above diagram induces the following homotopy commutative diagram:

$$\begin{array}{ccccc}
B\widetilde{H}^\wedge_p & \xrightarrow{(B\widetilde{i})^\wedge_p} & B\widetilde{G}^\wedge_p & \longrightarrow & BK'^\wedge_p \\
\downarrow & & \downarrow & & \downarrow \\
BH^\wedge_p & \xrightarrow{(Bi)^\wedge_p} & BG^\wedge_p & \longrightarrow & BK^\wedge_p
\end{array}$$

for a suitable compact connected Lie group K'. By Proposition 7.1 we see that $\widetilde{i}$ must be a product homomorphism: $\widetilde{i} = \widetilde{i}_s \times \widetilde{i}_t : H_s \times T_h \to G_s \times T_g$. We can show that $\widetilde{i}_s : H_s \to G_s$ is a monomorphism, and $\widetilde{i}_s(H_s) \triangleleft G_s$, since G_s is semi–simple. Consequently $\widetilde{i}(H_s \times T_h) \triangleleft G_s \times T_g$. Lemma 2.3 implies that $i(H)$ is a normal subgroup of G, which completes the proof of (1) in the general case.

(2) We will prove the un-completed version of (1). The argument is analoguous to the p–adic case. Again we first assume G is semi–simple. Likewise to the p–adic case, we obtain the following diagram

$$\begin{array}{ccccc} B\pi & \longrightarrow & B(\pi_1 G) & \longrightarrow & B(\pi_2 X) \\ \downarrow & & \downarrow & & \downarrow \\ BH' & \longrightarrow & B\widetilde{G} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ BH & \longrightarrow & BG & \longrightarrow & X \end{array}$$

where π is a suitable finite abelian group. The fibration $BH' \to B\widetilde{G} \to \widetilde{X}$ is trivial at any prime p. Thus, for each prime p, there is a subgroup $\widetilde{K}(p)$ of $\widetilde{G}$ such that $X \simeq B\widetilde{K}(p)^\wedge_p$. If $T_{H'}$ and $T_{\widetilde{G}}$ are maximal tori, there is an induced map $B(T_{\widetilde{G}}/T_{H'}) \to \widetilde{X}$, since the composition $BT_{H'} \to BT_{\widetilde{G}} \to B\widetilde{G} \to \widetilde{X}$ is nullhomotopic [9]. For each p, the induced map provides a maximal torus after p–adic completion, because $\widetilde{X}^\wedge_p \simeq BK(p)^\wedge_p$. Therefore, the homotopy fiber of the map $B(T_{\widetilde{G}}/T_{H'}) \to \widetilde{X}$ is simply-connected and the mod–p cohomology and rational cohomology look like the cohomology of a finite complex. Moreover, X and $B(T_{\widetilde{G}}/T_{H'})$ have the same rank. This means that $BT_{\widetilde{G}}/T_{H'} \to \widetilde{X}$ is integrally a maximal torus in the sense of [21].

Next we will show that the space $\widetilde{X}$ has a genus type of one of the $B\widetilde{K}(p)$'s. In fact, we can take $p = 2$; i.e. $X \in Genus(B\widetilde{K}(2))$. Suppose $\alpha_p : \widetilde{X}^\wedge_p \to B\widetilde{K}(p)$ is the homotopy equivalence. We see the following equivalences:

$$H^*(\widetilde{X}; \mathbb{Q}^\wedge) \cong H^*(B\widetilde{K}(p); \mathbb{Q}^\wedge) \cong H^*(B\widetilde{T}(p); \mathbb{Q}^\wedge)^{W(\widetilde{K}(p))}$$

for any p. Here $\widetilde{T}(p)$ is a maximal torus of the Lie group $\widetilde{K}(p)$, and $W(\widetilde{K}(p))$ is the Weyl group. If $T_{\widetilde{X}}$ is a maximal torus of $\widetilde{X}$, there is a map $\alpha'_p : BT_{\widetilde{X}}{}^\wedge_p \to B\widetilde{T}(p)^\wedge_p$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \widetilde{X}^\wedge_p & \xrightarrow{\alpha_p} & B\widetilde{K}(p)^\wedge_p \\ \uparrow & & \uparrow \\ BT_{\widetilde{X}}{}^\wedge_p & \xrightarrow{\alpha'_p} & B\widetilde{T}(p)^\wedge_p \end{array}$$

[21]. Consequently all of the $\mathbb{Q}^\wedge$–representations of the Weyl groups $W(\widetilde{K}(p))$ are equivalent each other. Here we need the following fact. Suppose K_1 and K_2 are products of a torus and simply-connected simple Lie groups. If the rational representaion of $W(K_1)$ is equivalent to that of $W(K_2)$, then the classifing space BK_1 is p–equivalent to BK_2 for any odd prime p. One can show this by using the classification of simply-connected simple Lie groups, since $W(K_1) \cong W(K_2)$ implies $K_1 = K_2$ up to $Sp(n)$ and $Spin(2n+1)$. Consequently $X^\wedge_p \simeq B\widetilde{K}(p)^\wedge_p \simeq B\widetilde{K}(2)^\wedge_p$ for p odd. Since $\widetilde{X}$ has a maximal torus, a result of Notbohm-Smith [22] shows that $\widetilde{X} \simeq B\widetilde{K}(2)$. Therefore $X \simeq BK$ if $K = \widetilde{K}(2)/\pi_2 X$.

