THE GENERA OF PRODUCTS OF QUATERNIONIC PROJECTIVE SPACES

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0. Introduction.

The completion genus set of the classifying space of a compact connected Lie group G, denoted by $Genus(BG)$, is the set of homotopy classes of spaces X with $(X)^\wedge_p \simeq (BG)^\wedge_p$ for any prime p. According to a result of Wilkerson [8], such spaces X are classified by the double coset space $Aut((BG)_0)\backslash Caut(((BG)^\wedge)_0)/Aut((BG)^\wedge)$. Here $Caut(((BG)^\wedge)_0)$ denotes the subgroup of $Aut(((BG)^\wedge)_0)$ which consists of homotopy classes of maps f such that each $\pi_*(f)$ is a $\mathbb{Q} \otimes \mathbb{Z}^\wedge$–module map on $\pi_*(((BG)^\wedge)_0)$. For $X \in Genus(BG)$, let A_X denote the corresponding element in the coset space, that is, the gluing map:

$$A_X \in Aut((BG)_0) \setminus Caut(((BG)^\wedge)_0)/Aut((BG)^\wedge)$$

Then the space X is the homotopy pullback as follows:

$$\begin{array}{ccc}
X & \longrightarrow & (BG)^\wedge \\
\Big\downarrow & & \Big\downarrow \\
 & & ((BG^\wedge)_0 \\
 & & \Big\downarrow{\scriptstyle A_X} \\
(BG)_0 & \longrightarrow & ((BG)^\wedge)_0
\end{array}$$

Of course, if A_X is the equivalence class of the identity map, then $X = BG$. We note, [5], that the genus set of BG is uncountably large whenever G is non-abelian.

The maps between classifying spaces $[BG, BK]$ have been investigated extensively, [1], [3], etc. We discuss a general problem: Determine the homotopy set $[X, Y]$ if $X \in Genus(BG)$ and $Y \in Genus(BK)$. For a

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX map $f : X \to Y$, let $f^\wedge : X^\wedge \to Y^\wedge$ be the completed map and let $f_0 : X_0 \to Y_0$ be the rationalization. We note here that the map $f^\wedge : X^\wedge \to Y^\wedge$ canonically splits into the product of maps $f_p^\wedge : X_p^\wedge \to Y_p^\wedge$, [7]. The maps $f^\wedge$ and f_0 induce the maps from $(X^\wedge)_0$ to $(Y^\wedge)_0$. We write these maps $C(f)$ and $R(f)$ respectively. Notice that $(X^\wedge)_0 \simeq ((BG)^\wedge)_0$ and $(Y^\wedge)_0 \simeq ((BK)^\wedge)_0$ for $X \in Genus(BG)$ and $Y \in Genus(BK)$. Using the above fibre square, we get the homotopy commutative diagram

$$\begin{array}{ccc} ((BG)^\wedge)_0 & \xrightarrow{C(f)} & ((BK)^\wedge)_0 \\ {\scriptstyle A_X}\downarrow & & \downarrow{\scriptstyle A_Y} \\ ((BG)^\wedge)_0 & \xrightarrow{R(f)} & ((BK)^\wedge)_0 \ , \end{array}$$

which we express in the equation

$$R(f) \cdot A_X = A_Y \cdot C(f) \ .$$

We discuss special cases about the homotopy sets $[X, Y]$ where $X \in Genus(BG)$ and $Y \in Genus(BK)$. First, suppose $G = K$ and G is simple. One can show that, for $X, Y \in Genus(BG)$, there is an essential map from X to Y, i.e. $[X, Y] \neq 0$, if and only if $X \simeq Y$. We sketch a proof of the statement that $[X, Y] \neq 0$ implies $X \simeq Y$; For an essential map $f : X \to Y$, the above equation has the following form

$$R(f) \cdot A_X = A_Y \cdot (B\tau \circ \psi^\alpha)$$

where τ is expressed as a product of outer automorphsims and ψ^α is an unstable Adams operation, [3]. Hence we see $\psi^\alpha = B\tau^{-1} A_Y^{-1} R(f) A_X \in Caut(((BG)^\wedge)_0)$ so that $\alpha = \prod_p \alpha_p \in \mathbb{Z}^\wedge \cap (\mathbb{Q}^\wedge)^*$. This implies $\alpha_p \in (\mathbb{Z}_p^\wedge)^*$ for sufficiently large p. Thus we can find $N \in \mathbb{N}$ such that $\frac{\alpha}{N} \in (\mathbb{Z}^\wedge)^*$. Consequently $X \simeq Y$ is shown by the following:

$$\begin{aligned} A_X &= (R(f)^{-1} \psi^N) A_Y (B\tau \psi^{\frac{\alpha}{N}}) \\ &\equiv A_Y \ \text{ in } \ Aut((BG)_0) \setminus Caut(((BG)^\wedge)_0) / Aut((BG)^\wedge) \end{aligned}$$

We note, however, that if G is not simple, the result can not be true , [2]. A counter-example is given by a fibration $BS^3 \to X \to BS^3$ where $X \in Genus(BS^3 \times BS^3)$ but $X \not\simeq BS^3 \times BS^3$. It is easy to see $[BS^3 \times BS^3, X] \neq 0$ and $[X, BS^3 \times BS^3] \neq 0$.

In this paper we will consider the case that G is a finite product of S^3's. ¿From now on, let $G = S^3 \times \cdots \times S^3$. We write $rank(X) = n$ if $X \in Genus(BG)$ and G is the product of n copies of S^3. For $X, Y \in Genus(BG)$

let $\epsilon_0(X,Y)$ denote the set of rational equivalences $f : X \to Y$, and $\epsilon_0(X)$ the monoid of rational self-equivalences $f : X \to X$. Namely $f \in \epsilon_0(X,Y)$ means that its rationalization $f_0 : X_0 \to Y_0$ is a homotopy equivalence. Similiarly $\epsilon_0(X_p^\wedge, Y_p^\wedge)$ and $\epsilon_0(X_p^\wedge)$ are defined.

We will investigate $\epsilon_0(X)$. When $X = BG$, it is known that $\epsilon_0(BG)$ is the set of monomial matrices whose nonzero entries are odd squares , and $\epsilon_0(BG_p^\wedge)$ consists of monomial matrices with entries given by p-adic squares. Moreover, for $p = 2$ these entries have to be 2–adic units [3] or [6].

For example, $\epsilon_0(BS^3 \times BS^3)$ consists of the following types of 2×2 matrices:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}$$

where a, b, c and d are squares of odd numbers. In general, if $f \in \epsilon_0(X)$, then we get the homotopy commutative diagram

$$\begin{array}{ccc} ((BG)^\wedge)_0 & \xrightarrow{C(f)} & ((BG)^\wedge)_0 \\ {\scriptstyle A_X}\downarrow & & \downarrow{\scriptstyle A_X} \\ ((BG)^\wedge)_0 & \xrightarrow{R(f)} & ((BG)^\wedge)_0 \ . \end{array}$$

Since $((BG)^\wedge)_0 = K(\bigoplus^n \mathbb{Q}^\wedge, 4)$ when G is the product of n copies of S^3, the maps $C(f)$, $R(f)$ and A_X can be regarded as $n \times n$ matrices over $\mathbb{Q}^\wedge$. Consequently the equation $R(f) \cdot A_X = A_X \cdot C(f)$ can be understood in terms of matrix multiplication.

