FAKE LIE GROUPS WITH MAXIMAL TORI. IV

BY
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1.Introduction.

This is a continuation of [N–S 1,2,3], which we refer to as I, II, and III. In this series of papers fake Lie groups and adic fake Lie groups are studied. Definitions, notation, and terminology about genus, adic genus, (adic) fake Lie groups, maximal tori, and Weyl groups may be found in I, II, and III.

One of the main goals was and is to determine all fake Lie groups allowing a maximal torus. In II and III, this is done for simply connected Lie groups and unitary groups. Here we will extend these results to the general case.

Theorem A. *Let X be a fake Lie group of type G with maximal torus $BT_X \longrightarrow BX$.*

(1) *There exists a compact connected Lie group H such that $BX \simeq BH$.*
(2) *If G is simply connected or simple, then $G = H$.*

As mentioned, the first part of (2) and (1) for unitary groups are already proved in III. Here, we will use a different approach for the proof, which we will explain in a moment.

In [N 2, 7.3], for $BX \in genus_0^\wedge(BG)$, it is proved that BX and BG have the same (local) genus iff X has a p–adic maximal torus $BTX_p^\wedge \longrightarrow BX_p^\wedge$ for each prime. Thus, theorem A is also true under the assumption of X being an adic fake Lie group of type G.

We define $FLGT(G)$ to be the set of equivalence classe of fake Lie groups of type G allowing an integral maximal torus. Two fake Lie groups X and Y are equivalent if $BX \simeq BY$.

We say that two maps $f : Y \longrightarrow Z$ and $g : U \longrightarrow V$ between nilpotent CW-complexes of finite type are in the same genus, if, for any prime p, there exists a homotopy commutative diagram between the p-localised spaces

$$\begin{array}{ccc} Y_{(p)} & \longrightarrow & U_{(p)} \\ {\scriptstyle f_{(p)}}\downarrow & & \downarrow{\scriptstyle g_{(p)}} \\ Z_{(p)} & \longrightarrow & V_{(p)} \end{array} ,$$

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where the horizontal arrows are homotopy equivalences. We write this as

$$(Y_{(p)} \longrightarrow Z_{(p)}) \simeq (U_{(p)} \longrightarrow V_{(p)}) \ .$$

By $genus(f : X \longrightarrow Y)$, we denote the set of homotopy classes of maps, which have the same genus as $f : X \longrightarrow Y$. The genus of a space X is given by $genus(\emptyset \longrightarrow X)$ or by $genus(id : X \longrightarrow X)$.

Now, let G be a compact connected Lie group, and X a fake Lie group of type G with maximal torus $BT_X \longrightarrow BX$. Then,

$$(BT_X \longrightarrow BX) \in genus(BT_G \longrightarrow BG) \ ,$$

where $T_G \longrightarrow G$ is the inclusion of the 'classical' maximal torus of G. For simply connected Lie groups, this follows from I 4.2, and in the general case from [N 2, 7.3 and the proof]. Because of the uniqueness of maximal tori of fake Lie groups (I 4.5), this establishes a map

$$FLGT(G) \longrightarrow genus(BT_G \longrightarrow BG) \ .$$

Since $genus(BT_G) = \{BT_G\}$, this is an isomorphism. The inverse map is the obvious one. We can reformulate theorem A to

Theorem A'. *Let G be a compact connected Lie group.*

(1) *Every element $(Y \longrightarrow Z) \in genus(BT_G \longrightarrow BG)$ is represented by a compact connected Lie group H and the associated maximal torus $BT_H \longrightarrow BH$.*
(2) *If G is simply connected or simple, the genus of $BT_G \longrightarrow BG$ is rigid; i.e. $genus(BT_G \longrightarrow BG) = \{BT_G \longrightarrow BG\}$.*

Remark. Part (2) is not true in general, even for semi simple Lie groups it is false as the following example demonstrates. Let $G = SU(pq)$, where p and q are different primes. We define the groups G/p, G/q and G/pq by dividing out the subgroups $\mathbb{Z}/p$, $\mathbb{Z}/q$ and $\mathbb{Z}/pq$ of the center of G. The genus of $BG/p \times BG/q$ containes the element $BG/pq \times BG$ and both Lie groups have a maximal torus. Another class of counterexamples is given by the fake unitary groups of III.

In the rest of the paper we prove theorem A', the second part in the next section and the first part in section 3.