In the general case, using a group extention $\overline{G}_s \to G \to \overline{T}_g$, we obtain the homotopy commutaive diagram

$$\begin{array}{ccccc} BH' & \longrightarrow & B\overline{G}_s & \longrightarrow & B\widetilde{K} \\ \downarrow & & \downarrow & & \downarrow \\ BH & \longrightarrow & BG & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B(H/H') & \longrightarrow & B\overline{T}_g & \longrightarrow & K(H^2(X;\mathbb{Z}),2) \end{array}$$

We can show that X has a maximal torus. By Theorem 2, there exists a compact connected Lie group K' such that $X \in Genus(BK')$ and hence, $X \simeq BK$ for a suitable compact connected Lie group K [18]. $\square$

5. Fibrations of type $Z \to BG \to BK$.

In this section we prove Theorem 3. First we have to investigate Adams fibers, which were defined, in the introduction, as the homotopy fibers of unstable Adams operations. Let G be a compact connected Lie group and $\alpha_G \to \widetilde{G} \xrightarrow{q_G} G$ a universal finite cover of G, where $\widetilde{G} = G_s \times T$ is a product of a simply connected Lie group G_s and a torus T. Note that $G_s = \prod_i G_i$ a product of simple simply connected Lie groups G_i. An unstable Adams operation $\psi^r : BG^\wedge_p \to BG^\wedge_p$ of degree $r \in \mathbb{Z}^\wedge_p$ gives rise to a diagram

$$(*) \qquad \begin{array}{ccc} B\widetilde{G}^\wedge_p & \xrightarrow{\psi^r} & B\widetilde{G}^\wedge_p \\ {\scriptstyle Bq_G}\downarrow & & {\scriptstyle Bq_G}\downarrow \\ BG^\wedge_p & \xrightarrow{\psi^r} & BG^\wedge_p \end{array} .$$

The map ψ^r exists if and only if r is a p–adic unit or p is coprime to the order of the Weyl group W_G [29] [30] [8]. The homotopy fiber $F(G,r)$ of ψ^r is nontrivial if $r = p^k u$, where u is a unit and $k \geq 1$. Thus, $F(G,r) \simeq F(G,p^k)$. If $k \geq 1$, then $(p, |W_G|) = 1$ and $Bq_G : B\widetilde{G}^\wedge_p \to BG^\wedge_p$ is an equivalence. This means that both Adams fibers $F(\widetilde{G},p^k)$ and $F(G,p^k)$ are equivalent. Moreover, $F(\widetilde{G},p^k) \simeq \prod_i F(G_i,p^k) \times B(\mathbb{Z}/p^k)^n$, where n is the dimension of T. This proves the following lemma:

Lemma 5.1. *Adams fibers of p–type are homotopy equivalent to a finite product of special Adams fibers of p–type and of the classifying space of a finite abelian p–group.*

Let E be the homotopy fiber of the completion map $BG \to BG^\wedge_p$. The homotopy groups of E are uniquely p–divisible. An Adams operation ψ^{p^k} induces an equivalence $\psi^{p^k} : E \xrightarrow{\simeq} E$. Therefore, in the commutative diagram of fibrations

$$(**) \qquad \begin{array}{ccccc} * & \longrightarrow & E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & BG & \xrightarrow{\psi^{p^k}} & BG \\ \simeq\downarrow & & \downarrow & & \downarrow \\ F(G,p^k) & \longrightarrow & BG_p^\wedge & \xrightarrow{\psi^{p^k}} & BG_p^\wedge \end{array}$$

the map $F \to F(G,p^k)$ is an equivalence. We use this equivalence to calculate the cohomology of Adams fibers. If $(p,|W_G|) = 1$, the mod–p cohomology and the p–adic cohomology of BG are polynomial, denoted by $H^*(BG;\mathbb{F}_p) \cong \mathbb{F}_p[x_1,\cdots,x_n]$ and $H^*(BG;\mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p^\wedge[x_1,\cdots,x_n]$, where x_i has degree $2r_i$. It follows that $H^*(G;\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(y_1,\cdots,y_n)$ and $H^*(G;\mathbb{Z}_p^\wedge) \cong \Lambda_{\mathbb{Z}_p^\wedge}(y_1,\cdots,y_n)$, which are exterior algebras with generators y_i of degree $2r_i - 1$. We use the same notation for the generators.

Proposition 5.2. *Let $k \geq 1$ and $(p,|W_G|) = 1$.*

(1) *$F(G,p^k)$ is a p–complete CW–complex of finite type.*

(2)
$$H^*(F(G,p^k);\mathbb{Z}) \cong \mathbb{Z}[x_1,\cdots,x_n]/(p^{kr_i}x_i : 1 \leq i \leq n) \ ,$$
where $(p^{kr_i}x_i : 1 \leq i \leq n)$ is the ideal generated by the elements $p^{kr_i}x_i$.

(3)
$$\begin{aligned} H^*(F(G,p^k);\mathbb{F}_p) &\cong H^*(G;\mathbb{F}_p) \otimes H^*(BG;\mathbb{F}_p) \\ &\cong \Lambda_{\mathbb{F}_p}(y_1,\cdots,y_n) \otimes \mathbb{Z}/p[x_1,\cdots,x_n] \ . \end{aligned}$$
as algebras. The Steenrod operations P^j, $j \geq 1$, act trivially on y_i and in the obvious way on $H^*(BG;\mathbb{F}_p)$. For the r_i-th order Bockstein β_{r_i} we have $\beta_{r_i}(y_i) = x_i$.

(4)
$$H^*(\Omega F(G,p^k);\mathbb{F}_p) \cong H^*(\Omega G;\mathbb{F}_p) \otimes H^*(G;\mathbb{F}_p)$$
as algebras. In particuliar, $H^*(\Omega F(G,p^k);\mathbb{F}_p)$ is not finite and $\Omega F(G,p^k)$ is not equivalent to a finite CW–complex.

(5) *If $F(G,p^k)$ is a special Adams fiber, i.e G is simple simply-connected , then $F(G,p^k)$ is not homotopy equivalent to the p–adic completion of the classifying space of a compact Lie group .*

Proof. For G a torus of dimension n, $F(G,p^k) \simeq B(\mathbb{Z}/p^k)^n$. Some straightforward and easy calculations prove everything in this case. Therefore, we only have to prove the statements for simply connected Lie groups. For abbreviation, we define $F := F(G,p^k)$.