For every $X \in Genus(BG)$ there are always the Adams operations of odd degree contained in $\epsilon_0(X)$. An Adams operation $\psi^k : BG_p^\wedge \to BG_p^\wedge$ is represented by a scalar matrix $k^2 \cdot Id$. The matrix is central, and we get the identity $\psi^k \cdot A_X = A_X \cdot \psi^k$, which establishes a self map $\psi^k : X \to X$. Because only odd degrees of Adams operations occur, we denote the set of Adams operations by $\mathbb{N}_{odd}$, the monoid of odd natural numbers, and express the above fact by $\mathbb{N}_{odd} \subset \epsilon_0(X)$.

There is a canonical embedding $\epsilon_0(X) \hookrightarrow \epsilon_0(X_p^\wedge)$. Since $X_p^\wedge \simeq BG_p^\wedge$, we see $\epsilon_0(X_p^\wedge) \cong \epsilon_0(BG_p^\wedge)$. If G is the product of n copies of S^3, there is a split short exact sequence of groups

$$D(p) \to \epsilon_0(BG_p^\wedge) \xrightarrow{r} \Sigma_n$$

where Σ_n denotes the symmetric group regarded as a subgroup of $GL(n, \mathbb{Z})$ by permutation representation. For example, if $n = 2$, the map r is as follows:

$$r \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The kernel $D(p)$ consists of diagonal matrices whose nonzero entries are squares in $\mathbb{Z}_p^\wedge$. Notice that $\epsilon_0(BG_2^\wedge) = Aut(BG_2^\wedge)$, and this group consists of non-singular monomial matrices. Hence, for $p = 2$, these squares have to be units. Since $\epsilon_0(X) \subset \epsilon_0(BG_p^\wedge)$, this exact sequence induces the following commutative diagram

$$\begin{array}{ccccc} D(p) & \longrightarrow & \epsilon_0(BG_p^\wedge) & \xrightarrow{r} & \Sigma_n \\ \cup & & \cup & & \cup \\ \delta_0(X,p) & \longrightarrow & \epsilon_0(X) & \longrightarrow & \sigma_0(X) \end{array}$$

where $\delta_0(X,p) = \epsilon_0(X) \cap Ker\ r$ is an abelian monoid and where $\sigma_0(X) = r(\epsilon_0(X))$.

Let $\delta_0(X) = \delta_0(X,p)$ for $p = 2$. If $X = BG$, the monoid $\delta_0(BG)$ is isomorphic to the direct sum of n copies of $\mathbb{N}_{odd}$. The following result shows the structure of $\delta_0(X)$.

Theorem 1. *Let $X \in Genus(BG)$ where G is a finite product of S^3. The monoid $\delta_0(X)$ consists of diagonal matrices whose non-zero entries are squares of odd numbers.*

The monoid $\delta_0(X)$ reveals the decomposability of X as the next result shows. We say $X \in Genus(BG)$ splits if there exist spaces $U \in Genus((BS^3)^k)$ and $V \in Genus((BS^3)^{n-k})$ such that $X \simeq U \times V$. Otherwise we call X indecomposable.

Theorem 2. *Let X be as in Theorem 1. Then the following holds:*

(1) *The space X splits into a product of indecomposable spaces $X_1 \times \cdots \times X_m$ with $X_i \in genus((BS^3)^{r_i})$ for some r_i.*
(2) *The monoid $\delta_0(X)$ is isomorphic to a direct sum of copies of $\mathbb{N}_{odd}$.*
(3) *The space X splits into m indecomposable spaces if and only if $\delta_0(X) \cong (\mathbb{N}_{odd})^m$.*

For X with $rank(X) = 2$, Theorem 3 classifies $\epsilon_0(X)$. Note that a map f with $f \cdot f = \psi^k$ is denoted by $\sqrt{\psi^k}$. Of course k has to be an odd number. The notation $< \mathbb{N}_{odd}, \sqrt{\psi^k} >$ means the monoid generated by $\mathbb{N}_{odd}$ and $\sqrt{\psi^k}$.

Theorem 3. *Let $X \in Genus(BS^3 \times BS^3)$. The monoid of rational quivalences $\epsilon_0(X)$ is isomorphic to one of the following four types of monoids:*

(i) $\mathbb{N}_{odd}$ *(ii)* $< \mathbb{N}_{odd}, \sqrt{\psi^k} >$ *(k: odd)*

(iii) $\mathbb{N}_{odd} \times \mathbb{N}_{odd}$ *(iv)* $(\mathbb{N}_{odd} \times \mathbb{N}_{odd}) \rtimes \Sigma_2$

All above monoids are realized as $\epsilon_0(X)$ for some $X \in Genus(BG)$.

Next we consider $\epsilon_0(X)$ in the general case that $G = S^3 \times \cdots \times S^3$.

Theorem 4. *Suppose*

$$X = X_1^{n_1} \times X_2^{n_2} \times \cdots \times X_s^{n_s}$$

where each X_i *is indecomposable and* $X_i \not\simeq X_j$ *for* $i \neq j$*. Then*

$$\epsilon_0(X) = \prod_{i=1}^{s} (\epsilon_0(X_i) \rtimes \Sigma_{n_i}).$$

If X is indecomposable, there is a strong relationship between $[X, X]$ and $[BG, BG]$.

Theorem 5. *Let* X *be as in Theorem 1. If* X *is indecomposable, for any* $f \in \epsilon_0(X)$*, there is a homotopy equivalence between* $X_2^{\wedge}$ *and* $BG_2^{\wedge}$ *so that we can find a self-map* h *of* BG *which makes the following diagram homotopy commutative:*

$$\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
X_2^{\wedge} & \xrightarrow{f_2^{\wedge}} & X_2^{\wedge} \\
\downarrow \simeq & & \downarrow \simeq \\
BG_2^{\wedge} & \longrightarrow & BG_2^{\wedge} \\
\uparrow & & \uparrow \\
BG & \xrightarrow{h} & BG
\end{array}$$

The following result shows that not every type of extension $\delta_0(X) \to \epsilon_0(X) \to \sigma_0(X)$ can be realized for some $X \in Genus(BG)$. If A_n denotes the alternating group, we have the following result:

Theorem 6. *Let* X *be as in Theorem 1. If* $A_n \subset \epsilon_0(X)$*, then* $\Sigma_n \subset \epsilon_0(X)$*.*

For instance, if an extension $\delta_0(X) \to \epsilon_0(X) \to A_n$ splits, one can show that up to conjugate the image in $\epsilon_0(X)$ is expressed as the permutation representation. The above theorem tells us that such a split extension does not exist.

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1. Rational self equivalences.

In this section we always assume that X and Y are in the genus of BG. Recall that $\epsilon_0(X,Y)$ denotes the set of all homotopy classes of maps $X \to Y$ which induce rational equivalences.