2. Rational self equivalences of $BT_G \longrightarrow BG$.

Given a set of primes P, we denote by X_P the localisation of X at P and by $X_{1/P}$ the localisation at the complementary set of primes. $X_0 = X_\varnothing$ is the rationalisation of X. The group of homotopy classes of self equivalences of the rationalization of a map $X \longrightarrow Y$ is denoted by $E_0(X \longrightarrow Y)$. $E_P(X \longrightarrow Y)$ is the subgroup consisting of rationalisations of self equivalences of $(X \longrightarrow Y)_P$.

For every compact connected Lie group G, there exists a finite covering $\widetilde{G} = G_s \times T \longrightarrow G$, where G_s is simply connected, and where T is a torus.

Lemma 2.1. $E_0(BT_G \longrightarrow BG) \cong E_0(BT_{G_s} \longrightarrow BG_s) \times E_0(BT)$.

Proof. It is

$$(BT_G \longrightarrow BG)_0 \simeq (BT_{G_s} \times BT \longrightarrow BG_s \times BT)_0 \ .$$

We can apply [N 3, 16.2] or [A–M, 2.25 and the proof] to show that every rational self equivalence splits into a product. □

2.2 Corollary.

$$genus(BT_{G_s} \times BT \longrightarrow BG_s \times BT) \cong genus(BT_{G_s} \longrightarrow BG_s) \ .$$

Proof. The elements in $genus(BT_{G_s} \times BT \longrightarrow BG_s \times BT)$ can be constructed as a homotopy inverse limit using glueing maps on the rationalization of $BT_{G_s} \times BT \longrightarrow BG_s \times BT$. By lemma 2.1, all the glueing maps split. The genus of BT is rigid. This completes the proof. □

Glover and Mislin studied the genus of G/T_G [G–M]. They explained in detail the relation between the genus of a space Y and the groups of self equivalences $E_0(Y)$ and $E_P(Y)$. Also, they set up a concept to prove the genus of G/T_G to be rigid. Papadima used this concept to complete the results of [G–M] and proved that, for a compact connected Lie group G, $genus(G/T_G) = \{G/T_G\}$. All these ideas extend to the case of the genus of $BT_G \longrightarrow BG$. The proof of theorem A' (2) is totally analogous to the proof of [P, 1.6 or 5.3]. We only need the following two lemmas.

Lemma 2.3. *Let G be compact connected Lie group. Then, there exists a finite set P of primes such that, for $(BT_X \longrightarrow BX) \in genus(BT_G \longrightarrow BG)$,*

$$(BTX \longrightarrow BX)_{1/P} \simeq (BT_G \longrightarrow BG)_{1/P} \ .$$

Proof. For every $(BT_X \longrightarrow BX) \in genus(BT_G \longrightarrow BG)$, there exists a finite set P of primes, such that $(BTX \longrightarrow BX)_{1/P} \simeq (BT_G \longrightarrow BG)_{1/P}$ (I 4.4). A careful analysis of the proof of I 4.4 shows that the following primes have to be inverted, namely primes dividing the order of the W_G, torsion primes of $H^*(BG;\mathbb{Z})$, and primes occuring in the denominators of the entries of the matrix and its inverse conjugating W_G into W_X. By the Jordan–Zassenhaus theorem [C–R, 79.1], there exist only finitely many conjugacy classes of integral lifts of the rational W_G–representation $H^2(BT_G;\mathbb{Q})$. The statement follows. □

For any set of primes, the fibration $G/T_G \longrightarrow BT_G \longrightarrow BG$ establishes a map

$$F_P : E_P(BT_G \longrightarrow BG) \longrightarrow E_P(G/T_G) \ .$$

2.4 Lemma. *Let G be simply connected or simple. For every finite set P of primes, F_P is an isomorphism.*

Proof. If G is simply connected or simple, $H^2(G/T_G;\mathbb{Q}) \cong H^2(BT_G;\mathbb{Q})$. Because BG_0 is a product of Eilenberg–McLane spaces, F_P is an injection. In [P], $E_0(G/T_G)$ is calculated. This computation and the results of [A–M, §2] and [W] show that F_P is surjective. A description of $E_0(BT_G \longrightarrow BG)$ may also be found in [J–M–O] or [Mø]. □

3. Proof of theorem A' (1).

For a compact connected Lie group G, there exists a finite covering

$$K \xrightarrow{\ i\ } \widetilde{G} = G_s \times T \longrightarrow G \ ,$$

where K is finite abelian, G_s simply connected, and T a torus. Passing to classifying spaces we yield a principal fibration with classifying map $BG \longrightarrow BBK$. The composition $BT_G \longrightarrow BG \longrightarrow BBK$ classifies the associated fibration of the classifying spaces of the maximal tori. We denote by K_p the p–Sylow subgroup of K. Then, $BK_p \simeq BK_{(p)}$.