Part (1) follows from the diagram (**). In order to calculate the cohomology we consider the diagram

$$\begin{array}{ccccc} G & \longrightarrow & F & \longrightarrow & BG \\ \| & & \downarrow & & \psi^{p^k}\downarrow \\ G & \longrightarrow & * & \longrightarrow & BG \ . \end{array}$$

For both p–adic and mod–p cohomology, in the Serre spectral sequence of the bottom fibration the generators y_i are mapped on x_i by the transgression. In mod–p cohomolgy, ψ^{p^k} induces the trivial map and the spectral sequences of the top row collapses. In p–adic cohomology, $\psi^{p^k *}(x_i) = p^{kr_i} x_i$ and the generators of $H^*(G; \mathbb{F}_p)$ are mapped on $p^{kr_i} x_i$ by the transgression in the spectral sequence of the top row. This shows that the isomorphism in (2) and (3) are isomorphisms of algebras. Because the reduced cohomology is pure torsion, we can deal with $\mathbb{Z}_p^\wedge$ instead of $\mathbb{Z}$ as coefficients.

In integral cohomology the class x_i generates a cyclic group of order p^{kr_i}. We see that $\beta_{r_i}(y_i) = x_i$. To calculate the action of the Steenrod powers we look at the mod–p Serre spectral sequence of the fibration $F \to BG \to BG$. The elements $y_i \in H^*(F; \mathbb{F}_p)$ are transgressive and mapped onto x_i + decomposables. Let us assume that $P^j(y_i) \neq 0$. Then $P^j(y_i)$ is transgressive. Because $(p, |W_G|) = 1$, we have $p > r_i$ for all i. In particuliar $P^j(y_i)$ is a sum of products of elements. Each summand contains at least one y_l as factor. Hence, some differential of degree less than $max(2r_i)$ acts nontriviallly on $P^j(y_i)$, which is a contradiction.

Statement (4) follows by similiar considerations. $H^*(\Omega G; \mathbb{F}_p) \cong \bigotimes_i D[z_i]$ is the tensor algebra of division algebras generated by classes z_i in degree $2r_i - 2$. We also have a diagram of fibrations

$$\begin{array}{ccccc} \Omega G & \longrightarrow & \Omega F & \longrightarrow & G \\ \| & & \downarrow & & \downarrow{\scriptstyle \psi^{p^k}} \\ \Omega G & \longrightarrow & * & \longrightarrow & G\,. \end{array}$$

In the Serre spectral sequence for mod–p cohomology of the bottom row all generators z_i are transgressive. Hence the spectral sequence for the top row collapses. This proves the first part of (4). The second part of (4) follows from the fact that $H^*(\Omega G; \mathbb{F}_p)$ is not finite.

Finally, let Γ be a compact Lie group , such that $F(G, p^k) \simeq B\Gamma_p^\wedge$. Then, Γ contains a nontrivial element of order p^l, $l \geq 1$, which gives a subgroup $\mathbb{Z}/p^l \subset \Gamma$. The composition $B\mathbb{Z}/p^l \to B\Gamma_p^\wedge \simeq F(G; p^k) \to BG_p^\wedge$ lifts to BG and is induced by a homomorphism $\rho : \mathbb{Z}/p^l \to G$ [7]. By [23], $H^*(B\mathbb{Z}/p^l; \mathbb{F}_p)$ is a finitely generated $H^*(B\Gamma; \mathbb{F}_p)$–module, and, by (3), a finitely generated $H^*(BG; \mathbb{F}_p)$–module. Thus, ρ is a monomorphism [23]. Because $\psi^{p^k} \circ B\rho$ is nullhomotopic, $l \leq k$. In particuliar, Γ is a finite group. Otherwise we would find elements of order p^l for every $l \in \mathbb{N}$. Now we choose $l = 1$, i.e. we restrict to the subgroup $\mathbb{Z}/p \subset \mathbb{Z}/p^l$ We get a diagram of fibrations

$$\begin{array}{ccccc} map(B\mathbb{Z}/p, B\Gamma_p^\wedge)_{Bi} & \longrightarrow & map(B\mathbb{Z}/p, BG_p^\wedge)_{B\rho} & \xrightarrow{(\psi^{p^k})^*} & map(B\mathbb{Z}/p, BG_p^\wedge)_{const} \\ {\scriptstyle ev}\downarrow & & {\scriptstyle ev}\downarrow & & {\scriptstyle ev}\downarrow \\ B\Gamma_p^\wedge & \longrightarrow & BG_p^\wedge & \xrightarrow{\psi^{p^k}} & BG_p^\wedge \end{array} .$$

By [7] and Proposition 7.4, $map(B\mathbb{Z}/p, B\Gamma_p^\wedge)_{Bi} \simeq BC_\Gamma(\mathbb{Z}/p)_p^\wedge$ and $map(B\mathbb{Z}/p, BG_p^\wedge)_{B\rho} \simeq BC_G(\rho)_p^\wedge$. The right vertical arrow is a homotopy equivalence. The left vertical arrow is rationally a homotopy equivalence, because Γ is a finite group. The homotopy fiber of $(\psi^{p^k})^*$ is connected and given by

$map(B\mathbb{Z}/p, B\Gamma_p^\wedge)_{Bi}$. If G is simple and simply-connected, the subgroup $\mathbb{Z}/p^l \subset G$ is not central, since $(p, |W_G|) = 1$. The centralizer $C_G(\mathbb{Z}/p)$ is connected, of maximal rank, $W_{C_G(\mathbb{Z}/p)} \subset W_G$ is a proper subgroup [19]. Hence, $BC_G(\rho) \to BG$ is not a rational homotopy equivalence. This is a contradiction and finishes the proof. □

