HOMOTOPY UNIQUENESS OF CLASSIFYING SPACES OF COMPACT CONNECTED LIE GROUPS AT PRIMES DIVIDING THE ORDER OF THE WEYL GROUP

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1. Introduction.

As a truism, Lie groups - in particular compact connected Lie groups - are very rigid objects. The perhaps best known instance of this rigidity was formulated in Hilbert's fifth problem and proved by Gleason, Montgomery and Zippin in the early fifties: It requires only very weak assumptions on the topology of a topological group to get a Lie group(for a survey see [H]).

Trying to distinguish two compact connected Lie groups, another kind of rigidity

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occurs. Very often, the rich structure of a Lie group is totally described by little information. For example, simply connected compact Lie groups or compact connected Lie groups up to the local isomorphism type are classified by pure combinatorical data, namely the Dynkin diagram. Semi simple Lie groups are distinguished by their normalizer of the maximal torus [C–W–W].

Similar phenomena seem to occur, if one considers the classifying space BG of a compact connected Lie group G. Surprisingly, pure algebraic data, given by cohomology or complex K–theory, is enough to distinguish BG as a space from other spaces. That is what most of the paper is about.

p–adic completion of spaces makes life a lot easier. Most of the results are about the p–adic completion $BG^\wedge_p$. For a large class of compact connected Lie groups, 'global' results are also obtained.

We are concerned with three concepts: The homotopy type, the p–adic type, and the mod–p type of a classifying space. We will explain these concepts in detail in a moment. The last two notions are purely algebraic. Each concept is weaker than the preceeding one. The main theorems will say that, under certain conditions, the first two are equivalent and characterize the homotopy type of $BG^\wedge_p$. That is what we understand by homotopy uniqueness.

We will use the following notation throughout: $T_G \hookrightarrow G$ denotes a fixed maximal torus of G, $N(T_G) \hookrightarrow G$ the normalizer of T_G, and W_G the Weyl group of G.

We say a p-complete space X has the *mod–p type* of BG, if there exists an isomorphism

$$\overline{\phi} : H^*(X;\mathbb{Z}/p) \longrightarrow H^*(BG;\mathbb{Z}/p)$$

as algebras over the Steenrod algebra. Under this condition, with an extra assumption for $p = 2$, Dwyer, Miller, and Wilkerson constructed a 'maximal torus' $f_T : BT_X \longrightarrow X$ of X, T_X a torus, and an action of W_G on BT_X. With the trivial action on X the map f_T is W_G–equivariant up to homotopy. There also exists a map $BT_G \longrightarrow BT_X$ such that

$$\begin{array}{ccc} BT_G & \longrightarrow & BT_X \\ \downarrow & & \downarrow \\ BG & \overset{\overline{\phi}}{\dashrightarrow} & X \end{array}$$

commutes and is W_G–equivariant in mod–p cohomology [D-M-W 2]. Here, W_G acts trivially on BG and X. The dotted arrow indicates that the map only exists in cohomology, but looks like being induced by a topological map. For details and exact definitions, see section 7.

For a space X with the mod–p type of BG, we say that X has the *p–adic type* of BG, if there exist a W_G–equivariant map $BT_{G}{}^\wedge_p \longrightarrow X$, and an isomorphism

$$\phi^\wedge_p : H^*(X;\mathbb{Z}^\wedge_p) \longrightarrow H^*(BG;\mathbb{Z}^\wedge_p)\ ,$$

compatible with $\overline{\phi}$ via the long exact sequence in cohomology associated to the sequence $\mathbb{Z}_p^\wedge \longrightarrow \mathbb{Z}_p^\wedge \longrightarrow \mathbb{Z}/p$, such that

$$\begin{array}{ccc} & BT_G & \\ \swarrow & & \searrow \\ BG & \overset{\phi_p^\wedge}{\dashrightarrow} & X \end{array}$$

commutes in p–adic cohomology. Roughly speaking, this means that the W_G–actions on BT_G and BT_X are conjugate over $\mathbb{Z}_p^\wedge$. On the other hand we can understand this as a topological realization of

$$H^*(X;\mathbb{Z}_p^\wedge) \xrightarrow{\phi_p^\wedge} H^*(BG;\mathbb{Z}_p^\wedge) \longrightarrow H^*(BT_G;\mathbb{Z}_p^\wedge)$$

with W_G as Weyl group.

For every compact connected Lie group G, there exists a finite covering

$$K \longrightarrow G_s \times T \longrightarrow G \ ,$$

where K is finite abelian, G_s is simply connected, and T is a torus. Such coverings we call finite universal. Only the cover $G_s \times T$ is uniquely determined, but not the map. We could compose the covering with a self covering of T. The factor G_s is called the simply connected part of G.

Definition. Let G be a compact connected Lie group.

(1) BG is called *p–torsion free*, if $H^*(BG;\mathbb{Z})$ or, equivalently $H^*(G;\mathbb{Z})$, contains no p–torsion.

(2) G is called *p–convenient* if BG is p–torsion free and if

$$H^*(BG_s;\mathbb{Z}/p) \cong H^*(BT_{G_s};\mathbb{Z}/p)^{W_G} \ .$$

(3) G is called *pseudo simply connected*, if G is a product of unitary groups, and simply connected Lie groups which are not isomorphic to $SU(n)$ (i.e. we replace $SU(n)$ by $U(n)$).

(4) G is called *pseudo projective* if there exists a homomorphism $Z(G_s) \longrightarrow T$ of the center of G_s into a torus T, such that $G \cong (G_s \times T)/Z(G_s)$.

The center of a pseudo projective Lie group is connected. This condition is also sufficient to characterize pseudo projective Lie groups. Both implications follow by easy arguments from the definition.

If $G := (G_s \times T)/Z(G_s)$ is p–convenient and pseudo projective, then $\pi_1(G) \cong \pi_2(BG) \cong H_2(BG;\mathbb{Z})$ is p–torsion free. This is equivalent to the condition that the kernel of the homomorphism $Z(G_s) \longrightarrow T$ is injective on p–torsion.

Now we are prepared to state our main theorems.

Theorem 1.1. *Let G be a compact connected Lie group and let X be a p–complete space with the mod–p type of BG.*

1. *If G is p–convenient, there exists a compact connected Lie group H such that BH has the same mod-p type as BG and X and such that X has the p–adic type of BH.*
2. *If G is p–convenient and simply connected, pseudo simply connected, or pseudo projective, or if G is a product of unitary groups, then X has the p–adic type of BG.*
3. *If p does not divide the order of the Weyl group W_G, then X has also the p–adic type of BG.*

Theorem 1.2. *Let G be a p–convenient compact connected Lie group or a product of unitary groups. If X has the p–adic type of BG, then X and $BG^{\wedge}_p$ are homotopy equivalent.*

Theorem 1.1 and 1.2 together imply the following corollary:

Corollary 1.3. *Let G satisfy one of the following conditions:*

1. *G is p–convenient and simply connected, pseudo simply connected, or pseudo projective.*
2. *G is a product of unitary groups.*
3. *p does not divide the order of W_G.*

If X has the mod–p type of BG, then X and $BG^{\wedge}_p$ are homotopy equivalent.

The condition of being p–convenient is essential for our proofs. A more natural condition for theorems of this type would be given by conditions like BG is mod–p polynomial, i.e. $H^*(BG;\mathbb{Z}/p)$ is a polynomial algebra, or BG is p–torsion free. For odd primes we can weaken our technical assumptions.

1.4 Proposition. *Let G be a compact connected Lie group. For p an odd prime, the following conditions are equivalent:*

1. *G is p–convenient.*
2. *BG is p–torsionfree.*
3. *BG is mod–p polynomial.*

For $p = 2$, the spaces $BSO(n)$, $n \geq 3$, and BG_2 are mod–2 polynomial, but not 2–torsion free. $BSp(n)$, $n \geq 1$, is 2–torsion free, but $Sp(n)$ not 2–convenient. Our methods do not cover the case $G = Sp(n)$ for $p = 2$. But we can handle the case of $U(2)$, respectively the case of products of unitary groups, which are not 2–convenient in general.

If p does not divide the order of W_G, the Lie group G is always p–convenient. In this case, $H^*(BG,\mathbb{Z}/p) \cong H^*(BT_G,\mathbb{Z}/p)^{W_G}$ and $H^*(BG_s \times BT,\mathbb{Z}/p) \cong H^*(BG,\mathbb{Z}/p)$ [Bo 2] [D].

The following table lists up the simple simply connected Lie groups and the primes for which the group is p–convenient, p–torsionfree, or which satisfy the condition $(p, |W_G|) = 1$.

Lie group	$((p, \|W_G\|) = 1)$	p–torsion free	p–convenient
$SU(2)$	p odd	all p	p odd
$SU(n),\ n \geq 3$	$p > n$	all p	all p
$Spin(n)$	$p > n$	p odd	p odd
$Sp(n)$	$p > n$	p odd	p odd
G_2	$p \geq 5$	p odd	p odd
F_4	$p \geq 5$	$p \geq 5$	$p \geq 5$
E_6	$p \geq 7$	$p \geq 5$	$p \geq 5$
E_7	$p \geq 11$	$p \geq 5$	$p \geq 5$
E_8	$p \geq 11$	$p \geq 7$	$p \geq 7$

For the calculation of the p–torsion see [Bo 3].

Products of p–convenient Lie groups are also p–convenient. We can characterize the p–convenient Lie group in terms of finite universal coverings.

1.5 Proposition. *Let $K \to G_s \times T \to G$ be a finite universal covering of the compact connected Lie group G. Then G is p–convenient if and only if G_s is p–convenient and $K_s := K \cap G_s \times \{1\}$ has order coprime to p.*

After having studied the local problem, it is natural to ask for the analogous global question. In some cases, we can offer a characterization of the integral homotopy type of BG.

Theorem 1.6. *Let G be a 2–convenient compact connected Lie group,such that BG is torsion free, or let G be a product of unitary groups. Let X be a simply connected* CW*–complex of finite type, such that $H^*(X;\mathbb{Z})$ is torsion free. If there exists an isomorphism*

$$K(Y) \cong K(BG)$$

as λ–rings, then Y and BG are homotopy equivalent.

Questions of this type were first studied by Dwyer, Miller, and Wilkerson. They proved the analogous statement to corollary 1.3 for $SU(2)$ and $SO(3)$ at all primes [D-M-W 1]. They also proved corollary 1.3 under the condition (3) [D-M-W 2]. McClure and Smith got the same result for $U(2)$ at $p = 2$ [M-S]. For $U(2)$, theorem 1.6 was already proved in [N-S 2].

The problem when proving theorems 1.1 and 1.2 is to pass from more or less pure algebraic data to geometric information. Let X be a space of the mod–p type of BG. Then, as mentioned above, there exists a maximal torus $f_T : BT_X \longrightarrow X$ and Weyl group action of W_G on $BT_X{}^\wedge_p$. The associated representation $L^*T_X{}^\wedge_p := H^2(BT_X;\mathbb{Z}^\wedge_p)$ of W_G is mod-p isomorphic to $L^*T_G{}^\wedge_p := H^2(BT_G;\mathbb{Z}^\wedge_p)$ (see proposition 7.2). To prove theorem 1.1 we have to study the p–adic liftings of the W_G–module $L^*T_G/p := H^2(BT_G;\mathbb{Z}/p)$.

The construction of a homotopy equivalence $BG_p^\wedge \longrightarrow X$ (i.e the proof of theorem 1.2), is based on the mod–p decomposition of BG of Jackowski, McClure, and Oliver [J-M-O] and on two ways to calculate $map(BV, BG)$, V an elementary abelian p–group, by Lannes–theory [L 2] and by using the centralizer of V in G [D-Z][L 2]. The structure of BG which has to be taken into account is too rich for constructing a map $BG \longrightarrow X$ directly. The mod–p decomposition of BG breaks $BG_p^\wedge$ into simpler spaces, namely the classifying spaces of p–toral stubborn subgroups (for the definition of stubborn see section 2). In section 2 we set up a program which reduces the construction of a homotopy equivalence $BG_p^\wedge \longrightarrow X$ to a series of propositions and lemmas.

This program goes through for pseudo simply connected Lie groups and products of unitary groups. In the case of simply connected Lie groups, all necessary propositions and lemmas should also be true, but the proofs are much harder. The difficulties come from dealing with $SU(n)$ (see proposition 6.7 and 12.1). In particular, $N(T_{U(n)})$, as a wreath product, is much simpler to handle than $N(T_{SU(n)})$, respectively the Weyl group action of $\Sigma_n = W_{U(n)} = W_{SU(n)}$ on $T_{U(n)}$,which is induced from the trivial action on S^1, is simpler than the one on $T_{SU(n)}$.

Every compact connected Lie group $G = (G_s \times T)/K$ is the quotient of the product of a simply connected Lie group G_s and a torus T by a finite abelian group K. This classification allows to pass first from pseudo simply connected to simply connected Lie groups, and second to general compact connected Lie groups which completes the proof of theorem 1.2.

The paper is organized as follows: Section 3 contains background including, what is needed from Lannes theory, [D-Z], and [No 1]. In section 4, p–convenient compact connected Lie group are discussed and the propositions 1.4 and 1.5 are proved. For later purpose we need information about the cohomology of Weyl groups in low dimensions. These calculations are done in section 5. In section 6 we study p-toral stubborn subgroups of pseudo simply connected Lie groups.

The construction of a maximal torus for spaces of the mod–p type of BG, as well as the proof of theorem 1.1 is contained in section 7. The necessary calculations about p–adic Weyl group representations are worked out in section 8 and 9.

For A an abelian p–toral group, there is a generalization of the main results of [D-Z] and [No 1] contained in section 10: namely a calculation of the mod–p cohomology of the components of $map(BA, X)$, where X is a space with the mod–p type of BG. The results of this section are independent of the previous ones, but are already qouted earlier, e.g. in section 9. We put section 10 in this place because we feel that the used arguments are similiar to the ones of section 7, but more complicated.

This algebraic replacement of [D-Z] is necessary for constructing a normalizer of the maximal torus of a space X of the p–adic type of BG, i.e. a map $BN(T_G) \longrightarrow X$ which looks cohomologically like $BN(T_G) \longrightarrow BG$. This is done in section 11 and 12.

After having formulated the classical Lie group theory like maximal tori, Weyl groups, and normalizers for spaces with the p–adic type of BG, we construct a homotopy equivalence $f_G : BG_p^\wedge \longrightarrow X$ in section 13–14, first for pseudo simply connected Lie groups and products of unitary groups, then for simply connected Lie groups, and finally in the general case. This finishes the proof of theorem 1.2.

In the last sections we discuss the integral homotopy type of BG and prove theorems 1.6.

Whenever we deal with completions, we mean completion in the sense of Bousfield and Kan [B-K].

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2. The program.

For a space X with the p–adic type of BG the construction of a map $f : BG \longrightarrow X$ is based on the mod–p approximation of BG of [J-M-O] via p–toral subgroups of G, i.e. extensions of a torus by a finite p–group. A p–toral subgroup P of G is called stubborn if the Weyl group $W(P) := N(P)/P$, the quotient of the normalizer of P in G by P, is finite and p–reduced, i.e. the intersection of all p–Sylow subgroups of $W(P)$ is trivial.

Let $\mathcal{R}_p(G) \subset \mathcal{O}(G)$ be the full subcategory of the orbit category $\mathcal{O}(G)$, whose objects are homogenous spaces G/P with P stubborn and p–toral. Let EG denote an acyclic free G–CW–complex.

2.1 Theorem [J-M-O 1]**.** *Let G be a connected Lie group. The canonical map*

$$\underset{\mathcal{R}_p(G)}{\underrightarrow{\text{holim}}}\ EG \times_G G/P \longrightarrow BG$$

is a p–local equivalence.

Remark. A map $X \longrightarrow Y$ is a p–local equivalence, if it induces an isomorphism $H^*(Y;\mathbb{Z}_{(p)}) \longrightarrow H^*(X;\mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ are the integers localised at p. Analogously we speak of mod–p and p–adic equivalences.

Because every p–toral subgroup $P \hookrightarrow G$ is conjugate to a subgroup of $N(T_G)$ [J-M-O 1], the theorem is still true if we take the full subcategory of those objects G/P of $\mathcal{R}_p(G)$, where P is in $N(T_G)$. This subcategory contains at least one object of every isomorphism class of objects of $\mathcal{R}_p(G)$. This full subcategory we call $\mathcal{R}_p(G)$ too.

Let X be a p-complete space with the p–adic type of BG, and let

$$\phi_p^\wedge : H^*(X;\mathbb{Z}_p^\wedge) \xrightarrow{\cong} H^*(BG;\mathbb{Z}_p^\wedge)$$

be an isomorphism of algebras such that

$$\begin{array}{ccc} & BT_G & \\ \swarrow & & \searrow \\ BG & \overset{\phi_p^\wedge}{- - - - \dashrightarrow} & X \end{array}$$

is W_G–equivariant and commutes in p–adic cohomology. We set up a program to construct a realization $f_G : BG \longrightarrow X$ of $\phi_p^\wedge$. By theorem 2.1, we have

$$\begin{aligned} map(BG, X) &\simeq map(\operatorname*{holim}_{\overrightarrow{\mathcal{R}_p(G)}} EG \times_G G/P, X) \\ &= \operatorname*{holim}_{\overleftarrow{\mathcal{R}_p(G)}} \; map(EG \times_G G/P, X) \; . \end{aligned}$$

To get the desired map $f_G : BG \longrightarrow X$, we first construct an extension $f_N : BN(T_G) \longrightarrow X$ of $f_T : BT_G \longrightarrow X$, which looks in mod–$p$ cohomology like the map $BN(T_G) \longrightarrow BG$. The map f_N is the homotopy theoretical version of the normalizer of the maximal torus of X.

2.2 Proposition. *Let G be a p–convenient pseudo simply connected Lie group or a product of unitary groups and let X be a p–complete space with the p–adic type of BG.*

(1) *There exists an extension*

$$\begin{array}{ccc} BN(T_G) & & \\ \uparrow & \searrow^{f_N} & \\ BT_G & \xrightarrow{f_T} & X \end{array} \quad .$$

(2) *The map f_N can be chosen such that the diagram*

$$\begin{array}{ccc} & BN(T_G) & \\ {}^{Bi}\swarrow & & \searrow^{f_N} \\ BG & \dashrightarrow & X \end{array}$$

commutes in mod–p cohomology.

Proof. (1) is proved in section 11, and (2) in section 12. $\square$

Now we are ready to begin with the construction of the map $f_G : BG \longrightarrow X$. Let P be a p–toral subgroup of G contained in $N(T_G)$. We define a map $f_P : BP \longrightarrow X$ by the composition $BP \longrightarrow BN(T_G) \xrightarrow{f_N} X$. Denote by $\{BP\}_{\mathcal{R}_p(G)}$ the diagram given by the Borel construction applied to the category $\mathcal{R}_p(G)$. The collection of the maps f_P is a map

$$\{BP\}_{\mathcal{R}_p(G)} \longrightarrow X$$

from the diagram to X, i.e. from any object to X. Because of the second part of the last proposition the diagram $\{BP\}_{\mathcal{R}_p(G)} \longrightarrow X$ commutes in mod–p cohomology. Surprisingly this piece of information is sufficient to prove the following proposition.

2.3 Proposition. *Let G be a p–convenient pseudo simply connected Lie group or a product of unitary groups. Then the diagram $\{BP\}_{\mathcal{R}_p(G)} \longrightarrow X$ commutes up to homotopy.*

Proof. See section 13. □

By theorem 2.1, $\operatorname{holim}_{\overrightarrow{\mathcal{R}_p(G)}} EG \times_G G/P \longrightarrow BG$ is a mod–p equivalence. The homotopy commutativity of the above diagram induces a map on the 1–skeleton of the homotopy colimit to X. The question of extending this map to the total homotopy colimit is decided by the obstruction groups

$$\lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^{i} \quad \pi_j(map(BP, X)_{f_P}) \ ,$$

where $\overleftarrow{\lim}^i$ is the i–th derived functor of the inverse limit functor [B-K] [Wo]. For $X = BG_p^\wedge$ these obstruction groups vanish, if G is simply connected [J-M-O 1], or if G is pseudo simply connected, which is an easy conclusion (see the proof of lemma 13.2). Therefore we compare the two functors

$$\Pi_i(X), \Pi_i(G) : \mathcal{R}_p(G) \longrightarrow \mathcal{A}b$$

given by

$$\Pi_i(X)(G/P) := \pi_i(map(BP, X)_{f_P})$$
$$\Pi_i(G)(G/P) := \pi_i(map(BP, BG_p^\wedge)_{(Bi_P)_p^\wedge}) \ .$$

$\mathcal{A}b$ denotes the category of abelian groups. We remark that $\Pi_1(G)(G/P) \cong \pi_1(BZ_P)$ is an abelian group as well as $\Pi_1(X)(G/P)$ (see section 13).

2.4 Proposition. *If G is a p–convenient pseudo simply connected Lie group or a product of unitary groups, there exists a natural transformation*

$$\mathcal{T} : \Pi_i(G) \longrightarrow \Pi_i(X)$$

which is an equivalence

Proof. See chapter 13. □

Now this proposition and lemma 13.2 imply that

$$0 = \lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^{i} \quad \pi_j(map(BP, BG_p^\wedge)_{(Bi_P)_p^\wedge}) \cong \lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^{i} \quad \pi_j(map(BP, X)_{f_P}) \ .$$

The obstruction groups vanish, and, for a pseudo simply connected Lie group or a product of unitary groups, we get the desired map

$$f : BG \longrightarrow X \ .$$

By construction, f realizes the cohomological isomorphism

$$\phi_p^\wedge : H^*(X; \mathbb{Z}_p^\wedge) \cong H^*(BT_G; \mathbb{Z}_p^\wedge)^{W_G} \cong H^*(BG; \mathbb{Z}/p)$$

we started with, and hence is a homotopy equivalence.

3. Background.

In this chapter we fix a prime p. The cohomology groups are always taken with coefficients in $\mathbb{Z}/p$ and $H^*(\ )$ always means $H^*(\ ;\mathbb{Z}/p)$. We denote by $\mathcal{K}$ the category of unstable algebras over the Steenrod algebra $\mathcal{A}_p$.

Let V be an elementary abelian p–group. An algebra A over $\mathcal{A}_p$ is called of finite type if A is finite in each dimension.

3.1 Theorem [L 2]**.** *If X is a p–complete space and $H^*(X)$ of finite type, then the canonical map*

$$[BV, X] \longrightarrow Hom_{\mathcal{A}_p}(H^*(X), H^*(BV))$$

is an isomorphism.

The evaluation map

$$BV \times map(BV, X) \longrightarrow X$$

induces a cohomological map

$$H^*(X) \longrightarrow H^*(BV) \otimes H^*(map(BV, X)) \ .$$

J. Lannes studied the functor $T^V : \mathcal{K} \longrightarrow \mathcal{K}$ which is the left adjoint of the functor $H^*(BV) \otimes_{\mathbb{Z}/p}$. Taking the adjoint of the evaluation map yields

$$T^V H^*(X) \longrightarrow H^*(map(BV, X)) \ .$$

For any map $g : BV \longrightarrow X$, there is an associated direct summand $T^V_{g^*} H^*(X)$ of $T^V H^*(X)$ which corresponds to the summand $H^*(map(BV, X)_g)$ of $H^*(map(BV, X))$. With respect to this splitting the above map is a direct sum of maps with coordinates

$$T^V_{g^*} H^*(X) \longrightarrow H^*(map(BV, X)_g) \ .$$

3.2 Theorem [L 2, 3.3.2]**.** *Let X be a space, such that $H^*(X)$ is of finite type. Let $g : BV \longrightarrow X$ be a map. The map*

$$T^V_{g^*} H^*(X) \longrightarrow H^*(map(BV, X^\wedge_p)_g)$$

is an isomorphism if $T^V_{g^*} H^*(X)$ is of finite type and one of the following three conditions is satisfied:

(1) *$T^V_{g^*} H^*(X)$ is zero in degree 1.*
(2) *$map(BV, X^\wedge_p)_g$ is 1–connected.*
(3) *There is a connected space Z with the property that $H^*(Z)$ is of finite type and a map*

$$Z \longrightarrow map(BV, X)_g \ ,$$

such that the associated map

$$T^V_{g^*} H^*(X) \longrightarrow H^*(Z)$$

is an isomorphism.

3.3 Theorem ([L 2, 3.4.3])**.** *In addition to the assumptions of theorem 2.2 let X be p–complete. Then the following conditions are equivalent:*

(1) $map(BV, X)_g$ *is p–complete.*
(2) $T^V_{g^*}H^*(X) \longrightarrow H^*(map(BV, X)_g)$ *is an isomorphism.*

If we consider a collection S of maps $BV \longrightarrow X$, then of course we get a direct summand $T^V_{S^*}H^*(X)$, where S^* is the collection of the associated cohomological maps. The theorems 3.2 and 3.3 are still true in this situation [L 2].

