ON THE FUNCTOR 'CLASSIFYING SPACE' FOR COMPACT LIE GROUPS

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ABSTRACT. Two compact connected Lie groups G and H are isomorphic as Lie groups if and only if the associated classifying spaces BG and BH are homotopy equivalent. The proof is based on the following fact: The functor 'classifying space' induces a map from group extension of the form $G \longrightarrow H \longrightarrow \Gamma$ to fibrations of the form $BG \longrightarrow X \longrightarrow B\Gamma$ which turns out to be a bijection for Γ a finite and G a compact connected Lie group.

1. Introduction.

In [16], Rector introduced the idea of studying compact connected Lie groups from the homotopy point of view, i.e. in terms of classifying spaces. As a Lie group, G has a very rich structure. How much of this structure is already contained in the classifying space BG ? This question has a long history and was again recently raised by G.Mislin at the Göttingen conference on classifying spaces in August 91 [19].

For G a finite group, or more generally for a discrete group, we have $G \cong \pi_1(BG)$. Two compact connected simple Lie groups G and H are isomorphic as Lie groups if and only if G and H are homotopy equivalent if and only if BG and BH are homotopy equivalent. The first equivalence is proved in [2], the second is a consequence of the first, because $\Omega BG \simeq G$. For semi simple Lie groups the first equivalence is not true. Counterexamples may be found in [2]. But it turns out that the first and the last condition are always equivalent.

Theorem 1.1. *Two compact Lie groups G and H are isomorphic as Lie groups if and only if BG and BH are homotopy equivalent.*

Results of this type were proved by Scheerer in the case of simply connected Lie groups [17], and recently but indepently by Osse as well as by Møller in the case of connected Lie groups [15, Theorem 2] [11, Proposition 3.3].

The proof of Theorem 1.1 is based on a comparison between group extensions of the form heorem 1.1$G \to H \to \Gamma$ and fibrations of the form $BG \to BH \to B\Gamma$, where G, H, and Γ are compact Lie groups, G connected and Γ finite.

For two CW–complexes F and B, we denote by $Fib(B, F)$ the set of equivalence classes of fibrations

$$F \to E \to B \ ,$$

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where the equivalence relation is given by fiber homotopy equivalences of the form

$$\begin{array}{ccccc} F & \longrightarrow & E_1 & \longrightarrow & B \\ \simeq\downarrow & & \simeq\downarrow & & \| \\ F & \longrightarrow & E_2 & \longrightarrow & B \end{array},$$

For two groups G and Γ, we denote by $Ext(\Gamma, G)$ the set of equivalence classes of group extensions $G \to H \to \Gamma$. The equivalence relation is the usual one, given by diagrams of the form

$$\begin{array}{ccccc} G & \longrightarrow & H_1 & \longrightarrow & \Gamma \\ \| & & \cong\downarrow & & \| \\ G & \longrightarrow & H_2 & \longrightarrow & \Gamma \end{array}.$$

The group of automorphisms $Aut(G)$ acts on $Ext(\Gamma, G)$. This action factors through the group of outer automorphisms $Out(G)$, because inner automorphisms, i.e conjugations by elements of G, act trivially. An outer automorphism of G induces a homotopy equivalence of BG. Therefore, the functor B induces a natural map

$$\mathcal{B} : Ext(\Gamma, G)/Out(G) \to Fib(B\Gamma, BG) ,$$

which turns out to be a bijection.

Theorem 1.2. *For a finite group Γ and a compact connected Lie group G, the map*

$$\mathcal{B} : Ext(\Gamma, G)/Out(G) \to Fib(B\Gamma, BG)$$

is a bijection.

Theorem 1.2 reduces the proof of Theorem 1.1 to the case of connected compact Lie groups as we show next.

Proof of Theorem 1.1. Let G and H be compact Lie groups and $f : BG \to BH$ a homotopy equivalence. Because f induces an isomorphism on the fundamental groups, we have $\pi_0(G) \cong \pi_0(H) =: \pi_0$. Moreover, the universal covers BG_e and BH_e are equivalent. G_e is the component of the unit of G. Thus, f induces an commutative diagram of fibrations

$$\begin{array}{ccccc} BG_e & \longrightarrow & BG & \longrightarrow & B\pi_0 \\ \simeq\downarrow & & \downarrow & & \| \\ BH_e & \longrightarrow & BH & \longrightarrow & B\pi_0 \end{array}.$$

By Theorem 1.3 (see below), we have $G_e \cong H_e$. Hence, the two groups G and H can be thought of as being elements in $Ext(B\pi_0, BG_e)$. Theorem 1.2 shows that $G \cong H$ as Lie groups. □

In the case of compact connected Lie groups, it turns out that the normalizer of the maximal torus also determines the Lie group up to isomorphisms. For a compact connected Lie group G we denote by T_G the maximal torus, by $N(T_G)$ the normalizer of T_G and by W_G the Weyl group of G.