Let G and K be compact connected Lie groups and let $Z \to BG_p^\wedge \xrightarrow{f} BK_p^\wedge$ be a fibration such that $H^*(\Omega Z; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is a finite $\mathbb{Q}_p^\wedge$–module. As a finitely generated connected Hopf algebra, $H^*(\Omega Z; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}_p^\wedge}(y_1, \cdots, y_r)$ is an exterior algebras generated by classes y_i of degree $2l_i - 1$. The functor $\otimes_\mathbb{Z}\mathbb{Q}$ is exact. The spectral sequence for p–adic cohomology of a fibration gives a spectral sequence converging to $H^*(\ ; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Therefore, $H^*(Z; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong \mathbb{Q}_p^\wedge[x_1, \cdots, x_r]$ is a polynomial algebra generated by the classes x_i of degree $2l_i$. Because BG and BK are rationally homotopy equivalent to a product of even dimensional Eilenberg–MacLane spaces, the spectral sequence for $H^*(\ ; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ of the fibration $Z \to BG_p^\wedge \to BK_p^\wedge$ collapses and $H^*(BG; \mathbb{Z}/p) \otimes \mathbb{Q} \cong H^*(Z; \mathbb{Z}_p^\wedge) \otimes H^*(BK; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. This implies that $H^*(BK_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \to H^*(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is a monomorphism and has a left inverse as map of algebras.

By [2] or [21], the map $f : BG_p^\wedge \to BK_p^\wedge$ gives rise to a commutative diagram

$$\begin{array}{ccc} BT_{G_p^\wedge} & \xrightarrow{f_T} & BT_{K_p^\wedge} \\ \downarrow & & \downarrow \\ BG_p^\wedge & \xrightarrow{f} & BK_p^\wedge \end{array} .$$

Passing to cohomology we obtain

$$\begin{array}{ccc} H^*(BT_{G_p^\wedge}; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} & \xleftarrow{f_T^*} & H^*(BT_{K_p^\wedge}; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \\ \uparrow & & \uparrow \\ H^*(BG_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} & \xleftarrow{f^*} & H^*(BK_p^\wedge; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \end{array} .$$

By [1], there exists a homomorphism $W_G \to W_K$ such that f_T^* is W_G–equivariant.

Now we assume that both $G = G_s \times T$ and $K = K_s \times S$ are products of a simply-connected Lie group G_s or K_s and a torus T or S. Moreover, $G_s = \prod_i G_i$ and $K_s = \prod_j K_j$ are products of simple simply connected Lie groups G_i or K_j. By the proof of Proposition 7.1, $G_s = G_1 \times G_2$ and $T = T_1 \times T_2$ such that f factors over $BG_{2p}^\wedge \times BT_{2p}^\wedge$. This establishes the diagram

$$\begin{array}{ccc} BG_p^\wedge \simeq BG_{1p}^\wedge \times BG_{2p}^\wedge \times BT_{1p}^\wedge \times BT_{2p}^\wedge & \xrightarrow{f} & BK_p^\wedge \\ \downarrow & & \| \\ BG_{2p}^\wedge \times BT_{2p}^\wedge & \xrightarrow{\overline{f}} & BK_p^\wedge \end{array} .$$

The map $\overline{f} : BG_{2p}^\wedge \times BT_{2p}^\wedge \to BK_p^\wedge$ induces an isomorphism in $H^*(\ ; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$–cohomology. Therefore, $T_2 \cong S$ and G_2 and K_s are of the same rational type, i.e the classifying spaces are rationally equivalent. $Spin(2n+1)$ and $Sp(n)$, $n \geq 3$ are the only nonisomorphic simple simply connected Lie groups of the same rational type.

Actually, $BSpin(2n+1)^\wedge_p \simeq BSp(n)^\wedge_p$ for odd primes, and there are no essential maps $BSpin(2n+1) \to BSp(n)$ and vice versa with $n \geq 3$ [9]. Thus, if we replace $Spin(2n+1)$ by $Sp(n)$ for odd primes, we can assume that $G_2 \cong K_s$ as Lie groups.

Self maps of $BK \simeq BK_s \times BS$, $K_s = \prod_j K_j$, are analysed in [12]. The result implies that

$$\overline{f} \simeq \sigma \circ (\prod_j B\alpha_j \circ e_j^{d_j} \circ \psi^{r_j}) \times A_T \ .$$

σ permutes equivalent factors of K_s, α_j is an outer autumorphism, e is an exceptional isogeny, which is always a mod–p equivalence if it occurs [1], $d_j \in \{0,1\}$, ψ^{r_j} is an unstable Adams operation of degree $r_j \in \mathbb{Z}^\wedge_p$, and A_T is a self map of $BS^\wedge_p$. Therefore, the homotopy fiber $hofib(\overline{f})$ of $\overline{f}$ is equivalent to a product $\prod_i F_i \times B\pi$ of special Adams fibers F_i of p–type and the classifying space $B\pi$ of a finite abelian p–group. F_i is the homotopy fiber of ψ^{r_i} and $B\pi$ the homotopy fiber of A_T.

We summarize these considerations in the following proposition, which is a special case of Theorem 3. Two compact connected Lie groups G and H are said to have the same p–adic type if $BG^\wedge_p \simeq BH^\wedge_p$.