Let V be an elementary abelian p–group and let $BV \xrightarrow{f} X \xrightarrow{g} Y$ be a pair of maps, such that the application of the Lannes functor calculates the mod–p cohomology of the mapping spaces $map(BV, X)_f$ and $map(BV, Y)_{gf}$. The following lemma, whose proof is straight forward, describes the induced map

$$H^*(map(BV, g)) : H^*(map(BV, Y)_{gf}) \longrightarrow H^*(map(BV, X)_f)$$

in terms of the T–functor.

3.4 Lemma. *Under the above assumptions, the map $H^*(map(BV, g))$ is given by the application of the Lannes–functor, i.e.*

$$H^*(map(BV, g)) = T^V_f(g) : H^*(map(BV, Y)_{gf}) \longrightarrow H^*(map(BV, X)_f \ .$$

In the case of X being the classifying space BG of a connected Lie group G, the functor T^V can be calculated.

For a p–toral group P we denote by $Rep(P, G)$ the set of representation classes $P \longrightarrow G$, i.e. G–conjugacy classes of homomorphism $P \longrightarrow G$. Passing to classifying spaces gives a map $Rep(P, G) \longrightarrow [BP, BG]$. For a particular representation $\rho : P \longrightarrow G$ let $C_G(\rho)$ denote the centralizer of ρ in G. The obvious product homomorphism $C_G(\rho) \times P \longrightarrow G$ has as adjoint on the level of classifying spaces the map $BC_G(\rho) \longrightarrow map(BP, BG)_{B\rho}$.

3.5 Theorem ([L 2, 3.4.5])**.** *Let V be an elementary abelian p–group. Then the maps*

$$T^V H^*(BG) \longrightarrow H^*(map(BV, BG)) \longrightarrow \coprod_{\rho \in Rep(V,G)} H^*(BC_G(\rho)) \ ,$$

and, if we choose a homomorphism $\rho : V \longrightarrow G$,

$$T^V_{B\rho^*}H^*(BG) \longrightarrow H^*(map(BV, BG)_{B\rho}) \longrightarrow H^*(BC_G(\rho))$$

are isomorphisms.

In [D-Z] and [No 1] mapping spaces are calculated, having the classifying space of a finite p–group respectively a p–toral group as source.

3.6 Theorem ([D-Z], [No 1]). *For a p–toral group P and a compact Lie group G the natural map*

$$Rep(P,G) \longrightarrow [BP,BG]$$

is an isomorphism, and the map

$$BC_G(\rho) \longrightarrow map(BP,BG)_{B\rho}$$

a mod–p equivalence.

For a finite extension of a torus as target, there is a stronger result.

3.7 Proposition. *Let $\rho : H \longrightarrow N$ be a homomorphism from a compact connected Lie group H to a finite extension N of a torus. Then*

$$BC_N(\rho) \longrightarrow map(BH,BN)_{B\rho}$$

is a homotopy equivalence.

Proof. BN fits into the fibration

$$BT \longrightarrow BN \longrightarrow BW \ ,$$

where $W := N/T$, and where T is the toral part of N, i.e. the component of the unit. Because H is connected, the image of ρ is contained in T. Let $Iso(\rho) \subset W$ be the group of those elements, which act trivially on the image of ρ. The Dwyer–Zabrodsky construction induces a commutative diagram

$$\begin{array}{ccccc} \prod_{w\in W/Iso(\rho)} BT & \longrightarrow & BC_N(\rho) & \longrightarrow & BW \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{w\in W/Iso(\rho)} map(BH,BT)_{w\circ B\rho} & \longrightarrow & map(BH,BN) & \longrightarrow & map(BH,BW)_c \ , \end{array}$$

where c denotes the constant map. Since the components of $BC_N(\rho)$ are given by $Iso(\rho)$, the upper row is a fibration. The bottom row is a fibration, because every map $BH \longrightarrow BT$ is induced by a homomorphism [L-M-S]. The left and right columns are equivalences, which follows from a short calculation respectively the Sullivan conjecture [M]. Therefore, the middle column is an equivalence which proves the statement. □

We work mainly with completed spaces. Hence, it is useful to know if the considered mapping spaces are complete.

3.8 Proposition [B-N]. *Let G and H be compact Lie groups, H connected. Then, for any map $f : BG \longrightarrow BH$, the component $map(BG, BH^\wedge_p)_{f^\wedge_p}$ is p–complete, and*

$$(map(BG,BH)_f)^\wedge_p \longrightarrow map(BG,BH^\wedge_p)_{f^\wedge_p}$$

is a homotopy equivalence.

This leads to a result slightly stronger than theorem 3.6 (see also [J-M-O 1]).

3.9 Theorem. *For any homomorphism $\rho : P \longrightarrow G$ of a p–toral group P into a compact connected Lie group G, the map*

$$BC_G(\rho)^\wedge_p \longrightarrow map(BP, BG^\wedge_p)_{B\rho^\wedge_p}$$

is a homotopy equivalence.

Now let π be a group acting on a space X. Let X^π denote the fixed–point set. The homotopy fixed–point set $X^{h\pi}$ is defined to be the mapping space $map_\pi(E\pi, Y)$ of equivariant maps. Here $E\pi$ is an acyclic free π–CW–complex of finite type. There is another description of $X^{h\pi}$. Consider the fibration

$$(*) \qquad X \longrightarrow E\pi \times_\pi X \longrightarrow B\pi \ .$$

Note the one to one relation between sections in this bundle and elements in $X^{h\pi}$. Thus, we can also define $X^{h\pi}$ as the fiber of the fibration

$$X^{h\pi} \longrightarrow map(B\pi, E\pi \times_\pi X)_{sec} \longrightarrow map(B\pi, B\pi)_{id} \ ,$$

where $map(-,-)_{sec}$ is the union of the components of the sections in the bundle $(*)$.

For an elementary abelian p–group V acting on X, the multiplication $BV \times BV \longrightarrow BV$ of the H–space BV induces an equivalence

$$\alpha : BV \longrightarrow map(BV, BV)_{id} \ .$$

This multiplication can also be used to construct a map

$$\beta : BV \times X^{hV} \longrightarrow map(BV, EV \times_V X)_{sec}$$

by defining $\beta(b, f) := f \circ \alpha(b)$. Here $f \in X^{hV}$ is interpreted as a section $BV \longrightarrow EV \times_V X$. The map β fits into the diagram

$$(**) \qquad \begin{array}{ccccc} X^{hV} & \longrightarrow & BV \times X^{hV} & \longrightarrow & BV \\ \Big\| & & \beta \Big\downarrow & & \alpha \Big\downarrow \\ X^{hV} & \longrightarrow & map(BV, EV \times_V X)_{sec} & \longrightarrow & map(BV, BV)_{id} \end{array}$$

This is a trivialization of the fibration in the bottom line.

Let $s : EV \longrightarrow X$ be an equivariant map respectively $s : BV \longrightarrow EV \times_V X$ a section. Applying the Lannes functor $T^V_{s^*}$ gives a map

$$(***) \qquad T^V_{s^*} H^*(EV \times_V X) \longrightarrow H^*(map(BV, EV \times_V X)_s) \ .$$

The natural isomorphisms

$$H^*(BV) \xrightarrow{\cong} T^V_{id^*} H^*(BV) \xrightarrow{\cong} H^*(map(BV, BV)_{id})$$

and the canonical map $H^*(BV)\longrightarrow\mathbb{Z}/p$ allow to apply the functor $-\otimes_{H^*(BV)}\mathbb{Z}/p$ to $(***)$. This yields a map

$$T^V_{s^*}H^*(EV\times_V X)\otimes_{H^*(BV)}\mathbb{Z}/p\longrightarrow H^*(map(BV,EV\times_V X)_s)\otimes_{H^*(BV)}\mathbb{Z}/p\ .$$

Because of the equivalences in $(**)$ the right side is isomorphic to $H^*(X_s^{hV})$. Here X_s^{hV} denotes the component of s of the homotopy fixed–point set X^{hV}. In [L 2, § 4] the left side is used to define a functor

$$HF^V: Top_V\longrightarrow\mathcal{K}: X\mapsto HF^V(X):=T^V_{sec^*}H^*(EV\times_V X)\otimes_{H^*(BV)}\mathbb{Z}/p$$

from the category of spaces with a V–action to the category $\mathcal{K}$. By sec we denote the set of homotopy classes of sections in the bundle $EV\times_V X\to BV$, and by sec^* the set of the associated induced maps in cohomology. For any subset $S\subset sec$, i.e. for any set of homotopy classes of homotopy fixed–points, there is an obvious map

$$HF^V_{S^*}(X)\longrightarrow H^*(X_S^{hV})\ .$$

3.10 Theorem ([L 2, 4.9.3]). *Let X be a space with a V–action, such that $H^*(X)$ is of finite type. Let $S\subset\pi_0(X^{hV})$ be a set of homotopy fixed–points. The map*

$$HF^V_{S^*}(X)\longrightarrow H^*((X_p^\wedge)_S^{hV})$$

is an isomorphism, if $H^{*V}_{S^*}(X)$ is of finite type and one of the following two conditions is satisfied:

(1) *$HF^V_{S^*}(X)$ is zero in degree 1.*
(2) *$(X_p^\wedge)_s^{hV}$ is 1–connected for each $s\in S$.*

3.11 Theorem ([L 2, 4.9.2]). *In addition to the assumptions of theorem 3.10 let X be p–complete. Then the following conditions are equivalent:*

(1) *X_S^{hV} is p–complete.*
(2) *$HF^V_{S^*}(X)\longrightarrow H^*(X_S^{hV})$ is an isomorphism.*

3.12 Remark. Let $K\longrightarrow G\longrightarrow H$ be an exact sequence of compact Lie groups. The 'extended' classifying space $\widetilde{BK}:=EG/K\simeq BK$ of K carries a free H action, and $\widetilde{BK}/H=BG$. For any space X, which H acts trivially on, we have

$$\begin{aligned}map(BG,X)&=map(\widetilde{BK}/H,X)\\&=map_H(\widetilde{BK},X)\\&\simeq map_H(EH\times\widetilde{BK},X)\\&=map_H(EH,map(\widetilde{BK},X))\\&=map(\widetilde{BK},X)^{hH}\ .\end{aligned}$$

This implies the following lemma:

3.13 Lemma. *Let $f_K : BK \longrightarrow X$ be a map and $map(BK, X)_{f_K}$ be simple, i.e the fundamental group acts trivial on the homotopy groups.*

(1) The obstructions for extending f_K to a map $f_G : BG \longrightarrow X$ are contained in the groups $H^{*+1}(BH, \pi_*(map(\widetilde{BK}, X)_{f_K}))$ (twisted coefficients).

(2) The homotopy classes of extensions $BG \longrightarrow X$ are classified by obstructions in $H^*(BH, \pi_*(map(\widetilde{BK}, X)_{f_K}))$ (twisted coeficients).

Proof. Extensions $BG \longrightarrow X$ are given by sections in the fibration

$$map(\widetilde{BK}, X)_{f_K} \longrightarrow map(BH, EH \times_H map(\widetilde{BK}, X)_{f_K}) \longrightarrow BH . \qquad \square$$

We finish this section with a purely algebraic result about the Lannes functor. Let A be an unstable algebra over $\mathcal{A}_p$, let W a finite group acting on A, and let $\phi : A \longrightarrow H^*(BV)$ an $\mathcal{A}_p$–algebra map. We call

$$Iso(\phi) := \{w \in W | \phi \circ w = \phi\}$$

the *isotropy group* of ϕ. Because $H^*(BV)$ is an injective object in $\mathcal{K}$, every map $\phi : A^W \longrightarrow H^*(BV)$ can be extended to a map $\phi : A \longrightarrow H^*(BV)$.

3.14 Theorem [L 1]**.**

$$T^V(A^W) = \coprod_{\phi \in Hom_{\mathcal{K}}(A, H^*(BV))/W} T^V_\phi A^W \cong \coprod_{\phi \in Hom_{\mathcal{K}}(A, H^*(BV))/W} (T^V_\phi A)^{Iso(\phi)} ,$$

or, in a compact version, $T^V(A^W) \cong (T^V A)^W$.

4. p-convenient compact connected Lie groups.

Let $K \to G \stackrel{q}{\to} H$ be an exact sequence of connected Lie groups, G and H connected and K finite. We have associated oriented fibrations $BK \to BG \stackrel{Bq}{\longrightarrow} BH$ and $BK \to BT_G \stackrel{Bq}{\longrightarrow} BT_H$.

4.1 Lemma. *If BH is p–torsion free, then BG is p–torsion free.*

Proof. By a theorem of Borel [Bo 3] BG is p–torsion free if and only if every elementary abelian p–subgroup of G is conjugate to a subgroup of T_G. Let V be an elemetary abelian p–subgroup of G. Then, up to conjugation, $q(V) \subset T_H$, and, up to conjugation, $V \subset q^{-1}(q(V)) \subset q^{-1}(T_H) = T_G$. $\square$

For any compact connected Lie group G there exists a finite covering

$$1 \longrightarrow K \longrightarrow \widetilde{G} = G_s \times T \longrightarrow G \longrightarrow 1$$

of compact Lie groups, where G_s is simply connected and T a torus (see for example [B-tD]). Such coverings we call finite universal. The cover $\widetilde{G}$ is unique up to isomorphism, but the homomorphism $\widetilde{G} \longrightarrow G$ is not unique. We can choose a finite self covering $T \longrightarrow T$ and can take the composition $G_s \times T \longrightarrow G_s \times T \longrightarrow G$. Among the finite universal coverings of G there is a minimal one, characterized by the condition that $K \longrightarrow G_s \times T \longrightarrow G_s$ is an injection. The minimal finite universal covering is a quotient of all the others.

Every finite universal covering establishes two associated commutative diagrams of exact sequences of connected Lie groups:

$$(*) \qquad \begin{array}{ccccc} K_s & \longrightarrow & G_s & \longrightarrow & G_s/K_s := \overline{G}_s \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & G_s \times T & \longrightarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ \overline{K} = K/K_s & \longrightarrow & T & \longrightarrow & T/\overline{K} := \overline{T} \end{array}$$

and

$$(**) \qquad \begin{array}{ccccc} K_s & = & K_s & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & G_s \times T & \longrightarrow & G \\ \downarrow & & \downarrow & & \| \\ \overline{K} & \longrightarrow & \overline{G}_s \times T & \longrightarrow & G \ , \end{array}$$

where $K_s := K \cap (G_s \times \{0\})$. Because $H^1(\overline{G}_s; \mathbb{Z}) = 0$, the sequence $\overline{G}_s \longrightarrow G \longrightarrow \overline{T}$ induces isomorphisms $H^1(\overline{T}; \mathbb{Z}) \cong H^1(G; \mathbb{Z})$ and $H^2(B\overline{T}; \mathbb{Z}) \cong H^2(BG; \mathbb{Z})$. On the other hand these isomorphisms determine the maps $G \longrightarrow \overline{T}$ and $BG \longrightarrow B\overline{T}$, and therefore, the fibration $B\overline{G}_s \longrightarrow BG \longrightarrow B\overline{T}$.

If BG is p–torsion free, then, by lemma 4.1, $BG_s \times BT$ and $B\overline{G}_s \times BT$ as well as BG_s and $B\overline{G}_s$ are p–torsionfree. The order $|K_s|$ of K_s is coprime to p, because $\pi_2(B\overline{G}_s) \cong H_2(B\overline{G}_s; \mathbb{Z}) \cong K_s$.

4.2 Lemma. *Let BG be p–torsion free. If G_s or $G_s \times T$ are p–convenient, then G also is p–convenient, and*

$$H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)^{W_G} \ .$$

Proof. Denote by $E_*^{*,*}(T_G)$ and $E_*^{*,*}(G)$ the terms of the mod–p cohomology Leray–Serre spectral sequences of the oriented fibrations

$$\begin{array}{ccccc} BT_{\overline{G}_s} & \longrightarrow & BT_G & \longrightarrow & B\overline{T} \\ \downarrow & & \downarrow & & \| \\ B\overline{G}_s & \longrightarrow & BG & \longrightarrow & B\overline{T} \ . \end{array}$$

By the above remarks $\overline{G}_s$ is p–torsionfree and the spectral sequences collapses, i.e $E_2^{*,*}(G) = E_\infty^{*,*}(G)$ and $E_2^{*,*}(T_G) = E_\infty^{*,*}(T_G)$.

If G_s is p–convenient, we have

$$\begin{aligned} E_\infty^{*,*}(G) &\cong H^*(B\overline{T};\mathbb{Z}/p)\otimes H^*(B\overline{G}_s;\mathbb{Z}/p) \\ &\cong H^*(B\overline{T};\mathbb{Z}/p)\otimes H^*(BT_{\overline{G}_s};\mathbb{Z}/p)^W \\ &\cong (H^*(B\overline{T};\mathbb{Z}/p)\otimes H^*(B\overline{G}_s;\mathbb{Z}/p))^W \\ &\cong E_\infty^{*,*}(T_G)^W \ . \end{aligned}$$

Using the 5–lemma and an induction over the extension problems of the two associated spectral sequences, one can show that $H^*(BG;\mathbb{Z}/p)\cong H^*(BT_G)^W$. □

In [D] is discussed, whether the canonical map

$$H^*(BT_G;\mathbb{Z}_{(p)})^{W_G}\otimes\mathbb{Z}/p \longrightarrow H^*(BT_G;\mathbb{Z}/p)^{W_G}$$

is an isomorphism. By the main theorem, proposition 6, and proposition 8 of [D], this is an isomorphism in the following cases:

$SU(n)\ ,\ n\geq 3$	all primes
$Spin(n),\ Sp(n),\ n\geq 1,\ G_2$	$p\geq 3$
$F_4,\ E_6,\ E_7$	$p\geq 5$
E_8	$p\geq 7$.

If p is odd, the pair (G,p), G simple and simply connected, belongs to the list if and only if BG is p–torsionfree. For $p=2$ the space $BSp(n)$ is 2-torsionfree, but $Sp(n)$ is not 2–convenient.

4.3 Proposition. *Let p be an odd prime, and let G be a compact connected Lie group. If BG is p–torsion free, then G is p–convenient, and*

$$H^*(BG;\mathbb{Z}/p)\cong H^*(BT_G;\mathbb{Z}/p)^{W_G}\ .$$

Proof. Let $K\longrightarrow G_s\times T\longrightarrow G$ be a finite universal covering of G, where G_s is simply connected and T is a torus. We can split $G_s = G_1\times\ldots\times G_r$ into a product of simple simply connected factors G_i. By lemma 4.1 G_s is p–torsionfree. The arguments of the proof of [Bo 2; 20.3] show that the map

$$H^*(BH;\mathbb{Z}_{(p)})\longrightarrow H^*(BT_H;\mathbb{Z}_{(p)})^{W_H}$$

is an isomorphism, if BH is p–torsionfree. The pairs (G_i,p) belong to the above list. We get

$$\begin{aligned} H^*(BG_s;\mathbb{Z}/p) &\cong H^*(BG_s;\mathbb{Z}_{(p)})\otimes\mathbb{Z}/p \\ &\cong H^*(BT_{G_s};\mathbb{Z}_{(p)})^{W_{G_s}}\otimes\mathbb{Z}/p \\ &\cong H^*(BT_{G_s};\mathbb{Z}/p)^{W_{G_s}}\ . \end{aligned}$$

Therefore G_s and $G_s\times T$ are p–convenient. By lemma 4.2 the statement follows. □

Proof of proposition 1.4.
The equivalence of (1) and (2) follows from proposition 4.3 and the equivalence of (2) and (3) from [Bo 1; proposition 5.2]. □

Proof of proposition 1.5.
Let $K \to G_s \times T \to G$ be a universal finite covering of G. Let $W := W_{G_s} \cong W_G$. The diagram (∗) establishes a fibration $B\overline{G}_s \to BG \to B\overline{T}$.

Let us assume that G_s is p–convenient and that the order of $K_s = K \cap G_s \times \{1\}$ is coprime to p. Because $H^*(B\overline{G}_s;\mathbb{Z}/p) \cong H^*(BG_s;\mathbb{Z}/p) \cong H^*(BT_{G_s};\mathbb{Z}/p)^W$, the spectral sequence of the fibration collapses because of degree reasons. This implies that $H^*(BG;\mathbb{Z}/p)$ is concentrated in even degrees and that G is p–convenient.

If G is p–convenientthen G_s is p-convenient by definition. If the order of K_s is divisible by p, the semi simple part $B\overline{G}_s$ has p–torsion. Lemma 4.1 applied to the bottom horizintal row of (∗∗) shows that BG has p–torsion which is a contradiction. □

For later purpose we are interested in centralizers of abelian p–toral subgroups of compact connected Lie groups. The next statement handles this question and has a purely albgraic analogous (see theorem 10.1).

4.4 Proposition. *Let A be a p–toral abelian compact Lie group and G a compact connected Lie group such that BG is p–torsion free. Then the following statements are true:*

(1) *$Rep(A, T_G) \to Rep(A, G)$ is onto.*
(2) *$Rep(A, T_G)/W_G \to Rep(A, G)$ is a bijection.*
(3) *For any homomorphism $\rho : A \to T_G$ the centralizer $C_G(\rho)$ is connected, and $BC_G(\rho)$ is p–torsion free.*

Proof. The first statement follows from [Bo 3; 5.2]. The other conditions we prove by induction on the number of topologically cyclic factors of A.

Let $A \cong S^1$ or $A \cong \mathbb{Z}/p^k$. Inparticuliar, be a toplogical cyclic compact p–toral group. Then part (2) follows from the isomorphism between the orbit space T_G/W_G and the set of conjugacy classes of elements in G (e.g. see [B-tD]).

Let $\rho : A \to T_G$ be an homomorphism and let $g \in C_G(\rho) =: C$ be a p–torsion element or a generator of a torus. Then the group $B :=< \rho(A), g >$, generated by $\rho(A)$ and g is p–toral and therefore, as a subgroup of G, subconjugated to T_G. We choose $h \in G$ such that $hBh^{-1} \subset T_G$. By (2) the conjugation by h on $\rho(A)$ can be realized by an element $n \in N(T_G)$. Now $n^{-1}h$ centralizes ρ and conjugates B into $T_G = T_C$. That is that every p–torsion element of C is subconjugated to T_C. By [J–M–O 1; Appendix] $\pi_0(C)$ is a finite p–group. Therefore the p–torsion elements of G are dense in G. This implies that G is connected which is the first part of (3).

Now let $V \subset C$ be an elementary abelian subgroup. As a subgroup of G, the group $< \rho(A), V >$ is subconjugated to T_G. We can apply the same trick as above to see that V is subconjugated to T_C as a subgroup of C. By [Bo 3] every elementar abelian p–subgroup of C is subconjugated to T_C if and only if BC is p–torsion free. This finishes the proof for cyclic groups.

Any p–toral abelian group A splits into a product $A \cong A_0 \times A_1$ such that $A_1 \cong S^1$ or $A_1 \cong \mathbb{Z}/p^k$. Let $\rho, \rho' : A \to T_G$ be two homorphisms which are conjugate in G. By induction hypothesis $\rho_0 := \rho|_{A_0}$ and $\rho'_0 := \rho'|_{A_0}$ differ on;y by an element in W_G. We can assume that $\rho_0 = \rho'_0$. Then ρ and ρ' correspond to homomophisms $\alpha, \alpha' : A_1 \to T_{C_G(\rho(A_0))} = T_G$ which are conjugated in $C := C_G(\rho(A_0)$. By induction hypothesis again α and α' differ only by an element in W_C. This implies part (2) in the general case.

The last two statements follow from the identity $C_G(\rho) = C_{C_G(\rho(A_0))}(\rho(A_1))$ and the induction hypothesis. □

We finish this section with the following two lemmas.

4.5 Lemma. *If G is p–convenient, the order of the cokernel of $Z(G) \hookrightarrow {T_G}^{W_G}$ is coprime to p.*

Proof. Let $i : A \hookrightarrow {T_G}^{W_G}$ be the p–toral part of ${T_G}^{W_G}$. By lemma 4.4 the centralizer $C := C_G(A)$ is connected and $W_C = W_G$. This implies that $C = G$ and that $A \subset Z(G)$. □

4.6 Lemma. *Let V_G be the maximal elementary abelian p–subgroup of T_G. If $H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)^{W_G}$, then $C_G(V_G) = T_G$.*

Proof. By lemma 4.4 $C_G(V_G)$ is connected and of the same rank as T_G. Let W_C be the Weylgroup of $C_G(V_G)$ which is also the isotropy group of the homomorphism $V_G \longrightarrow G$. Then W_C acts trivially on $H^*(BT_G; \mathbb{Z}/p)$ and $H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)^{W_G/W_C}$. This implies that $H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)$ is an integral Galois extension of degree $|W_G/W_C|$. But it is well known that this degree is given by the Euler characteristic of G/T_G which is the order of W_G. Therefore W_C is the trivial group which proves the statement. □

Remark. If p is odd the lemma is true for every compact connected Lie group G [D-M-W 2].

5. Subgroups of maximal rank and the cohomology of Weyl groups.

For studying the p-stubborn subgroups of the simple simply connected Lie groups, which we will do in the next chapter, it is very useful to have a list of subgroups of maximal rank satisfying certain conditions. For each prime p and each p-convenient simple simply connected Lie group G, we choose two compact connected subgroups H and K of G given by the following list. The first is maximal of maximal rank and contains the second, which only is of maximal rank. The number in brackets denotes the Weyl group index.