Theorem 1.3. *For two compact connected Lie groups G and H, the following statements are equivalent.*

(1) *G and H are isomorphic as Lie groups.*
(2) *$N(T_G)$ and $N(T_H)$ are isomorphic as Lie groups.*
(3) *$BN(T_G)$ and $BN(T_H)$ are homotopy equivalent.*
(4) *BG and BH are homotopy equivalent.*

The equivalence of (1) and (2) was proved by Curtis, Wiederhold and Williams for semi simple compact Lie groups [6]. In the general case this equivalence is implicitely contained in some recent work by Osse [15]. The equivalence of (2) and (3) follows from Proposition 2.3. Hence it remains to prove the implication from (4) to (1). This is based on the above mentioned result of Scheerer [17]. To give a proof of the equivalence of all four conditions independent of Osse's result we also show that (3) implies (4). This is also of interest because only arguments about classifying spaces are necessary to reduce this implication to the case of semi simple Lie groups. After having done this we can apply the result of Curtis, Wiederhold and Williams [6]. The proof, we will give here, is already worked out by the author in [14]

The paper is organized as follows: In Section 2 we discuss the relation between group extensions of compact Lie groups and the associated fibrations. In particuliar, we prove Theorem 1.2. Section 3 contains some well known facts about compact connected Lie groups . The final section is devoted to the proof of the Theorem 1.3.

When we deal with p–adic completion, we do this in the sense of Bousfield and Kan [4].

It is pleasure to thank G.Mislin for a helpful discussion on this subject.

2. Fibrations of classifying spaces and group extensions.

Let G be a compact connected Lie group. For the functor B we choose a construction which commutes with products [18]. Let $Z(G)$ denote the center of the compact connected Lie group G. Let $SHE(BG)$ denotes the component of the identity of the mapping space $HE(BG)$ of all self homotopy equivalences of BG. These both sets are topological group-like monoids. Passing to classifying spaces and taking the adjoint, the obvious homomorphism $Z(G) \times G \to G$ induces a map $ad : BZ(G) \to SHE(BG)$ which is a homomorphism of topological group like monoids. Because conjugation on G induces a self map of BG homotopic to the identity, there exists a homomorphism $Out(G) \to \pi_0(HE(BG))$ from the outer automorphism of G into the group of components of $HE(BG)$. The proof of Theorem 1.2, which is the main purpose of this section is based on the following statement which collects several results from different authors. For a space X we denote by $X^\wedge := \prod_p X_p^\wedge$ the product of all p–adic completions.

Theorem 2.1. *Let G be a connected compact Lie group. Then the following holds:*

(1) *The map ad induces homotopy equivalences*

$$ad^\wedge : BZ(G)^\wedge \xrightarrow{\simeq} SHE(BG)^\wedge$$
$$B(ad^\wedge) : B(BZ(G)^\wedge) \xrightarrow{\simeq} B(SHE(BG)^\wedge)$$
$$(Bad)^\wedge : (BBZ(G))^\wedge \xrightarrow{\simeq} (BSHE(BG))^\wedge \ .$$

(2) *We have*

$$SHE(BG)^\wedge \simeq SHE(BG^\wedge)$$

and

$$(BSHE(BG))^\wedge \simeq B(SHE(BG)^\wedge) \simeq BSHE(BG^\wedge) .$$

(3) *The homomorphism*

$$Out(G) \to \pi_0(HE(BG))$$

is an isomorphism.

Proof. The map ad is an equivalence after completion by [7, Theorem 3] [13, Theorem 1.7], and so is $B(ad^\wedge)$. Because all spaces are connected, the functor B commutes with completion [4, VI;6.5]. Hence, the map $(Bad)^\wedge$ is also an equivalence. Part (2) follows from [3] where it is shown that the map $SHE(BG) \to SHE(BG^\wedge)$ induces the desired first equivalence. This map is also a monoid homomorphism, which shows that $B(SHE(BG)^\wedge) \simeq BSHE(BG^\wedge)$. The last missing equivalence of (2) follows as in (1) by commuting the functor B and completion. The third statement is proved in [8, Corollary 2.7]. □

Every exact sequence

$$1 \to G \to H \to \Gamma \to 1$$

of compact Lie groups, G connected and Γ finite, gives rise to an homomorphism $\rho : \Gamma \to Out(G)$. The homomorphism ρ depends only the equivalence class of the exact sequence. It is well known that, for a compact connected Lie group, the projection $Aut(G) \longrightarrow Out(G)$ has a section $s : Out(G) \longrightarrow Aut(G)$. This might be seen as follows. the group $Inn(G) \cong G/Z(G)$ of the inner automorphism is isomorphic to the centerfree qoutient $\overline{G} := G/Z(G)$. By [9, IV, 6.2] the exact sequence $\overline{G} \to Aut(G) \to Out(G)$ splits and gives rise to the desired section. This section is not unique in general. We fix a chosen section s for the following. For any homomorphism $\rho : \Gamma \longrightarrow Out(G)$, the group extension $G \longrightarrow G \rtimes \Gamma \longrightarrow \Gamma$ realizes ρ, where Γ acts on G via ρ and the section s. We denote by $Ext_\rho(\Gamma, G)$ the set of all group extensions associated to a fixed homomorphism $\rho : \Gamma \to Out(G)$. Therefore,

$$Ext(\Gamma, G) = \coprod_{\rho \in Hom(\Gamma, Out(G))} Ext_\rho(\Gamma, G)$$

and

$$Ext(\Gamma, G)/Out(G) = \coprod_{\rho \in Rep(\Gamma, Out(G))} Ext_\rho(\Gamma, G)$$