1.1 Proposition. *Let $f \in \epsilon_0(X,Y)$. Then $f_p^\wedge : X_p^\wedge \to Y_p^\wedge$ is an equivalence for almost all primes.*

Proof. Recall that the map f induces the equation $R(f) \cdot A_X = A_Y \cdot C(f)$ as matrices. In the denominators and numerators of the entries of R(f) occur only a finite number of primes. For all the others, f induces an isomorphism in mod-p cohomology and hence, a p-adic equivalence. □

1.2 Lemma. *Let $f \in \epsilon_0(X,Y)$. Then there exist gluing maps A_X and A_Y such that $A_X = A_Y \cdot \psi^T$, i.e. $R(f) = id$ and $C(f)$ is a product of unstable Adams operations.*

Proof. There exist gluing maps A'_X and A'_Y and matrices $R'(f)$ and $C'(f)$ such that $R'(f)A'_X = A'_Y C'(f)$. As a self map of $BG_p^\wedge$ the map $C_p(f) = \sigma(p)\psi^{S(p)}$ is a product of a permutation and an Adams operation. We can split $S(p)$ into a product $S_1(p)S_2(p)$ such that the entries of $S_2(p)$ are powers of p, and that the entries of $S_1(p)$ are p–adic units. Here, the product is taken in the components. Because for almost all primes $C'_p(f)$ is an equivalence by Proposition 1.1, almost all tuples $S_2(p)$ are of the form $\{1,...,1\}$. Now let $T := \prod_p S_2(p)$ be the product of all tuples $S_2(p)$, which is finite, and let $Q(p) := T/S_2(p)$ be the quotient of T by $S_2(p)$. Then $Q(p)$ consists of p–adic units. Now we define $A_X := R'(f)A'_X$, $A_Y := A'_Y \prod_p(\sigma(p))\psi^{S_1(p)Q(p)^{-1}})$, $R(f) := id$ and $C(f) := \psi^T$. Then we see the equations $A_X = R'(f)A'_X = A'_Y C'(f) = A'_Y \prod(\sigma(p)\psi^{S_1(p)}\psi^{S_2(p)}) = A_Y \prod(\psi^{Q(p)}\psi^{S_2(p)}) = A_Y \psi^T$. This proves the statement. □

1.3 Corollary. *Let $f \in \epsilon_0(X,Y)$. Then there exists $g \in \epsilon_0(Y,X)$, such that $fg = gf = \psi^k$, i.e. both compositions are unstable Adams operations of the same degree.*

Proof. We choose gluing maps as in Lemma 1.2. Moreover, for $T = \{t_1,...,t_n\}$ we define $k := lcm(t_1,...,t_n)$, $S := k/T$, $R(g) := \psi^k$ and $C(g) := \psi^S$. Then, the equations $A_X\psi^S = A_X\psi^k\psi^{T^{-1}} = A_Y\psi^T\psi^k\psi^{T^{-1}} = \psi^k A_Y$ defines a rational equivalence $g : Y \to X$. Obviuosly, the compositions fg and gf are unstable Adams operations of degree k. □

The last result allows to speak of ψ^k–inverse maps.

1.4 Definition. The rational equivalence $g : Y \to X$ of the Corollary 1.3 is called the ψ^k–inverse of $f : X \to Y$.

Let $f \in \epsilon_0(X,Y)$, and let g be an ψ^k-inverse. Then, we have maps (which are not maps of monoids)

$$c : \epsilon_0(X) \to \epsilon_0(Y) : h \mapsto fhg \text{ and } c : \epsilon_0(Y) \to \epsilon_0(X) : k \mapsto gkf.$$

The composition is multiplication by ψ^{k^2}. Thus, both maps are injective. Choosing gluing maps A_X and A_Y as in Lemma 1.2, the matrices $C_p(f)$ and $C_p(g)$ are given by products of unstable Adams operations. Therefore, we get a commutative diagram

$$\begin{array}{ccccc} \delta_0(X,p) & \longrightarrow & \epsilon_0(X) & \longrightarrow & \Sigma_n \\ \downarrow & & \downarrow & & \| \\ \delta_0(Y,p) & \longrightarrow & \epsilon_0(Y) & \longrightarrow & \Sigma_n \end{array} .$$

As mentioned the conjugation c is not a map of monoids and, of course not an isomorphism in general. One would like to improve this by considering $\psi^{1/k}c$. Then, the composition becomes the identity. This doesn't work because we can't divide by ψ^k in the image of c. But if we restrict c to the kernels $\delta_0(X,p)$ and $\delta_0(Y,p)$, then we have only to deal with products of Adams operations, which is an abelian monoid. Therefore, for $h \in \delta_0(X,p)$, we have $\psi^{1/k}C_p(f)C_p(h)C_p(g) = C_p(h)\psi^{1/k}C_p(f)C_(g) = C_p(h)$, and $\psi^{1/k}fhg \in \epsilon_0(Y)$.

On the other hand, we can pass to the associated Grothendieck groups $K(\epsilon_0(X))$ and $K(\epsilon_0(Y))$, where we can also multiply by $\psi^{1/k}$. This proves the following statement

1.5 Proposition. *Let $f \in \epsilon_0(X,Y)$, and let g be a ψ^k inverse. Then there exist isomorphisms*

$$K(\epsilon_0(X)) \to K(\epsilon_0(Y)) \text{ and } \delta_0(X,p) \to \delta_0(Y,p) ,$$

given by conjugation.

2. Characteristic polynomials of self maps.

If $f \in \epsilon_0(X)$, we see, as before, that $R(f) \cdot A_X = A_X \cdot C(f)$ as matrices. Actually $R(f)$ is a matrix over $\mathbb{Q}$ and $C(f)$ is a matrix over $\mathbb{Z}^\wedge = \prod \mathbb{Z}_p^\wedge$. The map $C(f)$ canonically splits so that $C(f) = \prod C_p(f)$. Hence $C_p(f)$ is regarded as a monomial matrix over $\mathbb{Z}_p^\wedge$ whose nonzero entries are squares.

2.1 Lemma. *The matrices $R(f)$ and $C_p(f)$ have identical characteristic polynomials which are monic and have integral coefficents.*

Proof. The identity $R(f) \cdot A_X = A_X C(f)$ shows that all matrices are conjugate. Hence the characteristic polynamials are identical. It is a polynomial over $\mathbb{Z}_p^\wedge$ for all p and over $\mathbb{Q}$ and has therefore integral coefficients. Obviously it is monic. $\square$

2.2 Definition. Let $f : X \to X$ be an element of $\epsilon_0(X)$. Then we define the characteristic polynomial $\chi_f(t) \in \mathbb{Z}[t]$ of f as the characteristic polynomial of $R(f)$ or of $C_p(f)$. It is well defined because $R(f)$ and $C_p(f)$ are unique up to conjugation.

2.3 Lemma. *For every element $f \in \epsilon_0(X)$ the characteristic poynomial $\chi_f(t)$ is always of the form $\prod_i (t^{k_i} - a_i)$, where $\sum k_i = n$ and $a_i \in Z$ is an odd square.*

Proof. Recall that $\epsilon_0(BG_p^\wedge) = R \wr \Sigma_n$, where R consists of the squares of 2–adic units or nonzero p–adic squares if p is odd. In particuliar, $C_p(f) = \sigma(\psi^L)$ with $L = (\ell_1, ..., \ell_n)$. Here, σ is a permutation. The characteristic polynomial of $C(f)$ is of the desired form. To show that a_i is always a square, we first observe that an unstable Adams operation ψ^k induces in π_4 a multiplication by k^2. Therefore, a_i is always a square in $\mathbb{Z}_p^\wedge$ for all p, and hence a square over the integers. It is an odd square because only Adams operations of odd degrees are realizable as self maps over $BS^3{}_2^\wedge$. $\square$

Let $f \in \epsilon_0(X)$ and $\chi_f = \prod (t^{\ell_i} - a_i') = \prod (t^{k_i} - a_i)^{r_i}$ (in the last expression we just collected equal terms). Then there are two associated partitions of n , namely $P'(f) = \{\ell_1, ..., \ell_n\}$ and $P(f) = \{k_1 r_1, ..., k_n r_n\}$. We also associate to every permutation σ a partition $P(\sigma)$ given by the length of the cycles.