G	p	H	K
$Sp(n)$	p odd	$U(n)$ $[2^n]$	$U(n)$ $[2^n]$
$SO(2n+1)$	p odd	$U(n)$ $[2^n]$	$U(n)$ $[2^n]$
$SO(2n)$	p odd	$U(n)$ $[2^{n-1}]$	$U(n)$ $[2^{n-1}]$
G_2	$p=3$	$SU(3)$ $[2]$	$SU(3)$ $[2]$
G_2	$p \geq 5$	$SU(3)$ $[2]$	$U(2)$ $[6]$
F_4	$p \geq 5$	$SU(3) \times_{\mathbb{Z}/3} SU(3)$ $[2^5]$	T_{F_4} $[2^7 \cdot 3^2]$
E_6	$p \geq 5$	$SU(2) \times_{\mathbb{Z}/2} SU(6)$ $[2^2 \cdot 3^2]$	$S^1 \times_{\mathbb{Z}/2} U(5)$ $[2^4 \cdot 3^3]$
E_7	$p \geq 5$	$SU(8)/\mathbb{Z}/2$ $[2^3 \cdot 3^2]$	$U(7)/\mathbb{Z}/2$ $[2^6 \cdot 3^2]$
E_8	$p \geq 7$	$SU(9)/\mathbb{Z}/3$ $[2^7 \cdot 3 \cdot 5]$	$U(8)/\mathbb{Z}/3$ $[2^7 \cdot 3^3 \cdot 5]$

This list is taken from [J-M-O, §6], where a complete list of all maximal subgroups of maximal rank is given as well as the information about the Weyl group orders. See also [B–S] where the local isomorphy type of the all maximal subgroups of maximal rank is described. For p odd $BSpin(n)^\wedge_p \simeq BSO(n)^\wedge_p$. All the subgroups occuring in the list are quotients of compact connected Lie groups by a finite group of order coprime to p.

5.1 Definition. For a homomorphism $\rho : H \longrightarrow G$ between Lie groups we say H is a *mod-p subgroup* of G, if $H^*(BH;\mathbb{Z}/p)$ is a finite generated module over $H^*(BG;\mathbb{Z}/p)$.

If ρ is also a surjection we call ρ a *mod–p isomorphism of groups.*

Remark. For a homomorphism $\rho : H \longrightarrow G$ the group H is a mod–p subgroup iff $ker(\rho)$ is finite of order coprime to p [Q].

We collect the properties of the subgroups of the above list in the following lemma.

5.2 Lemma. *For every p-convenient simple simply connected Lie group G, $G \neq SU(n)$, there exist mod-p subgroups $K \longrightarrow H \longrightarrow G$ of G satisfying the following conditions:*

(1) *H is maximal of maximal rank and a unitary group or a product of special unitary groups. The index $[W_G : W_H]$ of the Weyl groups is coprime to p.*
(2) *K is of maximal rank and a product of unitary groups or $SU(p)$. The index $[W_G : W_K]$ is coprime to p. Moreover $K \longrightarrow C_G(V)$ is a mod–p isomorphism of groups, where $V \subset Z(K)$ is the maximal p-elementary abelian subgroup of the center $Z(K)$ of K.*

Proof. It is only left to show the last statement, namely $K \longrightarrow C_G(V)$ is a mod–p isomorphism of groups. All the other facts can be read of from the above list.

First we observe, that for a sequence $L \subset M \subset G$ of connected subgroups of G of maximal rank the index $[W_G : W_M]$ divides $[W_G : W_L]$.

Let $p \geq 5$, $G = E_6$, $H = SU(2) \times SU(6)$, $V = \mathbb{Z}/p \times \mathbb{Z}/p$, $\overline{H} = H/(\mathbb{Z}/2)$, $K = S^1 \times U(5)$, and $\overline{K} = K/(\mathbb{Z}/2)$. V is a central subgroup of $\overline{H}$ too. The centralizer $C_{E_6}(\mathbb{Z}/p \times \{0\})$ is connected ([B-S] or theorem 10.1) and containes $S^1 \times_{\mathbb{Z}/2} SU(6)$ by lemma 6.4. Using the complete list of maximal subgroups of maximal rank of [J-M-O 1] and the above condition on the Weyl group orders it follows that $\overline{H}$ is the only one containing $S^1 \times_{\mathbb{Z}/2} SU(6)$. Therefore $C_{E_6}(\mathbb{Z}/p \times \{0\}) = C_{\overline{H}}(\mathbb{Z}/p \times \{0\})$ and $C_{E_6}(V) = C_{\overline{H}}(V) = \overline{K}$ by lemma 6.4 again.

For F_4 and $p \geq 5$ lemma 4.5 proves the last statement. In all the other cases $V = \mathbb{Z}/p$ and the argument using the Weyl group index relation can be applied. $\square$

We end this section by listing the calculation of some cohomology groups of Weyl groups in low dimensions. There are canonical W_G–actions on $LT_{G}{}^\wedge_p := H_2(BT_G; \mathbb{Z}^\wedge_p)$, $LT_G/p^k := H_2(BT_G; \mathbb{Z}/p^k)$, $L^*T_{G}{}^\wedge_p := H^2(BT_G; \mathbb{Z}^\wedge_p)$, and $L^*T_G/p^k := H^2(BT_G; \mathbb{Z}/p^k)$.

5.3 Lemma. *Let G be a p–convenient connected Lie group.*

(1) *If p is odd all the groups*
$H^1(W_G; \mathbb{Z}/p^k)$, $H^2(W_G; \mathbb{Z}/p^k)$, $H^2(W_G; \mathbb{Z}^\wedge_p)$, $H^3(W_G; \mathbb{Z}^\wedge_p)$,
$H^1(W_G; L^*T_G/p^k)$ *and* $H^3(W_G; LT_{G}{}^\wedge_p)$
are trivial.

(2) *If p is odd and G pseudo simply connected, $H^1(W_G; LT_G/p^k) = 0$. we have* $H^1(W_G; L^*T_G/2^k) = 0$.

(3) $H^3(\Sigma_n; \mathbb{Z}^\wedge_2) = \mathbb{Z}/2$ *for $n \geq 4$ and vanishes otherwise.*

Proof. We can assume that $k = 1$. Because $L^*T_{U(n)}{}^\wedge_p$ and $LT_{U(n)}{}^\wedge_p$ are induced by the 1–dimensional trivial Σ_{n-1}–module,

$$\begin{aligned} H^j(\Sigma_n; L^*T_{U(n)}/p) &\cong H^j(\Sigma_{n-1}; \mathbb{Z}/p) \cong H^j(\Sigma_n; LT_{U(n)}/p) \\ &\cong 0 \quad \text{for } p \text{ odd and } j \leq 2 \\ H^j(\Sigma_n; LT_{U(n)}{}^\wedge_p) &\cong H^j(\Sigma_{n-1}; \mathbb{Z}^\wedge_p) \\ &\cong \begin{cases} 0 & , \quad j \leq 3 \text{ and } p \text{ odd or } p = 2,\ j = 3, \text{ and } n \leq 4 \\ \mathbb{Z}/2 & , \quad j = 3,\ p = 2, \text{ and } n \geq 5 \end{cases} \end{aligned}$$

For the calculation of the mod–p cohomology and homology, see [Na],[K-P]. This proves (1) and (2) for $G = U(n)$ and p odd and also proves (3).

Now let p be odd. The long exact sequences in cohomology, associated to the exact sequences

$$(*) \qquad \begin{array}{ccccc} \mathbb{Z}/p & \longrightarrow & L^*T_{U(n)}/p & \longrightarrow & L^*T_{SU(n)}/p \\ LT_{SU(n)}{}^\wedge_p & \longrightarrow & LT_{U(n)}{}^\wedge_p & \longrightarrow & \mathbb{Z}^\wedge_p \end{array}$$

imply (1) for $G = SU(n)$.

If G is p–convenient and simply connected, we choose a mod–p subgroup $H \longrightarrow G$

of G such that H is a product of $SU(n)$'s and $U(n)$'s (lemma 5.2). Then,

$$H^*(W_G; L^*T_G/p) \longrightarrow H^*(W_H, L^*T_G/p) \cong H^*(W_H; L^*T_H/p)$$

$$H^*(W_G; LT_{G}{}^{\wedge}_p) \longrightarrow H^*(W_H, LT_{G}{}^{\wedge}_p) \cong H^*(W_H; LT_{H}{}^{\wedge}_p)$$

are injections. A spectral sequence argument or a Künneth formula argument and an induction over the number of factors of H reduces the calculation to the case $H = U(n)$ or $H = SU(n)$, which we already proved.

For a general p–convenient compact connected Lie group, there are exact sequences

$$L^*T/p \longrightarrow L^*T_G/p \longrightarrow L^*T_{\overline{G}_s}/p$$

$$LT_{\overline{G}_s}{}^{\wedge}_p \longrightarrow LT_{G}{}^{\wedge}_p \longrightarrow LT^{\wedge}_p \ ,$$

where G_s is the simply connected part of G. The quotient $\overline{G}_s$ is p–convenient. Moreover, $L^*T_{\overline{G}_s}/p \cong L^*T_{G_s}/p$ and $LT_{\overline{G}_s}{}^{\wedge}_p \cong LT_{G_s}{}^{\wedge}_p$ because $G_s \longrightarrow \overline{G}_s$ is a mod–p isomorphism of groups. (See section 4). Again, the associated long exact sequences in cohomolgy complete the proof of (1).

Let p be odd and G pseudo simply connected. By lemma 5.2, we choose a mod–p subgroup $H \longrightarrow G$, such that H is a product of unitary groups and $SU(3)$'s. $SU(3)$ can only occur, if $p = 3$ and G_2 is among the factors of G. As already shown $H^1(W_{U(n)}; LT_{U(n)}/p) = 0$, and $H^1(W_{G_2}; LT_{G_2}/3)$ also vanishes. This follows because $W_{G_2} \cong \Sigma_3 \times \Sigma_2$ and because Σ_2 acts non trivially on $H^1(\Sigma_3; LT_{SU(3)}/3) \cong \mathbb{Z}/3$. This proves (2). □

6. p–stubborn subgroups of pseudo simply connected Lie groups.

In [O] is given a complete list of representatives of the conjugacy classes of p–stubborn groups of the classical Lie groups, i.e. $U(n)$, $SU(n)$, $Sp(n)$, and $SO(n)$ for all primes. We will recall these results for $U(n)$ and $SU(n)$.

Let Σ_{p^k} be the symmetric group of order $p^k!$. The permutations $\sigma_0, ..., \sigma_{k-1}$ are defined by

$$\sigma_r(i) = \begin{cases} i + p^r & if \;\; i \equiv 1, ..., (p-1)p^r \mod p^{r+1} \\ i - (p-1)p^r & if \;\; i \equiv (p-1)p^r + 1, ..., p^{r+1} \mod p^{r+1} \ . \end{cases}$$

$A_0, ..., A_{k-1} \in U(p^k)$ denote diagonal matrices with entries

$$(A_r)_{ii} = \zeta^{[(i-1)/p^r]} \qquad \zeta = exp(2\pi i/p) \ ,$$

and $B_0, ..., B_{k-1}$ are the permutation matrices of $\sigma_0, ..., \sigma_{k-1}$.

Definition. For each prime p and $k \geq 0$ the subgroups $E_{p^k} \subset \Sigma_{p^k}$ and $\Gamma^U_{p^k}$ are defined by setting

$$E_{p^k} = \langle \sigma_0, ..., \sigma_{k-1} \rangle \cong (\mathbb{Z}/p)^k \qquad \text{and}$$
$$\Gamma^U_{p^k} = \langle \lambda \cdot I, A_r, B_r \ : \ \lambda \in S^1, \ 0 \leq r \leq k-1 \rangle \subset U(p^k) \ .$$

6.1 Remark. The matrices A_r and B_s satisfy the commutator relations

$$[A_r, B_s] = [A_r, A_s] = [B_r, B_s] = 1 \text{ for } r \neq s \text{ and } [B_r, A_r] = \zeta \cdot I \ .$$

Also $\Gamma^U_{p^k}$ sits in the central extension

$$1 \longrightarrow S^1 \longrightarrow \Gamma^U_{p^k} \longrightarrow (\mathbb{Z}/p)^{2k} \longrightarrow 1$$

and in the splitting extension

$$1 \longrightarrow S^1 \times (\mathbb{Z}/p)^k \longrightarrow \Gamma^U_{p^k} \longrightarrow (\mathbb{Z}/p)^k \longrightarrow 1 \ ,$$

where $S^1 \times (\mathbb{Z}/p)^k = \Gamma^U_{p^k} \cap T_{U(p^k)}$.

A subgroup H of $U(n)$ is called *irreducible*, if the associated representation is irreducible.

6.2 Theorem ([O]).

(1) *For any $n \geq 1$, an irreducible p-toral subgroup $P \subset U(n)$ is stubborn iff it is conjugate to an iterated wreath product of the form*

$$P = \Gamma^U_{p^k} \wr E_{q_1} \wr \ldots \wr E_{q_r} \ ,$$

where $q_i = p^{t_i}$ and $n = p^{k+t_1+\ldots+t_r}$.

(2) *If $P \subset U(n)$ is an arbitrary p–stubborn subgroup then it is conjugate to a subgroup of the form $P_1 \times \ldots \times P_s$, where each P_i is an irreducible subgroup of $U(n_i)$ and where $\sum n_i = n$.*

(3) *A p–stubborn subgroup $P \subset U(n)$ is irreducible iff $C_{U(n)}(V) = U(n)$, where $V \subset Z(P)$ is the maximal p–elementary subgroup of the center of P.*

(4) *A p–toral subgroup $P \subset SU(n)$ is stubborn iff $\langle P, Z(U(n))\rangle$ is stubborn in $U(n)$ iff $P = P' \cap SU(n)$ for some p–stubborn subgroup $P' \subset U(n)$.*

Proof. Everything is proved in [O], but the third statement. That follows from the first two properties. □

Remark. This theorem describes *standard* models for every conjugacy class of p–stubborn subgroups in $U(n)$ and $SU(n)$.

6.3 Proposition ([J–M–O 1, proposition 1.6]).

(1) *For connected Lie groups G and H, $\mathcal{R}_p(G \times H) \cong \mathcal{R}_p(G) \times \mathcal{R}_p(H)$.*

(2) *For a finite covering $A \longrightarrow \widetilde{G} \longrightarrow G$ of compact connected Lie groups, we have $\mathcal{R}_p(\widetilde{G}) \cong \mathcal{R}_p(G)$. In particular, $\widetilde{P} \subset \widetilde{G}$ is p–stubborn iff $P = \widetilde{P}/(\widetilde{P} \cap A) \subset G$ is p–stubborn.*

(3) *If $P \subset G$ is p–stubborn then $C_G(P) = Z(P)$.*

Information about centralizers of p-toral subgroups of mod-p subgroups is provided by the next lemma.

6.4 Lemma. *Let $\rho : G \longrightarrow \overline{G}$ be a mod–p isomorphism of compact connected Lie groups.*

(1) *For any p-toral subgroup $\overline{P} \subset \overline{G}$ there exists a p-toral subgroup $P \subset G$, which is mod–p isomorphic to $\overline{P}$.*
(2) *$C_G(P) \longrightarrow C_{\overline{G}}(\overline{P})$ is a mod–p isomorphism.*
(3) *$Z(P) \longrightarrow Z(\overline{P})$ is a mod–p isomorphism.*

Proof. $Q := \rho^{-1}(\overline{P})$ is toral, i.e a finite extension of a torus. The p–toral Sylow subgroup P of Q maps onto $\overline{P}$.

$C_G(P) \longrightarrow C_{\overline{G}}(\overline{P})$ is a mod–p subgroup. Let l be the order of the kernel of ρ, let $x \in P$, and let $g \in \rho^{-1}(C_{\overline{G}}(\overline{P}))$. Then $gx^lg^{-1} = (gxg^{-1})^l = (xa)^l = x^l$, where $a \in ker(\rho)$. The map $P \longrightarrow P : x \mapsto x^l$ is a surjection because l is coprime to p. That is $C_G(P) \longrightarrow C_{\overline{G}}(\overline{P})$ is a surjection which proves (2).

(3) follows analogously by replacing G by P. □

The following lemma is the key to get control over the p–stubborn subgroups of all p–convenient simple simply connected Lie groups, in particular of the p–convenient exeptional Lie groups.

6.5 Lemma. *Let G be a compact connected Lie group and $P \subset G$ be p–stubborn. Then P is p-stubborn in $C_G(Z(P))$.*

Proof. P is contained in $K := C_G(Z(P))$. The normalizer $N_G(P)$ of P in G normalizes $Z(P)$ and K. The intersection $N_G(P) \cap K = N_K(P)$ is a normal subgroup in $N_G(P)$. Let Q be the intersection of all p–toral Sylow subgroups of $N_K(P)$. A conjugation by an element of $N_G(P)$ permutes the p–toral Sylow subgroups of $N_K(P)$. Thus Q is normal in $N_G(P)$. If Q/P is not trivial, i.e P is not stubborn in K, then P is also not stubborn in G. □

Now we are able to describe the p–stubborn subgroups of the simple simply connected Lie groups. By $im(\rho)$ we denote the image of a homomorphism $\rho : H \longrightarrow G$ between groups.

6.6 Proposition. *Let $P \subset G$ be a p–stubborn group of a p–convenient simple simply connected Lie group G.*

(1) *If $G \neq SU(n), G_2$, there exist a mod–p subgroup*

$$\rho : H \cong U(n_1)\times, ..., \times U(n_k) \longrightarrow G$$

and a p–stubborn subgroup $P' = P_1\times, ..., \times P_k$, such that P_i is irreducible in $U(n_i)$, $im(\rho) = C_G(Z(P))$, and $im(\rho|_{P'}) = P$.
(2) *For $G = G_2$ and $p = 3$, P is stubborn in $SU(3)$, and $Z(P) = Z(SU(3))$. For $G = G_2$ and $p \geq 5$ we have $P = T_{G_2}$.*

Proof. Let $\rho : K \longrightarrow G$ be a mod–p subgroup of G satisfying the conditions of lemma 5.2 (2), and let $\overline{K} = im(\rho)$. Since the Weylgroup index $[W_G : W_K]$ is coprime to p, the groups G and $\overline{K}$ have isomorphic and G and K mod–p isomorphic p–toral

Sylow subgroups. Therefore any p-stubborn subgroup $P \subset G$ sits in $\overline{K}$ up to conjugation. By lemma 6.4 there exists a p–toral subgroup $P' \subset K$ with $\rho(P') = P$ and $\rho(Z(P') = Z(P)$. In the sequence

$$Z(P) = C_G(P) \supseteq C_{\overline{K}}(P) \supseteq Z(P)$$

all the groups are equal. The first identity follows from lemma 6.3. We get $Z(P) \supseteq Z(\overline{K})$ and $C_G(Z(P)) = C_{\overline{K}}(Z(P))$. By lemma 6.5, P is p–stubborn in $C_{\overline{K}}(Z(P))$, and by proposition 6.3, P' is p–stubborn in $C_K(Z(P'))$.

For $G \neq SU(n), G_2$, the group K is a product of unitary groups. The centralizer of any subgroup of such a group always is of the same type. Thus $C_K(Z(P')) \cong U(n_1) \times ... \times U(n_r)$. By theorem 6.2 we have $P' = P_1 \times, ..., \times P_r$ and $P_i \subset U(n_i)$ is irreducible p–stubborn because $C_{U(n_i)}(Z(P_i)) = U(n_i)$.

For $G = G_2$ and $p \geq 5$ we have $P = T_{G_2}$ because $(p, |W_{G_2}|) = 1$. For $G = G_2$ and $p = 3$ we have $K = \overline{K} = SU(3)$. If $C_{G_2}(Z(P)) = C_{SU(3)}(Z(P)) \neq SU(3)$ this centralizer is isomorphic to $U(2)$ or $T_{G_2} = T_{SU(3)}$. In both cases $P = T_G = T_{SU(3)}$, because $|W_{C_{G_2}(Z(P))}| \leq 2$. This is a contradiction because W_{G_2} has a normal subgroup of order 3. Hence P is p–stubborn in $SU(3)$.

Theorem 6.2 finishes the proof. □

For the proof of theorem 1.2, we need certain properties of p–stubborn groups. With the help of the last proposition we can do this for a special class of compact connected Lie group, namely pseudo simply connected Lie groups.

6.7 Proposition. *Let G be a p–convenient pseudo simply connected Lie groupand $T_G \hookrightarrow G$ a fixed maximal torus. In the conjugacy class (P) of any p–stubborn subgroup of G exists a representative $i : P \hookrightarrow N(T_G) \hookrightarrow G$ satisfying the following conditions:*

(1) *$P_T := P \cap T_G = S \times V$, where S is a torus and V an elementary abelian p–group, and $Z(P) \subset P_T$.*
(2) *$C_G(P_T) = T_G$.*
(3) *For any extension $\alpha : P \longrightarrow G$ of $i : P_T \longrightarrow G$, we have $C_G(\alpha) = i(Z(P))$.*
(4) *The canonical map*

$$\pi_0(map(BP, BG^{\wedge}_p)_{B\alpha|BP_T=Bi}) \longrightarrow Hom(H^*(BG;\mathbb{Z}/p), H^*(BP;\mathbb{Z}/p))$$

is an injection.

Remark. By $map(BP, BG^{\wedge}_p)_{B\alpha|_{BP_T}=Bi}$ we denote the components of $map(BP, BG^{\wedge}_p)$ given by maps $B\alpha : BP \longrightarrow BG^{\wedge}_p$, such that $B\alpha|_{BP_T} \simeq Bi$. The homotopy classes of these extensions are classified by obstructions in the groups $H^*(BP/P_T; \pi_*(map(BP_T, BG^{\wedge}_p)_{Bi})$ (lemma 3.13). By (1) of the above theorem follows that there is only one obstruction group, namely $H^2(BP/P_T; \pi_2(map(BP_T, BG^{\wedge}_p)_{Bi})) \cong H^*(BP/P_T; \pi_2(BT_{G^{\wedge}_p}))$.

In the rest of this chapter we will prove this proposition in several steps.

Step 1. Let $G = U(p^k)$ and $P = \Gamma^U_{p^k}$.

Proof of (1) and (2). For abbreviation we set $\Gamma := \Gamma^U_{p^k}$. The condition (1) follows from remark 6.1, and $\Gamma_T \cong (\mathbb{Z}/p)^k \times S^1$. The j-th coordinate is generated by A_{j-1}, and the last coordinate is the the center of $U(p^k)$. The inclusion $i : \Gamma_T \longrightarrow T_{U(p^k)}$ is a splitting in 1–dimensional representations of the p^k–dimensional representation $\Gamma \longrightarrow U(p^k)$. The j-th summand is given by the homomorphism $i_j : \Gamma_T \longrightarrow S^1$ defined by

$$i_j(a_1, ..., a_k, \lambda) = \prod_s \zeta^{[(j-1)/p^s] \cdot a_s} \cdot \lambda \ , \qquad \zeta = exp(2\pi i/p) \ .$$

All these 1–dimensional representation are pairwise nonisomorphic (see the next lemma), which implies that $C_{U(p^k)}(\Gamma_T) = T_{U(p^k)}$. □

6.8 Lemma. *The two homorphism $i_j, i_l : \Gamma_T \longrightarrow S^1$ are equal iff $j = l$.*

Proof. If $i_j = i_l$ we have for all $(a_1, ..., a_k) \in (\mathbb{Z}/p)^k$

$$\prod_{s=1}^k \zeta^{[(j-1)/p^{s-1}] \cdot a_s} = \prod_{s=1}^k \zeta^{[(l-1)/p^{s-1}] \cdot a_s}$$

respectively

$$\sum_{s=1}^k [(j-1)/p^{s-1}] \cdot a_s \equiv \sum_{s=1}^k [(l-1)/p^{s-1}] \cdot a_s \ \ mod \ p \ .$$

This condition is satisfied iff $[(j-1)/p^{s-1}] \equiv [(l-1)/p^{s-1}] \ mod \ p$ for $1 \le s \le k$. This implies $j = l$, because $j, l \le k$. □

Proof of (3) and (4). Extensions $B\alpha : B\Gamma \longrightarrow BU(p^k)^\wedge_p$ of $Bi : B\Gamma_T \longrightarrow BU(p^k)$ are classified by obstructions in the groups $H^r(\Gamma/\Gamma_T; \pi_r(map(B\Gamma_T, BU(p^k)^\wedge_p))$ (see lemma 3.12). Because of theorem 3.9 and (1), there is only one obstruction group $H^2(\Gamma/\Gamma_T; \pi_2(BT_{U(p^k)})) \cong H^2(\Gamma/\Gamma_T; (\mathbb{Z}^\wedge_p)^{p^k})$. Here $\Gamma/\Gamma_T \cong E_{p^k} \cong (\mathbb{Z}/p)^{p^k}$ acts on $\mathbb{Z}_p^{\wedge p^k}$ via the inclusion $E_{p^k} \hookrightarrow \Sigma_{p^k}$ and the permutation representation. In particular Γ/Γ_T acts transitively on the standard basis of $\mathbb{Z}_p^{\wedge p^k}$. The canonical map $\mathbb{Z}^\wedge_p[\Gamma/\Gamma_T] \longrightarrow \mathbb{Z}_p^{\wedge p^k} : 1 \mapsto e_1$ is an isomorphism, where e_1 is the first basis vector of the standard basis. Hence the obstruction group $H^2(\Gamma/\Gamma_T; (\mathbb{Z}^\wedge_p)^{p^k})$ vanishes and the two extensions α and β are conjugate. This completes the proof of (3) and (4) as well as of step 1. □

Step 2. Let $P \subset G = SU(p)$ be a standard p–stubborn subgroup.