Proposition 5.3. *Let G and K be product groups of a simply connected Lie group and a torus.*

(1) *Let $Z \to BG^\wedge_p \xrightarrow{f} BK^\wedge_p$ be a fibration such that $H^*(\Omega Z; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}^\wedge_p$–vector space. Then, $G \cong G' \times G''$, such that G'' and K have the same p–adic type. Moreover, there exist a finite number of special Adams fibers F_i of p–type and a finite abelian p–group π such that*

$$Z \simeq BG'^\wedge_p \times \prod_i F_i \times B\pi \ .$$

(2) *Let $Z \to BG \xrightarrow{f} BK$ be a fibration such that $H^*(\Omega Z;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–vector space. Then $G \cong G' \times G''$ and $G'' \cong K$. Moreover, there exist a finite number of special Adams fibers F_i and a finite abelian group π such that*

$$Z \simeq BG' \times \prod_i F_i \times B\pi$$

Proof. There is something left to prove. The above considerations show that $G \cong G' \times G''$ such that G'' and K have the same p–adic type. Moreover, the map f splits over $BG''^\wedge_p$. This yields a diagram

$$\begin{array}{ccccc}
BG'^\wedge_p & = & BG'^\wedge_p & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
Z & \longrightarrow & BG^\wedge_p & \xrightarrow{f} & BK^\wedge_p \\
\downarrow & & \downarrow & & \| \\
F & \longrightarrow & BG''^\wedge_p & \longrightarrow & BK^\wedge_p \ ,
\end{array}$$

where $F \simeq \prod_i F_i \times B\pi$. Here, F_i is a special Adams fiber of p–type and π a finite abelian p–group. The middle vertical fibration is trivial and establishes an equivalence $Z \simeq BG'^\wedge_p \times F$, which proves (1).

The above considerations are also applicable to the uncompleted version. This completes the proof. □

Now we are prepared for the proof of Theorem 3.

Proof of Theorem 3. Let $Z \to BG^\wedge_p \xrightarrow{f} BK^\wedge_p$ be a fibration, such that $H^*(\Omega Z;\mathbb{Z}^\wedge_p)\otimes\mathbb{Q}$ is a finite dimensional $\mathbb{Q}^\wedge_p$–vector space. We can choose universal finite coverings $\alpha_G \to \widetilde{G} \xrightarrow{q_G} G$ and $\alpha_K \to \widetilde{K} \xrightarrow{q_K} K$ such that f lifts to a map $\widetilde{f} : B\widetilde{G} \to B\widetilde{K}$. We get a diagram

$$\begin{array}{ccccc}
\widetilde{Z} & \longrightarrow & B\widetilde{G}^\wedge_p & \xrightarrow{\widetilde{f}} & B\widetilde{K}^\wedge_p \\
\big\downarrow & & {\scriptstyle Bq_G}\big\downarrow & & {\scriptstyle Bq_K}\big\downarrow \\
Z & \longrightarrow & BG^\wedge_p & \xrightarrow{f} & BK^\wedge_p
\end{array}$$

Notice that both q_G and q_K are rationally equivalences. Thus we get

$$H^*(\Omega Z;\mathbb{Z}^\wedge_p)\otimes\mathbb{Q} \cong H^*(\Omega\widetilde{Z};\mathbb{Z}^\wedge_p)\otimes\mathbb{Q}\,.$$

Because all spaces are p–completed, we can assume that α_G and α_K are finite abelian p–groups.

We need a refinement of the splitting in Proposition 5.3. By that proposition, $\widetilde{G} \cong G' \times G''$, such that $G'' \cong G''_s \times T$, a product of a simply connected Lie group G''_s and a torus T, has the same p–type as $\widetilde{K}$. The extension map $\widetilde{f}$ factors over $\widetilde{f}'' : BG''^\wedge_{s\,p} \times BT^\wedge_p \to B\widetilde{K}^\wedge_p$. Up to permutations and equivalences $\widetilde{f}'' \simeq \prod \psi^{r_i} \times A_T$, $r_i \in \mathbb{Z}^\wedge_p$. We split $G''_s = G_2 \times G_3$ into a product. G_2 collects those factors of G''_s where r_i is a p–adic unit, and G_3 the others, i.e. $\widetilde{G} \cong G' \times G_2 \times G_3 \times T$. An Adams operation ψ^r only exists on $BH^\wedge_p$, H a compact connected Lie group, if r is a p–adic unit or if p is coprime to the order of W_H [29] [30] [8]. Therefore, the composition $\alpha_G \to \widetilde{G} \to G_3$ is constant.

Because $\widetilde{K}$ has the same p–type as G'', we get an analogous splitting $\widetilde{K} = K_2 \times K_3 \times T$, and $\alpha_K \to \widetilde{K} \to K_3$ is also constant. This yields a commutative diagram

$$\begin{array}{ccc}
B\widetilde{G}^\wedge_p & \xrightarrow{Bq_G} & BG^\wedge_p \\
\Big\| & & \Big\| \\
BG'^\wedge_p \times BG_{2\,p}^\wedge \times BG_{3\,p}^\wedge \times BT^\wedge_p & \xrightarrow{Bq_G} & B(G' \times G_2 \times T/\alpha_G)^\wedge_p \times BG_{3\,p}^\wedge \\
{\scriptstyle B\widetilde{pr}}\big\downarrow & & {\scriptstyle Bpr}\big\downarrow \\
BG_{2\,p}^\wedge \times BG_{3\,p}^\wedge \times BT^\wedge_p & \xrightarrow{B\overline{q}_G} & B(G_2 \times T/\alpha_G)^\wedge_p \times BG_{3\,p}^\wedge \\
{\scriptstyle \widetilde{f}''}\big\downarrow & & {\scriptstyle f''}\big\downarrow \\
B\widetilde{K}^\wedge_p = BK_{2\,p}^\wedge \times BK_{3\,p}^\wedge \times BT^\wedge_p & \xrightarrow{Bq_K} & B(K_2 \times T/\alpha_K)^\wedge_p \times BK_{3\,p}^\wedge
\end{array}\quad .$$

The map $\widetilde{f}'' \simeq \widetilde{f}''' \times f_3$, where

$$\widetilde{f}''' \simeq B\widetilde{\rho}^\wedge_p : B(G_2 \times T)^\wedge_p \to B(K_2 \times T)^\wedge_p$$

is induced by a homomorphism $\widetilde{\rho} : G_2 \times T \to K_2 \times T$, and where f_3 is a product of Adams operations. The map f'' exists according to a result of [20],. We have an anologous splitting of f'', namely $f'' \simeq f''' \times f_3$, where

$$f''' \simeq B\rho_p^\wedge : B((G_2 \times T)/\alpha_G)_p^\wedge \to B((K_2 \times T)/\alpha_K)_p^\wedge$$

is induced by a homomorphism $\rho : (G_2 \times T)/\alpha_G \to (K_2 \times T)/\alpha_K$. This implies that

$$\widetilde{Z} \simeq Bker(\widetilde{\rho} \circ \widetilde{pr})_p^\wedge \times F \ ,$$

(proposition 5.3) and therefore that

$$Z \simeq Bker(\rho \circ \overline{pr})_p^\wedge \times F \ .$$

Here, $ker(\ )$ denotes the kernel of a homomorphism, and F is a product of special Adams fibers of p–type, in particuliar the homotopy fiber of f_3.