split into disjoint unions of nonempty sets. Here, $Hom(\Gamma, G)$ denotes the set of homomorphism and $Rep(\Gamma, G)$ the set of representations, i.e. homomorphisms modulo conjugations. There is a similar splitting for $Fib(B\Gamma, BG)$, which we will explain next. By a result of Stasheff [20], fibrations of the form

$$BG \to Y \to B\Gamma$$

are classified up to fiber homotopy equivalences by homotopy classes of maps

$$B\Gamma \to BHE(BG)\ .$$

For the group of the components, we have $\pi_0(HE(BG)) \cong Out(G)$ (Theorem 2.1). Hence, there exists a fibration

$$BSHE(BG) \longrightarrow BHE(BG) \longrightarrow BOut(G)\ .$$

For every fibration $BG \longrightarrow Y \longrightarrow B\Gamma$, the composition

$$B\Gamma \longrightarrow BHE(BG) \longrightarrow BOut(G)$$

is homotopic to a map $B\rho$, where $\rho : \Gamma \longrightarrow Out(G)$ describes the action of $\Gamma \simeq \Omega B\Gamma$ on BG and therefore the action of Γ on the homotopy groups of BG. This homomorphism is only determined up to conjugation in Out(G), i.e. the composition determines a representation $\rho \in Rep(\Gamma, Out(G))$. Now we also can split

$$Fib(B\Gamma, BG) = \coprod_{\rho \in Rep(\Gamma, Out(G))} Fib_\rho(B\Gamma, BG)$$

into a disjoint union, where $Fib_\rho(B\Gamma, BG)$ denotes the set of fibrations whose classifying map $B\Gamma \to BHE(BG)$ is, up to homotopy, a lift of $B\rho : B\Gamma \to BOut(G)$.

Proposition 2.2. *For every homomorphism $\rho : \Gamma \to Out(G)$, the elements of $Fib_\rho(B\Gamma, BG)$ are classified by vertical homotopy classes of maps $B\Gamma \to BHE(BG)$ lying over $B\rho : B\Gamma \to BOut(BG)$.*

Proof. The section $BOut(G) \to BHE(BG)$ establishes a section in the fibration $map(B\Gamma, BHE(BG)) \to map(B\Gamma, BOut(G))$ of mapping spaces. Let F be the fiber over $B\rho$. Then, $\pi_0(F) \to \pi_0(map(B\Gamma, BHE(BG)))$ is an injection. The source describes the set of maps over $B\rho$, which are vertically homotopic, and the target the set of homotopic maps into $BHE(BG)$. This implies the statement. □

The splittings into disjoint unions of $Ext(\Gamma, G)/Out(G)$ and $Fib(B\Gamma, BG)$ are compatible with the map $\mathcal{B}$ of Theorem 1.2. For every ρ, we get a map

$$\mathcal{B}_\rho : Ext_\rho(\Gamma, G) \to Fib_\rho(B\Gamma, BG)\ .$$

The division by $Out(G)$ on $Ext(\Gamma, G)$ as in Theorem 1.2 is reflected by choosing a fixed homomorphism ρ for each representation $\Gamma \to Out(G)$. To prove Theorem 1.2 it is sufficient to show that the map $\mathcal{B}_\rho$ is a bijection for every homomorphism $\rho : \Gamma \to Out(G)$.

Now we can start with the proof of Theorem 1.2. This is done in several steps. First we consider the case of a torus T.

Proposition 2.3. *For a torus T and a finite group Γ, the map*

$$\begin{array}{cccc} \mathcal{B} : & Ext(\Gamma, T)/Aut(T) & \longrightarrow & Fib(B\Gamma, BT) \\ & T \to N \to \Gamma & \mapsto & BT \to BN \to B\Gamma \end{array}$$

is a bijection.

Proof. We only have to prove that, for a fixed homorphism $\rho : \Gamma \longrightarrow Aut(T)$, the map $\mathcal{B}_\rho : Ext_\rho(\Gamma, T) \longrightarrow Fib_\rho(B\Gamma, BT)$ is a bijection.

Elements of $Ext_\rho(\Gamma, T)$ are classified by obstruction classes in $H^2_\rho(\Gamma; T)$, where ρ indicates the action of Γ on T. We can do this in such a way that the associated cohomology class measures the difference to the extension given by the semi direct product [9, IV; 6.2].

Elements in $Fib_\rho(B\Gamma, BG)$ are classified by vertical homotopy classes of maps $B\Gamma \to BHE(BT)$ lying over $B\rho : B\Gamma \to Aut(T)$ (Proposition 2.2). The fiber $BSHE(BT) \simeq BBT \simeq K(\pi_1(T), 3)$ is an Eilenberg–MacLane space of dimension 3. The action of Γ on $\pi_3(BSHE(BT))$ is induced by ρ. By standard obstruction theory we can classify the different vertical homotopy classes of maps by cohomology classes in $H^3_\rho(B\Gamma; \pi_1(T)$. This obstruction class measures the difference to the classifying map of the fibration induced by the semi direct product $G \rtimes \Gamma$.