Proof. By theorem 6.2 the only p–stubborn subgroups of $SU(p)$ are $T_{SU(p)}$ (for $p \ge 5$), $P_1 = \Gamma^U_p \cap SU(p)$ and $P_2 = (S^1 \wr \mathbb{Z}/p) \cap SU(p)$. We have $P_{1T} = (\mathbb{Z}/p)^2$ because $\Gamma^U_{p\,T} = S^1 \times \mathbb{Z}/p$ and because $C_{SU(p)}(P_{1T}) = T_{SU(p)}$. For P_2 we get $P_{2T} = T_{SU(p)}$ and $C_{SU(p)}(P_{2T}) = T_{SU(p)}$. To calculate the extensions of

$BP_{iT} \longrightarrow BSU(p)$ we have only to consider the obstruction group $H^2(\mathbb{Z}/p; \mathbb{Z}_p^{\wedge p-1})$, where $\mathbb{Z}/p$ acts on $\mathbb{Z}_p^{\wedge p-1}$ via the action of Σ_p on $\pi_2(BT_{SU(p)})$. The exact sequence $\mathbb{Z}_p^{\wedge p-1} \longrightarrow \mathbb{Z}_p^{\wedge p} \longrightarrow \mathbb{Z}_p^{\wedge}$ of $\mathbb{Z}/p$–modules induces a long exact sequence

$$... \longrightarrow H^1(\mathbb{Z}/p; \mathbb{Z}_p^{\wedge}) \longrightarrow H^2(\mathbb{Z}/p; \mathbb{Z}_p^{\wedge p-1}) \longrightarrow H^2(\mathbb{Z}/p; \mathbb{Z}_p^{\wedge p}) \longrightarrow ... \quad .$$

The left and the right term vanish. Hence there exists only one extension and $C_{SU(p)}(P_i) = Z(P_i)$. For the p–stubborn subgroup $T_{SU(p)}$ all the statements are obvious. □

For the next step, namely the case of an arbitrary p–stubborn subgroup of $U(p^k)$, we need some technical lemmas.

6.9 Lemma. *Let $P \subset U(p^k)$ be a standard irreducible p–stubborn subgroup. Then P_T is normal in P, the qoutient P/P_T is an iterated wreath product of elementary abelian p–groups, and the homomorphism $P \longrightarrow P/P_T$ splits.*

Proof. By theorem 6.2, $P = \Gamma^U_{p^{r_0}} \wr E_{p^{r_1}} \wr .... \wr E_{p^{r_s}}$ and $P/P_T = \Gamma^U_{p^{r_0}}/\Gamma^U_{p^{r_0}\,T} \wr E_{p^{r_1}} \wr .... \wr E_{p^{r_s}}$ which is an iterated wreath product of elementary abelian p–groups. The homomorphism $\Gamma^U_{p^{r_0}} \longrightarrow \Gamma^U_{p^{r_0}}/\Gamma^U_{p^{r_0}\,T}$ splits by remark 6.1. Thus $P \longrightarrow P/P_T$ splits. □

6.10 Lemma. *Let $\alpha : Q \longrightarrow G$ be a homomorphism of an iterated wreath product of elementary abelian p–groups into a compact Lie group. If $H^*(B\alpha; \mathbb{Z}/p) = 0$, then α is constant.*

Proof. By theorem 3.1 the statement is true for elementary abelian p–groups. Now let $Q = Q_0 \wr (\mathbb{Z}/p)^k$. By induction hypothesis the homomorphism $\alpha|_{Q_0 \times ... \times Q_0} : Q_0 \times ... \times Q_0 \longrightarrow G$ is constant restricted to one factor, and hence constant on the product. α splits over $\overline{\alpha} : (\mathbb{Z}/p)^k \longrightarrow G$ which is trivial too, because $H^*(B\overline{\alpha}; \mathbb{Z}/p) = 0$. □

Step 3. $P \subset U(p^k)$ is a standard irreducible p–stubborn subgroup.

Proof. By proposition 6.2, $P = P' \wr E_{p^r}$, where $P' \subset U(p^{k-r})$ is also a standard irreducible p–stubborn subgroup. Therefore $P_T = (P'_T)^{p^r}$ is a product of a torus and an elementary abelian p–group. Since the splitting of P_T into the product comes from the homomorphism $U(p^{k-r})^{p^r} \longrightarrow U(p^k)$, we have $C_{U(p^k)}(P_T) = (C_{U(p^{k-r})}(P'_T))^{p^r} = (T_{U(p^{k-r})})^{p^r} = T_{U(p^k)}$ by induction hypothesis. This proves the first two parts.

Now let $\alpha, \beta : P \longrightarrow U(p^k)$ be two homomorphisms, such that $H^*(B\alpha; \mathbb{Z}/p) = H^*(B\beta; \mathbb{Z}/p)$ and $B\alpha|_{BP_T} = B\beta|_{BP_T} = Bi|_{BP_T}$, where $i : P \longrightarrow U(p^k)$ is the standard inclusion. Let

$$\alpha', \beta' : (P')^{p^r} \longrightarrow U(p^k)$$

be the restrictions of α and β.

We get $Z(P') = C_{U(p^{k-r})}(P') = Z(U(p^{k-r})) = S^1$ and $(U(p^{k-r}))^{p^r} = C_{U(p^k)}(V)$, where $V \subset (S^1)^{p^r} = (Z(P'))^{p^r} \subset P_T$ is the maximal elementary abelian p–subgroup. The homorphism α' and β' split over

$$\alpha'', \beta'' : P'^{p^r} \longrightarrow (U(p^{k-r}))^{p^r} \ .$$

The induced maps in cohomology are determined by the application of the Lannes functor $T^V_{Bi^*}$ (remark 3.4), in particular $H^*(B\alpha'') = H^*(B\beta'')$.

The homomorphism α'' can be described by a $(p^r \times p^r)$–matrix with homomorphisms $\alpha_{ij} : P'_i \longrightarrow U(p^{k-r})_j$ as entries. The indices i, j denote the components in the products. Analogously β'' is described by a matrix B with entries $\beta_{i,j} : P'_i \longrightarrow U(p^{k-r})_j$. By induction hypothesis $B\alpha_{ii} \simeq B\beta_{ii}$, i.e α_{ii} and β_{ii} are conjugate (theorem 3.6), and

$$C_{U(p^{k-r})_i}(\alpha_{ii}) = C_{U(p^{k-r})_i}(\beta_{ii}) = Z(P'_i) = Z(U(p^{k-r})_i) = S^1_i \ .$$

We can assume that $\alpha_{ii} = \beta_{ii}$. Because P'_i and P'_j commute, the homomorphisms α_{ij}, β_{ij} split over the inclusion $S^1_j = Z(U(p^{k-r})_j) \longrightarrow U(p^{k-r})_j$. Since $\alpha''|_{P_T} = \beta''|_{P_T}$, the homomorphisms $\alpha_{ij} \cdot \beta_{ij}{}^{-1} : P'_i \longrightarrow S^1_j$ factor over homomorphisms $\rho_{ij} : (P'/P'_T)_i \longrightarrow S^1_j$. and fit together to a map $\rho : (P'/P'_T)^{p^r} \longrightarrow (S^1)^{p^r}$. β'' can be described by the composition

$$(P')^{p^r} \xrightarrow{\Delta} (P')^{p^r} \times (P'/P'_T)^{p^r} \xrightarrow{\alpha'' \times \rho} U(p^{k-r})^{p^r} \times (S^1)^{p^r} \xrightarrow{\mu} U(p^{k-r})^{p^r},$$

where Δ is the diagonal composed with taking the quotient, and where μ is given by the canonical map $S^1 \times U(p^{k-r}) \longrightarrow U(p^{k-r})$. Because $H^*(B\alpha'') = H^*(B\beta'')$ and because ρ_{ii} is constant, the composition

$$H^*(BU(p^{k-r})^{p^r}) \xrightarrow{B\rho^*} H^*((BP'/P'_T)^{p^r}) \longrightarrow H^*(BP'^{p^r})$$

is trivial. By lemma 6.9 and lemma 6.10 the homorphism ρ is constant. That is that $\alpha'' = \beta''$, and $C_{U(p^k)}(\alpha') = C_{U(p^{k-r})^{p^r}}(\alpha'') = (Z(U(p^{k-r}))^{p^r} = (S^1)^{p^r}$.

The extensions $B\alpha : BP = BP' \wr E_{p^r} \longrightarrow BU(p^k)^\wedge_p$ of $B\alpha' : BP'^{p^r} \longrightarrow BU(p^k)^\wedge_p$ are classified by the obstructions in the groups

$$H^*(P/(P'^{p^r}); \pi_*(map(BP'^{p^r}, BU(p^k)^\wedge_p)_{B\alpha'^\wedge_p})$$

(lemma 3.13). By theorem 3.8

$$\begin{aligned} map(BP'^{p^r}, BU(p^k)^\wedge_p)_{B\alpha'} &= BC_{U(p^k)}(\alpha')^\wedge_p \\ &= BC_{U(p^{k-r})^{p^r}}(\alpha'')^\wedge_p \\ &= (BZ(U(p^{k-r})^\wedge_p))^{p^r} \\ &= (BS^1{}^\wedge_p)^{p^r} \ . \end{aligned}$$

The only obstruction group $H^2(E_{p^r}; \mathbb{Z}_p^{\wedge p^r})$ vanishes because $\mathbb{Z}_p^{\wedge p^r}$ is a free E_{p^r}–module (see the proof of step 1). There exists one extension α of α', and the centralizer $C_{U(p^k)}(P)$ is given by the fixed–point set

$$C_{U(p^k)}(P) = (C_{U(p^{k-r})^{p^r}}(P'^{p^r}))^{E_{p^r}} = (S^{1p^r})^{E_{p^r}} = S^1 = Z(P) \ . \quad \square$$

Step 4. Let $P \subset G$ be a p–stubborn subgroup of a pseudo simply connected Lie group.

Proof. According to proposition 6.6 we choose a mod–p subgroup

$$\rho : H = H_1 \times ... \times H_s \longrightarrow G$$

and a p–stubborn subgroup $P' = P_1 \times ... \times P_s$, such that H_i is isomorphic to a unitary group or to $SU(p)$, $P_i \subset H_i$ is p–stubborn, $P_i \subset H_i$ is irreducible if H_i is a unitary group, $\rho(P') = P$, and $H \longrightarrow \overline{H} := C_G(\rho|_V)$ is a mod–p isomorphism of groups, where $V \subset Z(P) = Z(\overline{H})$ is the maximal elementary abelian p–subgroup. Up to conjugacy $P_i \subset H_i$ is the standard model of theorem 6.2. P_T is a product of a torus and an elementary abelian p–group, which is the statement of (1).

Because $P_T \supset Z(P) \supset V$, step 3 induces the sequence

$$T_H = \prod_i T_{H_i} = \prod_i C_{H_i}(P_i) = C_H(P'_T) \longrightarrow C_{\overline{H}}(P_T) = C_G(P_T)$$

which is a mod–p isomorphism by lemma 6.4. Thus, $C_G(P_T) = T_G$ as required in (2).

Let $\alpha_G, \beta_G : P \longrightarrow G$ be two homomorphisms such that

$$H^*(B\alpha_G; \mathbb{Z}/p) = H^*(B\beta_G; \mathbb{Z}/p) \text{ and } \alpha_G|_{P_T} = \beta_G|_{P_T} = i|_{P_T} \ .$$

Both, α_G and β_G, split over homomorphisms $\alpha_{\overline{H}}, \beta_{\overline{H}} : P \longrightarrow \overline{H} = C_G(V)$. Applying the Lannes functor $T^V_{Bi^*}$ we get $H^*(B\alpha_{\overline{H}}) = H^*(B\beta_{\overline{H}})$ (remark 3.4). By lemma 6.4 there exist lifts $\alpha_H, \beta_H : P' \longrightarrow H$ of $\alpha_{\overline{H}}$ and $\beta_{\overline{H}}$. Of course $H^*(B\alpha_{\overline{H}}) = H^*(B\alpha_H)$ and $H^*(B\beta_{\overline{H}}) = H^*(B\beta_H)$. Now we are in a position to apply the methods of step 3. This implies $B\alpha_H \simeq B\beta_H$, $B\alpha_{\overline{H}} \simeq B\beta_{\overline{H}}$, and hence $B\alpha_G \simeq B\beta_G$. This proves (4). The condition (3) is satisfied, because

$$Z(P') = C_H(\alpha_H) \longrightarrow C_{\overline{H}}(\alpha_{\overline{H}}) = C_G(\alpha_G)$$

and

$$Z(P') \longrightarrow Z(P)$$

are mod–p isomorphisms by lemma 6.4 again and step 3. This finishes the proof of step 4 as well as the proof of proposition 6.7. □

7. Maximal tori and Weyl groups.

In [R] the concept of maximal tori and Weyl groups of a finite loop space is introduced (see also [NS 1,2,3], [R-S]). An algebraic version of this homotopy theoretical definition is given in [D-W]. For our purpose the following definition is convenient.

7.1 Definition. A *maximal torus* of a space X with the mod–p type of BG is a map

$$f_T : BT_X \longrightarrow X \quad \text{or} \quad f_{T_p^\wedge} : BT_{X_p^\wedge} \longrightarrow X ,$$

where T_X is a torus, such that

(1) $rank\ T_X = rank\ G$

(2) $H^*(BT_X;\mathbb{Z}/p)$ is a finite generated module over $H^*(X;\mathbb{Z}/p)$.

The *Weylgroup* of X is the group

$$W_X := \{[w] : BT_{X_p^\wedge} \longrightarrow BT_{X_p^\wedge} \ : \ f_{T_p^\wedge} \circ w \simeq f_{T_p^\wedge}\}$$

of homotopy classes of self maps of BT_X.

The definition of the Weyl group might depend on the chosen maximal torus $f_T : BT_X \longrightarrow X$. If p is odd or G is 2–convenient, similar methods as in the proof of [N-S 1; proposition 2.4] can be applied to show that any two maximal tori

$$f'_T, f''_T : BT_X \longrightarrow X$$

fit into a commutative diagram

$$\begin{array}{ccc} BT_{X_p^\wedge} & \xrightarrow{g} & BT_{X_p^\wedge} \\ {\scriptstyle f'_T}\downarrow & & \downarrow{\scriptstyle f''_T} \\ X & = & X , \end{array}$$

where g is an equivalence. Therefore W_X is well defined up to 'conjugation'.

Following ideas of Dwyer, Miller, and Wilkerson [D-M-W 2] we will construct a maximal torus of X. This construction we recall in detail because later we use similar methods in more complicated situations.

Now we assume that G satisfies the condition $H^*(BG;\mathbb{Z}/p) \cong H^*(BT_G;\mathbb{Z}/p)^{W_G}$. Let X be a p–complete space with the mod–p type of BG. By theorem 3.1 the composition

$$H^*(X;\mathbb{Z}/p) \xrightarrow{\phi} H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BT_G;\mathbb{Z}/p) \longrightarrow H^*(BV_G;\mathbb{Z}/p) .$$

has a topological realization $f_V : BV_G \longrightarrow X$. Applying the Lannes–functor we get the diagram

$$\begin{array}{ccccc} T^{V_G}_{f_V^*} H^*(X;\mathbb{Z}/p) & \xrightarrow{\cong} & T^{V_G}_{Bi_V^*} H^*(BG;\mathbb{Z}/p) & & \\ \downarrow & & \downarrow & & \\ H^*(map(BV_G, X)_{f_V};\mathbb{Z}/p)) & & H^*(map(BV_G, BG)_{Bi_V}) & \xrightarrow{\cong} & H^*(BT_G;\mathbb{Z}/p) . \end{array}$$

The last isomorphism comes from theorem 3.5 and lemma 4.5. The upper horizontal arrow is an isomorphism. The space BT_G is 1–connected. Thus, $T^{f_V^*}_{V_G} H^*(X;\mathbb{Z}/p)$

vanishes in degree 1, and the left vertical arrow is also an isomorphism (theorem 3.2). The mapping space $map(BV_G, X)_{f_V}$ is p–complete by theorem 3.3. Among the p–complete spaces, $BT_G{}^\wedge_p$ is determined up to homotopy by its mod–p cohomology. The evaluation map $e : BT_X{}^\wedge_p := map(BV_G, X)_{f_V} \longrightarrow X$, defined by choosing a suitable base point in BV_G, induces a map $f_T : BT_X{}^\wedge_p \longrightarrow X$ which plays the role of a maximal torus of X. Here T_X is a torus of the same rank as T_G.

The composition $BV_G \times BV_G \longrightarrow BV_G \longrightarrow X$ of the multiplication of the H–space BV_G and f_V has as adjoint the map

$$Bj : BV_G \longrightarrow map(BV_G, X)_{f_V} = BT_X \ .$$

There exists an equivalence $BT_G{}^\wedge_p \longrightarrow BT_X{}^\wedge_p$ which fits into the diagram

$$\begin{array}{ccc} BV_G & = & BV_G \\ \downarrow & & \downarrow \\ BT_G & \longrightarrow & BT_X{}^\wedge_p \\ \downarrow & & \downarrow \\ BG & \dashrightarrow & X \ , \end{array}$$

where the upper square commutes up to homotopy and the lower square in mod–p cohomology. The dotted arrow means a map which only exists in cohomology, in this case in mod–p cohomology.

The Weyl group W_G acts on BV_G and hence on $map(BV_G, X)$. The action on the components of $map(BV_G, X)$ is determined by pure cohomological properties, the action on $Hom_{\mathcal{A}_p}(H^*(X; \mathbb{Z}/p), H^*(BV_G; \mathbb{Z}/p))$ (theorem 3.1). Up to homotopy W_G fixes the map $Bi : BV_G \longrightarrow BG$ and therefore the component $map(BV_G, X)_{f_V}$ of $map(BV_G, X)$. We get an action of W_G on $BT_X{}^\wedge_p$ such that $f_T \circ w \simeq f_T$ for every $w \in W_G$. All the spaces in the above diagram carry an W_G–action (BG and X the trivial action), and the maps induce equivariant maps in mod–p cohomology.

We collect the results in the following proposition

7.2 Proposition. *Let G be a compact connected Lie group, and let X be a p–complete space with the mod–p type of BG. If $H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)^{W_G}$ then there exist a maximal torus $f_T : BT_X{}^\wedge_p \longrightarrow X$, a Weyl group action of W_G on BT_X, and a mod–p equivalence $BT_G \longrightarrow BT_X{}^\wedge_p$, equivariant in mod–$p$ cohomology, such that the diagram*

$$\begin{array}{ccc} BT_G & \longrightarrow & BT_X{}^\wedge_p \\ \downarrow & & \downarrow \\ BG & \dashrightarrow & X \end{array}$$

commutes in mod–p cohomology. Moreover, $H^*(X; \mathbb{Z}^\wedge_p) \cong H^*(BT_X; \mathbb{Z}^\wedge_p)^{W_G}$.

Proof. Only the last statement needs a comment. It follows from the isomorphism $H^*(X; \mathbb{Z}/p) \cong H^*(BT_X; \mathbb{Z}/p)^{W_G}$ and the Nakayama lemma. $\square$

Remark. All but the last statement is due to Dwyer, Miller and Wilkerson and is true under the weaker assumption $C_G(V_G) = T_G$, in particular, for all compact connected Lie groups at odd primes [D-M-W 2].

To prove theorem 1.1 we have to study the W_G–modules

$$L^*T_{X}{}^{\wedge}_p = H^2(BT_X;\mathbb{Z}^{\wedge}_p) \text{ and } L^*T_{G}{}^{\wedge}_p = H^2(BT_G;\mathbb{Z}^{\wedge}_p)\ .$$

7.3 Definition. A $\mathbb{Z}^{\wedge}_p[W_G]$–module L is called *p–reducible* if

$$P(L)^{W_G} \otimes \mathbb{Z}/p \cong P(L \otimes \mathbb{Z}/p)^{W_G}\ ,$$

where $P(-)$ denotes the symmetric part of the tensor algebra.

Proof of theorem 1.1.
The proof is based on results of the sections 8 and 9. By proposition 7.2 the modules $L^*T_{X}{}^{\wedge}_p$ and $L^*T_{G}{}^{\wedge}_p$ are p–reducible, and there mod–p reduction are isomorphic as $\mathbb{Z}/p[W_G]$–modules. Let G be p–convenient.

For $p \neq 2$ and G simply connected or pseudo simply connected, for $p = 2$ and G simply connected, and for $(p, |W_G|) = 1$ we can apply proposition 8.1. If G is pseudo projective we can use proposition 9.5, and if G is a product of unitary groups we can apply proposition 9.9. This proves that in these cases $L^*T_{G}{}^{\wedge}_p \cong L^*T_{X}{}^{\wedge}_p$ as $\mathbb{Z}^{\wedge}_p[W_G]$–modules.

In the general case there exists a compact connected Lie group H, such that BH has the mod–p type of BG and X, and such that $L^*T_{H}{}^{\wedge}_p \cong L^*T_{X}{}^{\wedge}_p$. This follows from proposition 9.2.

We can identify $BT_{G}{}^{\wedge}_p$, respectively $BT_{H}{}^{\wedge}_p$, and $BT_{X}{}^{\wedge}_p$ via this map. This proves that X has the p–adic type of BG respectively BH. By the Nakayama lemma follows that $H^*(X;\mathbb{Z}^{\wedge}_p) \cong H^*(BT_H;\mathbb{Z}^{\wedge}_p)^{W_G}$. $\square$.

8. p–adic Weyl group representations, for simply connected Lie groups.

The action of the Weyl group W_G on the maximal torus BT_G of a compact connected Lie group G gives a $\mathbb{Z}^{\wedge}_p[W_G]$–module $L^*T_{G}{}^{\wedge}_p := H^2(BT_G;\mathbb{Z}^{\wedge}_p)$. In this chapter we will study the p–adic liftings of the associated mod–p representation $H^2(BT_G;\mathbb{Z}/p)$ for a simply connected Lie group G. The general case is discussed in the next chapter.

Definitions and Notation. A $\mathbb{Z}^{\wedge}_p[W_G]$–module L is called *simply connected*, if $L^{W_G} = 0$. The module $L\otimes\mathbb{Z}/p^k$ is denoted by L/p^k, and $L/p^{k*} = Hom(L/p^k, \mathbb{Z}/p^k)$ is the dual module of L/p^k as a $\mathbb{Z}/p^k[W_G]$–module. $Gl(L/p^k)$ denotes the isomorphisms and $M(L/p^k)$ the endomorphisms of L/p^k. The symmetric part of the tensor algebra of L is given by $P(L)$.

8.1 Proposition. *Let G be a compact connected Lie groupand L be a torsion free $\mathbb{Z}_p^\wedge[W_G]$–module. Let*

$$\overline{\alpha} : L^*T_G/p \longrightarrow L/p$$

be a $\mathbb{Z}_p^\wedge[W_G]$–isomorphism. Then there exists a $\mathbb{Z}_p^\wedge[W_G]$–isomorphism

$$\alpha : L^*T_{G_p^\wedge} \longrightarrow L\ ,$$

which is a lift of $\overline{\alpha}$, if one of the following conditions is satisfied:

1. (1) *p is odd, and G is p–convenient and simply connected or pseudo simply connected.*
2. (2) *$p = 2$, L is p–reducible, and G is p–convenient and simply connected.*
3. (3) *$(p, |W_G|) = 1$.*

For the proof we need several lemmas.

A $\mathbb{Z}_p^\wedge[W_G]$–module L, which is free as $\mathbb{Z}_p^\wedge$–module of rank n, induces a homomorphism

$$\rho_L : W_G \longrightarrow Gl(L) \cong Gl(n, \mathbb{Z}_p^\wedge).$$

We denote by $\rho_L(k)$ the homorphism associated to the $\mathbb{Z}/p^k[W_G]$–module L/p^k.

The kernel of the restriction $Gl(n, \mathbb{Z}/p^{k+r}) \longrightarrow Gl(n, \mathbb{Z}/p^k)$ is the group

$$ker := \{id + p^k A \mid A \in M(n, \mathbb{Z}/p^r)\}\ ,$$

where $M(n, \mathbb{Z}/p^r) \cong M(L/p^r) \cong Hom(L/p^r, L/p^r) \cong L/p^{r*} \otimes L/p^r$ is the group of $n \times n$ matrices over $\mathbb{Z}/p^r$. The multiplication in ker is given by

$$(id + p^k A)(id + p^k B) = (id + p^k(A + B + p^k AB))\ .$$

Thus, if $r \leq k$, the map $A \mapsto id + p^k \cdot A$ induces an isomorphism $M(n, \mathbb{Z}/p^k) \cong ker$.

