CENTERS AND FINITE COVERINGS OF FINITE LOOP SPACES

BY
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Abstract. The homotopy theoretic analogue of a compact Lie group is a p–compact group, i.e. a space X with finite mod–p cohomology and a loop structure given by an equivalence of the form $X \simeq \Omega BX$. The 'classifying space' BX has to be a p–complete space. We are concerned with the notions of centers and finite coverings of connected p–compact groups. In particular , we prove in this category two well known results for compact Lie groups; namely that the center of a connected p–compact group is finite iff the fundamental group is finite and that every connected p–compact group has a finite covering which is a product of a simply connected p–compact group and a torus. The latter statement also translates to connected finite loop spaces.

1. Introduction.

A finite loop space X is a triple (X, BX, e), in which $e : X \to \Omega BX$ is an equivalence from the space X into the loop space ΩBX of the pointed space BX. The loop space X is called finite if X is homotopy equivalent to a finite CW–complex or if the integral homology $H_*(X;\mathbb{Z})$ is finitely generated as a graded abelian group. The latter condition is a little weaker, but sufficient for proving most of the nice results about finite loop spaces.

Finite loop spaces are considered to be the homotopy theoretic generalisation of compact Lie groups. For a compact Lie group G the triple (G, BG, e), consisting of the compact Lie group, the classifying space BG and the natural equivalence $e : G \to \Omega BG$, is a finite loop space. Following an old idea of Rector [R_1], namely passing from a group to the associated classifying space, one would like to develop Lie group theory in terms of classifying spaces. This would give the chance to extend all the beautiful results about Lie groups to the bigger class of finite loop spaces.

The maximal torus is one of the fundamental notions one has to define for finite loop spaces. A *maximal torus* of a finite loop space X is a map $f : BT_X \to BX$ from the classifying space BT_X of a torus T_X into BX such that the homotopy fiber X/T_X of f is equivalent to a finite CW–complex (or $H_*(X/T;\mathbb{Z})$ is a finitely generated graded abelian group) and such that T_X and X have the same rank. The rank of X is defined to be the number of generators of the exterior algebra $H^*(X;\mathbb{Q})$.

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Let $T_G \to G$ be a maximal torus of a compact Lie group G. Then, this definition is made up by extracting the basic properties of the associated fibration $G/T_G \to BT_G \to BG$. Later we will change our point of view and reformulate the definition of a maximal torus (see Section 2).

By work of Rector [R_2], with help from McGibbon [McG] at the prime 2, it turned out that there exist finite loop spaces which do not have a maximal torus. There even exists a conjecture that every finite loop space with maximal torus comes from a compact Lie group [W].

As usual completion makes life a lot easier. This also turns out to be true in the study of finite loop spaces. In a recent paper, Dwyer and Wilkerson [D-W] defined a *p–compact group* to be a loop space (X, BX, e) such that BX is p–complete and such that X is $\mathbb{F}_p$–finite, i.e. that $H^*(X;\mathbb{F}_p)$ is a finite dimensional graded $\mathbb{F}_p$–vector space. They studied p–compact groups in great detail.

Here is a warning: In general you don't get a p–compact group just by the completion of the classifying space of a finite loop space. For a p–compact group X, the fundamental group $\pi_1(BX)$ is a finite p–group, i.e. the group of the components of X is also a finite p–group. This is a restriction which comes into play. On the other hand the completion of the classifying space $B\Sigma_n$ of the symmetric group Σ_n at an odd prime gives a highly connected space whose loop space is not $\mathbb{F}_p$–finite. The 3-adic completion $(B\Sigma_3)^\wedge_3$ is 2–connected but not 3–connected as an easy calculation of the mod-3 cohomology via the Serre spectral sequence for the fibration $B\mathbb{Z}/3 \to B\Sigma_3 \to B\mathbb{Z}/2$ shows. But, by a theorem of Browder [B_2], the classifying space of a simply connected p–compact group is always 3–conected. Thus, $\Omega((B\Sigma_3)^w edge_3)$ is not a $\mathbb{F}_p$–finite. Nevertheless p–compact groups are the right object to study finite loop spaces. Let L be a finite loop space. The completion of the classifying space BL_p of the componets L_p lying over the p–Sylow subgroup of $\pi_0(L)$ gives a p–compact group. Moreover $BL_p \to BL$ is a finite covering. The existence of a transfer allows to carry over a lot of the cohomological properties of BL_p to BL (e.g. see [D-W, Section 2].

The results of Dwyer and Wilkerson [D-W] as well as our experience show that p–compact groups enjoy much of the rich internal structure of compact Lie groups. In particular , they showed that every p–compact group has a maximal torus and a Weyl group and that, for a connected p–compact group, the rational cohomology of the classifying space is given by the invariants of the Weyl group acting on the cohomology of the classifying space of the maximal torus. We will give an explicit formulation of their result in Section 2.

Following the spirit of that influential paper we are here concerned with the center of p–compact group and finite coverings of p–compact groups. To formulate our results we first have to translate some of the basic notions of group theory in terms of p–compact groups. In Section 2 we will recall the dictionary of [D-W] and add some more translations.

1.1 Definition. In this definition X denotes a p–compact group.

(1) A *p–compact torus of rank n* is a p–compact group (T, BT, e) such that BT is homotopy equivalent to an Eilenberg–MacLane space $K(\mathbb{Z}_p^{\wedge n}, 2)$.
(2) A p–compact group X is called *finite* if BX is equivalent to an Eilenberg–MacLane space $K(\pi, 1)$ of a finite p–group of degree 1.
(3) A *homomorphism* $g : Y \to X$ of p–compact groups is a pointed map $Bg : BY \to BX$. Two homorphisms $g_1, g_2 : Y \to X$ are *conjugate* if

the associated maps $Bg_1, Bg_2 : BY \to BX$ are freely homotopic.

(4) A homomorphism $g : Y \to X$ is a *monomorphism* or equivalently Y is a *subgroup* of X if the homotopy fiber X/Y of Bg is $\mathbb{F}_p$–finite.

(5) For $i = 1, 2$, let $g_i : Y_i \to X$ be subgroups of the p–compact group X. Then, Y_1 is *subconjugate* to Y_2 if there exists a homomorphism $h : Y_1 \to Y_2$ such that $g_2 h$ and g_1 are conjugate.

(6) A subgroup $g : Z \to X$ is *central* if the evaluation $ev : map(BZ, BX)_{Bg} \to BX$ is an equivalence.

(7) A central subgroup $Z(X) \to X$ is called the *center* of X if every central subgroup $Z \to X$ is subconjugate to $Z(X)$.

Before we state our first result we explain the motivation of some of these definitions. The third says that every homomorphism of groups is a loop map and that two conjugated homomorphism induce homotopic maps between the associated classifying spaces. Every inclusion $H \to G$ of compact Lie groups establishes a fibration $G/H \to BH \to BG$.

Analogously to the above definition of a maximal torus, the fourth part reflects the fundamental properties of this fibration.

The sixth definition goes back to a theorem of Dwyer and Zabrodsky [D-Z] on the one hand and the second author [N$_1$] on the other hand. For any homomorphism $\rho : P \to G$ of a p–toral group P, i.e. a finite extension of a torus by a finite p–group, into a compact Lie group G , there exists a map $BC_G(\rho) \to map(BP, BG)_{B\rho}$ which becomes an equivalence after completion. Here, $C_G(\rho)$ denotes the centralizer of ρ in G.

The definition of a center might not be something the reader expects. In the classical case conjugation acts trivially on the center. This also turns out to be true for p–compact groups (see Proposition 4.5 and 4.7).

1.2 Proposition. *Every p–compact group has a center.*

1.3 Theorem. *Let X be connected p–compact group. Then the center $Z(X) \to X$ is a finite subgroup of X if and only if the fundamental group $\pi_1(X)$ is finite.*

This theorem is the generalization of the analogous well known result about semisimple Lie groups. We can use this statement for the following definition. A connected p–compact group X is called *semisimple* if $\pi_1(X)$ is finite or, equivalently, if the center $Z(X)$ is a finite p–compact group.

In the classification of compact connected Lie groups, one first passes to a finite covering $\widetilde{G} \to G$ of a compact connected Lie group G, such that $\widetilde{G}$ is a product of a simply connected+ Lie group and a torus. Then one splits the simply connected Lie group into a product of simple simply connected Lie groups. Our next statement says that for p–compact groups at least the first step can always be carried out.

1.4 Theorem. *Let X be a connected p–compact group. Then there exist a simply connected p–compact group X_s, a p–compact torus T and a homomorphism $X_s \times T \to X$ which establishes a fibration*

$$BK \to BX_s \times BT \to BX .$$

Moreover, K is a finite p–compact group and $K \to X_s \times T \to X_s$ is a central monomorphism.

As mentioned earlier p–compact groups together with arithmetic square arguments will give you integral or global information. Our last statement is Theorem 1.4 in the global case.

1.5 Theorem. *Let L be a finite loop space. Then there exists a simply connected finite loop space L_s, an integral torus T and a finite covering $L_s \times T \to L$ which establishes a fibration*

$$BK \to BL_s \times BT \to BL \ .$$

Moreover, K is a finite abelian group.

These are the main results we can offer. On the way of proving these statements we have to formulate and to prove several well known results about compact Lie groups in the category of p–compact groups. Some of these are a triviality for Lie groups, but definitely not for p–compact groups (e.g see Section 2,3 and 4).

The paper is organized as follows: As already mentioned we recall the necessary basic notions and the dictionary of [D-W] in Section 2. A collection of well known results about compact Lie groups translated in terms of p–compact groups is the content of Section 3. Section 4 is devoted to the notion of the center and the proof of Proposition 1.2. In Section 5 we prove Theorems 1.3 and 1.4, and the last section contains a proof of Theorem 1.5.

Completion is always meant in the sense of Bousfield and Kan [B-K] and denoted by $U_p^\wedge$ for a space U.

We denote by $H^*_{\mathbb{Q}_p^\wedge}(\ ) := H^*(\ ;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ the cohomology with p–adic coefficients tensored over the integers with the rationals.

2. The dictionary.

In this section we recall the dictionary and some of the basic notion of [D-W]. The dictionary tells us how we have to translate notions of group theory and Lie group theory in terms of finite loop spaces or p–compact groups. This provides us also with an appropriate language to formulate our results and proofs. Most of the notions are motivated by passing from groups to classifying spaces and extracting the basic properties in similiar way as in the definitions of Section 1.

2.1 Homotopy fixed–points and proxy actions: Let G be a group acting on a space X. The homotopy fixed–point set $X^{hG} := map_G(EG, X)$ is defined to be the mapping space of G–equivariant maps from a contractible CW–complex EG with a free G–action into X. The homotopy fixed point set can also be interpreted as the space of sections in the fiber bundle $X \to X_{hG} \to BG$, where $X_{hG} := EG \times_G X$ is the homotopy orbit given by the Borel construction. The G–equivariant projection $EG \to *$ induces a map $X^G \cong map_G(*, X) \to X^{hG}$.

A homotopy equivalence $f : Y \to X$ of G–spaces, which is also G–equivariant, induces an homotopy equivalence $Y^{hG} \to X^{hG}$ between the homotopy fixed–point sets. This follows from the description as section spaces. This motivates the definition of proxy actions. A *proxy action* of G on a space X is a G–space Y together with a homotopy equivalence $Y \simeq X$. By X^{hG} we denote the homotopy fixed point set Y^{hG}.

Proxy actions very often come up in homotopy theory. For example, let G be a finite group and $F \to E \to BG$ a fibration. We can think of $E \to BG$ as the classifying map of a G–principal bundle $G \to F' \to E$. Then, we have $F' \simeq F$ and the G–action on F' realizes the proxy action of G on F.

For a fibration $F \to E \to X$ and every map $BG \to X$ the pull back diagram

$$\begin{array}{ccccc} F & \longrightarrow & F_{hG} & \longrightarrow & BG \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & E & \longrightarrow & X \end{array}$$

establishes a proxy action on F. We think of F_{hG} as the "Borel construction" of this proxy action. The homotopy fixed–point set F^{hG} is then given by the section space of the fibration $F \to F_{hG} \to BG$.

2.2 p**–compact groups :** There is an equivalent definition of a p–compact group [D-W , Lemma 2.1, Remark]: A finite loop space (X, BX, e) is a p–compact group, if X is $\mathbb{F}_p$–finite and p–complete and if $\pi_0(X)$ is a finite p–group. The rational rank of a p–compact group X is defined to be the number of exterior generators of $H^*_{\mathbb{Q}_p^\wedge}(X)$.

2.3 Isomorphisms and exact sequences: A homomorphism $Y \to X$ of p–compact groups is an *isomorphism* if $Bf : BY \to BX$ is an equivalence. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of finite loop spaces or p–compact groups is *exact* if the associated sequence $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration up to homotopy. In this case g is called an *epimorphism* and $Y \xrightarrow{f} X$ a normal subgroup.