The short exact sequence $\pi_1(T) \to \pi_1(T) \otimes \mathbb{R} \to T$ induces an isomorphism

$$H^2(\Gamma; T) \longrightarrow H^3(\Gamma; \pi_1(T)) \ .$$

this gives rise to a diagram

$$\begin{array}{ccc} Ext_\rho(\Gamma, T) & \xrightarrow{\mathcal{B}} & Fib_\rho(B\Gamma, BT) \\ \cong\big\downarrow & & \cong\big\downarrow \\ H^2(\Gamma; T) & \xrightarrow{\cong} & H^3(B\Gamma; \pi_2(BT)) \ , \end{array}$$

where the vertical arrows describe the classifications of the elements via obstruction classes. A straight forward comparison of the constructions of the obstruction classes show that the diagram commutes. Hence, $\mathcal{B}_\rho$ is also a bijection. □

Now, let A be a compact abelian Lie group, i.e. $A \cong A' \times T$ is a product of a finite abelian group A' and a torus T. Let Γ be a finite group acting on A. The exact sequence

$$T \longrightarrow A \longrightarrow A/T \cong A'$$

is a sequence of Γ–modules. For any group extension $A \longrightarrow G \longrightarrow \Gamma$, the subgroup T is normal in G. This leads to a diagram of exact sequences

$$\begin{array}{ccccc} T & = & T & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & G & \longrightarrow & \Gamma \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A/T & \longrightarrow & G/T & \longrightarrow & \Gamma \ . \end{array}$$

Proposition 2.4. *Let $A = A' \times T$ be a product of a finite abelian group A' and a torus T, and let Γ be a finite group. Then,*

$$\mathcal{B} : Ext(\Gamma, A)/Aut(A) \longrightarrow Fib(B\Gamma, BA)$$

is a bijection.

Proof. Let $A \longrightarrow G \longrightarrow \Gamma$ and $A \longrightarrow H \longrightarrow \Gamma$ be two elements of $Ext(\Gamma, G)$, such that the associated fibrations are equivalent. Then, there exists an equivalence of fibrations

$$\begin{array}{ccccc} BA & \longrightarrow & BG & \longrightarrow & B\Gamma \\ {\scriptstyle \phi_A}\downarrow & & {\scriptstyle \phi}\downarrow & & \| \\ BA & \longrightarrow & BH & \longrightarrow & B\Gamma \ . \end{array}$$

Because A is abelian, the map ϕ_A is induced by an automorphism of A. In particular, the composition $BT \xrightarrow{Bi} BA \xrightarrow{\phi_A} BA$ lifts to a map $\phi_T : BT \to BT$ such that $Bi\phi_T \simeq \phi_A Bi$. Because T is normal in G and H, we get a diagram

$$\begin{array}{ccccc} BT & \longrightarrow & BG & \longrightarrow & B(G/T) \\ {\scriptstyle \phi_T}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \overline{\phi}}\downarrow \\ BT & \longrightarrow & BH & \longrightarrow & B(H/T) \ . \end{array}$$

We have to show that the map $\overline{\phi}$ exists making the diagram commutative. Because H/T is a finite group, $map(BT, B(H/T))_c \simeq B(H/T)$, where c is the constant map. We will apply a version of [13; Proposition 1.1]. There is studied the situation of an exact sequence $K \to H \to \overline{H}$ of compact Lie groups, H and $\overline{H}$ connected, and of a map $BG \to BH$. But as the proof shows, we only have to assume that $map(BG, B\overline{H})_c \simeq B\overline{H}$ to get the same result, namely that $[B(G/T), B(H/T)] \to [BG, B(H/T)]$ is an injection and that the image consists of those maps $BG \to B(H/T)$, whose restriction to BT is homotopically trivial.

Hence, the map $\overline{\phi}$ exists as desired and is a homotopy equivalence. Because G/T and H/T are finite groups, the map $\overline{\phi}$ is given by an isomorphism $\alpha : G/T \cong H/T$ of groups.

Now we can think of G and H as elements in $Ext(G/T, T)$. The induced fibrations are equivalent. By proposition 2.3, we get an isomorphism of short exact sequences

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & G/T \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle \alpha}\downarrow \\ T & \longrightarrow & H & \longrightarrow & H/T \ . \end{array}$$

The composition $H \to H/T \xrightarrow{\alpha^{-1}} G/T \to \Gamma$ is the projection $H \to \Gamma$. All this establishes an isomorphism of short exact sequences

$$\begin{array}{ccccc} A & \longrightarrow & G & \longrightarrow & \Gamma \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow & & \| \\ A & \longrightarrow & H & \longrightarrow & \Gamma \end{array}$$

which shows that the two sequences give the same element in $Ext(\Gamma, A)/Aut(G)$. This proves that $\mathcal{B}$ is injective.

Now let $BG \to Y \to B\Gamma$ be any fibration. The long exact sequence of the homotopy groups shows that $\pi := \pi_1(Y)$ is a finite group and that $\pi_2(Y) \cong \pi_2(A) \cong$

$\pi_2(T)$. All higher homotopy groups vanish. This gives rise to a fibration $BT \to Y \to B\pi$. By Proposition 2.3, there exists a finite extension H of T such that $Y \simeq BH$. Because $BH \to B\pi$ is induced by a homomorphism $H \to \pi$, we have an exact sequence $A \to H \to \Gamma$. This shows that $\mathcal{B}$ is also a surjection. $\square$

Finally, we consider the case of a compact connected Lie group G. The group $Out(G)$ acts on the center $Z(G)$ of G. This gives rise to a homomorphism $Out(G) \to Aut(Z(G))$. For any homomorphism $\rho : \Gamma \longrightarrow Out(G)$, we get a homomorphism $\rho : \Gamma \to Aut(Z(G))$, also denoted by ρ. The associated semidirect products fit into a diagram

$$\begin{array}{ccccc} Z(G) & \longrightarrow & Z(G) \rtimes \Gamma & \longrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G & \longrightarrow & G \rtimes \Gamma & \longrightarrow & \Gamma \end{array},$$

where Γ acts on G via ρ and the chosen section $s : Out(G) \to Aut(G)$. Fixing these elements in $Ext_\rho(G)$ and $Ext_\rho(Z(G))$, the cohomology group $H^2(\Gamma; Z(G))$ acts transitively and freely on both extension sets [9, IV; 8.8]. Using these facts, one can prove the following well known proposition:

Proposition 2.5.