For $BT_X \longrightarrow BX \in genus(BT_G \longrightarrow BG)$, we can define p–local classifying maps $BT_{X\,(p)} \longrightarrow BX_{(p)} \longrightarrow BBK_p$. $BT_X \longrightarrow BX$ can be reconstructed from the localisations by a homotopy inverse limit. This yields maps $BT_X \longrightarrow BX \longrightarrow BBK$ classifying the diagram of fibrations

$$\begin{array}{ccccc} BK & \longrightarrow & Y & \longrightarrow & BT_X \\ \| & & \downarrow & & \downarrow \\ BK & \longrightarrow & Z & \longrightarrow & BX \ . \end{array}$$

$(Y \longrightarrow Z) \in genus(BT_{G_s} \times BT \longrightarrow BG_s \times BT)$. By lemma 3.1 and theorem A'(2), this genus is rigid. This establishes a diagram of fibrations

$$\begin{array}{ccccc} BK & \xrightarrow{B\rho} & BT_{\widetilde{G}} & \longrightarrow & BT_X \\ \| & & \downarrow & & \downarrow \\ BK & \xrightarrow{B\rho} & B\widetilde{G} & \longrightarrow & BX \ , \end{array}$$

where the first map in each row is induced by a homomorphism $\rho : K \longrightarrow T_{\widetilde{G}}$.

3.1 Lemma. *ρ is injective, and the image $im(\rho)$ of ρ is central in $\widetilde{G} = G_s \times T$.*

Proof. For every prime p, there exists a commutative diagram

$$\begin{array}{ccccc} BK_p & \xrightarrow{Bi} & BT_{\widetilde{G}(p)} & \longrightarrow & B\widetilde{G}_{(p)} \\ \| & & \downarrow h_{T_{\widetilde{G}}} & & \downarrow h_{\widetilde{G}} \\ BK_p & \xrightarrow{B\rho} & BT_{\widetilde{G}(p)} & \longrightarrow & B\widetilde{G}_{(p)} \end{array} .$$

The vertical arrows are equivalences. Self equivalences of $(BT_{G_s} \times BT \longrightarrow BG_s \times BT)_P$ split into products of p–local equivalences. This follows from lemma 2.1 and the p–local version of [N 2, 6.2]. Thus,

$$(h_{\widetilde{G}}, h_{\widetilde{G}}) = (h_{T_{G_s}}, h_{G_s}) \times h_T \ .$$

$h_{T_{G_s}} \circ Bi|_{BK_p}$ is induced by a monomomorphism $K_p \longrightarrow T_{G_s}$, whose image is central in G_s. This follows from lemma 2.4 and [P]. Hence, $\rho(K_p)$ for all p and $\rho(K)$ are central in $G_s \times T$. Moreover, ρ is injective. □

Now we can finish the proof of theorem A' (1).

Proof of theorem A'(1). In view of lemma 4.2 $H := \widetilde{G}/\rho(K)$ is a well defined compact connected Lie group. We will construct an equivalence $BH \longrightarrow BX$.

By construction, there exists for every prime p a p-local equivalence

$$(B\widetilde{G} \longrightarrow BX)_{(p)} \longrightarrow (B\widetilde{G} \longrightarrow BG)_{(p)} \ ,$$

which fits into a diagram

$$\begin{array}{ccccc} BK_p & = & BK_p & \longrightarrow & BK_p \\ \downarrow & & \downarrow & & \downarrow \\ B\widetilde{G} & = & B\widetilde{G}_{(p)} & \longrightarrow & B\widetilde{G}^\wedge_p \\ \downarrow & & \downarrow & & \downarrow \\ BH_{(p)} & & BX_{(p)} & \longrightarrow & BG_{(p)} \end{array} .$$

Going from the upper left corner to the lower right corner we see that BK_p is mapped to a point. The p-local version of [N 2; 1.1], which is implicit contained in the proof, allows in this situation to construct the dotted arrow. We get a p-local equivalence $BH_p \longrightarrow BX_p$ for every prime p. By construction, all these maps induce the same map in rational cohomology. Therefore, they are homotopic over the rationals. Glueing them together, we get a homotopy equivalence $BH \longrightarrow BX$, which extends to an equivalence $(BT_H \longrightarrow BH) \simeq (BT_X \longrightarrow BX)$. □

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