2.4 p**–compact toral groups:** We already defined what we understand by a p–compact torus and by a finite p–compact group. A *p–compact toral group* P is a p–compact group which fits into an exact sequence $T \to P \to \pi$ of p–compact groups, where T is a p–compact torus and π a finite p–compact group.

2.5 Elements of p**–compact groups:** An *element of order* p^n of a p–compact group X is a monomorphism $\mathbb{Z}/p^n \to X$. A *p-th root* of an element $f : \mathbb{Z}/p^n \to X$ is an element $f' : \mathbb{Z}/p^{n+1} \to X$ such that for the canonical homorphism $j : \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ the composition $f'j$ is conjugate to f. By [D-W, 2.5] any nontrivial p–compact group contains an element of order p.

2.6 Conjugation and subconjugation : Let $f : Y \to X$ be a monomorphism of p–compact groups and $i : P \to X$ a p–compact toral subgroup. In Section 1 we said that P is subconjugated to Y if there exists a homomorphism $j : P \to Y$ such that fj and i are conjugated. If we think of $BY \to BX$ as being a fibration, the induced map $Bi : BP \to BX$ establishes a proxy action on X/Y. The homotopy fixed–point set $(X/Y)^{hP}$ describes the lifts $BP \to BY$ over $Bi : BP \to BX$. Let L denote the set of homotopy classes of subconjugation of P into Y; i.e. we ask for homotopy clases of lifts $BP \to BY$. There exists a fibration

$$(X/Y)^{hP} \to map(BP, BY)_L \to map(BP, BX)_{Bi} \simeq BC_X(P)$$

which establishes an exact sequences of sets

$$\pi_1(BC_X(P)) \to \pi_0((X/Y)^{hP}) \to L\ .$$

The last map is onto, the first set is a group which acts on the middle set. That is to say that L is given by the set of orbits of the action of $\pi_1(BC_X(P))$ on $\pi_0((X/Y)^{hP})$.

2.7 Discrete approximations and closures: Let $T \cong (S^1)^n$ be a classical torus and let $\breve{T} := \{t \in T : t^{p^k} = 1 \text{ for some } k\}$. Then $\breve{T}$ is a discrete group, isomorphic to $(\mathbb{Z}/p^\infty)^n$, and the natural inclusion $\breve{T} \to T$ induces an $\mathbb{F}_p$–equivalence $B\breve{T} \to BT$. This is the generic example of a discrete approximation we have in our mind. Therefore in [D-W] a *p–discrete torus of rank n* is defined to be a discrete group isomorphic to $(\mathbb{Z}/p^\infty)^n$ and a *p–discrete toral group* to be an extension of a p–discrete torus by a finite p–group.

A homomorphism $f : \breve{P} \to P$ from a p–discrete toral group into a p–compact toral group is a *discrete approximation* if $Bf : B\breve{P} \to BP$ is an $\mathbb{F}_p$–equivalence. The p–compact toral group P is called the *closure* of $\breve{P}$. Every p-compact toral group has a p–discrete approximation [D-W, 6.8], and every p–discrete toral group has a functorial closure [D-W, 6.9 and 6.10].

Suppose that P and Q are p-compact toral groups with p–discrete approximations $\breve{P} \to P$ and $\breve{Q} \to Q$. For any homomorphism $f : P \to Q$ there exists [D-W, Remark 6.11] a group homomorphism $\breve{f} : \breve{P} \to \breve{Q}$ such that the diagram

$$\begin{array}{ccc} \breve{P} & \xrightarrow{\breve{f}} & \breve{Q} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle g} \\ P & \xrightarrow[f]{} & Q \end{array}$$

commutes up to conjugacy; in this situation we call $\breve{f}$ a discrete approximation to f. Note that the free homotopy set $[BP, BQ]$ of conjugacy classes of homomorphisms from P to Q is in a natural bijective correspondence with the set $\mathrm{Rep}(\breve{P}, \breve{Q})$ of conjugacy classes of homomorphisms from $\breve{P}$ to $\breve{Q}$.

2.8 Centralizers: Let $\rho : P \to G$ be a homomorphism from a classical p–toral group P into a compact Lie group G. By results of [D-Z] and [N_1] there exists a an $\mathbb{F}_p$–equivalence $BC_G(\rho) \to map(BP, BG)_{B\rho}$. If $\pi_0(G)$ is a finite p–group, using a result of [B-N], this can be translated to an equivalence $BC_G(\rho)^\wedge_p \to map(BP, BG^\wedge_p)_{B\rho^\wedge_p}$ (see also [J-M-O]). Therefore, for a homorphism $f : Y \to X$ between p–compact groups, we define the *centralizer* $C_X(f(Y))$ to be the loop space given by the triple

$$C_X(f(Y)) := (\Omega map(BY, BX)_{Bf}, map(BY, BX)_{Bf}, id) \ .$$

The evaluation $ev : BY \times map(BY, BX)_{Bf} \to BX$ establishes a homomorphism $Y \times C_X(f(Y)) \to X$ of loop spaces. If Y is a p–compact toral group the centralizer $C_X(f(Y))$ is again a p–compact group and the evaluation $C_X(f(Y)) \to X$ is a monomorphism [D-W, 5.1, 5.2 and 6.1].

2.9 Abelian p–compact group: A p–compact group A is called *abelian* if the evaluation induces an isomorphism $C_A(id) \to A$. In particular , the adjoint of the evaluation gives a multiplication $\mu : A \times A \to A$ which also is a homomorphism. Let $A \to X$ be a homomorphism from an abelian p–compact group into a p–compact group. Taking adjoints this multiplication establishes a natural homomorphism $A \to C_X(A)$ which shows that $A \to X$ is subconjugated to $C_X(A)$. An easy argument shows that every abelian p–toral group gives rise to an abelian p-compact toral group.

2.10 Maximal tori: The *maximal torus* of a p–compact group X is a monomorphism $T_X \to X$ of a p-compact torus into X such that the centralizer $C_X(T_X)$ is a p-compact toral group and such that $C_X(T_X)/T_X$ is homotopically discrete. The motivation of this definition comes from the fact that, for a compact connected Lie group G the maximal torus is self centralizing, and that therefore the centralizer of the maximal torus of a nonconnected compact Lie group is always a p–toral group whose component of the unit is given by the maximal torus.

2.11 Theorem [D-W, 8.11, 8.13 and 9.1]. *Let X be a p–compact group.*

1. *The p–compact group X has a maximal torus $T_X \to X$.*
2. *Any subtorus $T \to X$ of X is subconjugated to the maximal torus $T_X \to X$.*
3. *Any two maximal tori of X are conjugated.*
4. *If X is connected then $T_X \to C_X(T_X)$ is an isomorphism.*

2.12 Weyl spaces and Weyl groups: Let $T_X \to X$ be a maximal torus of a p–compact group. We think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_T(X)$ is defined to be the mapping space of all fiber maps over the identity on BX. Then each component of $\mathcal{W}_T(X)$ is contractible and the Weyl group $W_T(X) := \pi_0(\mathcal{W}_T(X))$ is a finite group under composition [D-W, 9.5].

The fibration $X/T_X \to BT_X \to BX$ establishes a proxy action of T_X on the homogeneous space X/T_X via $BT_X \to BX$. Every element of the Weyl space can be interpreted as a homotopy fixed–point of this proxy action. That is to say that $\mathcal{W}_T(X) = (X/T_X)^{hT_X}$.

Because all maximal tori of X are conjugated, the Weyl space as well as the Weyl group does not depend essentially on the chosen maximal torus. If T_X is understood we denote the Weyl space by $\mathcal{W}_X$ and the Weyl group by W_X.

2.13 Theorem [D-W, 9.5 and 9.7]. *Let $T_X \to X$ be the maximal torus of a connected p–compact group X.*

1. *The rank n of T_X is equal to the rank of X.*
2. *The order of the Weyl group W_X is equal to the Euler characteristic $\chi(X/T_X)$ of the homotopy fiber of $BT_X \to BX$.*
3. *The action of W_X on BT_X induces a representation*

$$W_X \to Aut(H^*_{\mathbb{Q}_p^\wedge}(BT_X)) \cong Gl(n, \mathbb{Q}_p^\wedge)$$

   which is a monomorphism whose image is generated by pseudoreflections.

4. *The map $H^*_{\mathbb{Q}_p^\wedge}(BX) \to H^*_{\mathbb{Q}_p^\wedge}(BT_X)^{W_X}$ is an isomorphism.*

This is the natural generalization of the well known results about compact connected Lie groups. One cannot expect that the Weyl group is always generated by honest reflections as examples of Clark and Ewing show [C-E].

2.14 Normalizers and p–normalizers of maximal tori: Let $i : T_X \to X$ be a maximal torus of a p–compact group X. Again we think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_X$ acts on BT_X via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to aut(BT_X)$ where $aut(BT_X)$ denotes the monoid of all self equivalences of BT_X. Passing to classifying spaces establishes a map $B\mathcal{W}_X \to Baut(BT_X)$ which we can be thought of as being a classifying map of fibration $BT_X \to BN(T_X) \to B\mathcal{W}_X$. The total space gives the the classifying space of the normalizer $N(T_X)$ of T_X. This construction is nothing but the Borel construction.

Let $\mathcal{W}_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a p–Sylow subgroup W_p of W_X. The restriction of the above construction to $\mathcal{W}_p$ gives the classifying space of the p–normalizer $N_p(T_X)$.

Since the action of $\mathcal{W}_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$. The restriction $N_p(T_X) \to X$ is a monomorphism [D-W 9.9].

There is a slightly different way to construct the normalizer for a connected p–compact group X. The Weyl group W_X acts only up to homotopy on BT_X. But because BT_X is an Eilenberg–MacLane space we can replace this "homotopy action" by a "real" action of W_X on BT_X. Moreover, we can assume that BT_X has a fixed–point which we choose as basepoint. Then the evaluation $ev : map(BT_X, BX)_{Bi} \to BX$ is a fibration and an W_X–equivariant map where W_X acts on the mapping space via the action on the source and on BX trivially. The equivalence $BT_X \simeq map(BT_X, BX)_{Bi}$ is another realisation of the homotopy action of W_X as a real action. Obviously the evaluation extends to a map $BN(T_X) := EW_X \times_{W_X} map(BT_X, BX)_{Bi} \to BX$. Analogously, we can define the p–normalizer using the action of W_p on BT_X. For a nonconnected p–compact group one has to consider The action of W_X on the component of the unit of $C_X(T_X)$ or on the universal cover of $BC_X(T_X)$ and then to carry out this construction.

Warning: The Borel construction $EW_X \times_{W_X} BT_X$ does not give the normalizer. This always establishes a splitting fibration $BT_X \to EW_X \times_{W_X} BT_X \to BW_X$ which is not true for the normalizer in general. The point is that one first has to turn the map $BT_X \to BX$ into a fibration.

The p–normalizer fits into an exact sequence $T_X \to N_p(T_X) \to W_p$ and is therefore a p-compact toral group.

2.15 Kernels and monomorphisms: Let $f : Y \to X$ be a homomorphism of p–compact groups, let $P := N_p(T_Y)$ be the p–normalizer of some maximal torus $T_Y \to Y$ and $\check{P} \to P$ the p–discrete approximation of P which is a "real" discrete group. Every element $a \in \check{P}$ generates a cyclic subgroup $\langle a \rangle \subset \check{P}$ of finite order and induces a sequence of homomorphism $Z/p^k \to \check{P} \to P \to Y \to X$ of p–compact groups (don't mind that $\check{P}$ is not a p–compact group). Then we define the *prekernel* by $preker(f) := \{a \in \check{P} : Bf|_{B\langle a \rangle} \simeq *\}$. This definition goes back to [I] and is denoted in [D-W] as the kernel of f. The set $preker(f)$ is a normal subgroup of $\check{P}$ [N_2] or [D-W]. We define the *kernel* $ker(f)$ of f to be the closure of $preker(f)$ which is then a "normal" p–compact toral subgroup of P, i.e. there exists a p-compact toral group $\overline{P}$ and an exact sequence $ker(f) \to P \to \overline{P}$ of p-compact toral groups.

This is proved in [D-W ,7.2]. But there is only treated the case of the p–discrete approximations. Passing to closures establishes the described result. For details see also [N_2], where only the case of Y being a compact Lie group is handled, but all the arguments also apply in our situation. The definition of $ker(f)$ does not depend essentially on the chosen maximal torus and p–normalizer, because all p–normalizers are conjugated. In [D-W; 8.11] this is proved for maximal tori, but similiar arguments also apply to p–normalizers.