(1) *There exists an isomorphism, induced by the inclusion* $Z(G) \to G$

$$\Phi : Ext_\rho(\Gamma, Z(G)) \longrightarrow Ext_\rho(\Gamma, G) \ .$$

(2) *There exists a commutaive diagram of exact sequences*

$$\begin{array}{ccccccc} \zeta & : & Z(G) & \longrightarrow & N & \longrightarrow & \Gamma \\ & & \downarrow & & \downarrow & & \| \\ \xi & : & G & \longrightarrow & H & \longrightarrow & \Gamma \end{array}$$

if and only if $\Phi(\zeta) = \xi$.

Next we will construct the analogous map on the level of fibrations, i.e. a map $Fib(B\Gamma, BZ(G)) \longrightarrow Fib(B\Gamma, BG)$. We will use the classification theorem of Stasheff [20]. We have to find a way to pass from $BHE(BZ(G))$ to $BHE(BG)$. The group $Out(G)$ acts on $SHE(BG)$ and on $BSHE(BG)$ via the chosen section $s : Out(G) \to Aut(G)$ and conjugation. We denote by $SHE_*(BG)$ and by $HE_*(BG)$ the subsets of all pointed maps, where the basepoint in BG is induced by the unit of G. Then, the action of $Out(G)$ can be restricted to $SHE_*(BG)$, because every automorphism of G induces a pointed self map of BG. The space EG denotes the acyclic free G-complex given by Milnor's construction.

Lemma 2.6.

(1)
$$BHE(BG) \simeq EOut(G) \times_{Out(G)} BSHE(BG) \ .$$

(2)
$$\begin{aligned} BHE(BZ(G)) &\simeq EAut(Z(G)) \times_{Aut(Z(G))} BSHE(BZ(G)) \\ &\simeq EAut(Z(G)) \times_{Aut(Z(G))} BBZ(G) \ . \end{aligned}$$

Proof. The action of $Out(G)$ on $SHE(BG)$ establishes a homomorphism of monoids

$$Out(G) \ltimes SHE(BG) \longrightarrow HE(BG) \ .$$

The classifying space $B(H \ltimes K)$ of a semidirect product is homotopy equivalent to the Borel construction $EH \times_H BK$. Passing to classifying spaces and using this equivalence, the above monoid homomorphism gives rise to the homotopy commutaive diagram of fibrtions

$$\begin{array}{ccccc} BSHE(BG) & \longrightarrow & EOut(G) \times_{Out(G)} BSHE(BG) & \longrightarrow & BOut(G) \\ \| & & \downarrow & & \| \\ BSHE(BG) & \longrightarrow & BHE(BG) & \longrightarrow & BOut(G) \ . \end{array}$$

Obviously, the middle arrow is an equivalence which is (1).

The proof of the first equivalence of (2) is analogous. The center $Z(G)$ of G is a product of a finite abelian group and a torus. Hence, $BZ(G) \longrightarrow SHE(BZ(G))$ is an integral equivalence. Moreover, this map is $Aut(Z(G))$-equivariant. All this establishes the second equivalence. □

Now we are prepared to prove the analogous of Proposition 2.5 for fibrations of the associated classifying spaces.

Proposition 2.7.

(1) *There exists a bijection, induced by the inclusion* $Z(G) \to G$*,*

$$\Psi : Fib_\rho(B\Gamma, BZ(G)) \longrightarrow Fib_\rho(B\Gamma, BG) \ .$$

(2) *There exists a commutative diagram of fibrations*

$$\begin{array}{ccccc} \epsilon : BZ(G) & \longrightarrow & X & \longrightarrow & B\Gamma \\ \downarrow & & \downarrow & & \| \\ \eta : \ BG & \longrightarrow & Y & \longrightarrow & B\Gamma \end{array}$$

if and only if $\Psi(\epsilon) = \eta$.

Proof. For every connected Lie group H, the universal fibration with fiber BH is given by

$$BH \to BHE_*(BH) \to BHE(BH)$$

[20]. For an abelian compact Lie group H, the map $BAut(H) \to BHE_*(BH)$ is an equivalence, because for any self map $f : BH \to BH$ the evaluation at the base point induces an equivalence $map(BH, BH)_f \simeq BH$.