The lifts in the diagram

$$\begin{array}{ccc} & & Gl(n, \mathbb{Z}/p^{k+r}) \\ & & \downarrow \\ W_G & \xrightarrow{\rho_L(k)} & Gl(n, \mathbb{Z}/p^k) \end{array}$$

are classified up to conjugation by the obstruction group

$$H^1(W_G; Hom(L/p^r, L/p^r))\ ,$$

where W_G acts on $Hom(L/p^r, L/p^r)$ via conjugation and the homomorphism $W_G \xrightarrow{\rho_L(r)} Gl(n, \mathbb{Z}/p^r)$. To prove lemma 8.1 we have to calculate this obstruction group.

Let G and H be two compact connected Lie groups. Then $L^*T_{G_p^\wedge} \times L^*T_{H_p^\wedge}$ as well as $L^*T_{G_p^\wedge}$ and $L^*T_{H_p^\wedge}$ are $W_G \times W_H$–modules, where W_H acts trivially on $L^*T_{G_p^\wedge}$ and W_G trivially on $L^*T_{H_p^\wedge}$.

8.2 Lemma. *Let G and H be two connected Lie groups.*

(1)
$$H^1(W_G \times W_H; Hom(L^*T_G/p^k, L^*T_G/p^k)) \cong H^1(W_G; Hom(L^*T_G/p^k, L^*T_G/p^k)) \oplus H^1(W_H; \mathbb{Z}/p^k) \otimes Hom(L^*T_G/p^k, L^*T_G/p^k)^{W_G}$$

If p is odd the second summand vanishes.

(2)
$$H^1(W_G \times W_H; Hom(L^*T_G/p^k, L^*T_H/p^k)) \cong (LT_G/p^k)^{W_G} \otimes H^1(W_H; L^*T_H/p^k) \oplus H^1(W_G; LT_G/p^k) \otimes (L^*T_H/p^k)^{W_H} \ .$$

If G and H are p–convenient and simply connected or if p is odd and G is p–convenient and pseudo simply connected then both summands vanish.

(3)
$$H^1(\Sigma_n; Hom(L^*T_{U(n)}/p^k, L^*T_{U(n)}/p^k)) \cong H^1(\Sigma_{n-1}; \mathbb{Z}/p^k) \oplus H^1(\Sigma_{n-2}; \mathbb{Z}/p^k)$$

(4) For $n \geq 3$ we have

$$H^1(\Sigma_n; Hom(L^*T_{SU(n)}/p^k, L^*T_{SU(n)}/p^k)) \cong H^1(\Sigma_{n-2}; \mathbb{Z}/p^k) \ .$$

(5) If G is p–convenient and pseudo simply connected or simply connected, and if p is odd,

$$H^1(W_G; Hom(L^*T_G/p^k, L^*T_G/p^k)) = 0 \ .$$

(6) If $(p, |W_G|) = 1$, then $H^1(W_G; Hom(L^*T_G/p^k, L^*T_G/p^k) = 0$.

Proof. The Hochschild–Serre spectral sequence and some easy calculations with the coefficients establish the two isomorphism of (1) and (2) . If p is odd $H^1(W_H, \mathbb{Z}/p^k)$ as well as $H^1(W_H; L^*T_H/p^k)$ vanish, and $H^1(W_G; LT_G/p^k)$ vanishes if G is pseudo simply connected. All this follows from lemma 5.3. Because WH is generaterd by elements of order 2 the first group $H^1(W_H, \mathbb{Z}/p^k)$ vanishes for every compact connected Lie group. If G and H are simply connected then $(L^*T_H/p^k)^{W_H} = 0 = (L^*T_H/p^k)^{W_H}$. These facts prove the vanishing of the summands which finishes the proof of (1) and (2) .

The representation $L^*T_{U(n)}/p^k \cong ind_{\Sigma_{n-1}}^{\Sigma_n} \mathbb{Z}/p^k$ is induced by the 1-dimensional trivial Σ_{n-1}–module $\mathbb{Z}/p^k$. Therefore

$$Hom(L^*T_{U(n)}/p^k, L^*T_{U(n)}/p^k) \cong ind_{\Sigma_{n-1}}^{\Sigma_n} Hom(\mathbb{Z}/p^k, L^*T_{U(n)}/p^k) \cong ind_{\Sigma_{n-1}}^{\Sigma_n} (L^*T_{U(n-1)}/p^k \oplus \mathbb{Z}/p^k) \ ,$$

and

$$H^1(\Sigma_n; Hom(L^*T_{U(n)}/p^k, L^*T_{U(n)}/p^k)) \cong H^1(\Sigma_{n-1}, L^*T_{U(n-1)}/p^k) \oplus H^1(\Sigma_{n-1}; \mathbb{Z}/p^k) \cong H^1(\Sigma_{n-2}; \mathbb{Z}/p^k) \oplus H^1(\Sigma_{n-1}; \mathbb{Z}/p^k) \ .$$

The exact sequence

$$0 \longrightarrow \mathbb{Z}/p^k \longrightarrow L^*T_{U(n)}/p^k \longrightarrow L^*T_{SU(n)}/p^k \longrightarrow 0$$

splits as sequence of $\mathbb{Z}/p^k$–modules and induces an exact sequence

$$0 \longrightarrow Hom(L^*T_{SU(n)}/p^k, L^*T_{SU(n)}/p^k) \longrightarrow Hom(L^*T_{U(n)}/p^k, L^*T_{SU(n)}/p^k)$$
$$\longrightarrow Hom(\mathbb{Z}/p^k, L^*T_{SU(n)}/p^k) \cong L^*T_{SU(n)}/p^k \longrightarrow 0 \ .$$

Because of lemma 5.3, the associated long exact sequence in cohomology shows that

$$\begin{aligned} &H^1(\Sigma_n; Hom(L^*T_{SU(n)}/p^k, L^*T_{SU(n)}/p^k)) \\ &\qquad \cong H^1(\Sigma_n; Hom(L^*T_{U(n)}/p^k, L^*T_{SU(n)}/p^k)) \\ &\qquad \cong H^1(\Sigma_{n-1}; Hom(\mathbb{Z}/p^k, L^*T_{SU(n)}/p^k)) \\ &\qquad \cong H^1(\Sigma_{n-1}, L^*T_{U(n-1)}/p^k) \\ &\qquad \cong H^1(\Sigma_{n-2}; \mathbb{Z}/p^k) \ . \end{aligned}$$

This proves (3) and (4).

To prove (5) we choose by remark 5.2 a mod–p subgroup $H \longrightarrow G$ of G, such that the Weyl group index is coprime to p and such that H is a product of unitary groups and special unitary groups. The restriction

$$H^1(W_G; Hom(L^*T_G/p^k, L^*T_G/p^k)) \to H^1(W_H; Hom(L^*T_H/p^k, L^*T_H/p^k))$$

is an injection. Let $H = H_1 \times ... \times H_r$ be the splitting into the factors; i.e $H_i \cong SU(n_i)$, $n_i \geq 3$, or $H_i \cong U(n_i)$. If p is odd $H^1(W_{H_i}; \mathbb{Z}/p^k) = 0$, and, by (1), (2), (3), and (4),

$$\begin{aligned} &H^1(W_H; Hom(L^*T_H/p^k, L^*T_H/p^k)) \\ &\qquad \cong \bigoplus_i H^1(W_{H_i}; Hom(L^*T_{H_i}/p^k, L^*T_{H_i}/p^k)) \\ &\qquad = 0 \end{aligned}$$

which finishes the proof of (5).

The statement (6) follows because $Hom(L^*T_G/p^k, L^*T_G/p^k)$ is a $\mathbb{Z}_p^\wedge$–modul and $(p, |W_G|) = 1$. □

8.3 Lemma. *Let G and H be 2–convenient compact connected Lie groups. If G is simply connected then*

$$H^1(W_G \times W_H; Hom(L^*T_G/4, L^*T_G/4))$$
$$\longrightarrow H^1(W_G \times W_H; Hom(L^*T_G/2, L^*T_G/2)$$

is the trivial map.

Proof. The only 2–convenientcompact connected Lie groups are qoutients of a product of $SU(n)$'s, $n \geq 3$ and a torus. Hence W_G and W_H are products of symmetric groups. Because of the last lemma we have only to show that

$$H^1(\Sigma_n; \mathbb{Z}/4) \longrightarrow H^1(\Sigma_n; \mathbb{Z}/2)$$

is the trivial map. This is obvious, because $H_1(\Sigma_n; \mathbb{Z}) = \mathbb{Z}/2$. □

Now we are in the position to prove proposition 8.1.

Proof of proposition 8.1.

Proof. The two $\mathbb{Z}_p^\wedge[W_G]$–modules $L^*T_{G_p^\wedge}$ and L induce two homomorphisms

$$\rho_{L^*T_{G_p^\wedge}}, \rho_L : W_G \longrightarrow Gl(n;\mathbb{Z}_p^\wedge) \ .$$

By assumption

$$\rho_{L^*T_{G_p^\wedge}}(1), \rho_L(1) : W_G \longrightarrow Gl(n;\mathbb{Z}/p)$$

are conjugate. We can assume they are equal. Because the obstruction groups $H^1(W_G; Hom(L^*T_G/p, L^*T_G/p))$ vanish for p odd and for $(p, |W_G|) = 1$ (lemma 8.2), all the lifts

$$\rho_{L^*T_{G_p^\wedge}}(k), \rho_L(k) : W_G \longrightarrow Gl(n;\mathbb{Z}_p^{\wedge k})$$

are conjugate. Thus, $\rho_{L^*T_{G_p^\wedge}}$ and ρ_L are conjugate. This proves 8.1 under the assumption (1) and (3).

Next we consider the case $G = SU(n)$, $n \geq 3$, and $p = 2$. We denote by $det : \Sigma_n \longrightarrow \mathbb{Z}_2^\wedge$ the composition of $\rho_{L^*T_{SU(n)_p^\wedge}}$ and the determinant. For $n \geq 4$ the group $H^1(\Sigma_{n-2};\mathbb{Z}/2) \cong \mathbb{Z}/2$, and for $n = 3$ the group vanishes; i.e. for $n = 3$ there is no obstruction and the lift $\rho_{L^*T_{G_p^\wedge}}(2)$ is unique up to conjugation, and for $n \geq 4$ there are two lifts of $\rho_{L^*T_{G_p^\wedge}}(1)$, given by $\rho_{L^*T_{G_p^\wedge}}(2)$ and $\rho_{L^*T_{G_p^\wedge}}(2) \otimes det$. But these two lifts can be distinguished by looking at the invariants

$$P(L^*T_{SU(n)}/4)^{\rho_{L^*T_{SU(n)_p^\wedge}}(2)(\Sigma_n)} \quad \text{and} \quad P(L^*TSU(n)/4)^{(\rho_{L^*T_{SU(n)_p^\wedge}}(2)\otimes det)(\Sigma_n)} \ ,$$

as a short calculation of the invariants of degree 6 shows. We assumed that $L/4$ is p–reducible, and hence, $L^*T_{SU(n)}/4 \cong L/4$.

If G is simply connected and 2–convenient G splits into factors $G = G_1 \times ... \times G_r$, where $G_i = SU(n_i)$ and $n_i \geq 3$. In this case, $L/4 \cong \overline{L}_1 \times ... \times \overline{L}_r$ by lemma 8.2, where $\overline{L}_i$ is a $\mathbb{Z}/4[W_{G_1} \times ... \times W_{G_r}]$–module. Because L is 2–reducible, $\overline{L}_1$ is 2–reducible as $W_{G_1} \times W_H$–module, where $H = G_2 \times ... G_r$. In the commutative diagram

$$(*) \qquad \begin{array}{ccc} P(\overline{L}_1)^{W_{G_1}\times W_H} & \xrightarrow{\alpha} & P(\overline{L}_1)^{W_G} \\ \downarrow \beta & & \downarrow \gamma \\ P(\overline{L}_1/2)^{W_{G_1}\times W_H} & \xrightarrow{\delta} & P(\overline{L}_1/2)^{W_G} \end{array}$$

δ and $\beta \otimes \mathbb{Z}/2$ are isomorphisms, and $\gamma \otimes \mathbb{Z}/2$ is an injection, hence also an isomorphism. This shows that $\overline{L}_1$ is 2–reducible as W_G–module. Therefore, by the above argument, $L^*T_{G_1{}_p^\wedge} \cong \overline{L}_1$ as W_G–modules, and α is an isomorphism by the Nakayama lemma.

The Weyl group W_H must act on $\overline{L}_1$ via a homorphism

$$W_H \longrightarrow C_{GL(\overline{L}_1)}(W_G) \cong C_{GL(L^*T_{G_1}/4)}(W_G) \cong \mathbb{Z}/4 \ .$$

That is to say that W_H acts via scalar multiplication. A non trivial action of W_H on $\overline{L}_1$ contradicts the fact that the map α in the diagram $(*)$ is an isomorphism. Thus $L^*T_{G_1}/4 \cong \overline{L}_1$ and $L^*T_G/4 \cong L/4$ as W_G–module.

Now, for an induction argument, we can assume that

$$L^*T_G/2^k \cong L/2^k, \quad \rho_{L^*T_{G_p^\wedge}}(k) = \rho_L(k)\ , \qquad \textit{and} \quad k \geq 2\ .$$

The two lifts $\rho_{L^*T_{G_p^\wedge}}(k+2)$ and $\rho_L(k+2)$ differ by an obstruction in $H^1(W_G; Hom(L^*T_G/4, L^*T_G/4))$. Because the map

$$H^1(W_G; Hom(L^*T_G/4, L^*T_G/4)) \longrightarrow H^1(W_G; Hom(L^*T_G/2, L^*T_G/2))$$

is trivial (lemma 8.3) the two lifts $\rho_{L^*T_{G_p^\wedge}}(k+1)$ and $\rho_L(k+1)$ are conjugate, i.e. $L^*T_G/2^{k+1} \cong L/2^{k+1}$ and $L^*T_{G_p^\wedge} \cong L$ as $\mathbb{Z}_p^\wedge[W_G]$–modules. This proves part (2). $\square$

9. p–adic Weyl group representations, for general compact connected Lie groups.

In this section we use the same notation as in the last one. G denotes in this chapter a compact connected Lie group. Uniqueness results about the p–adic liftings of the Weyl group representation L^*T_G/p are not true in general. In remark 9.6 we consrtruct a counterexample. Nevertheless we can describe all torsion free p–reducible p–adic representations L of W_G which are lifts of L^*T_G/p. The results to be applied in section 7 are the propositions 9.5 and 9.9.

Any compact connected Lie group G fits into an exact sequence $\overline{G}_s \longrightarrow G \longrightarrow T$, where G_s is simply connected and where T is a torus. If G is p–convenient G_s is also p–convenient, and $G_s \longrightarrow \overline{G}_s$ is a mod–p isomorphism of groups (see section 4). Moreover, $L^*T_{G_s}{}_p^\wedge \cong L^*T_{\overline{G}_s}{}_p^\wedge$ and $L^*T_{G_s}/p \cong L^*T_{\overline{G}_s}/p$.

9.1 Lemma. *Let G be a p–convenient compact connected Lie group. Let L be a torsion free p–reducible $\mathbb{Z}_p^\wedge[W_G]$–module. If $L/p \cong L^*T_G/p$ as $\mathbb{Z}_p^\wedge[W_G]$–module then there is an exact sequence*

$$1 \longrightarrow L^*T_p^\wedge \longrightarrow L \longrightarrow L^*T_{G_s}{}_p^\wedge \longrightarrow 1$$

Proof. The qoutient L/L^{W_G} is torsion free because L is. Thus, applying the functor $\otimes\mathbb{Z}/p$ to the exact sequence

$$0 \longrightarrow L^{W_G} \longrightarrow L \longrightarrow L/L^{W_G} \longrightarrow 0\ ,$$

establishes the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & L^{W_G}/p & \longrightarrow & L/p & \longrightarrow & (L/L^{W_G})/p & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & (L/p)^{W_G} & \longrightarrow & L/p & \longrightarrow & (L/p)/(L/p)^{W_G} & \longrightarrow 0 \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \longrightarrow & (L^*T_G/p)^{W_G} & \longrightarrow & L^*T_G/p & \longrightarrow & (L^*T_G/p)/(L^*T_G/p^{W_G}) & \longrightarrow 0
\end{array}$$

of exact rows. The exact sequence $\overline{G}_s \longrightarrow G \longrightarrow T$ establishes the isomorphism $(L^*T_G/p)^{W_G} \cong L^*T/p$. Because G is p–convenient we have

$$(L^*T_{G\,p}^{\wedge}/L^*T_{G\,p}^{\wedge\,W_G})/p \cong (L^*T_G/p)/(L^*T_G/p^{W_G}) \cong L^*T_{\overline{G}_s}/p \cong L^*T_{G_s}/p\ .$$

The two lower rows in the diagram are isomorphic by assumption. The two upper rows also are isomorphic, because L is p–reducible. We have $L^{W_G} \cong L^*T_p^{\wedge}$ and, by proposition 8.1, we get $L/L^{W_G} \cong L^*T_{G_s\,p}^{\wedge}$ as $\mathbb{Z}_p^{\wedge}[W_G]$–module. That is there exists an exact sequence

$$0 \longrightarrow L^*T_p^{\wedge} \longrightarrow L \longrightarrow L^*T_{G_s\,p}^{\wedge} \longrightarrow 0$$

of $\mathbb{Z}_p^{\wedge}[W_G]$–modules as desired. $\square$

Remark. If

$$\begin{aligned} Ext_{W_G}(L^*T_{G_s\,p}^{\wedge}, L^*T_p^{\wedge}) &\cong H^1(W_G; Hom(L^*T_{G_s\,p}^{\wedge}, L^*T_p^{\wedge})) \\ &\cong H^1(W_G; LT_{G_s\,p}^{\wedge}) \otimes L^*T_p^{\wedge} = 0\ , \end{aligned}$$

then $L \cong L^*T_{G\,p}^{\wedge}$ as $\mathbb{Z}_p^{\wedge}[W_G]$–modules. Thus, proposition 8.1 is true if $(p, |W_G|) = 1$, as already shown, or if G_s is a p–convenient simply connected Lie group , which contains no factor isomorphic to some $SU(n)$, i.e. G_s is also pseudo simply connected.

Next we will study the group of extensions $Ext_{W_G}(L^*T_{G_s\,p}^{\wedge}, L^*T_p^{\wedge})$ or, more generally, the extensions $Ext_{W_G}(L^*T_{G\,p}^{\wedge}, L^*T_p^{\wedge})$ for a compact connected Lie group G and a torus T. We denote by $GrExt(T,G)$ the set of exact sequences $G \longrightarrow G' \longrightarrow T$ of compact connected Lie groups dividing out the usual equivalence relations for extensions. There is a canonical map

$$\begin{array}{cccc} \mathcal{L}\ : & GrExt(T,G) & \longrightarrow & Ext_{W_G}(L^*T_{G\,p}^{\wedge}, L^*T_p^{\wedge}) \\ & (G \to G' \to T) & \mapsto & (L^*T_p^{\wedge} \to L^*T_{G'\,p}^{\wedge} \to L^*T_{G\,p}^{\wedge}) \end{array}$$

9.2 Proposition. *Let G be a compact connected Lie group. If the inclusion $Z(G) \longrightarrow T_G{}^{W_G}$ has a cokernel of order coprime to p the map $\mathcal{L}$ is a surjection.*

Remark. By lemma 4.4 the assumption of 9.2 are satisfied if G is p–convenient.

Proof. Let $E : L^*T_p^{\wedge} \longrightarrow L \longrightarrow L^*T_{G\,p}^{\wedge}$ be an exact sequence. Because $Ext_{W_G}(L^*T_{G\,p}^{\wedge}, L^*T_p^{\wedge}) \cong H^1(W_G; Hom(L^*T_{G\,p}^{\wedge}, L^*T_p^{\wedge}))$ is a finite group we can choose an injective self map $\alpha : L^*T_p^{\wedge} \longrightarrow L^*T_p^{\wedge}$ with finite cokernel K such that $\alpha_*(E)$ splits. Therefore, there exists a commutative diagram

$$\begin{array}{ccccc} L^*T_p^{\wedge} & \longrightarrow & L & \longrightarrow & L^*T_{G\,p}^{\wedge} \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \| \\ L^*T_p^{\wedge} & \longrightarrow & L^*T_p^{\wedge} \oplus L^*T_{G\,p}^{\wedge} & \longrightarrow & L^*T_{G\,p}^{\wedge} \\ \downarrow & & \downarrow & & \\ K & = & K\ . & & \end{array}$$

The equivariant maps $L^*T_p^\wedge \longrightarrow K \cong H^2(BBK;\mathbb{Z}_p^\wedge)$ and $L^*T_p^\wedge \oplus L^*T_{Gp}^\wedge \longrightarrow K \cong H^2(BBK;\mathbb{Z}_p^\wedge)$ are induced by homomorphisms $K\longrightarrow T$ and $K\longrightarrow T\times T_G$. Because K is a finite abelian p–group and because $H^*(BK,\mathbb{Z}/p)$ is a finitely generated $H^*(BT;\mathbb{Z}/p)$–module, both homomorphisms are injections [Q]. We have $K\subset T\times {T_G}^{W_G}$, and hence K maps into $Z(G)\subset T^{W_G}$. The compact connected Lie group $G' := (T\times G)/K$ fits into an exact sequence $G\longrightarrow G'\longrightarrow T$. By construction we get an isomorphism of extensions, namely

$$\begin{array}{ccccc} L^*T_p^\wedge & \longrightarrow & L & \longrightarrow & L^*T_{Gp}^\wedge \\ \| & & \downarrow & & \| \\ L^*T_p^\wedge & \longrightarrow & L^*T_{G'p}^\wedge & \longrightarrow & L^*T_{Gp}^\wedge \end{array} . \quad \square$$

Because $L^*T_{Gp}^\wedge$ is torsionfree the functor $\otimes\mathbb{Z}/p$ induces maps

$$\begin{aligned} \alpha : Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T_p^\wedge) &\longrightarrow Ext_{W_G}(L^*T_G/p, L^*T/p) \\ \beta : Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T/p) &\longrightarrow Ext_{W_G}(L^*T_G/p, L^*T/p)\ . \end{aligned}$$

The exact sequence

$$L^*T_p^\wedge \xrightarrow{m} L^*T_p^\wedge \xrightarrow{q_T} L^*T/p\ ,$$

where m is the multiplication with p, and the projection $q_G : L^*T_{Gp}^\wedge \longrightarrow L^*T_G/p$ establish the diagram

$$\begin{array}{ccc} Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T_p^\wedge) & & \\ \downarrow{\scriptstyle m_*} & & \\ Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T_p^\wedge) & \overset{\alpha}{\searrow} & \\ \downarrow{\scriptstyle q_{T*}} & & \\ Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T/p) & \underset{q_G^*}{\overset{\beta}{\rightleftarrows}} & Ext_{W_G}(L^*T_G/p, L^*T/p) \end{array} .$$

The map m_* is given by multiplication with p in the Ext–group, and β is a left inverse of q_G^*, i.e. $\beta q_G^* = id$. The commutativity of

$$\begin{array}{ccccc} L^*T_p^\wedge & \longrightarrow & L & \longrightarrow & L^*T_{Gp}^\wedge \\ {\scriptstyle q_T}\downarrow & & \downarrow & & \| \\ L^*T/p & \longrightarrow & L' & \longrightarrow & L^*T_{Gp}^\wedge \\ \| & & \downarrow & & \downarrow{\scriptstyle q_G} \\ L^*T/p & \longrightarrow & L/p & \longrightarrow & L^*T_G/p \end{array}$$

shows that $\beta q_{T*} = \alpha$ and $q_G^*\alpha = q_{T*}$. This implies the following lemma:

9.3 Lemma. $ker(\alpha) = ker(q_{T*}) = im(m_*)$.

Next we will show that the p–adic lifting of the Weyl group representation L^*T_G/p is unique for another class of Lie groups, namely for pseudo projective p–convenient Lie groups. We introduce the following definition.

Definition. An extension

$$E : L^*T_p^\wedge \longrightarrow L \longrightarrow L^*T_{Gp}^\wedge \in Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T_p^\wedge)$$

is called *initial* if, for every other extension $\hat{E} \in Ext_{W_G}(L^*T_{Gp}^\wedge, L^*T_p^\wedge)$, there exists a self map $g : L^*T_p^\wedge \longrightarrow L^*T_p^\wedge$ such that $\hat{E} = g_*E$.

The next lemma produces initial extensions.

9.4 Lemma. *Let G_s be simply connected. If $G \in GrExt(T, G_s)$ is pseudo projective the extension*

$$E : L^*T_p^\wedge \longrightarrow L^*T_{Gp}^\wedge \longrightarrow L^*T_{G_sp}^\wedge$$

is initial.