2.4 Lemma. *If $\chi_f = \prod_i (t^{\ell_i} - a_i)$, then $C_p(f) = \sigma \psi^K$, where $P(\sigma)$ is a subpartition of $P'(f)$. Every ℓ_i splits into a partition $\{m_i, ..., m_i\}$ where m_i occurs q_i times and $q_i m_i = \ell_i$.*

Proof. This follows from the calculation of $\chi_{\sigma\psi^K}(t)$. $\square$

2.5 Remark. The existence of roots of unity may cause a splitting in a subpartition, e.g. $\prod_{j=1}^k (t^\ell - \omega^j) = t^{\ell k} - 1$, where ω is a primitive k-th root of the unity. Because $\mathbb{Z}_2^\wedge$ contains no roots of unity besides ± 1, we have $P(\sigma) = P'(f)$ for $p = 2$.

2.6 Lemma. *If $C_2(f)$ is a diagonal matrix, then $C_p(f)$ also is a diagonal matrix for every prime.*

Proof. The assumptions imply that $\chi_f(t) = \prod_i (t - a_i)$ is a product of linear factors. Thus, the associated partition $P'(f)$ is given by $\{1, ..., 1\}$. By Lemma 2.4, all matrices $C_p(f)$ are diagonal.

2.7 Corollary. *The kernel $\delta_0(X)$ is a subset of $\delta_0(X,p)$ for all primes.*

Proof. Let $f \in \delta_0(X)$. Then $C_2(f) = \psi^T$ is a product of Adams operations and therefore a diagonal matrix. By Lemma 2.6 all the matrices $C_p(f)$ are diagonal. Thus, for all primes, we have $f \in \delta_0(X,p)$.

This result shows, as one have might expected, that most of the information about the spaces in the genus of BG is concentrated at the prime 2.

Proof of Theorem 1. If $f \in \delta_0(X)$, then the $\mathbb{Z}_2^\wedge$–matrix $C_2(f)$ is diagonal. By Lemma 2.6, the $\mathbb{Z}_p^\wedge$–matrix $C_p(f)$ is also diagonal. Since the characteristic polynomial χ_f splits into linear factors in $\mathbb{Z}_p^\wedge$ for any prime p, this monic integer-coefficient polynomial splits over $\mathbb{Z}$ as well. Recall that the polynomial ring $\mathbb{Z}_p^\wedge[t]$ is U.F.D.(unique factorization domain). Thus all $C_p(f)$ are diagonal matrices over $\mathbb{Z}$, indeed. Their main diagonals are the same up to permutation. Finally Lemma 2.3 shows that each entry of the main diagonal of $C_2(f)$ is the square of an odd integer. $\square$

3. Decomposition of spaces and a filtration on the genus of BG.

Recall $G = (S^3)^n$. For any partition $K = \{k_1, ..., k_r\}$ of n, there is an obvious inclusion

$$\Phi_K : \prod_i genus((BS^3)^{k_i}) \to Genus(BG) \ .$$

We say, that an element $X \in Genus(BG)$ has filtration K if X is in the image of Φ_K. The space X is indecompasable if X does not have the filtration K for any proper partition K of n. In this section we will discuss the relation between the filtration of X and the monoid $\delta_0(X)$ of self maps.

3.1 Proposition. *Let $f \in \delta_0(X)$ and let $\chi_f(t) = \prod_{i=1}^{\ell}(t-a_i)^{r_i}$ be a splitting into l pairwise coprime factors. Then the space X has filtration R with $R = \{r_1, ..., r_l\}$. In particular, $X \simeq X_1 \times \cdots \times X_\ell$ splits into a product of l spaces X_i with $X_i \in genus((BS^3)^{r_i})$.*

Proof. Because $f \in \delta_0(X)$ all the matrices $C_p(f)$ are diagonal matrices by Lemma 2.6. The characteristic polynomial therefore splits integrally into linear factors. Moreover, after reordering the entries, i.e. changing the gluing map, we can assume that all the matrices $C_p(f)$ are identical and of the form $\prod_i \psi^{a_i}$. Here, we have to interpret ψ^{a_i} as a diagonal matrix in $GL(r_i, \mathbb{Z}_p^\wedge)$ with constant entries. The rational matrix $R(f)$ is also diagonalizable even over $\mathbb{Q}$, because $\chi_f(t)$ has only integral zeros. Hence, after changing again the gluing map, we also can assume that $R(f) = C_p(f)$. That is to say that A_X centralizes $\prod_i \psi^{a_i}$. Hence $A_X = (A_1 \cdots A_\ell) \in \prod GL(r_i, \mathbb{Q}^\wedge)$ is a blockwise diagonal matrix. Therefore X has filtration R. □

3.2 Proposition. *A space X splits into ℓ factors if and only if $\delta_0(X)$ contains the direct sum of ℓ copies of $\mathbb{N}_{odd}$.*

Proof. Let us assume that $(\mathbb{N}_{odd})^\ell \subset \delta_0(X)$. Then we can find a map $f \in \delta_0(X)$ such that $\chi_f(t) = \prod_{i=1}^{\ell}(t - a_i)^{r_i}$ splits into ℓ pairwise coprime factors. By Proposition 3.1, this shows that X splits at least into ℓ factors.

Now we assume that X splits into a product of ℓ spaces $X_1 \times \cdots \times X_l$. Then Adams operations exist as self maps on each X_i and so does $\mathbb{N}_{odd}$. Thus, $(\mathbb{N}_{odd})^\ell$, as a set of diagonal matrices, is a subset of $\delta_0(X)$.

On the way of proving Theorem 2 we need the following special case.

3.3 Lemma. *A space $X \in Genus(BG)$ is indecomposable if and only if $\delta_0(X) = \mathbb{N}_{odd}$.*

Proof. The monoid $\mathbb{N}_{odd}$ is always a subset of $\delta_0(X)$. First suppose X is indecomposable and $f \in \delta_0(X)$. If f is not an Adams operation, then the characteristic polynomial $\chi_f(t) = \prod_i (t - a_i)^{r_i}$ splits into at least two coprime factors. By Proposition 3.2, this would imply that X splits, which is an contradiction. Hence $\mathbb{N}_{odd} = \delta_0(X)$.

Conversely, if X splits into at least two factors, then $\delta_0(X)$ contains at least $(\mathbb{N}_{odd})^2$ as a submonoid. □

Proof of Theorem 2. Proposition 3.1 shows that if X_i is a factor of $X \in genus((BS^3)^n)$, then $X_i \in genus((BS^3)^{r_i})$ for some r_i. Thus an argument using Proposition 3.2 and Lemma 3.3 implies Part (1).

Let us assume that $X \simeq X_1 \times \cdots \times X_m$ is a splitting into m indecomposables. By Proposition 3.2 there exists an inclusion $(\mathbb{N}_{odd})^m \subset \delta_0(X)$. We want to show that this is an isomorphism. Let $f \in \delta_0(X)$. Let A_i be the gluing map of X_i, and let $A = A_1 \cdots A_m$. Then f establishes an equation $R(f) \cdot A = A \cdot C(f)$. The matrices $C_p(f)$ are always diagonal matrices. Therefore, the matrix $R(f)$ has the same block structure as A and $R = R_1 \times \cdots \times R_m$, where each R_i has the same size as A_i. We can also write $C(f) = C_1 \times \cdots \times C_m$ where C_i also has the same size. That is to say that our above equation splits into the equations $R_i \cdot A_i = A_i \cdot C_i$ which establish self maps $f_i : X_i \to X_i$. The spaces X_i are indecomposable. Thus, by Lemma 3.3, the maps f_i are Adams operations. Therefore $f \in (\mathbb{N}_{odd})^m$. This proves part (2) as well as the one half of (3).