For later purpose, we observe that, for a homomorphism $\alpha : \Gamma \to \Gamma'$ between two compact connected Lie groups, $ker(\alpha)$ fits into an exact sequence

$$ker_e(\alpha) \to ker(\alpha) \to \pi_0(ker(\alpha)) \ ,$$

where $ker_e(\ )$ denotes the component of the unit. The group $\pi_0(ker(\alpha))$ is abelian and acts trivially on $ker_e(\alpha)$. This is obviously true for products of a simply connected Lie group and a torus, and therefore, using universal finite coverings, also true in the general case. Hence, $Bker_e(\alpha)$ and $Bker(\alpha)$ are rationally homotopy equivalent. In the above situation, $\pi_0(ker(\rho \circ pr))$ is an abelian p–group and $ker_e(\rho \circ pr) \cong G'/(\alpha_G \cap G')$.

If $H^*(\Omega Z; \mathbb{F}_p)$ is finite, Adams fibers of p–type can't occur (Proposition 5.2). This finishes the proof of part (1).

To prove the second statement, that is the uncompleted version, we also pass to universal finite coverings, yielding a diagram

$$\begin{array}{ccccc} \widetilde{Z} & \longrightarrow & B\widetilde{G} & \xrightarrow{\widetilde{f}} & B\widetilde{K} \\ \downarrow & & \downarrow {\scriptstyle Bq_G} & & \downarrow {\scriptstyle Bq_K} \\ Z & \longrightarrow & BG & \xrightarrow{f} & BK \ . \end{array}$$

We also have a splitting $\widetilde{G} = G' \times G''$, where $\widetilde{f}|_{BG'}$ is nullhomotopic, and where f factors over a rational equivalence $\widetilde{f}'' : BG'' \to B\widetilde{K}$. In fact, $\widetilde{f}''$ is a mod–p equivalence for almost all primes. Because BK and $B\widetilde{K}$ are simply-connected , completion maintains the fibrations [4]. By the proof of (1), the rationalisation $Z_{\mathbb{Q}}$ is equivalent to $BG'/(\alpha_G \cap G')_{\mathbb{Q}}$, and $Z_p^\wedge \simeq B\Gamma(p) \times F(p)$, where $F(p)$ a product of special Adams fibers of p–type, and where $\Gamma(p)$ is a compact Lie group. Moreover, there exists a finite abelian p–group $\pi(p)$ and an exact sequence $G'/(S_p\alpha_G \cap G') \to \Gamma(p) \to \pi(p)$, where $S_p\alpha_G$ denote the p–Sylow subgroup. Using the arithmetique square and the fact that the rationalisations of $B\pi(p)$ and $F(p)$ are trivial, one can show that $Z \simeq Z' \times F$. Here F is a product of special Adams fibers and Z' is the total space of a fibration $B(G'/(\alpha_G \cap G')) \to Z' \to B\pi$, where π is a finite abelian group. By [18], this implies that Z' is the classifying space of a compact Lie group.

If ΩZ is equivalent to a finite CW–complex, Adams fibers can't occur (Proposition 5.2 (4)). This finishes the proof. $\square$

6. Retracts of classifying spaces.

In this section we prove Theorem 5. As a remark, it will be shown that the Lie group K in Theorem 5 need not be a subgroup of G.

Proof of Theorem 5. Let $\alpha_G \to \widetilde{G} \to G$ be a universal finite covering of the compact connected Lie group G. If X is a retract of BG, an argument similar to the one used in the proof of the main theorm of [10] shows that one can find a retract $\widetilde{X}$ of $B\widetilde{G}$:

$$\begin{array}{ccccc} B\alpha_X & \longrightarrow & B\alpha_G & \longrightarrow & B\alpha_X \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow{i} & B\widetilde{G} & \xrightarrow{r} & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & BG & \longrightarrow & X \end{array}$$

where the composite $r \cdot i$ is homotopy equivalent to the identity map. Along the line of the proof again, one can show that for any prime p there is a simply connected Lie group $\widetilde{K}(p)$ which is a product of some factor subgroups of $\widetilde{G}$ such that $\widetilde{X}_p^\wedge \simeq B\widetilde{K}(p)_p^\wedge$.

First we claim that $\widetilde{X}$ has a maximal torus in the sense of [25] or [21]. The composite map $i \cdot r|_{BT_{\widetilde{G}}} : BT_{\widetilde{G}} \to B\widetilde{G}$ is of form $B\rho$ for a homomorphsim $\rho : T_{\widetilde{G}} \to \widetilde{G}$ [16]. We can show that the homotopy fibre of the map $r|_{B\rho(T_{\widetilde{G}})} \to \widetilde{X}$ is a retract of the finite CW–complex $\widetilde{G}/T_{\widetilde{G}}$ and hence homotopy equivalent to a finite CW–complex. Since $\rho(T_{\widetilde{G}})$ is a retract of $T_{\widetilde{G}}$, the monomorphism $H^*(X;\mathbb{Q}) \to H^*(B\rho(T_{\widetilde{G}});\mathbb{Q})$ is finite. Hence $rank(X) = rank(\rho(T_{\widetilde{G}}))$.