We say the $ker(f)$ is *trivial* if $Bker(f)$ is contractible. In Section 1 we defined f to be a monomorphism, if the homotopy fiber X/Y of Bf is $\mathbb{F}_p$–finite.

In classical group theory every homomorphism can be made into an monomorphism by dividing out the kernel. A similiar statement is true in the category of p–compact groups.

2.16 Proposition. *Let P be a p-compact toral group or a p–discrete toral group and $f : P \to X$ be a homomorphism into a p–compact group. Let $K := ker(f)$ be the kernel and $Q := P/K$ be the quotient. Then f factors over a homomorphism $\overline{f} : Q \to X$ with trivial kernel. Moreover, $map(BQ, BX)_{B\overline{f}} \to map(BP, BX)_{Bf}$ is a homotopy equivalence.*

Proof. For the case of a p–discrete toral group see [D-W , Lemma 7.5], and for the case of a p-compact toral group this follows by [N_2] or by using p–discrete approximations and taking closures. □

2.17 Theorem. *For a homomorphism $f : Y \to X$ of p–compact groups the following three conditions are equivalent:*

(1) *f is a monomorphism.*
(2) *$H^*(BY; \mathbb{F}_p)$ is a finitely generated $H^*(BX; \mathbb{F}_p)$–module.*
(3) *The kernel $ker(f)$ is trivial.*

Proof:. The equivalence of (1) and (2) is proved in [D-W , 9.11]. Let $P \to Y$ be a p–normalizer of a maximal torus of Y. By [D-W , 7.3] the restriction $f|_P$ is a monomorphism if and only if $ker(f)$ is trivial. This shows that (1) implies (3). Now let $ker(f)$ be trivial. Then, by what is already said, $H^*(BP; \mathbb{F}_p)$ is a finitely generated $H^*(BX; \mathbb{F}_p)$ module. The algebra $H^*(BX; \mathbb{F}_p)$ is noetherian [D-W ,2.3]. Therefore, the submodule $H^*(BY, \mathbb{F}_p) \subset H^*(BP; \mathbb{F}_p)$ [D-W , proof of Theorem 2.3] is also finitely generated over $H^*(BX; \mathbb{F}_p)$ which is condition (2). This completes a circle of implications. □

For a different proof of the equivalence of (2) and (3), which is not that much in the spirit of [D-W] see [N_2, Theorem 1.2]. There is only treated the case of Y being a compact Lie group. The major tool is a theorem of Quillen which says that, for a compact Lie group G, the cohomology $H^*(BG; \mathbb{F}_p)$ is detected up to nilpotent elements by elementary abelian subgroups. Because $H^*(BY; \mathbb{F}_p)$ is noetherian there is a similiar result in our case [R_3]. All the other arguments of [N_2] can be carried over to the case of p–compact groups.

In particular , Theorem 2.17 implies that the composition of two monomorphisms is always a monomorphism and that the first is a monomorphism if the composition is one.

For later purpose we will mention a slightly more general situation, where the kernel of a map $BX \to U$ of a map into a space U can be defined. A space U is called *$B\mathbb{Z}/p$–local* if the evaluation $ev : \mathrm{map}(BX, U) \to V$ is an equivalence, and *almost* $B\mathbb{Z}/p$*–local* if the evaluation induces an equivalence $map(B\mathbb{Z}/p, U)_{const} \simeq U$ between the component of the constant map and U. Then U is almost $B\mathbb{Z}/p$–local if and only if the loop space ΩU is $B\mathbb{Z}/p$–local. In [N_2] the definition of a kernel is given for maps $BG \to U$ where G is a compact Lie group and U a p–complete almost $B\mathbb{Z}/p$–local space. But all the arguments and all constructions also work for maps $BX \to U$ where X is a p–compact group and U is a p–complete almost $B\mathbb{Z}/p$–local space. In particular the kernel is a normal subgroup of $N_p(T_X)$.

2.18 Cohomological dimension : For an $\mathbb{F}_p$–finite space X, Dwyer and Wilkerson define the *mod–p cohomological dimension* $cd_{\mathbb{F}_p}(X)$ as the largest integer i such that $\widetilde{H}^i(X, \mathbb{F}_p)$ does not vanish. If the total reduced cohomology of X is zero, then $cd_{\mathbb{F}_p}(X) = -\infty$.

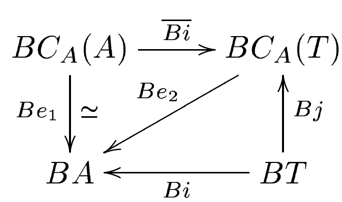
Analogously we define the *rational cohomological dimension* $cd_{\mathbb{Q}_p^\wedge}$ using the the cohomology theory $H^*_{\mathbb{Q}_p^\wedge}(\ )$. For a p–compact group X we get $cd_{\mathbb{Q}_p^\wedge}(X) = cd_{\mathbb{F}_p}(X)$ (see Lemma 3.2).

3. Lie theory for p–compact groups.

This section contains a collection of basic results to be used later. All of these results have Lie group analogues that are well-known if not blatantly obvious. We begin by investigating abelian p–compact groups and covering spaces of p–compact groups, then turn to monomorphisms into p-compact toral groups, mod p dimension, Weyl groups of nonconnected p–compact groups and finish by showing that the centralizer of a p-compact torus in a connected p–compact group is connected.

3.1 Proposition. *Any abelian p–compact group is isomorphic to a product of a p-compact torus and a finite abelian group.*

Proof. Let A be an abelian p–compact group and $i : T \to A$ a maximal torus. It suffices to show that A is a p-compact toral group for, by [D-W, Remark 8.5], A will then have the desired form. The centralizer $C_A(T)$ is a p-compact toral group by the definition of maximal torus; in fact the canonical lift (see 2.9) $j : T \to C_A(T)$ of i takes T isomorphically to the identity component of $C_A(T)$. Denoting precomposition with Bi by $\overline{Bi}$, we have a diagram



where e_1 and e_2 are evaluation homomorphisms. Both triangles in the diagram are commutative, i.e. $Be_2 \circ \overline{Bi} = Be_1$ and $Be_2 \circ Bj = Bi$. This implies that $\pi_*(Be_2)$ maps $\pi_*(BC_A(T))$ onto $\pi_*(BA)$ and that $\pi_2(Bi) : \pi_2(BT) \to \pi_2(BA)$ is an epimorphism with a right inverse. Hence $\pi_2(BA)$ is a free $\mathbb{Z}_p^\wedge$-module and A a p-compact toral group. □

The completed odd sphere $(S^{2n-1})_p^\wedge$, $n|p-1$, is homotopy commutative as an H-space but nonabelian as a p–compact group (when $n > 1$).

For later reference we record a lemma that can be extracted from Kane [K, §3-§4] (who credits Browder [B_1] with the original idea).

3.2 Lemma. *Let X be a connected H-space such that $\pi_i(X)$, $i \geq 1$, is a finitely generated $\mathbb{Z}_p^\wedge$-module and $H^*(X;\mathbb{F}_p)$ is finite. Then:*

(1) *Any connected covering space of X has the same properties.*
(2) *$H^*_{\mathbb{Q}_p}(X)$ is finite dimensional, $H^d_{\mathbb{Q}_p}(X) \cong \mathbb{Q}_p$ and $H^{>d}_{\mathbb{Q}_p}(X) = 0$ where $d = cd_{\mathbb{F}_p}(X)$.*

3.3 Corollary. *Suppose that X is a connected p–compact group, that Y is a connected space, and that $Y \to X$ is a covering map. Then Y is a p–compact group.*

Remark. In this corollary the p–adic integers are also allowed as fiber.

Proof. The given data amounts to a fibration

$$Y \to X \to BQ$$

where Q is a quotient of the finitely generated $\mathbb{Z}_p^\wedge$-module $\pi_1(X)$. The projection map in this fibration is a loop map, for $[BX, B^2Q] = [X, BQ]$, and therefore Y is a loop space. Lemma 3.2 shows that Y is in fact a p–compact group (see 2.2). $\square$

In Section 2 we explained what we mean by the p–discrete approximation $\check{f} : \check{X} \to \check{G}$ of a homomorphism $f : X \to G$ of p–compact toral groups.

3.4 Proposition. *Suppose that $f : X \to G$ is a homomorphism of p-compact toral groups and that $\check{f}$ is a discrete approximation to f. Then*

(1) *f is a monomorphism $\Leftrightarrow$ $\check{f}$ is a monomorphism*
(2) *f is an isomorphism $\Leftrightarrow$ $\check{f}$ is an isomorphism*
(3) *f is central $\Leftrightarrow$ $\check{f}$ is central*

Proof. (1) is a consequence of Theorem 2.15. The key observation is that $preker(f)$ is the usual algebraic kernel of $\check{f}$.

Statement (2) follows easily from the commutative diagram

$$\begin{array}{ccc} B\check{X} & \xrightarrow{B\check{f}} & B\check{G} \\ {\scriptstyle Bx}\downarrow & & \downarrow{\scriptstyle Bg} \\ BX & \xrightarrow[Bf]{} & BG \end{array}$$

where the vertical maps become homotopy equivalences after completion at p.

(3) Let $C_{\check{G}}(\check{X})$ denote the algebraic centralizer in $\check{G}$ of $\check{f}(\check{X})$. The homotopy fibre of $B\check{G} \to BG$ being $K(\pi_1(G) \otimes \mathbb{Q}, 1)$ implies that the homotopy fibre of

$$BC_G(X) = BC_G(\check{X}) = \mathrm{map}(B\check{X}, BG)_{B(fi)} \to \mathrm{map}(B\check{X}, B\check{G})_{B\check{f}} = BC_{\check{G}}(\check{X})$$

is $K(H^0(B\check{X}; \pi_1(G) \otimes \mathbb{Q}), 1)$; in particular, $C_{\check{G}}(\check{X})$ is a discrete approximation to $C_G(X)$. Hence $BC_G(X) \to BG$ is a homotopy equivalence if and only if $BC_{\check{G}}(\check{X}) \to B\check{G}$ is a homotopy equivalence if and only if $C_{\check{G}}(\check{X}) = \check{G}$. $\square$

3.5 Proposition. *Let X be a p–compact group, G a p-compact toral group, and $f : X \to G$ a monomorphism. Then:*

1. *(1) X is a p-compact toral group.*
2. *(2) If f is central, X is an abelian p–compact group.*
3. *(3) If G is a p-compact torus and X is connected, X is a p-compact torus.*
4. *(4) If G is a p-compact torus, so is G/X*

Proof. We first prove (3). Under the assumptions in (3), the homogeneous space G/X is connected and the fundamental group is a finitely generated $\mathbb{Z}_p^\wedge$-module. Let BY denote the universal covering space of G/X. The loop space $Y = \Omega BY$ is equivalent to a component of $\Omega\,(G/X)$ which is a covering space of the connected p–compact group X; hence Y is also a connected p–compact group by Corollary 3.3. Moreover, because BY is a covering of an $\mathbb{F}_p$-finite space, the Sullivan conjecture [M] shows that all homomorphisms $\mathbb{Z}/p \to Y$ are trivial. Thus Y is itself trivial (see 2.5). Consequently $G/X = K(\pi_1(G/X), 1)$ is, by $\mathbb{F}_p$-finiteness, a p-compact torus and so is X by the exact homotopy sequence.

(1) Let $f_0 : X_0 \to G_0$ be the restriction of f to the identity components. It suffices, by (3), to show that also f_0 is a monomorphism. But that follows from the fact that $ker(f_0) \to ker(f)$ is a monomorphism.

Now that we know X is a p-compact toral group, (2) follows from Proposition 3.4, because, with notation from that proposition, $\check{f}(\check{X}) \cong \check{X}$ is abelian if f is a central monomorphism. In this case, $\check{G}/\check{X}$ is easily seen to be a discrete approximation to the p–compact group G/X. As any quotient of a p-discrete torus is again a p-discrete torus [F, Theorem 23.1], this proves (4). □

The combination of Proposition 3.1 and Proposition 3.5 shows that if $f : X \to A$ is a monomorphism and A is an abelian p–compact group, so is X.

Specializing to the case of p–compact tori we obtain

3.6 Proposition. *Let S and T be p–compact tori and $f : S \to T$ a homomorphism.*

1. *(1) f is a monomorphism $\Leftrightarrow$ T/S is a p-compact torus $\Leftrightarrow$ $\pi_1(f)$ is a split injective homomorphism*
2. *(2) If $cd_{\mathbb{F}_p}(S) = cd_{\mathbb{F}_p}(T)$, then f is a monomorphism if and only if f is an isomorphism.*
3. *(3) If $\pi_1(f)$ is injective, then there exists a finite abelian p-group K and a factorization $f' : S/K \to T$ of f which is a monomorphism.*

Proof. (1) The proof of Proposition 3.5 shows that if $f : S \to T$ is a monomorphism then T/S is a p-compact torus; the converse is clear. The other biimplication is a direct consequence of the exact homotopy sequence.