Now let us assume that $\delta_0(X) = (\mathbb{N}_{odd})^m$. Then, by Proposition 3.2, $X = X_1 \times \cdots \times X_m$ splits into m spaces. If one of these is not indecompsable, we could split X further into more than m factors. By Proposition 3.2 again this is an contradiction, which finishes the proof. $\square$

3.4 Proposition. *Suppose that two spaces X and Y are contained in $Genus(BG)$. If there exists a map $f : X \to Y$ which is rationally an equivalence, then X and Y have the same filtration.*

Proof. Let us assume that X has filtration $R = \{r_1, \cdots, r_l\}$; i.e. $X = X_1 \times \cdots \times X_\ell$ splits into ℓ factors. Then there exists $f \in \delta_0(X)$ such that $\chi_f(t) = \prod_i (t-a_i)^{r_i}$ splits into l pairwise coprime factors. By Proposition 1.5 there exists a map $\delta_0(X) \to \delta_0(Y)$ which is given by conjugation. Therefore the image $g : Y \to Y$ of f has the same characteristic polynomial. Hence, by Proposition 3.1 Y has also filtration R. $\square$

4. Determination of $\epsilon_0(X)$ with $rank(X) = 2$.

We investigate the type of the gluing map A_X for $X \in Genus(BS^3 \times BS^3)$. For $f \in \epsilon_0(X)$, the trace of f, denoted by $tr(f)$, is defined as the trace of the matrix $R(f)$. Then the following two cases occur.

Case 1. Suppose there is $f \in \epsilon_0(X) - \mathbb{N}_{odd}$ with $tr(f) \neq 0$.
Take such an f. Suppose f is represented by $R \in Aut((BG)_0)$, $C \in Aut((BG)^\wedge)$ with $RA = AC$ where $A \in Caut((BG^\wedge)_0)$ represents X. Since $tr(f) = tr(R) \neq 0$ and $rank(X) = 2$, the 2×2 matrix C_p must be diagonal for all p:

$$A_p^{-1} R A_p = diag(\alpha_p^2, \beta_p^2) \quad \alpha_p, \beta_p \in \mathbb{Z}_p^\wedge$$

The characteristic polynomial for f then has the form

$$\chi_f(t) = (t - r_1)(t - r_2)$$

for some odd squares r_1 and r_2; replacing A_p by $A_p\tau$ if necessary (where τ is the involution), we may assume $r_1 = \alpha_p^2$, $r_2 = \beta_p^2$ for all p, i.e.

$$\forall p : A_p^{-1} R A_p = C_p = diag(r_1, r_2).$$

Since f is not an unstable Adams operation, $r_1 \neq r_2$, and R is diagonalizable; i.e.

$$\exists U \in GL(2, \mathbb{Q}) \ with \ U^{-1} R U = diag(r_1, r_2).$$

Since also the columns of A_p are eigenvectors for R with eigenvalues r_1 and r_2, respectively, they must be proportional to the columns of U;

$$A_p = U diag(\lambda_p, \mu_p)$$

for some $\lambda_p, \mu_p \in \mathbb{Q}_p^\wedge$. Since the entries of both A_p and U are p–adic units for $p >> 0$, so are λ_p and μ_p. Therefore $\lambda, \mu \in (\mathbb{Q}^\wedge)^*$. An equivalent representative for X is

$$U^{-1}A = diag(\lambda, \mu) \in GL(2, \mathbb{Q}^\wedge).$$

Case 2. Suppose $tr(f) = 0$ for any $f \in \epsilon_0(X) - \mathbb{N}_{odd}$.
For $f \in \epsilon_0(X) - \mathbb{N}_{odd}$ with $tr(f) = 0$ we have

$$\forall p : A_p^{-1} R A_p = diag(\alpha_p^2, \beta_p^2)\tau^{\epsilon_p}.$$

for some $\alpha = (\alpha_p), \beta = (\beta_p) \in \mathbb{Z}^\wedge \cap (\mathbb{Q}^\wedge)^*$ and $\epsilon_p = 0$ or 1. Since $tr(f) = 0$, the characteristic polynomial has the form

$$\chi_f(t) = (t - \gamma)(t + \gamma)$$

for some natural number $\gamma \in \mathbb{N}$. We may assume

$$\gamma = \begin{cases} \alpha_p^2 = -\beta_p^2 & \text{if} \quad \epsilon_p = 0 \\ \alpha_p\beta_p & \text{if} \quad \epsilon_p = 1 \end{cases}$$

Also, R is diagonalizable:

$$\exists U \in GL(2, \mathbb{Q}) \ with \ U^{-1}RU = diag(\gamma, -\gamma).$$

Suppose $\epsilon_p = 0$. Then $\alpha_p^2 = -\beta_p^2$ so $\sqrt{-1} \in \mathbb{Z}_p^\wedge$ and $p \equiv 1 \ mod \ 4$. As

$$A_p^{-1} R A_p = diag(\gamma, -\gamma)$$

the columns of A_p are eigenvectors for R, so

$$A_p = U diag(\lambda_p, \mu_p)$$

for some $\lambda_p, \mu_p \in \mathbb{Q}_p^\wedge (\in (\mathbb{Z}_p^\wedge)^* \ if \ p >> 0)$.
Suppose next $\epsilon_p = 1$. Put

$$M_p = \begin{pmatrix} \beta_p & \beta_p \\ \alpha_p & -\alpha_p \end{pmatrix} \in GL(2, \mathbb{Q}_p^\wedge)$$

Then

$$(A_pM_p)^{-1} R (A_pM_p) = M_p^{-1} diag(\alpha_p^2, \beta_p^2)\tau M_p = diag(\gamma, -\gamma)$$

so again we conclude that

$$A_pM_p = Udiag(\lambda_p, \mu_p)$$

where $\lambda_p, \mu_p \in \mathbb{Q}_p^\wedge (\in (\mathbb{Z}_p^\wedge)^* \ if \ p >> 0)$.
The matrix

$$U^{-1}A = \begin{cases} diag(\lambda,\mu) & \text{if} \quad \epsilon_p = 0 \quad (\text{then } p \equiv 1 \ mod \ 4) \\ diag(\lambda,\mu)M^{-1} & \text{if} \quad \epsilon_p = 1 \end{cases}$$

is another representative for X. Note that

$$M_p = \begin{pmatrix} \beta_p & \beta_p \\ \alpha_p & -\alpha_p \end{pmatrix} = \beta \begin{pmatrix} 1 & 1 \\ \frac{\alpha}{\beta} & \frac{-\alpha}{\beta} \end{pmatrix}$$

$$diag(\lambda,\mu)M_p^{-1} = diag(\lambda\beta^{-1}, \mu\beta^{-1}) \cdot \begin{pmatrix} 1 & 1 \\ \frac{\alpha}{\beta} & \frac{-\alpha}{\beta} \end{pmatrix}^{-1}$$

so by absorbing the β^{-1} into (λ, μ) we may always assume that M_p has the form

$$M_p = \begin{pmatrix} 1 & 1 \\ m_p & -m_p \end{pmatrix}$$

for some $m_p \in \mathbb{Q}_p^\wedge (\in (\mathbb{Z}_p^\wedge)^* \ if \ p >> 0)$.