Now let E be the pull back of $BHE(BZ(G)) \longrightarrow BAut(Z(G))$ along the map $BOut(G) \longrightarrow BAut(Z(G))$, i.e. $E \simeq EOut(G) \times_{Out(G)} BBZ(G))$. The map $BBZ(G) \to BSHE(BG)$ is $Out(G)$–equivariant map. Lemma 2.6 establishes a map

$$E \simeq EOut(G) \times_{Out(G)} BBZ(G) \to EOut(G) \times_{Out(G)} BSHE(BG) \simeq BHE(BG) \ ,$$

which fits into a diagram

$$(*) \qquad \begin{array}{ccccc} BBZ(G)) & \longrightarrow & E & \longrightarrow & BOut(G) \\ \downarrow & & \downarrow & & \| \\ BSHE(BG) & \longrightarrow & BHE(BG) & \longrightarrow & BOut(G)\ . \end{array}$$

Comparing the universal fibrations establishes a diagram

$$(**) \qquad \begin{array}{ccccc} BZ(G) & = & BZ(G) & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \\ BHE_*(BZ(G)) \simeq BAut(G) & \xleftarrow{Bs} & BOut(G) & \longrightarrow & BHE_*(BG) \\ \downarrow & & \downarrow & & \downarrow \\ BHE(BZ(G)) & \longleftarrow & E & \longrightarrow & BHE(BG)\ . \end{array}$$

The left side is a pull back.

Because ρ maps Γ into $Out(G)$, every element in $Fib_\rho(B\Gamma, BZ(G))$ is classified by a map $k; B\Gamma \longrightarrow E$ and is given by the pull back of $BOut(G) \to E$ along k. The map $E \to BHE(BG)$ induces the map

$$\Psi : Fib_\rho(B\Gamma, BZ(G)) \to Fib_\rho(B\Gamma, BG)\ .$$

The homotopy fiber F of $E \to BHE(BG)$ is connected, nilpotent and contractible after completion. This follows from the diagram (*) and because $(BBZ(G))^\wedge \to (BSHE(BG))^\wedge$ is an equivalence (Theorem 2.1). Inparticular, $\widetilde{H}^*(F, A) = 0$ for any finite abelian group A. By [4, V; 2.7 and 3.3], this implies that $\pi_*(F)$ is uniquely divisible. Hence, $H^i(B\Gamma; \pi_j(F)) = 0$ for any $i, j > 0$. Standard obstruction theory now shows that

$$\pi_0(map(B\Gamma, E)) \cong \pi_0(map(B\Gamma, BHE(BG)))\ ,$$

which proves the first part.

The second part follows from the diagram (**) and part (1). □

Finally, we can prove Theorem 1.2.

Proof of Theorem 1.2. We choose a homorphism $\rho : \Gamma \longrightarrow Out(G)$. Then, by Proposition 2.5 and Proposition 2.7, the diagram

$$\begin{array}{ccc} Ext_\rho(\Gamma, Z(G)) & \xrightarrow{\mathcal{B}} & Fib_\rho(B\Gamma, BZ(G)) \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Psi}\downarrow \\ Ext_\rho(\Gamma, G) & \xrightarrow{\mathcal{B}} & Fib_\rho(B\Gamma, BG) \end{array}$$

commutes, and Φ and Ψ are bijections. The upper horizontal map is also a bijection by Proposition 2.4. □

3. Compact connected Lie Groups.

For any compact connected Lie group G there exists a finite covering

$$1 \longrightarrow K \longrightarrow \widetilde{G} := G_s \times T \longrightarrow G \longrightarrow 1$$

of compact Lie groups, where G_s is simply connected and T a torus (e.g. see [5]). Such coverings we call universal finite. $\widetilde{G}$ is unique up to isomorphism, but the homomorphism $\widetilde{G} \longrightarrow G$ is not unique. We can choose a finite self covering $T \longrightarrow T$ and can take the composition $G_s \times T \longrightarrow G_s \times T \longrightarrow G$. Among the universal finite coverings of G, there is a minimal one, characterized by the condition that $K \longrightarrow G_s \times T \longrightarrow G_s$ is an injection. The minimal universal finite covering is a quotient of all the others.

Every universal finite covering establishes the following commutative diagrams of exact sequences of connected Lie groups:

$$(*)\qquad \begin{array}{ccccc} K_s & \longrightarrow & G_s & \longrightarrow & G_s/K_s := \overline{G}_s \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & G_s \times T & \longrightarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ \overline{K} = K/K_s & \longrightarrow & T & \longrightarrow & T/\overline{K} := \overline{T} \end{array} \quad ,$$

where $K_s := K \cap (G_s \times \{0\})$. We call $\overline{G}_s$ the semi simple part of G. Because $H^1(\overline{G}_s;\mathbb{Z}) = 0$, the sequence $\overline{G}_s \longrightarrow G \longrightarrow \overline{T}$ induces isomorphisms $H^1(\overline{T};\mathbb{Z}) \cong H^1(G;\mathbb{Z})$ and $H^2(B\overline{T};\mathbb{Z}) \cong H^2(BG;\mathbb{Z}) \cong H^2(BT_G;\mathbb{Z})^{W_G}$. These isomorphisms determine the maps $G \longrightarrow \overline{T}$ and $BG \longrightarrow B\overline{T}$, and therefore, the fibration $B\overline{G}_s \longrightarrow BG \longrightarrow B\overline{T}$.