(2) follows immediately from Proposition 3.4.

(3) Denoting by $K := ker(f)$ the kernel of f we get (2.15) a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & \nearrow_{f'} & \\ S/K & & \end{array}$$

of homomorphisms between p-compact torus groups where f' is a monomorphism. As $\pi_1(f)$ is assumed to be injective, the fundamental group functor shows that $\pi_1(K) = 0$, i.e. that K is a finite abelian p-group. □

3.7 Proposition. *Let $f : X \to Y$ be a monomorphism between two connected p–compact groups such that $H^*_{\mathbb{Q}_p}(f) : H^*_{\mathbb{Q}_p}(Y) \to H^*_{\mathbb{Q}_p}(X)$ is an isomorphism. Then f is a homotopy equivalence.*

Proof. Lemma 3.2 shows that $\mathrm{cd}_{\mathbb{F}_p}(X) = \mathrm{cd}_{\mathbb{F}_p}(Y)$ and therefore any monomorphism from X to Y is a homotopy equivalence [D-W, Proposition 6.14, Remark 6.15]. □

We are next aiming at the homotopy theoretic equivalent of the statement that connected abelian subgroups of compact connected Lie groups have connected centralizers.

In 3.8–3.11 below, X denotes a p–compact group.

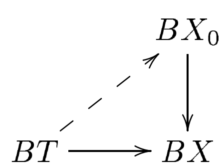
3.8 Proposition. *Suppose that $i : T \to X_0$ is a maximal torus for the connected component X_0 of X. Then $T \to X_0 \to X$ is a maximal torus for X and there exists a short exact sequence*

$$1 \to W_T(X_0) \to W_T(X) \xrightarrow{\lambda} \pi_0(X) \to 1$$

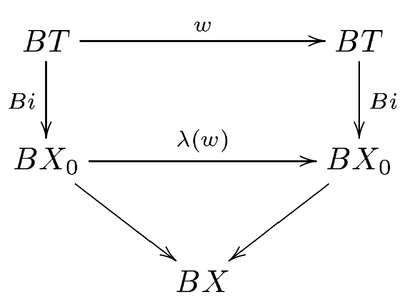
relating the Weyl groups.

Proof. Mapping BT into the universal covering map $BX_0 \to BX$ produces another covering map $BC_{X_0}(T) \to BC_X(T)$ showing that $C_X(T)$ is a p-compact toral group with $T \cong C_{X_0}(T)$ as its identity component.

Let $w : BT \to BT$ be an element in the Weyl space of $T \to X$. As both Bi and $Bi \circ w$ are lifts in the diagram



there exists by covering space theory a unique covering translation $\lambda(w) \in \pi_0(X)$ such that



commutes. Clearly, $\lambda : W_T(X) \to \pi_0(X)$ is a homomorphism with $W_T(X_0)$ as kernel. Surjectivity of λ follows from the fact that any morphism from any p-compact torus into X_0 factors through the maximal torus T (Theorem 2.11). □

So, for a p-compact toral group, the Weyl group agrees with the group of components.

In the following corollary, $N_p(T) \to X$ denotes the p-normalizer of a maximal torus $T \to X$.

3.9 Corollary. *The homomorphism $\pi_0(N_p(T)) \to \pi_0(X)$ is surjective.*

Proof. The p-normalizer is a p-compact toral group with T as its identity component and its group of components is a p-Sylow subgroup W_p of the Weyl group $W_T(X)$. When viewing $\pi_0(N_p(T)) = W_p$ as the group of covering translations of BX_0 over BX, the homomorphism $\pi_0(N_p(T)) \to \pi_0(X)$ becomes the restriction of λ to W_p. Since $\pi_0(X)$ is a finite p-group, the restriction of the epimorphism λ to W_p remains an epimorphism. $\square$

The next lemma can be viewed as a converse to [D-W, Proposition 5.5].

3.10 Lemma. *Suppose that, for any integer $n \geq 1$, any homomorphism $\mathbb{Z}/p^n \to X$ can be extended to $\mathbb{Z}/p^{n+1}$. Then X is connected.*

Proof. Assume X is not connected. Any discrete approximation $\check{N}$ to the p-normalizer $N_p(T)$ is an extension of a discrete approximation $\check{T}$ to the maximal torus of X by $\pi_0(N_p(T))$. Since $\pi_0(N_p(T))$ maps onto $\pi_0(X)$ by Corollary 3.9, $\check{N}$ contains some cyclic subgroup $\mathbb{Z}/p^n$ such that the homomorphism

$$\mathbb{Z}/p^n \hookrightarrow \check{N} \to \pi_0(N_p(T)) \to \pi_0(X)$$

is nontrivial. The corresponding homomorphism of p–compact groups

$$\mathbb{Z}/p^n \to N_p(T) \to X$$

is then nontrivial on π_0. This homomorphism can not be extended to the p-discrete torus $\mathbb{Z}/p^\infty$ for then it would factor through T (Theorem 2.11). $\square$

The proof of the final of the auxilliary results is very much in the spirit of the proofs of [D-W, Proposition 5.4, Proposition 5.5].

3.11 Proposition. *Let S be a p-compact torus and $S \to X$ a homomorphism. If X is connected, so is the centralizer $C_X(S)$ of S in X.*

Proof. Let n be an arbitrary natural number and $\mathbb{Z}/p^n \to C_X(S)$ a homomorphism. It suffices to show (Lemma 3.10) that this homomorphism extends to $\mathbb{Z}/p^{n+1}$, or, equivalently, that the adjoint $f : \mathbb{Z}/p^n \times S \to X$ extends to $\mathbb{Z}/p^{n+1} \times S$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(B(\mathbb{Z}/p^{n+1}) \times BS, BX) & \longrightarrow & \mathrm{map}(B(\mathbb{Z}/p^n) \times BS, BX) \\ \downarrow & & \downarrow \\ \mathrm{map}(B(\mathbb{Z}/p^{n+1}), BX) & \longrightarrow & \mathrm{map}(B(\mathbb{Z}/p^n), BX) \end{array}$$

of restriction fibrations. The homotopy fibre over $Bf|B(\mathbb{Z}/p^n)$ of the bottom map can [D-W, Lemma 10.6, Lemma 10.7] be identified to the homotopy fixed point set $(X^{p-1})^{h(\mathbb{Z}/p^{n+1})}$ for some proxy action of $\mathbb{Z}/p^{n+1}$ on X^{p-1}. This homotopy fibre is $\mathbb{F}_p$-finite [D-W, Theorem 4.5, Proposition 5.7] with Euler characteristic [D-W, Lemma 5.11] equal to p^r (here we use that X is connected) where r is the rational rank of X. Similarly, the homotopy fibre over Bf of the top map is homotopy equivalent to the homotopy fixed point set $((X^{p-1})^{h(\mathbb{Z}/p^{n+1})})^{h\check{S}}$ where

$\check{S}$ is a discrete approximation to S. We have just seen that $\left(X^{p-1}\right)^{h(\mathbb{Z}/p^{n+1})}$ is $\mathbb{F}_p$-finite with nonzero Euler characteristic so by [D-W, Theorem 4.7], and [D-W, Proposition 5.7] in order to handle the $\mathbb{F}_p$-completeness problem, the homotopy fibre of the top map is nonempty. □

In [D-W] the Euler characteristic of a homogenous space X/Y turns out to be a quite useful invariant in the study of p–compact groups.. In classical Lie group theory this invariant is not that much used. The next statement has a straight forward proof in classical Lie group theory using the the associated Lie algebras.

3.12 Proposition. *Let $f : Y \to X$ be a monomorphism of p–compact groups such that f induces an isomorphism $\pi_0(Y) \to \pi_0(X)$ between the components and such that the Euler characteristic $\chi(X/Y) = 1$. Then f is an isomorphism.*

The proof is based on two lemmas.

3.13 Lemma. *Let $Y \to X$ be a monomorphism of p–compact groups. If the Euler characteristic $\chi(X/Y) \not\equiv 0$ mod p then every p–compact toral subgroup $P \to X$ of X is subconjugate to Y.*

Proof. The homomorphism $P \to X$ establishes a proxy action on X/Y. If the homotopy fixed–point set X/Y^{hP} is non empty, i.e. if for example $\chi(X/Y^{hP}) \neq 0$, then P is subconjugated to Y. Let $\check{P} = \bigcup_k P_k \to P$ be a p–discrete approximation written as the union of finite p–groups. Then $\chi(X/Y^{hP_k}) \equiv \chi(X/Y) \not\equiv 0$ for every k [D-W, Theorem 4.6 and Proposition 5.7]. This implies that P_k is subconjugated to Y, and so is $\check{P}$. Passing to the closure, proves that P also is subconjugated to Y. □

3.14 Lemma. *Let $f : Y \to X$ be a monomorphism of p–compact groups. Then f induces an isomorphism $N(T_Y) \cong N(T_X)$ if and only if $\chi(X/Y) = 1$.*

Proof. By Lemma 3.13 the condition $\chi(X/Y) = 1$ implies that $N_p(T_Y) \cong N_p(T_X)$. In particular , $T := T_Y \cong T_X$, and $W_Y \to W_X$ is a monomorphism because the Weyl groups acts effictively on the maximal tori. The maps $T \to Y \to X$ defines a proxy action of T on the fibration $Y/T \to X/T \to X/Y$. In the associated fibration

$$Y/T^{hT} \to X/T^{hT} \to X/Y^{hT}$$

the first two terms are homotopically discrete. We have $Y/T^{hT} \simeq W_Y$ and $X/T^{hT} \simeq W_X$. Therefore X/Y^{hT} is also homotopically discrete and $1 = \chi(X/Y^{hT}) = |W_X/W_Y|$ This implies that $W_Y \cong W_X$ and that $N(T_Y) \cong N(T_X)$.

An isomorphism $N := N(T_Y) \cong N(T_X)$ of loop spaces establishes the diagram

$$\begin{array}{ccccc}
 & & Y/N & \longrightarrow & X/N & \longrightarrow & X/Y \\
 & & \downarrow & & \downarrow & & \\
 & & BN & \xrightarrow{=} & BN & & \\
 & & \downarrow & & \downarrow & & \\
X/Y & \longrightarrow & BY & \longrightarrow & BX\,. & &
\end{array}$$

The fibration $Y/N \to BN \to BY$ is oriented because $\pi_1(BN) \to \pi_1(BY)$ is an epimorphism. Thus, the top horizontal fibration is also oriented. Now the multiplicativity of the Euler characteristic shows that $1 = \chi(X/N) = \chi(Y/N)\chi(X/Y) = \chi(X/Y)$, which proves the second half of the statement. $\square$

Proof of 3.12. The monomorphism $f : Y \to X$ establishes a diagram of fibrations

$$\begin{array}{ccccc}
X_0/Y_0 & \xrightarrow{\simeq} & X/Y & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
BY_0 & \longrightarrow & BY & \longrightarrow & B\pi_0(Y) \\
\downarrow{\scriptstyle Bf_0} & & \downarrow{\scriptstyle Bf} & & \downarrow{\scriptstyle \simeq} \\
BX_0 & \longrightarrow & BX & \longrightarrow & B\pi_0(X)
\end{array} \quad .$$

Here, Y_0 and X_0 denote the components of the unit. The right lower vertical arrow is an equivalence by assumption. Thus, the upper left arrow is also an equivalence. Hence $Y \to X$ is an isomorphism if and only if $Y_0 \to X_0$ is an isomorphism. For the latter map the Euler characteristic condition is also satisfied. By Lemma 3.14, the monomorphism f_0 induces an isomorphism $W_{Y_0} \cong W_{X_0}$ between the Weyl groups. Moreover, Y_0 and X_0 are connected. Hence, by Theorem 2.11, $H^*_{\mathbb{Q}_p^\wedge}(BX_0) \cong H^*_{\mathbb{Q}_p^\wedge}(BY_0)$. By Proposition 3.7 this implies that $f_0 : Y_0 \to X_0$ is an isomorphism and so is $f : Y \to X$. $\square$

This finishes the collection of assorted basic facts about p–compact groups extending the astonishing similarity with compact Lie groups.

4. The center of a p–compact group.

In this section we define the center of a p–compact group and show that any central monomorphism factors through this center.

Throughout this section, X and Z denote p–compact groups. Let $i : T \to X$ be a maximal torus for X; its centralizer $C_X(T)$ is a p-compact toral group with T as its identity component.

4.1 Lemma. *Let $f : Z \to X$ be a central monomorphism.*

(1) *There exists a central monomorphism $g : Z \to C_X(T)$ such that*

$$\begin{array}{ccc}
 & & C_X(T) \\
 & {\scriptstyle g}\nearrow & \downarrow \\
Z & \xrightarrow[f]{} & X
\end{array}$$

commutes up to conjugacy; in particular, Z is abelian.