4.1 Lemma. *Suppose $q \not\equiv 1 \ mod \ 4$. If $C_q(f) = diag(\alpha^2, \beta^2)$, then, for any p, $C_p(f) = diag(k^2, \ell^2)$ for some integers $k, \ell \in \mathbb{Z}$.*

Proof. First let $C_p = C_p(f)$, $C_q = C_q(f)$ and $R = R(f)$. Since $q \not\equiv 1 \ mod \ 4$, we see $tr(C_q) = \alpha^2 + \beta^2 \neq 0$. This implies that C_p and R must be diagonal. Here consider the characteristic polynomials. Since $\chi(C_q) = \chi(R)$ and the the polynomial ring $\mathbb{Q}_q^\wedge[t]$ is UFD, α^2 is contained in the localized integer $\mathbb{Z}_{(q)}$. Analogous results for the other primes enable us to see $\alpha^2 \in \mathbb{Z}$. Suppose that $\alpha^2 = \pm p_1^{e_1}...p_r^{e_r}$ and that p is one of the p_i's $(1 \leq i \leq r)$. When $C_p = diag(\alpha_p^2, \beta_p^2)$, we can write $\alpha_p = p^k u$ for some $u \in (\mathbb{Z}_p^\wedge)^*$. Since $\alpha^2 = \alpha_p^2 = p^{2k}u^2$, we see each e_i must be an even number. Therefore $\alpha^2 = \pm(p_1^{\frac{e_1}{2}}...p_r^{\frac{e_r}{2}})^2$. If $\alpha^2 = -(p_1^{\frac{e_1}{2}}...p_r^{\frac{e_r}{2}})^2$, then $\alpha^2 + (p_1^{\frac{e_1}{2}}....p_r^{\frac{e_r}{2}})^2 = 0$. This is a contradiction, since $\sqrt{-1} \notin \mathbb{Z}_q^\wedge$ for $q \not\equiv 1 \ mod \ 4$. □

4.2 Lemma. *Let $X \in Genus(BS^3 \times BS^3)$ and X is indecomposable. Suppose $f, g \in \epsilon_0(X)$ are represented at $p = 2$ as follows: $C_2(f) = \begin{pmatrix} 0 & \alpha^2 \\ \frac{k^2}{\alpha^2} & 0 \end{pmatrix}$, $C_2(g) = \begin{pmatrix} 0 & \beta^2 \\ \frac{\ell^2}{\beta^2} & 0 \end{pmatrix}$. If g is divisible by ψ^n for some $n \neq 1$, then $f = g$.*

Proof. Notice that the following product

$$\begin{pmatrix} 0 & \alpha^2 \\ \frac{k^2}{\alpha^2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \beta^2 \\ \frac{\ell^2}{\beta^2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\alpha^2\ell^2}{\beta^2} & 0 \\ 0 & \frac{k^2\beta^2}{\alpha^2} \end{pmatrix}$$

is scalar, since X is indecomposable. It follows that $\frac{\alpha^2\ell^2}{\beta^2} = \frac{k^2\beta^2}{\alpha^2}$, and hence $\alpha^2 = \frac{k}{\ell}\beta^2$ and $\beta^2 = \frac{\ell}{k}\alpha^2$. If $k = \ell$, then $\alpha^2 = \beta^2$ and hence $f = g$. We now assume $k \neq \ell$. Since the product is $diag(k\ell, k\ell)$, we see $k\ell$ is a square. Consequently either $\frac{k}{\ell}$ or $\frac{\ell}{k}$ contains square which is not equal to 1. Note that

$$C_p(g) = \begin{pmatrix} 0 & \frac{\ell}{k}\alpha_p^2 \\ \frac{k\ell}{\alpha_p^2} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}$$

This implies g is divisible by ψ^n. This contradiction completes the proof. □

4.3 Lemma. *For any odd positive integer k there is $X \in Genus(BS^3 \times BS^3)$ such that $\epsilon_0(X) =< \mathbb{N}_{odd}, \sqrt{\psi^k} >$.*

Proof. Express the p part of a gluing map A_X by A_p. Take

$$A_p^{-1} = \begin{pmatrix} a & b \\ \frac{bk^2}{\alpha} & \frac{a}{\alpha} \end{pmatrix}$$

for some $a \neq 0, b \neq 0$. Then

$$\begin{pmatrix} 0 & 1 \\ k^2 & 0 \end{pmatrix} = A_p \begin{pmatrix} 0 & \alpha \\ \frac{k^2}{\alpha} & 0 \end{pmatrix} A_p^{-1}$$

for the fixed α. Note that

$$A_p = \frac{\alpha}{a^2 - b^2 k} \begin{pmatrix} \frac{a}{\alpha} & -b \\ \frac{-bk^2}{\alpha} & a \end{pmatrix}$$

and X is indecomposable. We see that $A_p \cdot diag(s^2, t^2) \cdot A_p^{-1} \in GL(2, \mathbb{Q})$ if and only if $s^2 = t^2$, since

$$\begin{pmatrix} s^2 & 0 \\ 0 & t^2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ \frac{bk^2}{\alpha} & \frac{a}{\alpha} \end{pmatrix} = \begin{pmatrix} s^2 a & s^2 b \\ \frac{t^2 bk^2}{\alpha} & \frac{t^2 a}{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ \frac{bk^2}{\alpha} & \frac{a}{\alpha} \end{pmatrix} \cdot \begin{pmatrix} s^2 & 0 \\ 0 & t^2 \end{pmatrix} = \begin{pmatrix} s^2 a & t^2 b \\ * & * \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ \frac{bk^2}{\alpha} & \frac{a}{\alpha} \end{pmatrix} \cdot \begin{pmatrix} t^2 & 0 \\ 0 & s^2 \end{pmatrix} = \begin{pmatrix} t^2 a & s^2 b \\ * & * \end{pmatrix}$$

This shows $\epsilon_0(X) =< \mathbb{N}_{odd}, \sqrt{\psi^k} >$. □

Proof of Theorem 3. Case 1. Suppose there is $f \in \epsilon_0(X) - \mathbb{N}_{odd}$ with $tr(f) \neq 0$. Then $X = Y_1 \times Y_2$. If there is $g \in \epsilon_0(X) - \mathbb{N}_{odd}$ with $tr(g) = 0$, Lemma 4.1 implies that $C_2(g)$ is not diagonal so that there is an essential map between Y_1 and Y_2. Thus $Y_1 = Y_2$ and hence $\epsilon_0(X) = \epsilon_0(BG) = (\mathbb{N}_{odd} \times \mathbb{N}_{odd}) \rtimes \Sigma_2$. Otherwise $Y_1 \neq Y_2$ and $\epsilon_0(X) = \mathbb{N}_{odd} \times \mathbb{N}_{odd}$.
Case 2. Suppose $tr(f) = 0$ for all $f \in \epsilon_0(X) - \mathbb{N}_{odd}$. Then $f^2 = \psi^k$ for some k. Lemma 4.2 implies $\epsilon_0(X) =< \mathbb{N}_{odd}, \sqrt{\psi^k} >$.

Using Lemma 4.3, one can show that each of the monoids is realized as $\epsilon_0(X)$ for some $X \in Genus(BG)$. $\square$

5. Integrality of rational equivalences.

5.1 Lemma. *For $X, Y \in Genus(BG)$, if $\epsilon_0(X, Y)$ is non-empty, then $X \simeq Y$.*

Proof. For $f \in \epsilon_0(X, Y)$, we have an equation

$$R(f) \cdot A_X = A_Y \cdot \sigma \cdot (\prod_i \psi^{\alpha_i}) \ .$$

where σ is a permutation. An argument similar to the one we used in the introduction to show a result associated with a simple group will complete the proof. $\square$

Proof of Theorem 4. Let $f \in \epsilon_0(X)$. Identify f as $C_2(f)$ so that regard f as a monomial $\mathbb{Z}_2^\wedge$–matrix relative to a basis $\mathfrak{B}$. Consider the subbasis $\mathfrak{B}_i$ corresponding to X_i $(1 \leq i \leq s)$ so that

$$\mathfrak{B} = \bigcup_{i=1}^{s} (\bigcup^{n_i} \mathfrak{B}_i)$$

It is convenient to write as follows:

$$\bigcup^{n_i} \mathfrak{B}_i = \mathfrak{B}_{i,1} \bigcup \mathfrak{B}_{i,2} \bigcup \cdots \bigcup \mathfrak{B}_{i,n_i}$$