Proof. By proposition 9.2 every extension $E' : L^*T_p^\wedge \longrightarrow L \longrightarrow L^*T_{G_sp}^\wedge$ is induced by a group extension $G_s \longrightarrow G' \longrightarrow T$ or alternatively by an exact sequence

$$K \longrightarrow G_s \times T \longrightarrow G' \ ,$$

where the composition $K \longrightarrow G_s \times T \longrightarrow T$ is an injection. The inclusion $K \longrightarrow Z(G_s)$ can be extended to

$$\begin{array}{ccc} K & \longrightarrow & Z(G_s) \\ \downarrow & & \downarrow \\ T & \longrightarrow & T \ . \end{array}$$

This establishes a finite covering $G' \longrightarrow G$. By construction we get

$$\begin{array}{ccccc} L^*T_p^\wedge & \longrightarrow & L^*T_{Gp}^\wedge & \longrightarrow & L^*T_{G_sp}^\wedge \\ \downarrow & & \downarrow & & \| \\ L^*T_p^\wedge & \longrightarrow & L^*T_{G'p}^\wedge = L & \longrightarrow & L^*T_{G_sp}^\wedge \end{array}$$

which shows that E is initial. $\square$

Remember that a compact connected Lie group is pseudo projective if $Z(G)$ is connected.

9.5 Proposition. *Let G_s be a p–convenient simply connected Lie group, and let $G \in GrExt(T, G_s)$ be pseudo projective. Let L be a torsion free p–reducible $\mathbb{Z}_p^\wedge[W_G]$–module, and let $\overline{\alpha} : L/p \longrightarrow L^*T_G/p$ be a $\mathbb{Z}/p[W_G]$–isomorphism. Then there exists a $\mathbb{Z}_p^\wedge[W_G]$–isomorphism*

$$\alpha : L \longrightarrow L^*T_{Gp}^\wedge \ ,$$

which is a lift of $\overline{\alpha}$*.*

Proof. Let $E_G : L^*T_p^\wedge \longrightarrow L^*T_{G_p^\wedge} \longrightarrow L^*T_{G_s p}^\wedge$ be the sequence associated to G, and $E : L^*T_p^\wedge \longrightarrow L \longrightarrow L^*T_{G_s p}^\wedge$ the sequence associated to L (lemma 9.1). The difference $E_1 := E - E_G$ is mod–p trivial. Thus, by lemmma 9.3, there is an extension E_2 such that $E_1 = m_*(E_2)$. The map $m : L^*T_p^\wedge \longrightarrow L^*T_p^\wedge$ is the multiplication with p. The exact sequence E_G is initial (lemma 9.4). Hence $E_2 = a_*(E_G)$ and $E_1 = m_* a_*(E_G)$ for a suitable homomorphism $a : L^*T_p^\wedge \longrightarrow L^*T_p^\wedge$. This yields

$$\begin{array}{rcccccc}
E_G : & L^*T_p^\wedge & \longrightarrow & L^*T_{G_p^\wedge} & \longrightarrow & L^*T_{G_s p}^\wedge \\
 & \downarrow{\scriptstyle (id,am)} & & \downarrow & & \downarrow{\scriptstyle \Delta} \\
E_G + E_1 : & L^*T_p^\wedge \oplus L^*T_p^\wedge & \longrightarrow & L^*T_{G_p^\wedge} \oplus L_1 & \longrightarrow & L^*T_{G_s p}^\wedge \oplus L^*T_{G_s p}^\wedge \\
 & \downarrow & & \downarrow & & \| \\
 & L^*T_p^\wedge & \longrightarrow & \overline{L} & \longrightarrow & L^*T_{G_s p}^\wedge \oplus L^*T_{G_s p}^\wedge \\
 & \| & & \uparrow & & \uparrow{\scriptstyle \Delta} \\
E : & L^*T_p^\wedge & \longrightarrow & L & \longrightarrow & L^*T_{G_s p}^\wedge \ ,
\end{array}$$

where the three lower rows represent the sum $E_G + E_1 = E$ and where Δ is the diagonal map. Therefore $E = (id + am)_*(E_G)$, and, because $id + am \equiv id \bmod p$, the map $L^*T_{G_p^\wedge} \longrightarrow L$ is an isomorphism. □

9.6 Remark. Let $\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 = Z(SU(p^2))$ and $\mathbb{Z}/p \hookrightarrow S^1$ be the standard inclusions, and let $G = (SU(p^2) \times S^1)/\mathbb{Z}/p$. G is a p–fold cover of $U(p^2)$. This fits into

$$\begin{array}{ccccc}
 & & B\mathbb{Z}/p & = & B\mathbb{Z}/p \\
 & & \downarrow & & \downarrow \\
BSU(p^2) & \longrightarrow & BG & \longrightarrow & BS^1 \\
\| & & \downarrow & & \downarrow{\scriptstyle f} \\
BSU(p^2) & \longrightarrow & BU(p^2) & \longrightarrow & BS^1
\end{array}$$

and establishes

$$\begin{array}{ccccc}
L^*S^1{}_p^\wedge & \longrightarrow & L^*T_{G_p^\wedge} & \longrightarrow & L^*T_{SU(p^2)_p^\wedge} \\
{\scriptstyle f^*}\uparrow & & \uparrow & & \| \\
L^*S^1{}_p^\wedge & \longrightarrow & L^*T_{U(p^2)_p^\wedge} & \longrightarrow & L^*T_{SU(p^2)_p^\wedge} \ ,
\end{array}$$

where f^* is multiplication by p. Therefore, by lemma 9.3, the upper row splits after applying the functor $\otimes \mathbb{Z}/p$. That is to say that $L^*T_G/p \cong L^*T_{SU(p^2)}/p \oplus L^*S^1/p$ as Σ_{p^2}–modules. Because G is p–convenient BG and $BSU(p^2) \times BS^1$ have the same mod–p type, but not the same p–adic type. This is the generic example of compact Lie groups having the same mod–p type but different p–adic type, i.e. all examples come from p–fold coverings.

For $p = 2$, the last proposition only covers the case of products of unitary groups $U(n)$, $n \geq 3$, which are all pseudo projective, but not the group $U(2)$ which is not 2–convenient. Nevertheless, with one extra assumption, the same statement is still true for products of all unitary groups at $p = 2$ as we will see. First we consider the case of a product of $U(2)$'s.

9.7 Proposition. *Let $G = U(2)^r$ and let L be a torsion free 2–reducible $\mathbb{Z}_2^\wedge[W_G]$–module. Let*

$$\overline{\alpha} : L^*T_G/2 \rightarrow L/2$$

be a $\mathbb{Z}_2^\wedge[W_G]$–isomorphism. Then there exists a $\mathbb{Z}_2^\wedge[W_G]$–isomorphism

$$\alpha : L^*T_G \rightarrow L$$

which is a lift of $\overline{\alpha}$.

Proof. The two representations L^*T_G and L establish homomorphisms $\rho_L, \rho_{L^*T_G}$: $W_G \rightarrow Gl(2r, \mathbb{Z}_2^\wedge)$. Because L is 2–reducible and because $P(L)^{W_G}/2 \cong P(L/2)^{W_G}$ is a polynomial algebra, the 2-adic invariants $P(L)^{W_G}$ are also a polynomial algebra as well as $P(L)^{W_G} \otimes \mathbb{Q} \cong P(L \otimes \mathbb{Q})^{W_G}$. Therefore the rational representation $\rho_L : W_G \rightarrow Gl(2r, \mathbb{Q}_2^\wedge)$ represents W_G as a pseudo reflection group. We can choose $s_1, ..., s_r \in (W_G) \cong (\mathbb{Z}/2)^r$ which generate W_G and which are pseudo reflections with respect to ρ_L. The only pseudo reflections in the image of W_G in $Gl(2r, \mathbb{Z}_2^\wedge$ are given by the generators of the Weyl groups of the factors of G. Therefore, after reordering, the element s_i generates the Weyl group of the i-th factor.

Let $W' \subset W$ be the subgroup generated by $s_2, ..., s_r$. The fixed–points of a pseudo reflection acting on a $\mathbb{Z}_2^\wedge$–module or $\mathbb{Q}_2^\wedge$–module has codimension 1 or codimension 0. Therefore, $\mathrm{rk}(L^{W'}) \geq 2r - r + 1 = r + 1$ and $\mathrm{rk}(L/L^{W'}) \leq r - 1$. Here rk( ) denotes the rank of free modules. We notice that the qoutient $L/L^{W'}$ is torsion free.

In the commutative diadram of exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{W'} & \longrightarrow & L & \longrightarrow & L/L^{W'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^{W'}/2 & \longrightarrow & L/2 & \longrightarrow & (L/L^{W'})/2 & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & L/2^{W'} & \longrightarrow & L/2 & \longrightarrow & (L/2)/(L/2^{W'}) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & L^*T_G/2^{W'} & \longrightarrow & L^*T_G/2 & \longrightarrow & (L^*T_G/2)/(L^*T_G/2^{W'}) & \longrightarrow & 0
\end{array}$$

the right vertical composition of maps is an epimorphism. Hence, $\mathrm{rk}_{\mathbb{Z}_2^\wedge} L/L^{W'} = \mathrm{rk}_{\mathbb{Z}/2}(L^*T_G/2)/((L^*T_G/2)^{W'}) = r-1$. This implies that $(L/L^{W'})/2 \to (L/2)/((L/2)^{W'})$ is an isomorphism as well as $L^{W'}/2 \to (L/2)^{W'}$.

According to the splitting $G = U(2)^r$ we choose a basis $\overline{B} = \{\overline{x}_1, \overline{y}_1, ..., \overline{x}_r, \overline{y}_r\}$ of $L/2$ characterized by the properties $s_i(\overline{x}_j) = \overline{x}_j$ and $s_i(\overline{y}_j) = \overline{y}_j$ for $i \neq j$ and the property $s_i(\overline{x}_i) = \overline{y}_i$. By the above considerations there exist lifts $x_1, y_1 \in L^{W'}$ of $\overline{x}_1$ and $\overline{y}_1$, such that $s_1(x_1) = y_1$. Analogously we can choose lifts x_i, y_i of $\overline{x}_i$ and $\overline{y}_i$ for all i such that $B = \{x_1, y_1, ..., x_r, y_r\}$ is a basis of L, such that$s_i(x_i) = y_i$ and such that, for $i \neq j$, $s_i(x_j) = x_j$ and $s_i(y_j) = y_j$. This implies that there exist a $\mathbb{Z}_2^\wedge[W_G]$–isomorphism $L^*T_G \to L$ which is a lift of $\overline{\alpha}$. □

Let X be a space with the mod–p type of $BG \times BH$. The assumption that G and H are p–convenient is too strong for what follows. We only assume that

$$(*) \quad H^*(BG; \mathbb{Z}/p) \cong H^*(BT_G; \mathbb{Z}/p)^{W_G} \text{ and } H^*(BH; \mathbb{Z}/p) \cong H^*(BT_H; \mathbb{Z}/p)^{W_H} .$$

Let $BT_X \longrightarrow X$ be the maximal torus of X, let $V_G \subset T_G$ and $V_H \subset T_H$ be the maximal elementary abelian p–subgroups of the maximal tori T_G and T_H, and let $g : BV_H \longrightarrow BV_H \times BV_G \longrightarrow X$ be the obvious map (see section 7). The isotropy group of g is W_G. Hence, for $Y := map(BV_H, X)_g$,

$$H^*(Y; \mathbb{Z}/p) \cong H^*(BT_X; \mathbb{Z}/p)^{W_G} \cong H^*(BT_G \times BT_H; \mathbb{Z}/p)^{W_G}$$

(theorem 10.1). In particular, $H^*(Y; \mathbb{Z}/p)$ is concentrated in even degrees, and therefore, $H^*(Y; \mathbb{Z}_p^\wedge) \longrightarrow H^*(Y; \mathbb{Z}/p)$ is a surjection.

9.8 Lemma. *If G and H are two compact connected Lie groups satifying $(*)$ then $L^*T_X{}_p^\wedge$ is p–reducible as W_G–module.*

Proof. We have $H^*(BT_X;{}_p^\wedge) \cong P(L^*T_X{}_p^\wedge)$ which is the symmetric part of the tensor algbra of $L^*T_X{}_p^\wedge$. This induces maps

$$H^*(Y; \mathbb{Z}_p^\wedge)/p \longrightarrow P(L^*T_X{}_p^\wedge)^{W_G}/p \hookrightarrow P(L^*T_X/p)^{W_G} \cong H^*(Y; \mathbb{Z}/p) .$$

The composition is an isomorphism. This implies the statement. □

Now we are in the position to prove the following result.

9.9 Proposition. *Let G be a product of unitary groups. If X is a space with the mod–2 type of BG, then $L^*T_X{}_p^\wedge \cong L^*T_G{}_p^\wedge$ as W_G–modules.*

Proof. The group $G = G' \times G''$ splits into a product, where G' contains all factors isomorphic to $U(2)$ and G'' all the other factors $U(n)$, $n \geq 3$. Then $L^*T_X{}_2^\wedge$ is 2–reducible as $W_{G'}$–module (lemma 9.7), and $L^*T_X{}_2^{\wedge W_{G'}}$ has as $W_{G''}$–module the mod–2 type of $L^*Z(G') \oplus L^*T_{G''}{}_2^\wedge$. Moreover, $Z(G') \times G''$ is pseudo projective, and therefore, $L^*T_X{}^\wedge_2{}^{W_{G'}} \cong L^*Z(G')^\wedge_2 \oplus L^*T_{G''}{}^\wedge_2$ as $W_{G''}$–modules (proposition 9.5).

Now we consider the diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^*T_{G''}{}^\wedge_2 & \longrightarrow & L^*T_X{}^\wedge_2 & \longrightarrow & L^*T_X{}^\wedge_2/L^*T_{G''}{}^\wedge_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^*T_{G''}/2 & \longrightarrow & L^*T_X/2 & \longrightarrow & (L^*T_X{}^\wedge_2/L^*T_{G''}{}^\wedge_2)/2 & \longrightarrow & 0 \\
 & & \| & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & L^*T_{G''}/2 & \longrightarrow & L^*T_{G'}/2 \oplus L^*T_{G''}/2 & \longrightarrow & L^*T_{G'}/2 & \longrightarrow & 0\ .
\end{array}$$

The middle arrow is also an exact sequence because $L^*T_X{}^\wedge_2/L^*T_{G''}{}^\wedge_2$ is torsion free. Moreover, the analogous diagram for the polynomial algebras of the modules shows that $L^*T_X{}^\wedge_2/L^*T_{G''}{}^\wedge_2$ is 2–reducible as $W_{G'}$–module. By proposition 9.7 this implies that $L^*T_X{}^\wedge_2/L^*T_{G''}{}^\wedge_2 \cong L^*T_{G'}{}^\wedge_2$ as $W_{G'}$–modules.

L^*T_X can be considered as an element in

$$\begin{aligned}
Ext_{W_{G'}\times W_{G''}}(L^*T_{G'}{}^\wedge_2, L^*T_{G''}{}^\wedge_2) &\cong H^1(W_{G'} \times W_{G''}; Hom(L^*T_{G'}{}^\wedge_2, L^*T_{G''}{}^\wedge_2)) \\
&\cong LT_{G'}{}^\wedge_2{}^{W_{G'}} \otimes H^1(W_{G''}; L^*T_{G''}{}^\wedge_2) \\
&\qquad \oplus H^1(W_{G'}; LT_{G'}{}^\wedge_2) \otimes L^*T_{G''}{}^\wedge_2{}^{W_{G''}} \\
&= 0\ .
\end{aligned}$$

This follows by lemma 8.2 and because $H^1(W_{G'}; LT_{G'}{}^\wedge_2) = H^1(W_{G''}; L^*T_{G''}{}^\wedge_2) = 0$. Finally we get $L^*T_X{}^\wedge_2 \cong L^*T_{G'}{}^\wedge_2 \oplus L^*T_{G''}{}^\wedge_2 = L^*T_G{}^\wedge_2$ as WG–modules. □

10. Mapping spaces.

In this section G is a compact connected Lie group satifying the condition $H^*(BG;\mathbb{Z}/p) \cong H^*(BT_G;\mathbb{Z}/p)^{W_G}$. Let X be a space with the mod–p type of BG. In section 7 we constructed a maximal torus $f_T : BT_X \longrightarrow X$, respectively $f_T : BT_X{}^\wedge_p \longrightarrow X$, with Weyl group W_G such that the diagram

$$\begin{array}{ccc}
BT_G{}^\wedge_p & \overset{\simeq}{\longrightarrow} & BT_X{}^\wedge_p \\
\downarrow & & \downarrow \\
BG^\wedge_p & \dashrightarrow & X
\end{array}$$

commutes in mod–p cohomology. In this chapter we always work with mod–p cohomology and define $H^*(\ ) := H^*(\ ;\mathbb{Z}/p)$.

Let A be an abelian p–toral group. Then W_G acts on $map(BA, BT_X)$. This action induces a map

$$[BA, BT_X{}^\wedge_p]/W_G \longrightarrow [BA, X]\ .$$

For a map $g : BA \longrightarrow BT_X{}^\wedge_p$ we define

$$Iso(g) := \{w \in W_G | w \circ g \simeq g\}$$

to be the *isotropy group* of g. The composition of g with the maximal tori of X is also denoted by g.

10.1 Theorem. *Let G be a compact connected Lie group such that $H^*(BG;\mathbb{Z}/p) \cong H^*(BT_G;\mathbb{Z}/p)^{W_G}$. Let X be a space with the mod–p type of BG, and let A be an abelian p–toral group. Then the following hold:*

(1) $[BA, BT_X{}^\wedge_p] \longrightarrow [BA, X]$ *is a surjection.*
(2) $[BA, BT_X{}^\wedge_p]/W_G \longrightarrow [BA, X]$ *is a bijection.*
(3) *For any* $g : BA \longrightarrow BT_X{}^\wedge_p$,

$$H^*(map(BA, X)_g) \cong H^*(map(BA, BT_X{}^\wedge_p)_g)^{Iso(g)}$$

and $map(BA, X)_g$ *is p-complete.*

10.2 Remark.

(1) By 10.1 (1) there exists for any map $g : BA \longrightarrow X$ a lift $g' : BA \longrightarrow BT_X{}^\wedge_p$, which, by 10.1 (2), is unique up to conjugation by elements of the Weyl group. Thus $Iso(g') \subset W_G$ is uniquely determined up to conjugation by g.

(2) The isomorphism of 10.1 (3) is induced by the maximal torus $BT_X{}^\wedge_p \longrightarrow X$. Of course, $BG^\wedge_p$ satisfies the assumptions of the theorem. Let X have the p–adic type of BG. Part (3) and theorem 3.8 induce canonical isomorphisms

$$H^*(map(BA, X)_g) \cong H^*(map(BA, BG^\wedge_p)_g) \cong H^*(map(BA, BG)_g) \ .$$

(3) Let $g : BA \longrightarrow BT_G$ be a map and $g \simeq B\alpha$ for a suitable homomorphism $\alpha : A \longrightarrow T_G$ (theorem 3.6). $\pi_0(C_G(\alpha))$ is a p–group [J-M-O; A.4]. Therefore, by 10.1 (3), the centralizer $C_G(\alpha)$ is connected and p–convenient, and $W_{C_G(\alpha)} = Iso(g)$.

(4) Theorem 10.1 is true under some weaker assumptions as the proof will show. We only have to assume that X is a complete space with maximal torus $f : BT_X{}^\wedge_p \to X$ and Weyl group W_X such that $H^*(X;\mathbb{Z}/p) \cong H^*(BT_X{}^\wedge_p;\mathbb{Z}/p)^{W_X}$; i.e. it is not necessary that X is of the mod p–type of the classifying space of a compact connected Lie group.

We can reformulate theorem 10.1, using the identity

$$H^*(map(BA, BT_X{}^\wedge_p))^{W_G} \cong \prod_{g \in [BA, BT_X]/W_G} H^*(map(BA, BT_X{}^\wedge_p)_g)^{Iso(g)} \ .$$

This identity is in anology to theorem 3.14.

10.3 Theorem. *Under the assumption of 10.1, the maximal torus* $BT_X{}^\wedge_p \longrightarrow X$ *induces an isomorphism*

$$H^*(map(BA, X)) \cong H^*(map(BA, BT_X{}^\wedge_p))^{W_G} \ ,$$

and $map(BA, X)$ *is p–complete, i.e. every component is p–complete.*

For the next statement, we assume that X has the p–adic type of BG and that there exists an extension

$$BN(T_G) \longrightarrow X \ ,$$

of the maximal torus $BT_G \longrightarrow X$, such that the diagram

$$\begin{array}{ccc} & BN(T_G) & \\ \swarrow & & \searrow \\ BG & \dashrightarrow & X \end{array}$$

commutes in mod–p cohomology. For any abelian abelian p–toral group A and any map $g : BA \to BT_G$, theorem 10.3 establishes the following sequence of isomorphisms

$$\begin{aligned} H^*(map(BA, X)_g) &\cong H^*(map(BA, BT_{X}{}^{\wedge}_p)_g)^{W_G} \\ &\cong H^*(map(BA, BT_{G}{}^{\wedge}_p)_g)^{W_G} \\ &\cong H^*(map(BA, BG^{\wedge}_p)_g) \ , \end{aligned}$$

and therefore, a dotted arrow $map(BA, BG)_g \dashrightarrow map(BA, X)_g$

10.4 Proposition. *For any map $g : BA \longrightarrow BT_G$, the diagram*

$$\begin{array}{ccc} & map(BA, BN(T_G))_g & \\ \swarrow & & \searrow \\ map(BA, BG)_g & \dashrightarrow & map(BA, X)_g \end{array}$$

commutes in mod–p cohomology.

In the rest of this chapter we will prove theorem 10.3 and proposition 10.4.

10.5 Lemma. *It is sufficient to prove 10.3 and 10.4 for finite abelian p–groups.*

Proof. Let A be a p–toral abelian group. We denote by $A_k \subset A$ the elements of order p^k and define $A_\infty := \bigcup A_k$. Then A_k is a finite p–group. The map $BA_\infty \longrightarrow BA$ is a mod–p equivalence, which implies that $map(BA, X) \simeq \underleftarrow{holim}\, map(BA_k, X)$ and $map(BA, BT_X) \simeq \underleftarrow{holim}\, map(BA_k, BT_X)$. The inclusions $BA_{k-1} \longrightarrow BA_k$ induce a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & map(BA_k, BT_{X}{}^{\wedge}_p) & \longrightarrow & map(BA_{k-1}, BT_{X}{}^{\wedge}_p) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & map(BA_k, X) & \longrightarrow & map(BA_{k-1}, X) & \longrightarrow & \cdots. \end{array}$$

By theorem 10.3 for finite abelian p–groups we have $H^1(map(BA_k, BT_{X}{}^{\wedge}_p)) = H^1(map(BA_k, X)) = 0$.

The two function spaces $map(BA_r, BT_X{}_p^\wedge)$ and $map(BA_r, X)$ are p–complete. Hence, both mapping spaces are also 1-connected [B-K; I.6]. The Milnor sequence for calculating components of homotopy inverse limits therefore reduces to

$$\begin{array}{ccc} [BA, BT_X{}_p^\wedge] & \xrightarrow{\cong} & \varprojlim\, [BA_k, BT_X{}_p^\wedge] \\ \downarrow & & \downarrow \\ [BA, X] & \xrightarrow{\cong} & \varprojlim\, [BA_k, X] \end{array} .$$

The obvious map

$$[BA, BT_X{}_p^\wedge]/W_G \longrightarrow \varprojlim\, ([BA_k, BT_X{}_p^\wedge]/W_G)$$

is a bijection. That is to say that

$$H^0(map(BA, X)) \cong H^0(map(BA, BT_X{}_p^\wedge))^{W_G} .$$

Let $g : BA \longrightarrow BT_X{}_p^\wedge$ be a map and denote by g_k the restriction $g|_{BA_k}$. The sequence

$$\ldots \supset Iso(g_{k-1}) \supset Iso(g_k) \supset \ldots$$

stabilizes, since W_G is a finite group. We can choose k' (big enough), such that $Iso(g) = Iso(g_k)$ for $k \geq k'$. Thus

$$map(BA_{k+1}, X)_{g_{k+1}} \longrightarrow map(BA_k, X)_{g_k}$$

are mod–p equivalences and therefore equivalences for $k \geq k'$ because all the spaces are p–complete. This implies

$$\begin{aligned} H^*(map(BA, X)_g) &\cong H^*(map(BA_{k'}, X)_{g_{k'}}) \\ &\cong H^*(map(BA_{k'}, BT_X{}_p^\wedge)_{g_{k'}})^{Iso(g_{k'})} \\ &\cong H^*(map(BA, BT_X{}_p^\wedge)_g)^{Iso(g)} , \end{aligned}$$

which finishes the proof of first part of the statement.