By the above diagram $(*)$, the map ${T_G}^{W_G} \longrightarrow \overline{T}$ is an epimorphism. Restricting the projection $G \longrightarrow \overline{T}$ to $N(T_G)$ or ${T_G}^{W_G}$ and passing to the classifying spaces yields a commutative diagram of fibrations

$$\begin{array}{ccccc} B({T_{\overline{G}_s}}^{W_{\overline{G}_s}}) & \longrightarrow & B({T_G}^{W_G}) & \longrightarrow & B\overline{T} \\ \downarrow & & \downarrow & & \| \\ BN(T_{\overline{G}_s}) & \longrightarrow & BN(T_G) & \longrightarrow & B\overline{T} \\ \downarrow & & \downarrow & & \| \\ B\overline{G}_s & \longrightarrow & BG & \longrightarrow & B\overline{T} \ . \end{array}$$

In the next lemma we have to deral with completed spaces.

Lemma 3.1. *The canonical inclusion* $Z(\overline{G}_s) \longrightarrow {T_{\overline{G}_s}}^{W_{\overline{G}_s}}$ *induces an injection*

$$Fib(BT^\wedge_p, B\overline{G}_{s\,p}^\wedge) \longrightarrow Fib(BT^\wedge_p, B({T_{\overline{G}_s}}^{W_{\overline{G}_s}})^\wedge_p) \ .$$

Proof. Elements in $Fib(BT^\wedge_p, B\overline{G}_{s\,p}^\wedge)$ are classified by homotopy classes of maps $BT^\wedge_p \longrightarrow BHE(B\overline{G}_{s\,p}^\wedge)$ [20]. Because $BT^\wedge_p$ is simply connected, every homotopy class lifts to a homotopy class $BT_p^\wedge \longrightarrow BSHE(B\overline{G}_{s}{}_p^\wedge)$.

The composition of equivalences (Theorem 2.1) and maps

$$BSHE(B\overline{G}_{s}{}_p^\wedge) \overset{\simeq}{\longleftarrow} BBZ(\overline{G}_s)_p^\wedge \to BB(T_{\overline{G}_s}{}^{W_{\overline{G}_s}})_p^\wedge \overset{\simeq}{\longrightarrow} BSHE(B(T_{\overline{G}_s}{}^{W_{\overline{G}_s}}))_p^\wedge$$

establishes the desired map.

Since $\overline{G}_s$ is semi simple, $Z(\overline{G}_s)$ and $T_{\overline{G}_s}{}^{W_{\overline{G}_s}}$ are finite abelian groups. Therefore,

$$\begin{aligned}[BT_p^\wedge, BBZ(\overline{G}_s)_p^\wedge] &\cong H^2(BT; S_pZ(\overline{G}_s)) \\ &\longrightarrow H^2(BT; S_p(T_{\overline{G}_s}{}^{W_G})) \cong [BT_p^\wedge, BB(T_{\overline{G}_s}{}^{W_G})_p^\wedge]\end{aligned}$$

is an injection. Here, $S_pZ(G_s)$ denotes the p–Sylow subgroup. □

4. The proof of Theorem 1.3.

For the proof of Theorem 1.3, we need a result about self maps of the classifying space of a compact connected Lie group.

Lemma 4.1. *Let $f : BG_s \times BT \longrightarrow BG_s \times BT$ be a self map, where G_s is a simply connected Lie group and where T is a torus. If f is rationally an equivalence, there exist a map $f_s : BG_s \longrightarrow BG_s$ and a homomorphism $\alpha : T \longrightarrow T$, such that $f \simeq f_s \times B\alpha$. Both, f_s and $B\alpha$ are rationally equivalences.*

Proof. For a rational equivalence $f : BG_s \times BT \longrightarrow BG_s \times BT$, there exist homorphisms $\beta : T_{G_s} \times T \longrightarrow T_{G_s} \times T$ and $\gamma : W_{G_s} \longrightarrow W_{G_s}$, such that β is W_{G_s}–equivariant with respect to γ, i.e. $\beta w = \gamma(w)\beta$ for $w \in W_{G_s}$, and such that

$$\begin{array}{ccc} BT_{G_s} \times BT & \xrightarrow{B\beta} & BT_{G_s} \times BT \\ \downarrow & & \downarrow \\ BG_s \times BT & \xrightarrow{f} & BG_s \times BT \end{array}$$

commutes up to homotopy. All this follows from [1, Corollary 1.8 and Theorem 2.21] and [12, Theorem 1.1]. Moreover, $B\beta$ is a rational equivalence and γ an isomorphism.

We can describe β by a matrix

$$A = \begin{pmatrix} \beta_{s,s} & \beta_{T,s} \\ \beta_{s,T} & \beta_{T,T} \end{pmatrix} ,$$

where $\beta_{s,T} : T_{G_s} \to T_{G_s} \times T \overset{\beta}{\to} T_{G_s} \times T \to T$. All the other entries are given by analogous compositions, and all entries are W_{G_s}–equivariant. Because, up to homotopy, every map $BG_s \longrightarrow BT$ is trivial, the homomorphism $\beta_{s,T}$ is constant. Because W_{G_s} acts trivially on T, we get $\beta_{T,s} = \beta_{T,s}w = \gamma(w)\beta_{T,s}$ for all $w \in W_{G_s}$. This shows that the image of $\beta_{T,s}$ is contained in the finite group $T_{G_s}{}^{W_{G_s}}$, and hence, that $\beta_{T,s}$ is constant.