Next we claim that the space $\widetilde{X}$ has a genus type of $B\widetilde{K}(2)$. We see that

$$H^*(\widetilde{X};\mathbb{Q}^\wedge) \cong H^*(B\widetilde{K}(p);\mathbb{Q}^\wedge) \cong H^*(B\widetilde{T}(p);\mathbb{Q}^\wedge)^{W(\widetilde{K}(p))}$$

for any p. Here $\widetilde{T}(p)$ is a maximal torus of $\widetilde{K}(p)$. An argument similar to the one used in the proof of Theorem 1 (2) implies

$$X_p^\wedge \simeq B\widetilde{K}(p)_p^\wedge \simeq B\widetilde{K}(2)_p^\wedge$$

for p odd. If $\widetilde{K} = \widetilde{K}(2)$, then $\widetilde{X} \simeq B\widetilde{K}$. We now have a fibration $B\alpha_X \to B\widetilde{K} \to X$. Let f be the map $B\alpha_X \to B\widetilde{K}$, and let $S_p\alpha_X$ be the p–Sylow subgroup of the finite abelian group α_X. Notice that $BS_p\alpha_X \to B\widetilde{K}_p^\wedge \to X_p^\wedge$ is a fibration. Since $S_p\alpha_X$ is a p–group, there is a monomorphism $i_p : S_p\alpha_X \to \widetilde{K}$ such that $f_p^\wedge \simeq (Bi_p)_p^\wedge$. Theorem 4 shows $i_p(S_p\alpha_X)$ is central in $\widetilde{K}$. Consequently, the map $f : B\alpha_X \to B\widetilde{K}$ factorizes over an homotopy equivalence $X \simeq BK$ if $K = \widetilde{K}/\alpha_X$. $\square$

Remark. There are compact connected semi-simple Lie groups K and G which satisfy the following:

(1) BK is a retract of BG.

(2) K is not a subgroup of G.

The following is such an example. Let $\widetilde{K} = SU(n)$ and $\widetilde{G} = SU(n) \times SU(n)$. The center of the special unitary group $SU(n)$ is isomorphic to $\mathbb{Z}/n$, and consists of the scalar matrices

$$\begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix} \text{with } \zeta^n = 1$$

For a positive integer k, let $\eta_k : \mathbb{Z}/k \to \widetilde{G}$ be the monomorphism as follows:

$$\eta_k(\zeta) = \begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix} \times \begin{pmatrix} \zeta^k & & \\ & \ddots & \\ & & \zeta^k \end{pmatrix}$$

Now we define $G = \widetilde{G}/\eta_k(\mathbb{Z}/n)$ and $K = \widetilde{K}/\mathbb{Z}/n = PSU(n)$. When k is prime to $n!$, we have the commutative diagram:

$$\begin{array}{ccc} B\widetilde{K} & \xrightarrow{id \times \psi^k} & B\widetilde{G} \\ \downarrow & & \downarrow \\ BK & \xrightarrow{f} & BG \end{array}$$

where ψ^k is an unstable Adams operation and f is the induced map. If $q : B\widetilde{G} \to B\widetilde{K}$ is the projection to the first factor subgroup of $\widetilde{G}$, then $q \cdot (id \times \psi^k) \simeq id$. Thus BK is a retract of BG. However, K is not a subgroup of G if $k \neq 1$.

7. Appendix.

This section contains some results about mapping spaces of classifying spaces of compact Lie groups needed in the proofs of the main results.

For $R = \mathbb{Z}$ or $R = \mathbb{Z}_p^\wedge$ we denote by X_R the space X itself or the p–adic completion of X respectively.

Proposition 7.1. *Let $R = \mathbb{Z}$ or $R = \mathbb{Z}_p^\wedge$. Let $G = G_s \times T_1$ and $K = K_s \times T_2$ be products of a a simply–connected compact Lie group and a torus. Let $f : BG_R \to BK_R$ be a map such that $f^* : H^*(BK_R; R) \otimes \mathbb{Q} \to H^*(BG_R; R) \otimes \mathbb{Q}$ is a monomorphsim. Then we can find two subgroups G' and G'' of G satisfying the following:*

1. $G = G' \times G''$
2. *G' and G'' are also products of a simply connected Lie group and a torus.*
3. *The restriction $f|_{BG''}$ is nullhomotopic so that f factors over $f' : BG'_R \to BK_R$*
4. *$f'^* : H^*(BK_R; R) \otimes \mathbb{Q} \to H^*(BG'_R; R) \otimes \mathbb{Q}$ is a monomorphism*
5. *G' and K have the same rank.*

If f^* has a left inverse as a map between algebras, then f'^* is an isomorphism.

Corollary 7.2. *Let G be a product of a simply–connected compact Lie group and a torus, and let H and K be (arbitrary) connected compact Lie groups.*

(1) *Any fibration of the form $(BH)^\wedge_p \to (BG)^\wedge_p \to (BK)^\wedge_p$ is fiber homotopically trivial.*
(2) *Any integral fibration of the form $BH \to BG \to BK$ is fiber homotopically trivial.*

Proof. The Serre spectral sequence for rational cohomology of the fibration (completed or un-completed) collapses since everything is concentrated in even degrees. By the last proposition, this shows, using the notation from above, that $H^*(BK_R; R)\mathbb{Q} \otimes H^*(BH_R; R) \otimes \mathbb{Q} \cong H^*(BG_R; R) \otimes \mathbb{Q}$. Hence, the maps $BH_R \to BG''_R$ as well as $BG'_R \to BK_R$ are rationally equivalences. Both map can only have a finite kernel. The first map makes $H^*(BH; \mathbb{Z}/p)$ into a finitely generated $H^*(BG''; \mathbb{Z}/p)$ module. By [23] this implies that $BH_R \to BG''_R$ has a trivial kernel and is therefore an equivalence. This shows that the fibration is trivial.