Let $\overline{f}$ be the image of f under the projection $\epsilon_0(X) \longrightarrow \sigma_0(X)$. Recall that $\sigma_0(X)$ is a subgroup of Σ_n. Suppose the order of $\overline{f}$ is m. For a subbasis $\mathfrak{B}_{i,j}$, if $\mathbf{e} \in \mathfrak{B}_{i,j}$ and $f^m = f \circ \cdots \circ f$, then $f^m(\mathbf{e}) = k\mathbf{e}$ for suitable k. Fix such a basis element $\mathbf{e_0} \in \mathfrak{B}_{i,j}$. We will inductively define m maps $\{f_i\}$ in $\epsilon_0(X)$. Let $f_1 = f$ and, for $\mathbf{b} \in \mathfrak{B}$, we define

$$f_i(\mathbf{b}) = \begin{cases} \ell \cdot f(\mathbf{b}) & \text{if } f_{i-1}(\mathbf{e_0}) \in < \mathfrak{B}_{i_0,j_0} > \text{ and } \mathbf{b} \in \mathfrak{B}_{i_0,j_0} \\ f(\mathbf{b}) & \text{otherwise} \end{cases}$$

where $<\mathfrak{B}_{i_0,j_0}>$ means the vector space spanned by $\mathfrak{B}_{i_0,j_0}$. Let $f_\psi = f_m \circ \cdots \circ f_1$. Thus $f_\psi(\mathbf{e_0}) = \ell^{m-1}k \cdot \mathbf{e_0}$. For $\mathbf{e} \in \mathfrak{B}_{i,j}$ we claim that $f_{i-1}(\mathbf{e_0}) \in <\mathfrak{B}_{i_0,j_0}>$ if and only if $f_{i-1}(\mathbf{e}) \in <\mathfrak{B}_{i_0,j_0}>$. If not, we would have $f_\psi(\mathbf{e}) = \ell^q k \cdot \mathbf{e}$ for some $q < m-1$. This is a contradiction, since each X_i is indecomposable. Consequently $C_2(f)$ is expressed as a block-wise monomial matrix. Each non-singular block induces a rational equivalence. By Lemma 5.1, we obtain the desired result. □

Remark. There are essential self–maps of an indecomposable space X which are not rational equivalences. Here is an example: When $BS^3 \to X \xrightarrow{f} BS^3$ is a fibration and $X \not\simeq BS^3 \times BS^3$, the space X must be indecomposable, since S^3 is a simple Lie group. It is not hard to construct an essential self–map of X as a composite map $X \xrightarrow{f} BS^3 \to X$.

For $f \in \epsilon_0(X)$, the induced $\mathbb{Z}_2^\wedge$–matrix $C_2(f)$ is monomial. With respact to a suitable basis, i.e. up to $Aut(BG_2^\wedge)$, we can write

$$C_2(f) = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_m \end{pmatrix}$$

where each D_i is an $n_i \times n_i$ monomial matrix of the following form for $i \leq m-1$:

$$D_i = \begin{pmatrix} 0 & & & a_i \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

and D_m is a diagonal matrix. Consequently, taking a suitable representative for A_X, we may assume that $C_2(f)$ has the above form.

5.2 Lemma. *Let $P(x) \in \mathbb{Z}[x]$. If $P(x) = \prod_{i=1}^r P_i(x)$ where each $P_i(x)$ is a monic polynomial over $\mathbb{Q}$, then $P_i(x) \in \mathbb{Z}[x]$ for any i.*

Proof of Theorem 5. Since X is indecomposable, we see $\chi_f = \prod_{j=1}^r (t^{n_j} + b_j)^{e_j}$ with $n_1 > n_2 > \cdots > n_r$, where each a_i, which is an entry of the above monomial matrix D_i, is equal to one of $\{-b_j\}$'s. It suffices to show $b_i \in \mathbb{Q}$. The coefficient of $t^{(\sum_{j=1}^r n_j e_j - n_r)}$–term, which is the second largest term of χ_f is $e_r b_r$. Since $e_r b_r \in \mathbb{Z}$, it follows that $b_r \in \mathbb{Q}$. Inductively, we see that each $b_j \in \mathbb{Q}$. Consequently $a_i \in \mathbb{Z}_{(2)}$ for all i. Since $\chi_f \in \mathbb{Z}[t]$, Lemma 5.2 implies $a_i \in \mathbb{Z}$. Thus $C_2(f)$ is a $\mathbb{Z}$–matrix □

6. Some results about $\sigma_0(X)$, the types of rational equivalences.

If $X \in Genus(BG)$, then we obtain the short exact sequence of the monoids

$$\delta_0(X) \to \epsilon_0(X) \to \sigma_0(X)$$

Here $\sigma_0(X)$ is a subgroup of the symmetric group Σ_n. For example, we see $\sigma_0(BG) = \Sigma_n$. It is natural to ask if any subgroup of Σ_n is realizable as $\sigma_0(X)$ for some $X \in Genus(BG)$. The answer is no. We will show that if $A_n \subset \epsilon_0(X)$, then $\Sigma_n \subset \epsilon_0(X)$.

Let $\rho : A_n \longrightarrow GL(n, \mathbb{Z})$ be the permutation representation and let U belong to $GL(n, \mathbb{Q}_p^\wedge)$. We recall that $U^{-1}\rho(x)U$ is a monomial matrix for any $x \in A_n$ if and only if the set of columns of U, say $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$, is A_n–invariant up to scalar multiplication. Let $< \mathbf{u}_i >$ denote the $\mathbb{Q}_p^\wedge$–line containing $\mathbf{u}_i$. Then the set of lines $\{< \mathbf{u}_1 >, \cdots, < \mathbf{u}_n >\}$ is invariant under the A_n–action. We will determine all of the A_n–invariant sets $S = \{< \mathbf{v}_1 >, \cdots, < \mathbf{v}_n >\}$, where $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is linearly independent. To do so, we first investigate the orbits $A_n\cdot < \mathbf{v} >$ for $\mathbf{v} \in \bigoplus^n \mathbb{Q}_p^\wedge$ with $|A_n\cdot < \mathbf{v} > | \leq n$. Of course such a orbit can be a part of S.

6.1 Lemma. *For positive integers $\{n_i\}$ we have $n_1! \cdots n_k! \leq ((\sum_{i=1}^k n_i) - k + 1)!$.*

Proof. If $k = 1$, both sides are $n_1!$. So this statement holds. If $k = 2$, then $(n_1 + n_2 - 1)! = (n_1 + n_2 - 1)(n_1 + n_2 - 2) \cdots (n_1 + 1) \geq n_2! n_1!$. Assume now that the statement holds up to $k - 1$ with $k \geq 2$. Then $n_1! \cdots n_k! = n_1! \cdots n_{k-1}! n_k! \leq ((\sum_{i=1}^{k-1} n_i) - (k-1) + 1)! n_k! \leq ((\sum_{i=1}^{k-1} n_i) - (k - 1) + n_k)! = ((\sum_{i=1}^{k} n_i) - k + 1)!$. $\square$

6.2 Lemma. *Suppose $n \geq 5$ and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is the canonical basis of $\bigoplus^n \mathbb{Q}_p^\wedge$. If $|A_n\cdot < \mathbf{v} > | \leq n$, then up to permutation $< \mathbf{v} >=< \mathbf{e}_1 >$ or $< a\mathbf{e}_1 + b\mathbf{e}_2 + \cdots + b\mathbf{e}_n >$ for some a and b.*

Proof. If $\mathbf{v} = \sum_{i=1}^n \lambda_i \mathbf{e}_i$ and k of the coefficients $\{\lambda_i\}_{i=1}^n$ are zero, then the isotopy subgroup $(A_n)_{<\mathbf{v}>}$ is included in $(\Sigma_k \times \Sigma_{n-k}) \cap A_n$. Hence

$$\begin{aligned} |A_n\cdot < \mathbf{v} > | &= \frac{|A_n|}{|(A_n)_{<\mathbf{v}>}|} \\ &\geq \frac{\frac{n!}{2}}{\frac{k!(n-k)!}{2}} \\ &= \binom{n}{k} \end{aligned}$$

Thus $|A_n\cdot < \mathbf{v} > | \leq n$ implies $k = 0, 1, n - 1$.