To prove 10.4, we consider a map $g : BA \longrightarrow BT_G$. According to remark 10.2 (3), we can speak of the centralizers $C_G(g_k)$ and $C_G(g) = \bigcap_k C_G(g_k)$, which are compact Lie groups. The sequence $C_G(g_{k+1}) \subset C_G(g_k)$ stabilizes. Again we can choose k' big enough such that $C_G(g) = C_G(g_{k'})$ and, analogously, $C_{N(T_G)}(g) = C_{N(T_G)}(g_{k'})$. Theorems 10.1 and 3.6 and the above considerations show that the maps

$$\begin{array}{ccc} map(BA, BN(T_G))_g & \longrightarrow & map(BA_{k'}, BN(T_G))_{g_{k'}} \\ map(BA, BG)_g & \longrightarrow & map(BA_{k'}, BG)_{g_{k'}} \\ map(BA, X)_g & \longrightarrow & map(BA_{k'}, X)_{g_{k'}} \end{array}$$

are mod–p equivalences which reduces 10.4 to the case of a finite group. □

Let A be a finite abelian p–group. We can choose a subgroup $A_0 \subset A$ of index p and get an exact sequence

$$1 \longrightarrow A_0 \longrightarrow A \longrightarrow \mathbb{Z}/p \longrightarrow 1 \ .$$

We will prove 10.3 and 10.4 for a finite group A by an induction over the order of A. In remark 3.12 we defined $\widetilde{BA_0} := EA/A_0 \simeq BA_0$ which carries a free $\mathbb{Z}/p$–action. To this end we introduce the following notation and abbreviations:

$$\begin{aligned} MX &:= map(BA, X) \\ MX_0 &:= map(\widetilde{BA_0}, X) \\ MT &:= map(BA, BT_{X_p^\wedge}) \\ MT_0 &:= map(\widetilde{BA_0}, BT_{X_p^\wedge}) \\ W &:= W_G \ . \end{aligned}$$

Then $MX \cong {MX_0}^{h\mathbb{Z}/p}$ and $MT \cong {MT_0}^{h\mathbb{Z}/p}$. Because the actions of W and $\mathbb{Z}/p$ on MT_0 commute the group W acts on the Borel product $E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0$.

Now let us assume that 10.3 and 10.4 is true for A_0.

10.6 Lemma.

(1) *The fibration*

$$MT_0 \longrightarrow E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0 \longrightarrow B\mathbb{Z}/p$$

is fiber homotopic trivial.

(2) $H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MX_0) \cong H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0)^W$.

Proof. (1) Every map $g_0 : BA_0 \longrightarrow BT_{X_p^\wedge}$ is, up to homotopy, induced by a homomorphism. For every g_0 we can choose an extension $g : BA \longrightarrow BT_{X_p^\wedge}$ such that $g|_{BA_0} = g_0$. $\mathbb{Z}/p$ acts trivially on $\overline{MT} := \bigcup_g map(BA, BT_{X_p^\wedge})_g$ and on $E\mathbb{Z}/p \times_{\mathbb{Z}/p} \overline{MT} = B\mathbb{Z}/p \times \overline{MT}$. By theorem 3.9 the restriction from BA to $\widetilde{BA_0}$ induces a homotopy equivalence

$$\overline{MT} \longrightarrow MT_0 \ .$$

Moreover, this map is $\mathbb{Z}/p$–equivariant, and fits into a commutative diagram

$$\begin{array}{ccccc} \overline{MT} & \longrightarrow & B\mathbb{Z}/p \times \overline{MT} & \longrightarrow & B\mathbb{Z}/p \\ \downarrow & & \downarrow & & \| \\ MT_0 & \longrightarrow & E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0 & \longrightarrow & B\mathbb{Z}/p \end{array}$$

of homotopy equivalent fibrations, which proves (1).

(2) By the assumptions $H^*(MX_0) \cong H^*(MT_0)^W$. Moreover, this is a direct summand in $H^*(MT_0)$ considered as vector spaces. The diagram

$$\begin{array}{ccccc} MT_0 & \longrightarrow & E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0 & \longrightarrow & B\mathbb{Z}/p \\ \downarrow & & \downarrow & & \| \\ MX_0 & \longrightarrow & E\mathbb{Z}/p \times_{\mathbb{Z}/p} MX_0 & \longrightarrow & B\mathbb{Z}/p \end{array}$$

and (1) show that both fibrations are oriented and that all differentials in the associated Serre spectral sequences are trivial. For every n we have

$$\begin{aligned} H^n(B\mathbb{Z}/p; H^*(MX_0)) &\cong H^*(MX_0) \\ &\cong H^*(MT_0)^W \\ &\cong H^n(B\mathbb{Z}/p; H^*(MT_0))^W \ . \end{aligned}$$

The isomorphism between the first and last group is compatible with the inclusion $H^n(B\mathbb{Z}/p; H^*(MX_0)) \longrightarrow H^n(B\mathbb{Z}/p; H^*(MT_0))$ because W acts on the space level. Applying the 5–lemma to the extension problems yields the desired isomorphism

$$H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MX_0) \cong H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0)^W \ . \qquad \square$$

Proof of 10.3 for A a finite abelian p–group.
By induction hypothesis $H^*(MX_0) \cong H^*(MT_0)^W$. To calculate $MX = {MX_0}^{h\mathbb{Z}/p}$ we apply the Lannes functor for homotopy fixed–point sets (see section 3). Because MT is p–complete and $H^1(MT) = 0$, the W–equivariant map

$$HF^{\mathbb{Z}/p}(MT_0) \longrightarrow H^*(MT)$$

is an isomorphism (theorem 3.10). Using this fact and theorem 3.14 we get

$$\begin{aligned} HF^{\mathbb{Z}/p}(MX_0) &\cong T^{\mathbb{Z}/p}_{sec^*} H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MX_0) \otimes_{H^*(B\mathbb{Z}/p)} \mathbb{Z}/p \\ &\cong T^{\mathbb{Z}/p}_{sec^*}(H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0)^W) \otimes_{H^*(B\mathbb{Z}/p)} \mathbb{Z}/p \\ &\cong T^{\mathbb{Z}/p}_{sec^*}((H^*(B\mathbb{Z}/p) \otimes H^*(MT_0))^W) \otimes_{H^*(B\mathbb{Z}/p)} \mathbb{Z}/p \\ &\cong (T^{\mathbb{Z}/p}_{id}(H^*(B\mathbb{Z}/p)) \otimes T^{\mathbb{Z}/p}(H^*(MT_0)^W) \otimes_{H^*(B\mathbb{Z}/p)} \mathbb{Z}/p \\ &\cong H^*(B\mathbb{Z}/p) \otimes T^{\mathbb{Z}/p}(H^*(MT_0))^W \otimes_{H^*(B\mathbb{Z}/p)} \mathbb{Z}/p \\ &\cong H^*(MT)^W \ , \end{aligned}$$

where sec^* denotes the cohomological maps induced by sections in the bundle $E\mathbb{Z}/p \times_{\mathbb{Z}/p} MT_0$. The third isomorphism follows from lemma 10.6 which says that the bundle $E\mathbb{Z}/p \times MT_0 \to B\mathbb{Z}/p$ is fiber homotopic trivial. The fourth isomorphism follows because T commutes with tensor products and because every section in the trivial bundle $B\mathbb{Z}/p \times MT_O \to B\mathbb{Z}/p$ is given by a pair $(id_{B\mathbb{Z}/p}, g)$ of maps where $g : B\mathbb{Z}/p \to MT_0$. Hence $HF^{\mathbb{Z}}/p(MX_0)$ vanishes in degree 1. By theorem 3.10 again

$$H^*(MT)^W \cong HF^{\mathbb{Z}/p}(MX_0) \longrightarrow H^*(MX)$$

is an isomorphism, and by theorem 3.11, MX is p–complete which finishes the proof of 10.3. □

Proof of 10.4 for A a finite abelian p–group.
Let $g \simeq B\alpha : BA \to BT_G$ be a map and denote by $S_pN(T_G) \subset N(T_G)$ a p–toral Sylow subgroup. Then $C_G(\alpha)$ is connected (remark 10.2) and

$$C_{S_pN(T_G)}(\alpha) \subset C_{N(T_G)}(\alpha) = N(T_{C_G(\alpha)})$$

is an inclusion of index coprime to p and therefore, also a p–toral Sylow subgroup. In particular,

$$H^*(BC_{N(T_G)}(\alpha)) \longrightarrow H^*(BC_{S_pN(T_G)}(\alpha))$$

and, by theorem 3.6,

$$H^*(map(BA, BN(T_G))) \longrightarrow H^*(map(BA, BS_pN(T_G)))$$

are injections. In order to prove 10.4 it is sufficient to replace $N(T_G)$ by $S_pN(T_G)$.

$BC_{S_pN(T_G)}(\alpha)^\wedge_p \simeq (map(BA, BS_pN(T_G))_g)^\wedge_p$ are p–complete (theorem 3.9), and

$$\begin{aligned} map(BA, BS_pN(T_G))_{g}{}^\wedge_p &\simeq map(BA, BS_pN(T_G)^\wedge_p)_{g^\wedge_p} \\ &\simeq (map(BA_0, BS_pN(T_G)^\wedge_p)_{g_0{}^\wedge_p})^{h\mathbb{Z}/p} \\ &\simeq (map(BA_0, BS_pN(T_G))_{g_0}{}^\wedge_p)^{h\mathbb{Z}/p}_{g^\wedge_p} \end{aligned}$$

is the homotopy fixed–point set of a p–complete space. Therefore and because of theorem 3.11

$$\begin{aligned} HF^{\mathbb{Z}/p}_{g^*}(map(BA_0, BS_pN(T_G))_{g_0}) &\cong HF^{\mathbb{Z}/p}_{g^*}(map(BA_0, BS_pN(T_G)^\wedge_p)_{g_0{}^\wedge_p}) \\ &\cong H^*(map(BA, BS_pN(T_G)^\wedge_p)_{g^\wedge_p}) \\ &\cong H^*(map(BA, BS_pN(T_G))_g)\ . \end{aligned}$$

The diagram
(*)

$$\begin{array}{ccc} & H^*(map(BA, BS_pN(T_G))_g) & \\ \nearrow & & \nwarrow \\ H^*(map(BA, X)_g) & \longrightarrow & H^*(map(BA, BG)_g) \end{array}$$

comes up by repeated applications of the functor $HF^{\mathbb{Z}/p}$ to the commutative diagram

$$\begin{array}{ccc} & H^*(BS_pN(T_G)) & \\ \nearrow & & \nwarrow \\ H^*(X) & \longrightarrow & H^*(BG) \end{array} \quad .$$

That is that the diagram (∗) also commutes (remark 3.4) which finishes the proof of 10.4. □

11. The normalizer of the maximal torus I.

Let G be a p–convenient compact connected Lie group, and let X be a space of the p–adic type of BG. In this section we construct an extension of the maximal torus

$$f_T : BT_G \longrightarrow X$$

to a map

$$f_N : BN(T_G) \longrightarrow X$$

of the classifying space of the the normalizer $N(T_G)$ of T_G.

The space $BN(T_G)$ is a two stage Posnikov system given by the fibration

$$BT_G \longrightarrow BN(T_G) \longrightarrow BW_G \ .$$

We have to find a suitable element in $map(BN(T_G), X) \simeq map(BT_G, X)^{hW_G}$ (remark 3.12); i.e. a section in the bundle

$$EW_G \times_{W_G} map(\widetilde{BT_G}, X)_{f_T} \longrightarrow BW_G \ .$$

The obstructions are lying in

$$H^{*+1}(BW_G; \pi_*(map(\widetilde{BT_G}, X)_{f_T}) \cong H^3(BW_G; LT_{G_p^\wedge})$$

(twisted coefficients). The isomorphism follows from theorem 10.1 and the identity $\pi_2(BT_{G_p^\wedge}) \cong H_2(BT_G; \mathbb{Z}_p^\wedge) = LT_{G_p^\wedge}$. By lemma 5.3 the obstruction group vanishes for odd primes which is sufficient to prove the following statement for odd primes.

11.1 Proposition. *Let G be a compact connected Lie group, and let X be a space with the p–adic type of BG. If p is odd and G is p–convenient or if $p = 2$ and G is a product of unitary groups, there is an extension*

$$\begin{array}{ccc} BT_G & \xrightarrow{f_T} & X \\ \downarrow & \nearrow_{f_N} & \\ BN(T_G) & & \end{array}$$

of the maximal torus $BT_G \longrightarrow X$ *to* $BN(T_G) \longrightarrow X$.

Because the obstruction group does not vanish for $p = 2$, we need a different approach. The proof in this case is postponed to the end of the section.

Choosing a fixed–point of the W_G–action on BT_G the evaluation at this fixpoint

$$map(BT_{G\,p}^{\wedge}, X)_{f_T} \longrightarrow X$$

is equivariant with respect to the trivial action of W_G on X. The group W_G acts freely on the product $BT_{G\,p}^{\wedge} \simeq EW_G \times map(BT_{G\,p}^{\wedge}, X)_{f_T}$. We get a well defined map on the orbit space $Y := (EW_G \times map(BT_{G\,p}^{\wedge}, X)_{f_T})/W_G$ and an extension

$$\begin{array}{ccc} BT_{G\,p}^{\wedge} \simeq map(BT_{G\,p}^{\wedge}, X)_{f_T} & \xrightarrow{f_T} & X \\ \downarrow & \nearrow & \\ Y & & \end{array}$$

We will show that Y is nothing but the fiberwise completion $BN(T_G)^{\circ}_p$ of $BN(T_G)$ of the fibration $BT_G \longrightarrow BN(T_G) \longrightarrow BW_G$ (see [B-K; I.8]). By construction Y fits into a fibration

$$BT_{G\,p}^{\wedge} \longrightarrow Y \longrightarrow BW_G\ .$$

We will consider a more general situation. Let W be a finite group, acting on a torus T via homomorphisms. Fibrations of the form

$$BT_p^{\wedge} \longrightarrow Y \longrightarrow BW$$

can be classified by homotopy classes of maps

$$BW \longrightarrow BHE(BT_p^{\wedge})\ ,$$

where $HE(BT_p^{\wedge})$ is the monoid of self equivalences of $BT_p^{\wedge}$ [St]. Denote by $SHE(BT_p^{\wedge})$ the component of the identity. Then $Gl(LT_p^{\wedge}) \cong \pi_0(HE(BT_p^{\wedge}))$ is the group of the components. The Dwyer–Zabrodsky map

$$BT_p^{\wedge} \longrightarrow SHE(BT_p^{\wedge})$$

is a homotopy equivalence. Considering BT as a group the map is given by mapping each element to the associated left translation. Therefore, it is a homomorphism of monoids. Moreover, it is W–equivariant because W acts via homomorphisms. We get an W–equivariant equivalence

$$BBT_p^{\wedge} \longrightarrow BSHE(BT_p^{\wedge})\ .$$

Up to homotopy, the composition $BW \longrightarrow BHE(BT_p^{\wedge}) \longrightarrow BGl(LT_p^{\wedge})$ is induced by an homomorphism which is given by the action of W on $BT_p^{\wedge}$ associated to the fibration. The difference between two clasifying maps $\phi, \psi : BW \longrightarrow BHE(BT^\wedge_p)$ with the same action on $BT^\wedge_p$ is measured by an obstruction class

$$d(\phi,\psi) \in H^3(BW;\pi_3(BSHE(BT^\wedge_p) \cong H^3(BW;\pi_2(BT^\wedge_p) \cong H^3(W;LT^\wedge_p)$$

for lifting homotopies in the fibration

$$BSHE(BT^\wedge_p) \longrightarrow BHE(BT^\wedge_p) \longrightarrow BGl(LT^\wedge_p)\ .$$

Notice that $H^3(W;LT^\wedge_p) \cong H^2(W;T)^\wedge_p$ which, forgetting the completion, describes the equivalence classes of group extensions $T \to H \to W$.

Let $s : BW \longrightarrow BHE(BT^\wedge_p)$ be the classifying map of the fibration $BT^\wedge_p \longrightarrow B(T \rtimes W)^\circ_p \longrightarrow BW$.

Definition. By $Fib(BW, BT^\wedge_p)$ we denote the set of fibrations divided out the equivalence relations given by

$$\begin{array}{ccccc} BT^\wedge_p & \longrightarrow & Y_1 & \longrightarrow & BW \\ \| & & \downarrow & & \| \\ BT^\wedge_p & \longrightarrow & Y_2 & \longrightarrow & BW\ . \end{array}$$

11.2 Lemma. *The map*

$$\begin{array}{ccc} Fib(BW, BT^\wedge_p) & \longrightarrow & H^3(W;LT^\wedge_p) \\ (\phi : BW \to BHE(BT^\wedge_p)) & \mapsto & d(s,\phi) \end{array}$$

is a bijection.

Proof. Every cohomology class can be realized as an obstruction. □

11.3 Corollary. *The canonical map*

$$\begin{array}{cccc} \mathcal{B} : & GrExt(W,T) & \longrightarrow & Fib(BW, BT^\wedge_p) \\ & T \to N \to W & \mapsto & BT^\wedge_p \to BN^\circ_p \to BW \end{array}$$

is a surjection.

Proof. Group extensions are described by cohomology classes in $H^2(W;T) \cong H^3(W;LT)$. The canonical map $H^2(W;T) \longrightarrow H^3(W;LT^\wedge_p)$ is a surjection and reflects the map $\mathcal{B}$ on the cohomological level. □

11.4 Remark. Obviously there is an integral version of corollary 11.3. That is that

$$GrExt(W,T) \longrightarrow Fib(BW,BT)$$

is a bijection.

Now let us assume that $G = G_1 \times G_2$ is a product of unitary groups. W_G acts trivially on the centers $Z(G)$ and $Z_i := Z(G_i)$, which are all tori. The canonical map

$$Fib(BW_G, BZ_{1}{}^{\wedge}_{p}) \longrightarrow H^3(W_G, LZ_{1}{}^{\wedge}_{p})$$

is a bijection and establishes a map

$$Fib(BW_G, BZ_{1}{}^{\wedge}_{p}) \longrightarrow Fib(BW_G, BT_{G_1}{}^{\wedge}_{p})$$

given by the inclusion $Z_1 \longrightarrow T_{G_1}$.

The inclusion $S^1 = Z(U(n)) \hookrightarrow T_{U(n)}$ induces

$$\begin{aligned} H^3(\Sigma_n; \mathbb{Z}^{\wedge}_p) &\cong H^3(\Sigma_n; LS^1{}^{\wedge}_p) \\ &\longrightarrow H^3(\Sigma_n; LT_{U(n)}{}^{\wedge}_p) \cong H^3(\Sigma_{n-1}; LS^1{}^{\wedge}_p) \cong H^3(\Sigma_{n-1}; \mathbb{Z}^{\wedge}_p) \end{aligned}$$

This composition is given by the restriction and therefore an isomorphism for $n \neq 4$ and a surjection for $n = 4$. In dimensions ≤ 2 this restriction induces an isomorphism for $n \neq 2, 4$ and a surjection in all cases.

Now a spectral sequence argument shows that

$$H^3(W_G; LZ_{1}{}^{\wedge}_{p}) \longrightarrow H^3(W_G; LT_{G_1}{}^{\wedge}_{p})$$

and

$$Fib(BW_G, BZ_{1}{}^{\wedge}_{p}) \longrightarrow Fib(BW_G, BT_{G_1}{}^{\wedge}_{p})$$

are surjective. They are isomorphisms, if G_1 has no factor isomorphic to $U(4)$ or $U(2)$.

11.5 Lemma. *Let G be a product of unitary groups. Let*

$$BT_G{}^{\wedge}_2 \longrightarrow Y \longrightarrow BW_G \ \in \ Fib(BW_G, BT_G{}^{\wedge}_2)$$

be a fibration such that $H^2(Y; \mathbb{Z}/2) \longrightarrow H^2(BT_G{}^{\wedge}_2; \mathbb{Z}/2)^{W_G}$ is a surjection. Then there exists an equivalence of fibrations

$$\begin{array}{ccccc} BT_G{}^{\wedge}_2 & \longrightarrow & Y & \longrightarrow & BW_G \\ \| & & \downarrow & & \| \\ BT_G{}^{\wedge}_2 & \longrightarrow & B(T_G \rtimes W_G)^{\circ}_2 & \longrightarrow & BW_G \ . \end{array}$$

Proof. Let $G = U(n_1) \times \ldots \times U(n_k)$. By corollary 11.3 there exists a group extension

$$E \ : \ T_G \longrightarrow N \longrightarrow W_G$$

such that

$$\begin{array}{ccccc} BT_G{}^\wedge_2 & \longrightarrow & BN^\circ_2 & \longrightarrow & BW_G \\ \| & & \downarrow & & \downarrow \\ BT_G{}^\wedge_2 & \longrightarrow & Y & \longrightarrow & BW_G \end{array}$$

is an equivalence of fibrations. We have to show that $BN^\circ_2 \simeq B(T_G \rtimes W_G)^\circ_2$.

The isomorphism

$$H^2(W_G; T_G) \cong \bigoplus_i H^2(W_G; T_{U(n_i)})$$

establishes group extensions

$$E_i \ : \ T_{U(n_i)} \longrightarrow N_i \longrightarrow W_G \ .$$

E is given by the semi direct product, i.e $N \cong T_G \rtimes W_G$, if and only if E_i is isomorphic to the semi direct product for all i. Moreover, we have a map

$$\begin{array}{ccccc} BT_G{}^\wedge_2 & \longrightarrow & BN^\circ_2 & \longrightarrow & BW_G \\ \downarrow & & \downarrow & & \| \\ BT_{U(n_i)}{}^\wedge_2 & \longrightarrow & BN_i{}^\circ_2 & \longrightarrow & BW_G \end{array}$$

between the two fibrations.

$H^2(Y; \mathbb{Z}/2) \cong H^2(BN^\circ_2; \mathbb{Z}/2) \longrightarrow H^2(BT_G; \mathbb{Z}/2)^{W_G}$ is an epimorphism. That is to say that the differential

$$d^2 : H^0(BW_G; H^2(BT_G; \mathbb{Z}/2)) \cong H^2(BT_G; \mathbb{Z}/2)^{W_G} \longrightarrow H^3(BW_G; \mathbb{Z}/2)$$

of the Leray–Serre spectral sequence is trivial. Because

$$H^0(W_G; H^2(BT_{U(n_i)}; \mathbb{Z}/2)) \longrightarrow H^0(W_G; H^2(BT_G; \mathbb{Z}/2))$$

is an injection the differential

$$d^2 : H^2(BT_{U(n_i)}; \mathbb{Z}/2)^{W_G} \longrightarrow H^3(BW_G; \mathbb{Z}/2)$$

is also trivial.

By the above remark there is a commutative diagram

$$\begin{array}{ccccc} BS^1{}^\wedge_2 & \longrightarrow & Y_i & \longrightarrow & BW_G \\ \downarrow & & \downarrow & & \| \\ BT_{U(n_i)}{}^\wedge_2 & \longrightarrow & BN_i{}^\circ_2 & \longrightarrow & BW_G \ . \end{array}$$

Because the action of W_G on BS^1 is trivial the classifying map of the top fibration lifts to a map $BW_G \longrightarrow BSHE(BS^1{}^\wedge_2) \simeq BBS^1{}^\wedge_2$. This map is totally determined by the first transgression

$$d^2 : H^2(BS^1{}^\wedge_2; \mathbb{Z}^\wedge_2) \longrightarrow H^3(BW_G; \mathbb{Z}^\wedge_2)$$

in the Serre spectral sequence of the 2–adic cohomology and, because $H^3(B\Sigma_n; \mathbb{Z}^\wedge_2)$ and $H^3(W_G; \mathbb{Z}^\wedge_2)$ are annihilated by 2, also by the differential

$$d^2 : H^2(BS^1{}^\wedge_2; \mathbb{Z}/2) \longrightarrow H^3(B\Sigma_n; \mathbb{Z}/2) \ .$$

Now we first assume that n is odd. Then $H^2(BT_{U(n)}; \mathbb{Z}/2)^{W_G} \cong H^2(BS^1, \mathbb{Z}/2)$, and the above transgression is given by the differential

$$d^2 : H^2(BT_{U(n)}{}^\wedge_2; \mathbb{Z}^\wedge_2)^{W_G} \longrightarrow H^3(BW_G; \mathbb{Z}^\wedge_2)$$

which, by assumption, vanishes. Thus, the fibration $BS^1{}^\wedge_2 \longrightarrow Y_i \longrightarrow BW_G$ is trivial, and the fibration $BT_{U(n_i)}{}^\wedge_2 \to BN_{i}{}^\circ_2 \to BW_G$ is given by the fiber wise completion $B(T_{U(n_i)} \rtimes W_G)^\circ_2$.