(2) *The composition $U \to Z \xrightarrow{f} X$ is a central monomorphism for any p–compact group U and any monomorphism $U \to Z$.*

Proof. (1) Choose, as in [D-W, Lemma 8.6], a homomorphism $h : Z \times X \to X$ with $f = h|(Z \times *)$ and $h|(* \times X)$ equal to the identity map on X. Let $g : Z \to C_X(T)$

and $j : T \to C_X(Z)$ be the adjoints of $Z \times T \to Z \times X \xrightarrow{h} X$. Note that g is a lift of f and j is a lift of i; in particular, both g and j are monomorphisms so Z is a p-compact toral group by Proposition 3.5. Centrality of g is now a consequence [D-W, Lemma 8.6] of the commutative diagram

$$\begin{array}{ccccc}
BZ & & \xrightarrow{\;Bg\;} & & \\
\downarrow & & & \searrow & \\
BZ \times \operatorname{map}(BT, BC_X(Z))_{Bj} & \longrightarrow & \operatorname{map}(BT, BX)_{Bi} & = & BC_X(T) \\
\uparrow & & \nearrow \simeq & & \\
\operatorname{map}(BT, BC_X(Z))_{Bj} & & & &
\end{array}$$

where the horizontal arrow takes (z, v), $z \in BZ$ and $v : BT \to BC_X(T) = \operatorname{map}(BZ, BX)_{Bf}$, to the map $BT \ni t \to v(t)(z)$. The upward slanting arrow is induced by the homotopy equivalence $BC_X(Z) \to BX$. The central subgroup Z is abelian by Propositon 3.5.

(2) Note first that also U is abelian, in particular a p-compact toral group, by Proposition 3.1 and Proposition 3.5. The commutative diagram

$$\begin{array}{ccc}
BC_X(Z) & \longrightarrow & BC_X(U) \\
 & {\scriptstyle\simeq}\searrow \quad \swarrow & \\
 & BX &
\end{array}$$

of restriction homomorphisms, shows that the right evaluation fibration admits a section. By [D-W, Lema 8.6], this implies that $U \to X$ is central. □

Let now $\check{Z} \to Z$ and $\check{C} \to C_X(T)$ be discrete approximations. For any subgroup $A \subset Z(\check{C})$, let $A \to X$ denote the homomorphism of loop spaces defined as the composite $A \hookrightarrow Z(\check{C}) \to \check{C} \to C_X(T) \to X$. As usual, if also $B < Z(\check{C})$, AB denotes the subgroup generated by A and B.

4.2 Lemma. *Suppose that A and B are subgroups of $Z(\check{C})$ such that the homomorphisms $A \to X$ and $B \to X$ are central. Then also $AB \to X$ is central.*

Proof. The abelian group structure on the p-discrete toral group $Z(\check{C})$ can be used to define an epimorphism $A \times B \to AB$. We have $C_X(A \times B) \cong C_X(AB)$ by [D-W, Lemma 7.5]. Furthermore, $C_X(A \times B) \cong C_{C_X(A)}(B) \cong C_X(B) \cong X$ by adjointness and centrality. □

4.3 Definition. The p-discrete center of X is the set

$$\check{Z}(X) := \{t \in Z(\check{C}) \mid \langle t \rangle \to X \text{ is central}\}.$$

where $\langle t \rangle$ stands for the finite cyclic subgroup of $Z(\check{C})$ generated by t. By Lemma 4.2, $\check{Z}(X)$ is actually a subgroup of $Z(\check{C})$; in particular $\check{Z}(X)$ is an abelian p-discrete toral group. The center of X, denoted $Z(X)$, is defined as the closure of $\check{Z}(X)$.

The center $Z(X)$ of X enjoys a pleasant universal property.

4.4 Theorem. *Let X be a p–compact group.*

(1) *The center $Z(X)$ is an abelian p–compact group and $Z(X) \to X$ is a central monomorphism.*
(2) *For any central monomorphism $f : Z \to X$ there exists a monomorphism $g : Z \to Z(X)$ such that*

$$\begin{array}{ccc} & & Z(X) \\ & \overset{g}{\nearrow} & \downarrow \\ Z & \xrightarrow[f]{} & X \end{array}$$

commutes up to conjugacy.

Proof. The center $Z(X)$ is abelian by its very definition and the homomorphism $Z(X) \to C_X(T) \to X$ is, as the composition of two monomorphisms, a monomorphism. To see that this homomorphism is central, choose [D-W, Proposition 6.7, Proposition 6.21] a finite subgroup $A < \check{Z}(X) < Z(\check{C})$ such that the restriction homomorphism $C_X(Z(X)) \to C_X(\check{Z}(X)) \to C_X(A)$ is an isomorphism. As A is finite and for each element $t \in A$, $\langle t \rangle \to X$ is central, a finite induction using Lemma 4.2, shows that $A \to X$ is central. Hence $C_X(Z(X)) \cong C_X(A) \cong X$, i.e. $Z(X) \to X$ is central.

Any discrete approximation $\check{g} : \check{Z} \to \check{C}$ to $g : Z \to C_X(T)$ factors through the center $Z(\check{C})$ of $\check{C}$ by Proposition 3.4. By Lemma 4.1, the homomorphism

$$\langle z \rangle \to \check{Z} \xrightarrow{\check{g}} \check{C} \to X$$

is central for any $z \in \check{Z}$, i.e. $\check{g}(z) \in \check{Z}(X)$ for all $z \in \check{Z}$. This means that $\check{g}$ factors through $\check{Z}(X)$ so g factors through $\mathrm{Cl}(\check{Z}(X)) = Z(X)$. □

Let, for example G be a p-compact toral group. Since the evaluation homomorphism $C_G(G) \to G$ is a central monomorphism [D-W, Proposition 5.1, Proposition 5.2, Theorem 6.1], it factors through the center $Z(G)$. On the other hand, as $Z(G)$ is a p-compact toral group (even abelian), the central monomorphism $Z(G) \to G$ admits a factorization through $C_G(G)$ (see 2.9). It follows (use e.g. Proposition 3.6) that the abelian p–compact groups $Z(G)$ and $C_G(G)$ are isomorphic. (It is a tempting conjecture that such an isomorphism exists for any p–compact group. In the case of compact connected Lie groups this is proved by [J-M-O].)

Let G be a connected compact Lie group with Lie group theoretic center $Z(G)$. The central monomorphism $Z(G^\wedge_p) \to G^\wedge_p$ factors [D-W, Lemma 8.6] through the centralizer $BC_{G^\wedge_p}(G^\wedge_p) = \mathrm{map}(BG^\wedge_p, BG^\wedge_p)_{B1}$ which is homotopy equivalent [J-M-O] to $BZ(G)^\wedge_p$. On the other hand $Z(G)^\wedge_p \to G^\wedge_p$ is obviously a central subgroup [D-W, Lemma 8.6]. By the universal property of the center (Theorem 4.4) it follows that $Z(G^\wedge_p) \cong Z(G)^\wedge_p$.

Let $j : Z \to X$ be a p–compact toral subgroup. The pull back diagram

(*)
$$\begin{array}{ccc} (X/Z)_{hZ} & \longrightarrow & BZ \\ \downarrow & & \downarrow{\scriptstyle Bj} \\ BZ & \xrightarrow{Bj} & BX \end{array}$$

establishes a proxy action of Z on X/Z.

4.5 Proposition. *Let $j : Z \to X$ be an abelian p–compact toral subgroup. Then Z is central if and only if the fibration $X/Z \to (X/Z)_{hZ} \to BZ$ is fiber homotopically trivial. Moreover, if this is the case, we have $(X/Z)^{hZ} \simeq X/Z$.*

Proof. The identity $id : Z \to Z$ subconjugates Z into Z. This implies that there exists a natural section $s : BZ \to (X/Z)_{hZ}$ of the fibration $(X/Z)_{hZ} \to BZ$. We can apply the functor $map(BZ, \ )$ to the pull back diagram $(*)$ which yields another pullback diagram

$$(**) \qquad \begin{array}{ccc} M & \longrightarrow & map(BZ, BZ)_{id} \\ \downarrow & & \downarrow \\ map(BZ, BZ)_{id} & \longrightarrow & map(BZ, BX)_{Bj} \end{array} .$$

The space $M \subset map(BZ, (X/Z)_{hZ})$ consists of some components and contains at least the component of s. The mapping spaces $map(BZ, BZ)_{id}$ are homotopy equivalent to BZ via the evaluation.

If $j : Z \to X$ is central we have $map(BZ, BX)_{Bj} \simeq BX$ which implies that $M \simeq (X/Z)_{hZ}$ and that $map(BZ, (X/Z)_{hZ})_s \subset M$ is one component of $(X/Z)_{hZ}$. Using this fact and taking the adjoint we can construct the middle arrow in the diagram of fibrations

$$\begin{array}{ccccc} X/Z & \longrightarrow & X/Z \times BZ & \longrightarrow & BZ \\ \| & & \downarrow & & \| \\ X/Z & \longrightarrow & (X/Z)_{hZ} & \longrightarrow & BZ \end{array} .$$

By construction the diagram commutes and gives the desired trivialization.

If the fibration $X/Z \to (X/Z)_{hZ} \to BZ$ is fiber homotopically trivial, there exists a unique section $s_c : BZ \to (X/Z)_{hZ}$ for every element $c \in \pi_0(X/Z) \cong \pi_0((X/Z)_{hZ})$, and the mapping space $\coprod_c \mathrm{map}(BZ, (X/Z)_{hZ})_{s_c}$ is equivalent to $(X/Z)_{hZ}$. Hence, in the pull back diagram $(**)$ we have $M \simeq (X/Z)_{hZ}$ This implies that $map(BZ, BX)_{Bj} \simeq BX$ and that Z is a central subgroup.

The last statement of the proposition follows from the Sullivan conjecture [M]. This finishes the proof. $\square$

The homotopy fixed point set $(X/Z)^{hZ}$ measures the different ways you can subconjugate Z into Z. Because $Z \to X$ is central, the fundamental group $\pi_1(BC_X(Z)) \cong \pi_1(BX)$ acts transitively on $\pi_0(X/Z^{hZ}) \cong \pi_0(X/Z)$. The last statement and the remarks of 2.6 say that, up to homotopy there is only one way to do it. That is that "conjugation by elements" of X acts trivially on the center.

Asume from now on that X is connected. Then the maximal torus $i : T \to X$ is self-centralizing, i.e. $T = C_X(T)$ (Theorem 2.11). Thus any central monomorphism $f : Z \to X$ will (Lemma 4.1) factor through a monomorphism $g : Z \to T$. These monomorphisms extend [D-W, Proposition 8.3] to a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{g} & T & \longrightarrow & T/Z \\ \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i/Z} \\ Z & \xrightarrow{f} & X & \longrightarrow & X/Z \end{array}$$

where the rows are exact sequences of p–compact groups.

4.6 Proposition. *Let $f : Z \to X$ be a central monomorphism into a connected p–compact group X. Then:*

1. *$i/Z : T/Z \to X/Z$ is a maximal torus for X/Z.*
2. *X/T and $\frac{X/Z}{T/Z}$ are homotopy equivalent homogeneous spaces.*
3. *$W_T(X)$ and $W_{T/Z}(X/Z)$ are isomorphic groups.*
4. *The center of X/Z is $Z(X)/Z$.*

Going to extremes, we take the central monomorphism $Z(X) \to X$ and form the p–compact group $PX = X/Z(X)$ with the maximal torus $Pi : PT = T/Z(X) \to X/Z(X) = PX$.

4.7 Corollary. *The center of PX is trivial.*

For the proof of Proposition 4.6 we need some lemmas. For a central monomorphism $Z \to X$ into a p–compact group X the composition $BT \to BX \to BX/Z$ establishes a proxy action of T on BZ. The homotopy fixed–point set for proxy action is defined in 2.1.

4.8 Lemma. *Let $Z \to X$ be a central monomorphism into a connected p–compact group X. Then*

1. *$BZ_{hT} = BZ \times BT$ and $BZ^{hT} = map(BT, BZ)$.*
2. *$(X/T)_{hZ} = X/T \times BZ$ and $(X/T)^{hZ} = X/T$.*

Proof. (1) This follows immediately from the commutative diagram

$$\begin{array}{ccccc} BZ \times BT & \longrightarrow & BT & \longrightarrow & BX \\ {\scriptstyle pr_2}\downarrow & & \downarrow & & \downarrow \\ BT & \longrightarrow & B(T/Z) & \longrightarrow & B(X/Z) \end{array}$$

where the left square is induced from the commutative square

$$\begin{array}{ccc} \check{Z} \times \check{T} & \longrightarrow & \check{T} \\ \downarrow & & \downarrow \\ \check{T} & \longrightarrow & \check{T}/\check{Z} \end{array}$$

with the top homomorphism given by group multiplication. The expression for the homotopy fixed point set now follows from the Sullivan conjecture [M].