If $k = n - 1$, then $< \mathbf{v} >=< \mathbf{e}_i >$ for some i. So we're done. Suppose

$k = 1$. Without loss of generality, we may assume $\mathbf{v} = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_{n-1}\mathbf{e}_{n-1}$. Note that $A_{n-1} \subset (A_n)_{<\mathbf{v}>}$ since $|A_n \cdot < \mathbf{v} > | \leq n$. We will consider two cases.

Case 1. n is even.

Notice that, since the cycle $\tau = (1\ 2\ \cdots n-1) \in (A_n)_{<\mathbf{v}>}$ we see $\tau \cdot \mathbf{v} = \alpha\mathbf{v}$ for some nonzero elements α in $\mathbb{Q}_p^\wedge$, and $\tau \cdot \mathbf{v} = \lambda_1\mathbf{e}_2 + \lambda_2\mathbf{e}_3 + \cdots + \lambda_{n-2}\mathbf{e}_{n-1} + \lambda_{n-1}\mathbf{e}_1$. Hence $\alpha\lambda_i = \lambda_{i-1} (2 \leq i \leq n-1)$ and $\alpha\lambda_1 = \lambda_{n-1}$. This implies that if $\beta = \frac{1}{\alpha}$, then $\lambda_i = \beta^{i-1}\lambda_1$ and $\beta^{n-1} = 1$. Thus $< \mathbf{v} > = < \mathbf{e}_1 + \beta\mathbf{e}_2 + \cdots + \beta^{n-2}\mathbf{e}_{n-1} >$. Since $(1\ 2)(3\ 4) \in (A_n)_{<\mathbf{v}>}$ and $n \geq 6$, we can show $\beta = 1$ using an analogous result.

Case 2. n is odd.

Since $\tau = (1\ 2\ \cdots n-2) \in (A_n)_{<\mathbf{v}>}$, we can show $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2}$. The element $(1\ 2)(n-2\ n-1) \in (A_n)_{<\mathbf{v}>}$ enable us to see $\lambda_1 = \lambda_{n-1}$. We therefore conclude that $< \mathbf{v} > = < \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{n-1} >$ up to permutation.

It remains to show the case $k = 0$. For $\mathbf{v} = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n$, we consider the set of the $n-1$ coefficients $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}\}$. Suppose this set consists of k elements $\{\mu_1, \cdots, \mu_k\}$. Let $n_i = card\{\lambda_j | \lambda_j = \mu_i\}$ $(1 \leq i \leq k)$. Notice that if $\sigma \in \Sigma_{n-1}$ and $\sigma \cdot < \mathbf{v} > = < \mathbf{v} >$, then $\sigma \cdot \mathbf{v} = \mathbf{v}$. Hence, using Lemma 6.1, we have the following:

$$\begin{aligned} |A_n \cdot < \mathbf{v} > | &= \frac{|A_n|}{|(A_n)_{<\mathbf{v}>}|} \geq \frac{|A_n|}{|(\Sigma_n)_{<\mathbf{v}>}|} = \frac{1}{2}\frac{|\Sigma_n|}{|(\Sigma_n)_{<\mathbf{v}>}|} \\ &\geq \frac{1}{2}|\Sigma_{n-1} \cdot \mathbf{v}| = \frac{1}{2}\frac{(n-1)!}{n_1! \cdots n_k!} \geq \frac{1}{2}\frac{(n-1)!}{(n-k)!} \\ &= \frac{1}{2}(n-1)\cdots(n-k+1). \end{aligned}$$

Consequently if $k \geq 3$, then $|A_n \cdot < \mathbf{v} > | > n$ since $n \geq 5$. We may conclude that the set of all coefficients $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ contains at most two elements. It is not hard to show $< \mathbf{v} > = < a\mathbf{e}_1 + b\mathbf{e}_2 + \cdots + b\mathbf{e}_n >$ up to permutation. This completes the proof. □

6.3 Lemma. *Let $\rho : \Sigma_n \longrightarrow GL(n, \mathbb{Z})$ be the permutation representation and let U belong to $GL(n, \mathbb{Q}_p^\wedge)$. If $U^{-1}\rho(x)U$ is a monomial matrix with entries in $\mathbb{Z}_p^\wedge$ for any $x \in A_n$, then $U^{-1}\rho(y)U$ is a monomial matrix with entries also in $\mathbb{Z}_p^\wedge$ for any $y \in \Sigma_n$.*

Proof. According to Lemma 6.2, the matrix U can be taken as

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \text{ or } \begin{pmatrix} a & b & \dots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a \end{pmatrix}$$

up to permutation. Suppose first that

$$U = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}.$$

Let $\tau = (i\ j)(k\ \ell) \in A_n$. Then the (i,j)–th entry of the monomial matrix $U^{-1}\tau U$ is equal to $\frac{\alpha_j}{\alpha_i}$. Since $det(U^{-1}\tau U) = 1$, any entry of $U^{-1}\tau U$ is a unit in $\mathbb{Z}_p^\wedge$. Consequently for any j there is a unit $u_j \in \mathbb{Z}_p^\wedge$ such that $\alpha_j = \alpha_1 u_j$. Hence $U = \alpha_1 diag(1, u_2, \cdots, U_n)$. If $U_1 = \frac{1}{\alpha_1}U \in GL(n, \mathbb{Z}_p^\wedge)$, then $U^{-1}\rho(y)U = U_1^{-1}\rho(y)U_1 \in GL(n, \mathbb{Z}_p^\wedge)$ for any $y \in \Sigma_n$.
Next suppose

$$U = \begin{pmatrix} a & b & \dots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a \end{pmatrix}.$$

Then $U^{-1}\rho(y)U = \rho(y)$ for any $y \in \Sigma_n$. □

Proof of Theorem 6. Note (RA = AC equation) that $R(x)A_2 = A_2 x$ for $x \in A_n$. We recall that two $\mathbb{Q}$–representaions are similar if and only if they afford the same character, since $char(\mathbb{Q}) = 0$. Thus there is $Q \in GL(n, \mathbb{Q})$ such that $Q^{-1}xQ = R(x)$ for any $x \in A_n$. Consequently QA_2 belongs to the centralizer of the alternating group, and hence of the symmetric group. Notice, for other p, that $C_p(x) = A_p^{-1}Q^{-1}xQA_p$ is a monomial matrix with entries in $\mathbb{Z}_p^\wedge$ for any $x \in A_n$. Lemma 6.3 shows that if $C_p(y) = A_p^{-1}Q^{-1}yQA_p$ and $R(y) = Q^{-1}yQ$, then $R(y)A_P = A_p C_p(y)$ for any $y \in \Sigma_n$. This means $\Sigma_n \subset \epsilon_0(X)$ since $C_p(y) \in GL(n, \mathbb{Z}_p^\wedge)$. □

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