Now we define f_s to be the composition

$$BG_s \longrightarrow BG_s \times BT \xrightarrow{f} BG_s \times BT \longrightarrow BG_s$$

and set $\alpha := \beta_{T,T}$. Then, $f_s \times B\alpha$ and f induce the same self map of the maximal torus. Thus, both maps are homotopic [8, Corollary 1.10] [13, Theorem 1.7]. □

Proof of Theorem 1.3.
As already mentioned in the introduction, we only show that (4) implies (1) and that (3) implies (4).

First let us assume that (4) is satisfied. If $BG \simeq BH$, G and H are locally isomorphic [17]; i.e. the universal finite covers of G and H are isomorphic as Lie groups. Let $G_s \times T$ be this universal finite cover.

The equivalence $f : BG \xrightarrow{\simeq} BH$ lifts to a map $\tilde{f} : BG_s \times BT \longrightarrow BG_s \times BT$ [13, Corollary 2.3], yielding a diagram

$$\begin{array}{ccccc} BK_G & \longrightarrow & BG_s \times BT & \longrightarrow & BG \\ \downarrow & & \tilde{f}\downarrow & & f\downarrow\simeq \\ BK_H & \longrightarrow & BG_s \times BT & \longrightarrow & BH \; . \end{array}$$

The universal finite cover of H can be chosen to be minimal, i.e. the homomorphism $K_H \longrightarrow G_s \times T \longrightarrow G_s$ is an injection (see section 4). Up to homotopy,

$$\tilde{f} = f_s \times B\alpha : BG_s \times BT \longrightarrow BG_s \times BT$$

is a product (Lemma 4.1). The restriction $f_s|_{BK_G}$ is induced by a homomorphism. Hence, the kernel of $K_G \to G_s$ is contained in the kernel of $K_H \to G_s$ and therefore trivial. That is that the universal finite cover $G_s \times T \to G$ is also minimal. Applying the same considerations to the homotopy inverse $f^{-1} : BH \xrightarrow{\simeq} BG$ shows, that $\tilde{f} = f_s \times B\alpha$ is a homotopy equivalence and that $K_G \cong K_H$.

Because f_s is an self equivalence of BG_s, it is homotopic to $B\beta$ for a suitable outer automorphism β of G_s (Theorem 2.1). The isomorphism of groups

$$\beta \times \alpha : G_s \times T \longrightarrow G_s \times T$$

maps K_G onto K_H. Therefore, $G \cong H$ as Lie groups which is condition (1).

Now we assume that $BN(T_G) \simeq BN(T_H)$. We will show that $BG \simeq BH$, which is condition (4). The conditions (2) and(3) are equivalent (Proposition 2.3), which says that $W_G \cong W_H$, and $T_G \cong T_H$ as W_G–modules. Thus,

$$H^2(BG;\mathbb{Z}) \cong H^2(BT_G;\mathbb{Z})^{W_G} \cong H^2(BT_H;\mathbb{Z})^{W_H} \cong H^2(BH;\mathbb{Z}) \; .$$

Because $H^2(BG,\mathbb{Z})$ and $H^2(BH;\mathbb{Z})$ determine the semi simple parts of G and H (see section 4), we yield a diagram of fibrations

$$(*) \qquad \begin{array}{ccccccc} B\overline{G}_s & \longleftarrow & BN(T_{\overline{G}_s}) & \simeq & BN(T_{\overline{H}_s}) & \longrightarrow & B\overline{H}_s \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BG & \longleftarrow & BN(T_G) & \simeq & BN(T_H) & \longrightarrow & BH \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BT & = & BT & = & BT & = & BT \; . \end{array}$$

The groups $\overline{G}_s$ and $\overline{H}_s$ are semi simple, and, by (2), they have isomorphic normalizers. Therefore, $\overline{G}_s \cong \overline{H}_s$ as Lie groups [6]. Moreover, the two fibrations

$$\begin{array}{ccccc} B({T_{\overline{G}_s}}^{W_G}) & \longrightarrow & B({T_G}^{W_G}) & \longrightarrow & BT \\ \simeq\downarrow & & \simeq\downarrow & & \| \\ B({T_{\overline{H}_s}}^{W_H}) & \longrightarrow & B({T_H}^{W_H}) & \longrightarrow & BT \end{array}$$

are also isomorphic. Because BT is simply connected, localization and completion preserve the fibrations. By Lemma 3.1, the left and right columns in $(*)$ are equivalent as fibrations after completion. Hence, there exists an equivalence $f_p^\wedge : BG_p^\wedge \longrightarrow BH_p^\wedge$, which fits into

$$\begin{array}{ccc} BT_{G_p^\wedge} & \xrightarrow{\simeq} & BT_{H_p^\wedge} \\ \downarrow & & \downarrow \\ BG_p^\wedge & \xrightarrow{f_p^\wedge} & BH_p^\wedge \end{array} .$$

The top horizontal arrow is induced by a homotopy equivalence $f_T : BT_G \to BT_H$ which comes from the equivalence $BN(T_G) \simeq BN(T_H)$. Since BG and $BN(T_G)$ are rationally equivalent, we also have an equivalence $f_{\mathbb{Q}} : BG_{\mathbb{Q}} \longrightarrow BH_{\mathbb{Q}}$, which is compatible with the map $f_T : BT_G \longrightarrow BT_H$. Therefore, the coherence conditions for glueing together all these maps via the arithmetic square are satisfied. This yields an equivalence $BG \simeq BH$. and finishes the proof of Theorem 1.3. □

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