(2) Evaluation yields a commutative diagram

$$\begin{array}{ccc} map(BZ, BT)_{Bg} \times BZ & \longrightarrow & BT \\ \downarrow & & \downarrow \\ map(BZ, BX)_{Bf} \times BZ & \longrightarrow & BX \end{array}$$

equivalent to a commutative diagram

$$\begin{array}{ccc} BT \times BZ & \longrightarrow & BT \\ \downarrow & & \downarrow \\ BX \times BZ & \longrightarrow & BX \end{array}$$

of classifying spaces. Restrict to the fibre of the fibration to the left and obtain a trivialization

$$\begin{array}{ccc} X/T \times BZ & \longrightarrow & BT \\ {\scriptstyle pr_2}\downarrow & & \downarrow \\ BZ & \xrightarrow[Bf]{} & BX \end{array}$$

of $(X/T)_{hZ}$. The expression for the homotopy fixed point set now follows from the Sullivan conjecture [M]. □

Recall from Proposition 1.6, that $\pi_1(g) : \pi_1(Z) \to \pi_1(T)$ is a (split) monomorphism. Let $\pi_1(T)^W$ denote the set of invariant elements for the action of the Weyl group $W := W_T(X)$ on $\pi_2(BT) = \pi_1(T)$.

4.9 Lemma. $\pi_1(Z) \subset \pi_1(T)^W$.

Proof. For any element w of the Weyl space, i.e. for any map $w : BT \to BT$ over BX, the two maps $Bg,\ w \circ Bg : BZ \to BT$ are both lifts of $Bf : BZ \to BX$. Since the space of lifts, $(X/T)^{hZ} = X/T$, is connected, Bg and $w \circ Bg$ are homotopic over BX. In particular, $\pi_2(Bg) = \pi_2(w \circ Bg) = \pi_2(w) \circ \pi_2(Bg)$. □

Proof of 4.6.(1)–(3). First note that T/Z is a p-compact torus by Proposition 1.5 and that the centralizer $C_{X/Z}(T/Z) \cong C_{X/Z}(T)$ is connected by Proposition 1.11. Next map BT into the fibration $BX \to B(X/Z)$ to obtain [D-W, Lemma 10.6] the fibration

$$BZ^{hT} \to \operatorname{map}(BT, BX) \to \operatorname{map}(BT, B(X/Z))$$

containing the subfibration (the base space here is 1-connected)

$$BZ \to BC_X(T) \to BC_{X/Z}(T)$$

with connected total space; here we used Lemma 4.8 to identify the fibre. A comparison of fibrations now shows that $C_{X/Z}(T) \cong C_X(T)/Z \cong T/Z$ and thus T/Z is a maximal torus for X/Z.

The commutative diagram immediately above Proposition 4.6 induces a homotopy equivalence

$$X/T \xrightarrow{\simeq} \frac{X/Z}{T/Z}$$

of homogeneous spaces and shows that

$$\begin{array}{ccc} BT & \longrightarrow & B(T/Z) \\ {\scriptstyle Bi}\downarrow & & \downarrow{\scriptstyle B(i/Z)} \\ BX & \longrightarrow & B(X/Z) \end{array}$$

is a pull back. Naturality of pull backs now determines a homomorphism $W_{T/Z}(X/Z) \to W_T(X)$ of Weyl groups. This map is injective for $W_T(X)$ acts (Lemma 4.9) on $\pi_1(T)/\pi_1(Z) = \pi_1(T/Z)$ where $W_{T/Z}(X/Z)$ is faithfully presented (Theorem 2.13). But as $|W_{T/Z}(X/Z)| = \chi\left(\frac{X/Z}{T/Z}\right) = \chi(X/T) = |W_T(X)|$ by (Theorem 2.13) it is in fact a group isomorphism. □

In the above proof of 4.6 and elsewhere in this paper, we need to restrict fibration maps to connected components of the total spaces. To that end, we make a general remark: Let

$$\coprod_{c\in\pi_0(F)} F_c \to E \to B$$

be a fibration of based spaces; the fibre F is written as the disjoint union of its connected components F_c. Let E_0 and B_0 be the base point components of E and B and let $\partial : \pi_1(B) \to \pi_0(F)$ be the boundary map in the exact homotopy sequence. Then

$$\coprod_{c\in\partial\pi_1(B)} F_c \to E_0 \to B_0$$

is again a fibration.

Yet two more lemmas, not without independent interest, however, are needed before the proof of the final assertion of Proposition 4.6.

4.10 Lemma. *Let $Z \to X$ be a central monomorphism into a connected p–compact group X. Then the induced map of (based or free) homotopy sets*

$$[BG, BZ] \to [BG, BX]$$

is injective for any p-compact toral group G.

Proof. Because BX is simply connected, and because $\pi_1(BZ)$ acts trivially on the homotopy classes of pointed maps $BG \to BZ$, it suffices to consider the case of based maps. The fibres of $[BG, BZ] \to [BG, BX]$ are the orbits of a group action

$$[BG, \Omega B(X/Z)] \times [BG, BZ] \to [BG, BZ]$$

associated to the fibration $BZ \to BX \to B(X/Z)$. But the group $[BG, \Omega B(X/Z)]$ $= [BG, X/Z]$ is trivial by the Sullivan conjecture [M]. □

4.11 Lemma. *Let $Z \to X$ a central monomorphism into a connected p–compact group X. Then the homomorphism*

$$Z \to X \to X/A$$

is also central for any central monomorphism $A \to X$.

Proof. Both monomorphisms $A \to X$ and $Z \to X$ factor through the center $Z(X)$ by Theorem 4.4. Exactly as in the proof of Lemma 4.8, consider the trivialization of BA_{hZ}

$$\begin{array}{ccccc} BZ \times BA & \xrightarrow{B\mu} & BZ(X) & \xrightarrow{Bz} & BX \\ \downarrow{\scriptstyle pr_1} & & \downarrow & & \downarrow \\ BZ & \longrightarrow & B(Z(X)/A) & \longrightarrow & B(X/A) \end{array}$$

where $z : Z(X) \to X$ is the canonical monomorphism and μ is the restriction $Z \times A \to Z(X) \times Z(X) \to Z(X)$ to $Z \times A$ of the abelian structure on $Z(X)$.

For any $\varphi \in [BZ, BA]$, define $1 + \varphi \in [BZ, BZ(X)]$ to be the composite map

$$BZ \xrightarrow{\Delta} BZ \times BZ \xrightarrow{1\times\varphi} BZ \times BA \xrightarrow{B\mu} BZ(X)$$

where Δ is the diagonal. Identifying the homotopy sets involved with the corresponding sets of homomorphisms of discrete approximations, one sees that $\varphi \to 1+\varphi$ is injective.

Using the above trivialization of BA_{hZ} to describe the fibre BA^{hZ} as $\mathrm{map}(BZ, BA)$ we obtain the fibration

$$\mathrm{map}(BZ, BA) \to \mathrm{map}(BZ, BX) \to \mathrm{map}(BZ, B(X/A))$$

by mapping BZ into the fibration defining $B(X/A)$. The homotopy sequence ends with the exact sequence

$$\pi_1(\mathrm{map}(BZ, B(X/A))) \xrightarrow{\partial} [BZ, BA] \to [BZ, BX]$$

of sets. The last map, given by $\varphi \to Bz \circ (1+\varphi)$, is an injection by the above remarks and Lemma 4.10. Thus the boundary map ∂ is constant so the above fibration contains the subfibration

$$BA \to BC_X(Z) \to BC_{X/A}(Z)$$

of connected spaces. Since $BC_X(Z) \simeq BX$ by centrality, comparison shows that $BC_{X/A}(Z) \simeq B(X/A)$, i.e. that Z is central X/A. $\square$

(The connectedness condition in Lemma 4.10 and Lemma 4.11 can be relaxed a little. In 4.10 (4.11) it suffices to require that $\pi_0(Z) \to \pi_0(X)$ ($\pi_0(Z(X)) \to \pi_0(X)$) be surjective.)

We are now ready for the proof of the final statement of Proposition 4.6.

Proof of 4.6.(4). The p-discrete center of X/Z is a subgroup of the discrete approximation $\check{T}/\check{Z}$ to T/Z. Suppose $t \in \check{T}$ is such that $t\check{Z} \in \check{Z}(X/Z)$ meaning that the homomorphism $\langle t\check{Z}\rangle \to T/Z \to X/Z$ is central. So is then $\langle t\rangle \to X \to X/Z$ by [D-W, Lemma 7.5]. Mapping $B(\langle t\rangle)$ into the fibration $BX \to B(X/Z)$ yields the fibration

$$BZ^{h\langle t\rangle} \to \mathrm{map}(B(\langle t\rangle), BX) \to \mathrm{map}(B(\langle t\rangle), B(X/Z)).$$

This time the base space component $BC_{X/Z}(\langle t\rangle)$ is simply connected being homotopy equivalent to $B(X/Z)$ by centrality. Thus this fibration contains the subfibration

$$BZ \to BC_X(\langle t\rangle) \to B(X/Z)$$

showing, by comparison, that $BC_X(\langle t\rangle) \simeq BX$ or, equivalently, that t is in $\check{Z}(X)$. Hence $\check{Z}(X/Z) \subset \check{Z}(X)/\check{Z}$.

Conversely, if $t \in \check{Z}(X)$, then $\langle t\rangle \to T \to X$ is central. So is then $\langle t\rangle \to X \to X/Z$ by Lemma 4.11 and $\langle t\check{Z}\rangle \to T/Z \to X/Z$ by [D-W, Lemma 7.5]. Thus $\check{Z}(X)/\check{Z} \subset \check{Z}(X/Z)$. $\square$

Next we will generalize another basic property of compact Lie groups, namely that any finite normal subgroup of a compact connected Lie group is central.

4.12 Proposition. *Let $K \xrightarrow{\iota} X \to Y$ be an exact sequence of p–compact groups, where K is finite and X is connected, i.e. $\iota : K \to X$ is a normal subgroup. Then, $\iota : K \to X$ is a central.*

Proof. We apply the functor $map(BK, \ )$ to the fibration $BK \to BX \to BY$. This yields a diagram of fibrations

$$\begin{array}{ccccc}
F & \longrightarrow & M & \xrightarrow{ev} & BK \\
\| & & \downarrow & & \downarrow \\
F & \longrightarrow & map(BK, BX)_{B\iota} & \xrightarrow{ev} & BX \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & map(BK, BY)_{const} & \xrightarrow[\simeq]{ev} & BY
\end{array} \quad .$$

The space M consists of some, but a finite number, components of $map(BK, BK)$ and contains at least the component of the identity. Because K is a finite group, the fiber F of the top row is homotopically discrete given by a disjoint union of a finite number of qoutients of K. The homogenous space $X/C_X(\iota(K)) \simeq F$ is connected because X is connected. Therefore $X/C_X(\iota(K)) \simeq *$ which implies that $K \to X$ is central. $\square$

Finally we will look at the center from two more points of view, which also could be used for the definition.

The first views the center as a maximal central subgroup. We say the center $Z(X)$ of X is a central subgroup which satisfies the condition of Theorem 4.4 (2). For the construction of the center one considers the set of all conjugacy classes of central subgroups. This is partially ordered by the relation given by subconjugation. All central subgroups are subconjugated to $C_X(T)$ which is a p-compact toral group. We can choose a maximal element $Z(X)$ in the poset of all central subgroups. If there exists a central subgroup Z of X which is not subconjugated to $Z(X)$, we can construct a proper extension of $Z(X)$ which is also central by using the arguments of the proof of Lemma 4.2. This contradicts the maximality of $Z(X)$. The universal property of the center ensures that both definitions give the same.

The second views the center as the kernel of an adjoint representation. For a compact Lie group G the center $Z(G)$ is also given as the kernel of the adjoint representation $G \to Aut(G)$ of G into the automorphisms of G given by conjugation. Similiarly one can proceed for p–compact groups.

The free loop space fibration

$$X \simeq \Omega BX \to \Lambda BX := \mathrm{map}(S^1, BX) \to BX$$

of a p–compact group X has a classifying map $\lambda : BX \to Baut(X)$ which is called the classifying map of the adjoint representaion of X. For a compact Lie group this construction gives the induced map $BG \to BAut(G) \to Baut(G)$.

To speak about kernels it is necessary that the target $Baut(X)$ of the map λ is p–complete and is almost $B\mathbb{Z}/p$–local. The latter condition means that the evaluation induces an equivalence $\mathrm{map}(B\mathbb{Z}/p, Baut(X))_{const} \simeq Baut(X)$ (see Section 2).