The proof of Proposition 7.1 is based on the following lemma:

Lemma 7.3. *Let $\mathbb{F}$ be a field of characteristic zero, let $W = \prod_i W_i$ be a finite product of finite groups W_i, $1 \le i \le n$, and let M_i be an irreducible $\mathbb{F}[W_i]$–module. Then, $M := \bigoplus_i M_i$ is a direct sum of irreducible $\mathbb{F}[W]$–modules and, for every $\mathbb{F}[W]$–submodule $N \subset M$, there exist a subset $I \subset \{1, ..., n\}$ such that N maps onto $\bigoplus_{i \in I} M_i$ and such that the projection $N \subset M \to M_i$ is trivial for $i \notin I$. In particuliar, $N \cong \bigoplus_{i \in I} M_i$.*

Proof. Obviously, each M_i is an irreducible W–module. Notice that N splits into irreducible $\mathbb{F}[W]$–modules N_j i.e. $N \cong \bigoplus_j N_j$. The composition $N \subset M \to M_i$ is either trivial or an epimorphism so that each summand N_j is isomorphic to a summand M_{i_j}. Without loss of generality we can assume that all compositions are epimorphisms. The action of W on M can be described by diagonal matrices $D(w_1, \cdots, w_n)$ for $w_i \in W_i$, and the map $\bigoplus_j N_j \to \bigoplus_j M_{i_j}$ by a matrix $\Phi = (\phi_{j,i})$. At least one entry in each column and each row is nontrivial and hence an isomorphism. Let N_j be a summand on which all factors W_i, $i \neq i_0$, act trivially. Then,

$$\begin{pmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_n \end{pmatrix} \cdot \begin{pmatrix} \phi_{j,1} \\ \phi_{j,2} \\ \vdots \\ \phi_{j,n} \end{pmatrix} = \begin{pmatrix} \phi_{j,1} \\ \phi_{j,2} \\ \vdots \\ \phi_{j,n} \end{pmatrix} \cdot w_{i_o} \quad .$$

If, for $k \neq i_o$, $\phi_{j,k}$ is nontrivial, then $w_{i_o} = \phi_{j,k}^{-1} \cdot w_k \cdot \phi_{j,k}$ for all $w_{i_0} \in W_{i_0}$ and $w_k \in W_k$, which is a contradiction. Therefore, $\phi_{j,k} = 0$ for $k \neq i_0$ and ϕ_{j,i_0} is an isomorphism. Moreover Φ is a square matrix. In particuliar, it is a monomial matrix, i.e every row and every column has exactly one nontrivial entry, which is an isomorphism. This finishes the proof. □

Proof of Proposition 7.1. First we assume that $G = G_s$ and $K = K_s$, i.e. G and K are simply-connected. Let $G = \prod G_i$ and $K = \prod K_j$ be the splitting into simple simply connected Lie group. By [16] or [21], the map f lifts to a map $f_T : (BT_G)_R \to (BT_K)_R$ between the classifyng spaces of the maximal tori. Moreover, by [1] there exists a homorphism $W_G \to W_K$ such that the induced map

$$f_T^* : H^*((BT_K)_R; R) \otimes \mathbb{Q} \to H^*((BT_G)_R; R) \otimes \mathbb{Q}$$

is W_G–equivariant. The isomorphism

$$H^2((BT_G)_R;R) \cong \bigoplus_i H^2((BT_{G_i})_R;R)\otimes \mathbb{Q}$$

is a splitting into irreducible $\mathbb{Q}_R[W_G]$–modules of the form as assumed in the previous lemma. Thus there exists a splitting $G = G' \times G''$ such that

$$H^2((BT_K)_R;R)\otimes\mathbb{Q} \cong H^2((BT_{G'})_R;R)\otimes\mathbb{Q}\ ,$$

i.e. G' and K have the same rank, and $f|_{BG''_R}$ is trivial in $H^*(\ ;R)\otimes\mathbb{Q}$–cohomology and therefore nullhomotopic [12]. By a standard trick we can divide out BG'' (e.g. see [9]), i.e. f factors over a map $f' : BG'_R \to BK_R$. The induced map in $H^*(\ ;R)\otimes$ $\mathbb{Q}$–cohomology is still a monomorphism and has a left inverse as algebra map if f^* does. Because both polynomial rings $H^*(BG'_R;R)\otimes\mathbb{Q}$ and $H^*(BH_R;R)\otimes\mathbb{Q}$ have the same transcendence degree over $\mathbb{Q}_R$, f'^* is an isomorphism.

Now we consider the general case, i.e $G = G_s \times T_1$ and $K = K_s \times T_2$. Then $f \simeq f_s \times f_T$, where $f_s : BG_{sR} \to BK_{sR}$ and $f_T : BT_{1R} \to BT_{2R}$. This follows from the proof of [19, Theorem 16.2], which is the uncompleted version of proposition 7.1 for self maps. The same arguments also apply in our case. $H^*(f_s;R)\otimes\mathbb{Q}$ and $H^*(f_T;R)\otimes\mathbb{Q}$ are monomorphisms. Then, $G_s = G'_s \times G''_s$ splits into two groups with the desired properties. By similiar arguments, T_1 splits into a product of two tori with the same properties. This finishes the proof. □

The following proposition as well as the proof was pointed out to the second author by G.Mislin.

Proposition 7.4. *Let V be an elementary abelian p–group and G a compact Lie group. Then, for any homomorphism $\rho : V \to G$,*

$$(map(BV,BG)_{B\rho})^\wedge_p \simeq map(BV,BG^\wedge_p)_{B\rho}\ .$$

Proof. By [13, 3.4.5] or [7], $BC_G(\rho)^\wedge_p \to (map(BV,BG)_{B\rho})^\wedge_p$ is a homotopy equivalence. The classifying space BG is p–good in the sense of [4]. By [13, 3.3.2 and 3.4.3], the mapping space $map(BV,BG^\wedge_p)_{B\rho}$ is p–complete, and $BC_G(\rho)^\wedge_p \to$ $map(BV,BG^\wedge_p)_{B\rho}$ is a homotopy equivalence. □

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