4.13 Proposition. *Let Y be a p–complete $\mathbb{F}_p$–finite space. Then the following holds:*

(1) *$Baut(Y)$ is almost $B\mathbb{Z}/p$–local.*
(2) *If Y is a loop space in addition, then $Baut(Y)$ is also p–complete.*

Proof. For (1) it is sufficient to show that $aut(Y) \simeq \Omega Baut(Y)$ is $B\mathbb{Z}/p$–local. Taking adjoints we get

$$\begin{aligned} \mathrm{map}(B\mathbb{Z}/p, aut(Y)) &\simeq \mathrm{map}(B\mathbb{Z}/p \times Y, Y)_F \\ &\simeq \mathrm{map}(Y, \mathrm{map}(B\mathbb{Z}/p, Y))_{\overline{F}} \\ &\simeq aut(Y,Y)\ . \end{aligned}$$

Here F denotes the set of homotopy classes of maps $f : B\mathbb{Z}/p \times Y \to Y$ such that $f|_Y$ is a homotopy equivalence, and $\overline{F}$ is the set of homotopy classes of the adjoints of F. The last equivalence is a consequence of the Sullivan conjecture [M].

Condition (2) follows from a combination of [B-K; VI 5.4, 7.1, 7.2]. Roughly speaking this says that each component of $aut(Y)$ is p–complete because Y is p–complete and that $\pi_0(aut(Y))$ behaves nicely, i.e. that $\pi_0(aut(Y))$ is $Ext-p-complete$, because Y is a loop space in addition. In this situation completion commutes with passing to the classifying space and therefore BY is also p–complete. □

Because a p–compact group enjoys the properties of the last proposition we can speak about the kernel $K := ker(\lambda) \to N_p(T_X) \to X$ of the map λ.

4.14 Proposition. *The subgroup $j : K \to X$ is the center of X.*

Proof. We have to show two things, namely that K is a central subgroup of X and that every central subgroup of X is subconjugated to K. The universal property of the center, stated in Theorem 4.4, then proves the statement.

Let $Z \to X$ be a central subgroup. The product map $BZ \times BX \to BX$ establishes a map $BZ \times \Omega BX \to \Lambda BX$ which fits into a pull back diagram of fibrations

$$\begin{array}{ccc} BZ \times \Omega BX & \longrightarrow & BZ \\ \downarrow & & \downarrow \\ \Lambda BX & \longrightarrow & BX \end{array} .$$

The upper row is the trivial fibration and shows that the composition $BZ \to BX \to Baut(X)$ is null homotopic. The central subgroup $Z \to X$ is subconjugated to $N_p(T_X)$ and therefore also subconjugated to K.

As a subgroup of $N_p(T_X)$ the kernel K is a p-compact toral group. The proxy action of K on X established by the pull back diagram

$$\begin{array}{ccc} BK \times X & \longrightarrow & BK \\ \downarrow & & \downarrow{\scriptstyle Bj} \\ \Lambda BX & \longrightarrow & BX \end{array}$$

is trivial. Hence, we have $X^{hK} \simeq X$. Taking adjoints establishes the equivalences

$$\begin{aligned} \Omega\mathrm{map}(BK, BX)_{Bj} &\simeq \mathrm{map}(BK \times S^1, BX)_{f|_{BK\times *}=Bj} \\ &\simeq \mathrm{map}(BK, \mathrm{map}(S^1, BX))_{f\circ ev=Bj} \\ &\simeq \Gamma(BK \times X \to BK) \\ &\simeq X^{hK}\ . \end{aligned}$$

Here $\Gamma(\ )$ denotes the section space of the bundle. This shows that the evaluation $ev : \mathrm{map}(BK, BX)_{Bj} \to BX$ is a homotopy equivalence and that therefore $K \to X$ is central. $\square$

5. The finite covering.

Throughout this section, X denotes a connected p–compact group with maximal torus $i : T \to X$, Weyl group $W := W_T(X)$ and center $Z(X)$.

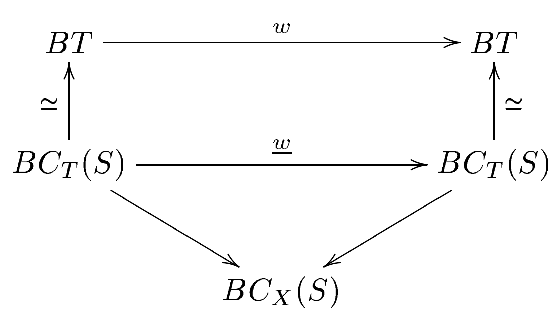
Our first goal is to to obtain a description, rationally at least, of the subgroup $\pi_1(Z(X))$ of $\pi_1(T)$. It was shown in Lemma 4.9 that the fundamental group of the center is contained in the W-invariant subgroup of the fundamental group of T.

5.1 Proposition. *The index of $\pi_1(Z(X))$ in $\pi_1(T)^W$ is finite.*

Proof. We show that $\dim_{\mathbb{Q}_p}(\pi_1(Z(X)) \otimes \mathbb{Q}) \geq \dim_{\mathbb{Q}_p}(\pi_1(T)^W \otimes \mathbb{Q})$.

Let S be a p-compact torus with mod p-dimension equal to the rank of the free finitely generated $\mathbb{Z}_p^\wedge$-module $\pi_1(T)^W$. There exists, since $[BS, BT] = \mathrm{Hom}(\pi_1(S), \pi_1(T))$, a homomorphism $e : S \to T$ such that the image of the induced monomorphism $\pi_1(e)$ is $\pi_1(T)^W$.

Composition with $i : T \to X$ produces a homomorphism $\underline{i} : C_T(S) \to C_X(S)$ of centralizers. An adjointness argument, bearing in mind that $C_X(T) \cong T$, shows that $C_T(S) \cong T$ (Theorem 2.11) is a maximal torus for the connected (Proposition 3.10) p–compact group $C_X(S)$. Consider the homomorphism $W_T(C_X(S)) \to W_T(X)$ of Weyl groups determined by the diagram $T \to C_X(S) \to X$. Both Weyl groups are faithfully presented in $\pi_1(T)$ (Theorem 2.13), so this homomorphism is injective. It is also surjective. To see this, note that because $\pi_1(S) = \pi_1(T)^W$ is invariant under W, $w \circ e \simeq w$ for any fibre self-map w of BT over BX; in other words, the mapping space component $BC_T(S) = \mathrm{map}(BS, BT)_{Be}$ is mapped to itself under post-composition with w. Hence we obtain a commutative diagram



showing that w is in the Weyl group $W_{C_T(S)}(C_X(S))$. Theorem 2.13 now implies that the monomorphism (see 2.8) $C_X(S) \to X$ induces an isomorphism $H^*_{\mathbb{Q}_p}(X) \xrightarrow{\cong} H^*_{\mathbb{Q}_p}(C_X(S))$ and therefore (Proposition 3.7) $C_X(S) \to X$ is an isomorphism. This means that $ie : S \to X$ is central.

As shown in Proposition 3.6, $e : S \to T$ factors through a monomorphism $e' : S/K \to T$ for some finite abelian p-group K. The composition of e' with $i : T \to X$ remains central [D-W, Lemma 7.5] and is, as the composition of monomorphisms, a monomorphism. Thus, by the universal property of the center (Theorem 4.4), there exists a monomorphism $S/K \to Z(X)$ and hence (Proposition 3.6) $\dim_{\mathbb{Q}_p}(\pi_1(Z(X)) \otimes \mathbb{Q}) \geq \dim_{\mathbb{Q}_p}(\pi_1(S/K) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}_p}(\pi_1(S) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}_p}(\pi_1(T)^W \otimes \mathbb{Q})$ as required. $\square$

5.2 Corollary. *The monomorphism $Z(X) \to X$ induces an isomorphism $\pi_1(Z(X)) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_1(X) \otimes \mathbb{Q}$ of vector spaces over $\mathbb{Q}_p$.*

Proof. We have

$$\begin{aligned}\pi_2(BZ(X)) \otimes \mathbb{Q} &\cong (\pi_2(BT) \otimes \mathbb{Q})^{W_T(X)} \cong H^2_{\mathbb{Q}_p}(BT)^{W_T(X)} \cong H^2_{\mathbb{Q}_p}(BX) \\ &\cong \pi_2(BX) \otimes \mathbb{Q}\end{aligned}$$

by Proposition 5.1 and Theorem 2.13. □

The following theorem is an immediate consequence of Corollary 5.2.

5.3 Theorem. *The center of X is isomorphic to a finite abelian p-group if and only if the fundamental group $\pi_1(X)$ is finite.*

Recall from Lemma 3.3 that connected covering spaces of connected p–compact groups are p–compact groups. Thus, in particular, the universal covering space $X\langle 1\rangle$ of X is a p–compact group.

Choose [D-W, Lemma 8.6] a homomorphism $X \times Z(X) \to X$ extending the identity map on X and the central monomorphism $Z(X) \to X$. The composite homomorphism

$$\pi : X\langle 1\rangle \times Z(X)_0 \to X \times Z(X)_0 \to X \times Z(X) \to X$$

is investigated more carefully in the following main result.

5.4 Theorem. *For any connected p–compact group X, there exists a short exact sequence of the form*

$$K \xrightarrow{\iota} X\langle 1\rangle \times Z(X)_0 \xrightarrow{\pi} X$$

where K is a finite abelian p-group and $K \xrightarrow{\iota} X\langle 1\rangle \times Z(X)_0 \xrightarrow{\mathrm{pr}_1} X\langle 1\rangle$ is a central monomorphism.

Proof. The exact homotopy sequence for π together with Corollary 5.2 immediately show the existence of the short exact sequence and also that K is a finite abelian p-group. Proposition 4.12 and Lemma 4.11 show that $\iota : K \to X\langle 1\rangle \times Z(X)_0$ and $\mathrm{pr}_1 \circ \iota : K \to X\langle 1\rangle$ are central homomorphisms.

The commutative diagram

$$\begin{array}{ccccc}
X\langle 1\rangle/K & \longrightarrow & B(Z(X)_0) & \longrightarrow & BX \\
\downarrow & & \downarrow & & \| \\
BK & \xrightarrow{B\iota} & BX\langle 1\rangle \times B(Z(X)_0) & \xrightarrow{B\pi} & BX \\
{\scriptstyle pr_1 \circ B\iota}\downarrow & & {\scriptstyle pr_1}\downarrow & & \downarrow \\
BX\langle 1\rangle & = & BX\langle 1\rangle & \longrightarrow & *
\end{array}$$

of interlocking fibrations shows that $X\langle 1\rangle/K$ is homotopy equivalent to $X/Z(X)_0$, in particular $\mathbb{F}_p$-finite. Thus $\mathrm{pr}_1 \circ \iota : K \to X\langle 1\rangle$ is a monomorphism. □

5.5 Corollary. *Let $i_1 : S \to X\langle 1\rangle$ be a maximal torus for the universal covering p–compact group $X\langle 1\rangle$. Then:*

(1) $(i_1 \times 1_{Z(X)_0})/K : (S \times Z(X)_0)/K \to (X\langle 1\rangle \times Z(X)_0)/K$ *is a maximal torus for* $X = (X\langle 1\rangle \times Z(X)_0)/K$.

(2) $Z(X) = (Z(X\langle 1\rangle) \times Z(X)_0)/K$.

Proof. (1) By [D-W, Lemma 7.5] and since any homomorphism into the abelian p–compact group $Z(X)_0$ is central,

$$\begin{aligned} C_{X\langle 1\rangle \times Z(X)_0}(S \times Z(X)_0) &= C_{X\langle 1\rangle}(S \times Z(X)_0) \times C_{Z(X)_0}(S \times Z(X)_0) \\ &= C_{X\langle 1\rangle}(S) \times Z(X)_0 \end{aligned}$$

is a p-compact toral group with $S \times Z(X)_0$ as its identity component. Thus $S \times Z(X)_0$ is a maximal torus for $X\langle 1\rangle$. (More generally, the maximal torus of a product is the product of the maximal tori.) Now point (1) follows from Proposition 4.6.

(2) Let $\check{S} \times \check{Z}_0 \to S \times Z(X)_0$ be a discrete approximation. For any pair $(s,z) \in \check{S} \times \check{Z}_0$,

$$C_{\langle (s,z)\rangle}(X\langle 1\rangle \times Z(X)_0) = C_{\langle s\rangle}(X\langle 1\rangle) \times Z(X)_0$$

by a computation similar to the one above. Thus $\check{Z}(X\langle 1\rangle \times Z(X)_0) = \check{Z}(X\langle 1\rangle) \times \check{Z}_0$ and point (2) follows from Proposition 4.6. □

A fundamental theorem of Browder [B$_2$] says, when translated into the present context, that the first nonzero homotopy group of a connected p–compact group occurs in an odd dimension. So for example, $\pi_2(X) = \pi_2(X\langle 1\rangle) = 0$ always.

5.6 Corollary. *Let X be a connected p–compact group with maximal torus $i : T \to X$ and Weyl group W. Then*

(1) *The homomorphism $\pi_1(i) : \pi_1(T) \to \pi_1(X)$ is surjective and the rank of the kernel equals the rational rank of the universal covering p–compact group $X\langle 1\rangle$ of X.*

(2) *X/T is simply connected and $\pi_2(X/T)$ is a free finitely generated $\mathbb{Z}_p^\wedge$-module.*

(3) *X and $X\langle 1\rangle$ have isomorphic Weyl groups.*

(4) $H^*_{\mathbb{Q}_p}(BX) \cong H^*_{\mathbb{Q}_p}(BX\langle 1\rangle)^W \otimes H^*_{\mathbb{Q}_p}(B(Z(X)_0))$

Proof. By Corollary 5.5, the maximal torus of X has the form $T = (S \times Z(X)_0)/K$ where $S \to X\langle 1\rangle$ is a maximal torus for $X\langle 1\rangle$. Proposition 4.6 tells that

$$X/T \simeq \frac{X\langle 1\rangle \times Z(X)_0}{S \times Z(X)_0} \simeq X\langle 1\rangle/S$$

which is simply connected and has second homotopy group isomorphic to $\pi_1(S)$ since $\pi_2(X\langle 1\rangle) = 0$ by Browder's theorem. This proves (1), and (2) is just a reformulation of (1) using the exact homotopy sequence.

In order to prove (3), note that

$$\left(\frac{X\langle 1\rangle \times Z(X)_0}{S \times Z(X)_0}\right)^{h(S\times Z(X)_0)} \simeq \left((X\langle 1\rangle/S)^{hS}\right)^{hZ(X)_0} \simeq (X\langle 1\rangle/S)^{hS}$$

where the last homotopy equivalence comes from the fact that the action of the divisible abelian group $Z(X)_0$ on the homotopically discrete Weyl space of $X\langle 1\rangle$ must be essentially trivial; cfr. [D-W, Proposition 8.10]. Taking groups of components, we obtain the first of the isomorphisms

$$W_S(X\langle 1\rangle) \cong W_{S\times Z(X)_0}(X\langle 1\rangle \times Z(X)_0) \cong W_T(X)$$

while the second one follows from Proposition 4.6.

The final assertion follows by expressing $H^*_{\mathbb{Q}_p}(BX)$ as a ring of invariants (Theorem 2.13). □

In classical Lie group theory, the order of the center of a simply connected Lie group divides the order of the Weyl group. In particular , at large primes, every compact connected Lie group splits into a product of a simply connected one and a torus. The same statement also is true for p–compact groups.

5.7 Theorem. *Let X be a connected p–compact group. If $(p, |W_X|) = 1$, then $X \cong X\langle 1\rangle \times Z(X)_0$ is isomorphic to the product of the universal cover $X\langle 1\rangle$ and the connected component $Z(X)_0$ of the center $Z(X)$ of X.*

Proof. Because p is coprime to the order of W_X, we have an isomorphism $H^*(BX;\mathbb{Z}_p^\wedge) \cong H^*(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$ [D-M-W]. In particular , $H^*(BX;\mathbb{Z}_p^\wedge)$ is torsion-free. Moreover, the W_X–module $H^2(BT_X,\mathbb{Z}_p^\wedge) \cong M_1 \oplus M_2$ splits into a direct sum where M_1 is a fixed–point free W_X–module and $M_2 \cong H^2(BT_X,\mathbb{Z}_p^\wedge)^{W_X}$ is given by the fixed–points. Classifying spaces of p–compact tori are Eilenberg–MacLane spaces. Therefore we can realize the summands by maps $Bj_i : BT_i \to BT_X$, $i = 1, 2$. Both tori, T_1 and T_2, inherit an W_X–action and the maps can be realized by equivariant maps.

Next we want to show that $T_2 \to T_X \to X$ is central. The centralizer $C_X(T_2)$ is connected and of maximal rank (Proposition 3.11). By construction $W_{C_X(T_2)} = W_X$. Hence, by Theorem 2.13, the map $BC_X(T_2) \to BX$ is rationally an equivalenc, and by Proposition 3.7 a homotopy equivalence. This shows that $T_2 \to X$ is central. In particular , we have $T_2 \cong Z(X)_0$.

Let $det : X \to T$ be the generalized determinant. That is that T is a p-compact torus of the same rank as the free $\mathbb{Z}_p^\wedge$–module $H^2(X;\mathbb{Z}_p^\wedge)$ and that $Bdet$ is given by an isomorphism $H^2(X;\mathbb{Z}_p^\wedge) \cong H^2(BT;\mathbb{Z}_p^\wedge)$. By the above remarks the fiber of $Bdet$ is given by the universal cover $X\langle 1\rangle$. The composition $BZ(X)_0 \to BX \xrightarrow{Bdet} BT$ is a homotopy equivalence, because $H^2(BX;\mathbb{Z}_p^\wedge) \cong H^2(BT_X;\mathbb{Z}_p^\wedge)^{W_X}$. We identify $Z(X)_0$ and T via this equivalence. Hence $Bdet$ has a left inverse given by the central map $BZ(X)_0 \to BX$. The adjoint of $BX\langle 1\rangle \to BX \simeq \mathrm{map}(BZ(X)_0, BX)_{Bj_2}$ establishes an equivalence of fibrations

$$\begin{array}{ccccc} BX\langle 1\rangle & \longrightarrow & BX\langle 1\rangle \times BZ(X)_0 & \longrightarrow & BZ(X)_0 \\ \| & & \downarrow \simeq & & \downarrow \simeq \\ BX\langle 1\rangle & \longrightarrow & BX & \xrightarrow{Bdet} & BT \end{array} .$$

This finishes the proof. □

The proof of Theorem 5.7 obviously has the following corollary:

5.8 Corollary. *Let X be a connected p–compact group. If p does not divide the order of the Weyl group, then the center $Z(X)$ is connected. In particular , if X is semisimple, then $Z(X)$ is trivial.*

6. Finite coverings of connected finite loop spaces.

In this section we will prove Theorem 1.5 which says that every connected finite loop space has a finite covering which splits into a product of a simply connected finite loop space and a torus. To do this we use the results of the last section which give us a splitting at each prime. An arithmetic square argument will complete the proof.

Let $L = (L, BL, e)$ be a connected finite loop space. Then completion at a prime p gives a p–compact group $L_p^\wedge = (L_p^\wedge, BL_p^\wedge, e_p^\wedge)$. The rational cohomology $H^*(BL;\mathbb{Q}) \cong \mathbb{Q}[x_1, ..., x_n]$ is a polynomial algebra of generators x_i of even degree $2r_i$. We define $d(X) := \prod_i r_i$.

The 2-dimensional cohomology $H^2(BL;\mathbb{Z})$ is torsionfree of rank s, because BL is simply connected. Let T be a torus of the same rank and let $Bdet : BL \to BT$ be the generalized determinant established by a chosen isomorphism $H^2(BL;\mathbb{Z}) \cong H^2(BT;\mathbb{Z})$.

6.1 Proposition. *There exists an unstable Adams operation $\psi^k : BT \to BT$ and a map $BT \to BL$ such that the diagram*

$$\begin{array}{ccc} & \nearrow & BL \\ & & \downarrow{\scriptstyle Bdet} \\ BT & \xrightarrow{\psi^k} & BT \end{array}$$

commutes up to homotopy. Moreover, for every prime p, completion establishes a central subgroup $T_p^\wedge \to L_p^\wedge$

Proof. Let p be a prime. Passing to completion and by Theorem 5.4 we get a commutative diagram

$$\begin{array}{ccccc} BZ(L_p^\wedge)_0 & \longrightarrow & BL_p^\wedge\langle 1\rangle \times BZ(L_p^\wedge)_0 & \longrightarrow & BL \\ \uparrow & & & & \downarrow{\scriptstyle Bdet} \\ BT_p^\wedge & & \xrightarrow{\psi^{k_p}} & & BT_p^\wedge \end{array} .$$

The composition $Bg : BZ(L_p^\wedge)_0 \to BT_p^\wedge$ of the upper row and $Bdet$ is rationally an equivalence because $BL_p^\wedge\langle 1\rangle$ is 3–connected. In particular , the fiber is given by the kernel of g which is a finite abelian p–group K_p. Hence, there exists only one obstruction for a left inverse of Bg contained in $H^2(BT_p^\wedge;\pi_1(K_p)) \cong H^2(BT_p^\wedge;K_p)$. Let $k_p := |K_p|$. By an Adams map $\psi^{k_p} : BT_p^\wedge \to BT_p^\wedge$ this obstruction is mapped to zero which proves the existence of the left vertical arrow.

By Theorem 2.13 and by a theorem of Chevalley [C] the order of the Weyl group $W_{L_p^\wedge}$ is equal to $d(L)$ for every prime. If p is coprime to $d(X)$, then $BL_p^\wedge$ is equivalent to the product of $BL_p^\wedge\langle 1\rangle \times BZ(L_p^\wedge)_0$, and the left vertical arrow exists with $k_p = 1$. That is to say that only for a finite number of primes k_p is unequal to 1. The product $k := \prod_p k_p$ is a finite number. Unstable Adams operations of any degree can be realized as self maps of BT and commute up to homotopy. This establishes a commutative diagram

$$\begin{array}{ccc} & & BL^\wedge \\ & \nearrow & \downarrow{\scriptstyle Bdet} \\ BT^\wedge & \xrightarrow{\psi^k} & BT^\wedge \end{array} \quad .$$

Here, $BL^\wedge := \prod_p BL_p^\wedge$ denotes the product of all p–adic completions. The map ψ^k can be realized as aself map of BT.

Rationally BL is a product of rational Eilenberg-MacLane spaces. The map $Bdet^* : H^*(BT;\mathbb{Q}) \to H^*(BL;\mathbb{Q})$ is an isomorphism on the 2-dimensional generators of the polynomial ring $H^*(BL;\mathbb{Q})$. Rationally, the map $\psi^k : BT \to BT$ is a homotopy equivalence with inverse $\psi^{1/k}$. Therefore, there exists a right inverse $\phi : BT_{\mathbb{Q}} \to BL_{\mathbb{Q}}$ of $\psi^{1/k}Bdet^*$.This establishes a diagram commutative up to homotopy

$$\begin{array}{ccc} & & BL_{\mathbb{Q}} \\ & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle Bdet} \\ BT_{\mathbb{Q}} & \xrightarrow{\psi^k} & BT_{\mathbb{Q}} \end{array} \quad .$$

Rationally all spaces are products of Eilenberg-MacLane spaces and all maps are classified up to homotopy by cohomolgy. Thus, the coherence conditions for using the arithmetic square are satisfied by construction. This establishes the desired diagram of the statement. The centrality of the lift $BT \to BL$ is already proved. □

The universal cover $L\langle 1\rangle \to L$ of a connected finite loop space L is also a finite loop space, and passes to map $BL\langle 1\rangle \to BL$ between the classifying spaces. This follows analogously as in Corollary 3.3. Actually, the proof of Proposition 3.2 and Corollary 3.3 is the p–adic version of an integral argument.

Proof of Theorem 1.5. Let $L\langle 1\rangle \to L$ be the universal cover of the finite loop space L. By Proposition 6.1 and the proof we can choose an Adams operation $\psi^k : BT \to BT$ and a central lift $Bg : BT^\wedge \to BL^\wedge$. The adjoint of $BL\langle 1\rangle^\wedge \to BL^\wedge \simeq \mathrm{map}(BT^\wedge, BL^\wedge)_{Bg}$ establishes a diagram commutative up to homotopy

$$\begin{array}{ccc} BL\langle 1\rangle^\wedge \times BT^\wedge & \longrightarrow & BL^\wedge \\ \downarrow & & \downarrow{\scriptstyle Bdet} \\ BT^\wedge & \xrightarrow{\psi^k} & BT^\wedge \end{array} \quad .$$

The left vertical map is the projection on the second factor.

Rationally the map $BL\langle 1\rangle_{\mathbb{Q}} \to BL_{\mathbb{Q}}$ extends to a map $BL\langle 1\rangle_{\mathbb{Q}} \times BT_{\mathbb{Q}} \to BL_{\mathbb{Q}}$ where the restriction on the second factor is given by the left inverse of $\psi^{1/k}Bdet$. This establishes the analogous diagram for the rationalisations of the spaces. Again, the coherence conditions for glueing together are satisfied because over the adeles the homotopy classes of the maps are controlled by cohomology. This proves the statement. □

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