Now let n_i be even, and let $G = U(n_i) \times G_i$. Let $S^1 \longrightarrow T_{U(n_i)}$ be the inclusion in the last factor and $\mathbb{Z}/2 \longrightarrow S^1$ the standard inclusion. Then

$$C_{N_i}(\mathbb{Z}/2)/C_{T_{U(n_i)}}(\mathbb{Z}/2) = C_{N_i}(\mathbb{Z}/2)/T_{U(n_i)} = \Sigma_{n_i - 1} \times G_i \ ,$$

and $S^1 \longrightarrow C_N(\mathbb{Z}/2)$ is central. We define $\overline{N}_i := C_{N_i}(\mathbb{Z}/2)/S^1$ and get a commutative diagram of fibrations

$$\begin{array}{ccccc}
BT_{U(n_i-1)}{}^\wedge_2 & \longrightarrow & B\overline{N}_{i}{}^\circ_2 & \longrightarrow & B\Sigma_{n_i-1} \times BW_{G_i} \\
\uparrow & & \uparrow & & \| \\
BT_{U(n_i)}{}^\wedge_2 & \longrightarrow & BC_{N_i}(\mathbb{Z}/2)^\circ_2 & \longrightarrow & B\Sigma_{n_i-1} \times BW_{G_i} \\
\| & & \downarrow & & \downarrow \\
BT_{U(n_i)}{}^\wedge_2 & \longrightarrow & BN_{i}{}^\circ_2 & \longrightarrow & B\Sigma_{n_i} \times BW_{G_i}
\end{array} \ .$$

A comparison of the two lower rows shows that the transgression

$$d^2 : H^2(BT_{U(n_i)}; \mathbb{Z}/2)^{\Sigma_{n_i-1} \times W_{G_i}} \longrightarrow H^3(B\Sigma_{n_i-1} \times BW_{G_i}; \mathbb{Z}/2)$$

is trivial, and a comparison between the two upper rows implies that the differential of the top fibration

$$d^2 : H^2(BT_{U(n_i-1)}; \mathbb{Z}/2)^{\Sigma_{n_i-1} \times W_{G_i}} \longrightarrow H^3(B\Sigma_{n_i-1} \times BW_{G_i}; \mathbb{Z}/2)$$

is also trivial. Because $n_i - 1$ is odd the top fibration is given by the semi direct product.

On the level of the cohomology groups, describing the group extensions, the above construction goes along with the maps

$$H^2(\Sigma_{n_i}\times W_{G_i};T_{U(n_i)}) \longrightarrow H^2(\Sigma_{n_i-1}\times W_{G_i};T_{U(n_i)}) \longrightarrow H^2(\Sigma_{n_i}\times W_{G_i};T_{U(n_i-1)})\ .$$

The composition

$$\begin{aligned} H^2(\Sigma_{n-1};R) \xrightarrow{\cong} H^2(\Sigma_n;R^n) \longrightarrow H^2(\Sigma_{n-1};R^n) \\ \longrightarrow H^2(\Sigma_{n-1};R^{n-1}) \xrightarrow{\cong} H^2(\Sigma_{n-2};R) \end{aligned}$$

is given by the restriction. This is an isomorphism for $n = 4, 2,\ n \geq 6$, and $R = S^1$ or $R = \mathbb{Z}/2$ [Na]. Hence, by a spectral sequence argument follows that the above composition is an isomorphism for even n_i's.

Passing to the obstruction groups, describing the associated fibrations, we see that

$$BT_{U(n_i)}{}^{\wedge}_2 \longrightarrow BN_{i}{}^{\circ}_2 \longrightarrow B\Sigma_{n_i} \times W_{G_i}$$

is given by the semi direct product if and only if the fibration

$$BT_{U(n_i-1)}{}^{\wedge}_2 \longrightarrow B\overline{N}_{i}{}^{\circ}_2 \longrightarrow B\Sigma_{n_i-1} \times W_{G_i}$$

is given by the semi direct product which we already proved. $\square$

Now we are in the position to finish the proof of proposition 11.1.

Proof of proposition 11.1. It is only left to consider one case, namely G is a product of unitary groups and $p = 2$. The composition of maps

$$BT_G{}^{\wedge}_2 \longrightarrow Y \longrightarrow X$$

shows that $H^*(Y;\mathbb{Z}/2) \longrightarrow H^*(BT_G{}^{\wedge}_2;\mathbb{Z}/2)^{W_G}$ is surjective. By lemma 11.5 the fibration

$$BT_G{}^{\wedge}_2 \longrightarrow Y \longrightarrow BW_G$$

is given by the fiberwise completion of the classifying spaces of the exact sequence

$$T_G \longrightarrow T_G \rtimes W_G \longrightarrow W_G\ .$$

The observation $T_G \rtimes W_G = N(T_G)$ completes the proof. $\square$

12. The normalizer of the maximal torus II.

In this section G is a compact connected Lie group which is p–convenient for p odd and a product of unitary groups for $p = 2$. Let X be a space with the p–adic type of BG.

In section 11 we constructed an extension $f_N : BN(T_G) \longrightarrow X$ of the maximal torus $f_T : BT_G \longrightarrow X$, which fits into the diagram

$$\begin{array}{ccccc} BT_G & \xrightarrow{Bi^*} & BN(T_G) & \xrightarrow{Bi^*} & BG \\ & \searrow^{f_T} & & \searrow^{f_N} & \downarrow \\ & & & & X \end{array} ,$$

where the outer triangle commutes in p–adic cohomology. We consider the question, whether the inner triangle commutes in mod-p cohomology.

The extension $BN(T_G) \longrightarrow X$ induces a lift in

$$(*) \qquad \begin{array}{ccc} & & H^*(BN(T_G);\mathbb{Z}/p) \\ & \nearrow^{a} & \downarrow Bi^* \\ H^*(BG;\mathbb{Z}/p) & \xrightarrow{Bi^*} & H^*(BT_G;\mathbb{Z}/p) . \end{array}$$

Here we work in the category of unstable algebras over the Steenrod algebra. The lift a is given by the composition

$$H^*(BG;\mathbb{Z}/p) \xleftarrow{\overline{\phi}} H^*(X;\mathbb{Z}/p) \xrightarrow{f_N^*} H^*(BN(T_G);\mathbb{Z}/p) ,$$

where $\overline{\phi}$ is an isomorphism. The map $Bi^* : H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BT_G;\mathbb{Z}/p)$ is induced by the standard inclusion. Another lift comes from the standard inclusion $Bi^* : BN(T_G) \longrightarrow BG$.

For any homomorphism $\psi : W_G \longrightarrow {T_G}^{W_G}$, we define $j_\psi : N(T_G) \longrightarrow N(T_G)$ to be the automorphism

$$N(T_G) \xrightarrow{\Delta} N(T_G) \times W_G \xrightarrow{id \times \psi} N(T_G) \times {T_G}^{W_G} \xrightarrow{\mu} N(T_G) ,$$

where Δ is the diagonal composed with the projection on W_G and μ the multiplication. Obviously, $j_\psi|_{T_G} = id|_{T_G}$, and $N(T_G) \xrightarrow{j_\psi} N(T_G) \xrightarrow{i} G$ induces another lift in the diagram $(*)$. We will prove the following statement:

12.1 Proposition. *Let G be a p–convenient pseudo simply connected Lie group or a product of unitary groups, and let*

$$a : H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BN(T_G);\mathbb{Z}/p)$$

be a lift of $H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BT_G;\mathbb{Z}/p)$.

(1) *If p is odd then $a = Bi^*$; i.e. there is only one algebraic lift, given by the standard inclusion.*

(2) *If $p = 2$ there exists a homomorphism $\psi : W_G \longrightarrow {T_G}^{W_G}$ such that $a = Bj_\psi^* Bi^*$; i.e a can be realized by an self automorphism of $N(T_G)$ composed with the standard inclusion.*

12.2 Corollary. *Let G be a p–convenient pseudo simply connected Lie group or a product of unitary groups. Then there exists an extension $f_N : N(T_G) \longrightarrow X$ of $f_T : BT_G \longrightarrow X$ such that*

$$\begin{array}{ccccc} BT_G Bi^* & \longrightarrow & BN(T_G) & \xrightarrow{Bi^*} & BG \\ & {\scriptstyle f_T} \searrow & & {\scriptstyle f_N} \searrow & \downarrow \\ & & & & X \end{array} ,$$

commutes in mod–p cohomology.

Proof. By proposition 11.1 we choose an extension $f'_N : BN(T_G) \longrightarrow X$ of $f_T : BT_G \longrightarrow X$. If p is odd we can directly apply the above theorem.

If $p = 2$ we have ${f'_N}^* \overline{\phi}^{-1} = Bj_\psi^* Bi^*$ for a suitable homomorphism $\psi : W_G \longrightarrow {T_G}^{W_G}$. By construction j_ψ is an isomorphism. Hence $f_N := f'_N B(j_\psi^{-1})$ satisfies the statemant. □

For the following we assume that $G = U(n_1) \times \ldots \times U(n_r)$ is a product of unitary groups. We fix a lift $a : H^*(BG; \mathbb{Z}/p) \longrightarrow H^*(BN(T_G); \mathbb{Z}/p)$ of the standard inclusion into $H^*(BT_G; \mathbb{Z}/p)$.

The next lemma says that, for the proof of proposition 12.1, it suffices to look at compositions

$$H^*(BG; \mathbb{Z}/p) \xrightarrow{\ a\ } H^*(BN(T_G); \mathbb{Z}/p) \xrightarrow{\ Bj^*\ } H^*(BV; \mathbb{Z}/p)$$

for every elementary abelian p–subgroup $j : V \longrightarrow N(T_G)$.

12.3 Lemma. *Let G be a product of unitary groups. Then $H^*(BN(T_G); \mathbb{Z}/p)$ is detected by elementary abelian subgroups.*

Proof. [G-L-Z]. □

Every element $x \in N(T_{U(p)})$ can be written in the form $x = (\lambda_1, \ldots, \lambda_p; \tau)$, $\lambda_i \in S^1$ and $\tau \in \Sigma_p$. Let $\sigma \in \Sigma_p$ be the permutation represented by the cycle $(1, 2, \ldots, p)$, let $V_2 \subset N(T_{U(p)})$ be the group generated by $(1, \ldots, 1; \sigma)$, and let $V_1 \subset N(T_{U(p)})$ be the subgroup generated by $diag(\omega, \ldots, \omega)$, $\omega = exp(2\pi i/p)$. Here $diag(\ )$ denotes the canonical element in $T_{U(p)} \subset N(T_{U(p)})$. Then $V_1 \times V_2 \longrightarrow N(T_{U(p)})$ is a subgroup.

12.4 Lemma. *The elements*

$$(\lambda_1, \ldots, \lambda_p; \sigma),\ (\omega\lambda_1, \ldots, \omega\lambda_p; \sigma),\ (\lambda_1, \ldots, \lambda_p; \sigma^l),\ (\mu, \ldots, \mu; \sigma)$$

are conjugate in $N(T_{U(p)})$ if $(l, p) = 1$ and $\mu^p = \prod_i \lambda_i$.

Proof. Choose $\alpha_1 = 1$ and $\alpha_i = \alpha_{i-1}\lambda_{i-1}\mu^{-1}$. Then $(\alpha_1, \ldots, \alpha_p; 1)$ conjugates the first element into the last one. σ and σ^l have the same cycle type. They are conjugate in Σ_p. With these operation we can construct all necessary conjugations. □

For abbreviation, we set

$$H_n(k) := U(p)^k \times (S^1)^{n-pk} \subset U(n) ,$$

$$W_n(k) := (V_1 \times V_2)^k \times (\mathbb{Z}/p)^{n-pk} \subset H_n(k) \subset U(n) ,$$

$$V_n(k) := (V_1)^k \times (\mathbb{Z}/p)^{n-pk} = W_n(k) \cap T_{U(n)} \subset H_n(k) \subset U(n) .$$

12.5 Lemma. *Every element $x \in N(T_G)$ of order p is conjugate to an element in a suitable subgroup*

$$W_{n_1}(k_1) \times ... \times W_{n_r}(k_r) \subset N(T_{U(n_1)}) \times ... \times N(T_{U(n_r)}) = N(T_G) \ .$$

Proof. x can be written in the form $\prod_i(\lambda_i;\tau_i)$, $\lambda_i \in T_{U(n_i)}$ and $\tau_i \in \Sigma_{n_i}$. τ_i is represented by a product of cycles of length p. An application of lemma 12.4 finishes the proof. □

Let $j : V \longrightarrow N(T_G)$ be an elementary abelian p–subgroup of $N(T_G)$. By the theorems 3.1 and 3.6 the map Bj^*a can be realized by a homomorphism $\rho_V : V \longrightarrow G$ which is unique up to conjugation. The character $\chi_{\rho_V}(x)$ depends on the chosen element $x \in N(T_G)$ and on a, but not on the subgroup V containing x, nor on the homomorphism ρ_V. Therefore $\chi(x) := \chi_a(x) := \chi_{\rho_V}(x)$ is well defined for all elements $x \in N(T_G)$ of order p. By $Tr(x)$ we denote the trace of the associated matrix, given by $N(T_G) \longrightarrow G = U(n_1) \times ... \times U(n_r)$. Now we are prepared for the proof of theorem 12.1.

Proof of proposition 12.1 for G a product of unitary groups.
Let $G = U(n_1) \times ... \times U(n_r)$. Let $j : V' \longrightarrow N(T_G)$ be an elementary abelian p–subgroup of $N(T_G)$. Because G is a product of unitary groups, it is a question of characters to determine $\rho_{V'}$. Thus we have only to consider cyclic subgroups of order p.

Let $x \in N(T_G)$ be an element of order p. Up to conjugation

$$x \in W_{N,K} := W_{n_1}(k_1) \times ... \times W_{n_r}(k_r) \xrightarrow{\ j\ } N(T_G)$$

for suitable $k_1, ..., k_r$. We choose $k_1, ..., k_r$ minimal; i.e x is not contained in a group $W_{n_1}(l_1) \times ... \times W_{n_r}(l_r)$ of the same form such that $l_i \leq k_i$ for all i and at least for one i there is strict inequality. Because the cohomological restriction of $a : H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BN(T_G);\mathbb{Z}/p)$ to $H^*(BT_G;\mathbb{Z}/p)$ is the standard inclusion, we can assume that $\rho_{W_{N,K}}|_{V_{N,K}}$ is the standard inclusion, where $V_{N,K} := V_{n_1}(k_1) \times ... \times V_{n_r}(k_r)$.

We set

$$H_{N,K} := H_{n_1}(k_1) \times ... \times H_{n_r}(k_r) = C_G(V) \ .$$

Applying the Lannes functor $T^V_{Bi^*}$ to

$$H^*(BG;\mathbb{Z}/p) \xrightarrow{\ a\ } H^*(BN(T_G);\mathbb{Z}/p) \longrightarrow H^*(BT_G;\mathbb{Z}/p)$$

gives

$$H^*(BH_{N,K};\mathbb{Z}/p) \xrightarrow{\ a\ } H^*(BN(T_H);\mathbb{Z}/p) \longrightarrow H^*(BT_H;\mathbb{Z}/p) = H^*(BT_G;\mathbb{Z}/p)$$

(theorem 3.5) which shows that $\rho_{W_{N,K}}$ can be taken to have image in $H_{N,K}$.

$x \in W_{N,K}$ can be thought of having the form

$$x = (x_1, ..., x_r)$$
$$x_i = (\underbrace{(1, ..., 1; \sigma), ..., (1, ..., 1; \sigma)}_{k_i \text{ times}}, \lambda_1(i), ..., \lambda_{n_i - pk_i}(i)) \ \in \ W_{n_i}(k_i) \ .$$

We also consider $y = (y_1, ..., y_r)$ with

$$y_i = (\underbrace{(1, ..., 1; \sigma), ..., (1, ..., 1; \sigma)}_{k_i \text{ times}}, 1, ..., 1) \ \in \ W_{n_i}(k_i) \ ,$$

and write $\rho_{W_{N,K}}(y) = (B_1, ..., B_r) \in H_{N,K}$ with

$$B_i = (A_1(i), ..., A_{k_i}(i), \beta_1(i), ..., \beta_{n_i - pk_i}(i)) \ \in \ H_{n_i}(k_i) \ ,$$

where $A_j(i) \in U(p)$ and $\beta_l(i) \in S^1$. This implies

$$\rho_{W_{N,K}}(x) = (D_1, ..., D_r)$$
$$D_i = (A_1(i), ..., A_{k_i}(i), \beta_1(i)\lambda_1(i), ..., \beta_{n_i - pk_i}(i)\lambda_{n_i - pk_i}(i))$$
$$= B_i diag(1, ..., 1, \lambda_1(i), ..., \lambda_{n_i - pk_i}(i)) \ .$$

Since y_i and $y_i' := y_i diag(\omega, ..., \omega, 1, ..., 1)$ (ω pk_i–times) are conjugate in $N(T_{H_{n_i}(k_i)})$ (lemma 12.4), $\chi(y) = \chi(y_1, ..., y_i', ..., y_r)$. This shows that $Tr(A_1(i)) = 0$. Analogously follows that $Tr(A_j(i)) = 0$ for all j and i.

We first assume p is odd. Since y_i is conjugate to $(y_i)^2$ in $N(T_{H_{n_i}(k_i)})$ and $y_i diag(1, ..., 1, \omega)$ to $(y_i)^2 diag(1, ..., 1, \omega)$, we get $\sum_j \beta_j(i) = \sum_j \beta_j(i)^2$ and $\beta_{n_i - pk_i}(i)(\omega - 1) = \beta_{n_i - pk_i}(i)^2(\omega - 1)$. This implies $\beta_{n_i - pk_i}(i) = 1$. An analogous argument shows that $\beta_j(i) = 1$ for all j and i.

Let $\pi_l : G \longrightarrow U(n_l)$ be the projection onto the l–th factor. Because $\rho_{W_{N,K}}|_{V_{N,K}}$ is the standard inclusion, it follows immediately that

$$\chi(x) = \chi_{\pi_l \rho_{W_{N,K}}}(x) = \sum_{j=1}^{n_l} \lambda_j(l) \ ,$$

which is the character of the inclusion

$$V(x) \hookrightarrow W_{N,K} \stackrel{i}{\longrightarrow} G \stackrel{\pi_l}{\longrightarrow} U(n_l) \ ,$$

where $V(x)$ is the cyclic group of order p generated by x. Thus, $\pi_l \rho_{W_{N,K}} j$ is conjugate to $\pi_l i|_{V(x)}$ and $\rho_{W_{N,K}} j = \rho_{V(x)}$ is conjugate to the standard inclusion.

Now we assume $p = 2$. We have $\chi(x) = Tr(\rho_{W_{N,K}}(x)) = \sum_{i,j} \beta_j(i)\lambda_j(i)$. If we permute the $\lambda_j(i)$'s for a fixed i the new element is conjugate to x in $N(T_G)$. We can vary x to realize different values for the $\lambda_j(i)$'s. Therefore, $\beta_j(i) = \beta_l(i)$ for all j and

l. As elements of order 2, $\beta_j(i) = \pm 1$. This implies that $\rho_{W_{N,K}}(x) = (D_1, ..., D_r)$ with

$$D_i = (A_1(i), ..., A_{k_i}(i), \varepsilon_i(x)\lambda_1(i), ..., \varepsilon_i(x)\lambda_{n_i - pk_i}(i)) \ .$$

The function $\varepsilon_i : W_G \longrightarrow \{\pm 1\}$ is constant on conjugate elements and multiplicative if restricted to an 2–elementary abelian subgroup. We choose for every i a transposition $\tau_i \in \Sigma_{n_i}$ and set

$$z(\tau_i) := (z_1, ... z_r) \ , \ z_i \in N(T_{U(n_i)})$$
$$z_j := \begin{cases} diag(1, ..., 1) & if \ \ j \neq i \\ (1, ..., 1; \tau_i) & if \ \ j = i \ . \end{cases}$$

We define a homomorphism

$$\psi = (\psi_{j,k}) : W_G = \prod_{j=1}^{r} \Sigma_{n_j} \longrightarrow {T_G}^{W_G} = \prod_{k=1}^{r} S^1$$

by setting

$$\psi_{j,k} : \Sigma_{n_j} \longrightarrow S^1 \ : \ \sigma \ \mapsto \begin{cases} sign(\sigma) \ \ if \ \ \varepsilon_j(z(\tau_k)) = -1 \\ \ \ +1 \ \ \ \ if \ \ \varepsilon_j(z(\tau_k)) = +1 \ . \end{cases}$$

Again we denote by $\pi_l : G \longrightarrow U(n_l)$ the projection on the l–th factor, and by $V(x) \subset W$ the subgroup generated by x. Then, by construction, $\pi_l \rho_{V(x)}$ and the map $V(x) \hookrightarrow N(T_G) \xrightarrow{j_\psi} N(T_G) \xrightarrow{i} G \xrightarrow{\pi_l} U(n_l)$ have the same character for all $x \in N(T_G)$. That is to say that both homomorphisms are conjugate. To finish the proof we can proceed as for p odd. $\square$

In order to prove theorem 12.1 in the other case, namely for a p–convenient pseudo simply connected Lie group G, we will use the mod–p subgroup $H \longrightarrow G$ of G given by proposition 5.2. Unfortunately, for $p = 3$, H might have a factor $SU(p)$ besides unitary groups. We have to prove the theorem for products of unitary groups and $SU(p)$'s. For odd primes the p–toral Sylow subgroup of $N(T_{SU(p)})$ is $T_{SU(p)} \rtimes \mathbb{Z}/p$. Thus, lemma 12.4 is still true.

If we define

$$H'_p(0) := T_{SU(p)} \subset SU(p)$$
$$H'_p(1) := SU(p)$$
$$W'_p(1) := V_1 \times V_2 \subset SU(p)$$
$$W'_p(0) := T_{SU(p)} =: V'_P(0)$$
$$V'_p(1) := V_1 \subset W'_p(1) \cap T_{SU(p)} \subset H'_p(1) \subset SU(p) \ ,$$

few minor changes make the proof work. Only the replacement of lemma 12.3 needs some comments.

12.6 Lemma. *For p odd, $H^*(BN(T_{SU(p)});\mathbb{Z}/p)$ is detected by elementary abelian subgroups.*

Proof. We consider the Gysin sequence

$$\ldots \longrightarrow H^{*-2}(BN(T_{SU(p)});\mathbb{Z}/p) \longrightarrow H^*(BN(T_{U(p)});\mathbb{Z}/p)$$
$$\xrightarrow{Bi^*} H^*(BN(T_{SU(p)});\mathbb{Z}/p) \xrightarrow{d} H^{*-1}(BN(T_{U(p)});\mathbb{Z}/p) \longrightarrow \ldots$$

of the fibration

$$S^1 \to BN(T_{SU(p)}) \xrightarrow{Bi} BN(T_{U(p)}) \ .$$

Let $x \in H^*(BN(T_{U(p)});\mathbb{Z}/p)$, such that $d(x) \neq 0$. We choose an elementary abelian p–subgroup $V \longrightarrow BN(T_{U(p)})$, detecting $d(x)$. Let $\langle V, Z(U(p))\rangle$ be the group generated by V and the center $Z(U(p))$ of $U(p)$. Then $W := \langle V, Z(U(p))\rangle \cap N(T_{SU(p)})$ is an elementary abelian group, fits into the fibration

$$S^1 \to BW \to B\langle V, Z(U(p))\rangle \ ,$$

and detects x.

If $d(x) = 0$ then $x = Bi^*(y)$ for $y \in H^*(BN(T_{U(p)});\mathbb{Z}/p)$. Because $S^1 \wr \mathbb{Z}/p$ is the p–toral Sylow group of $N(T_{U(p)})$, the spectral sequence of $B(S^1 \wr \mathbb{Z}/p)$ shows that every class in $H^*(BN(T_{U(p)});\mathbb{Z}/p)$ is detected by $W_p(1)$ which, for p odd, is contained in $SU(p)$, or by $(\mathbb{Z}/p)^p \subset T_{U(p)}$. In the first case $W_p(1)$ detects x, and in the second case $(\mathbb{Z}/p)^{p-1}$. □

Proof of theorem 12.1 in the p–convenient pseudo simply connected case.

We can assume that p is odd because every 2–convenient pseudo simply connected Lie group is a product of unitary groups. By lemma 5.2, there is a p-elementary subgroup $i : V \longrightarrow T_G \longrightarrow G$ of G, such that $H := C_G(V)$ is mod–p isomorphic as group to an product of unitary groups and $SU(p)$'s. Moreover the index
$[N(T_G) : N(T_H)]$ is coprime to p. Applying the Lannes functor $T^V_{Bi^*}$ to a lift
$a : H^*(BG;\mathbb{Z}/p) \longrightarrow H^*(BN(T_G);\mathbb{Z}/p)$ establishes a commutative diagram

$$\begin{array}{ccccc}
H^*(BG;\mathbb{Z}/p) & \xrightarrow{a} & H^*(BN(T_G);\mathbb{Z}/p) & \longrightarrow & H^*(BT_G;\mathbb{Z}/p) \\
\downarrow & & \downarrow & & \| \\
H^*(BH;\mathbb{Z}/p) & \xrightarrow{b} & H^*(BN(T_H);\mathbb{Z}/p) & \longrightarrow & H^*(BT_H;\mathbb{Z}/p)
\end{array} ,$$

where $b = T^V_{Bi^*}(a)$. The composition in the bottom line is induced by

$$BT_H{}^\wedge_p \longrightarrow map(BV, BT_G{}^\wedge_p)_{Bi} \longrightarrow map(BV, BG^\wedge_p)_{Bi} \simeq BH^\wedge_p \ ,$$

which can be considered as the standard inclusion (see remark 3.4). The two left vertical arrows are injections, and b is induced by the standard inclusion. Therefore, a is also given by the standard inclusion. □

13. The map $BG \longrightarrow X$ part I: G pseudo simply connected.

In this section G is a p–convenient pseudo simply connected Lie group for p odd, and a product of unitary groups for $p = 2$. For $p = 2$ this includes the case of pseudo simply connected Lie groups. As usual, X is a space with the p–adic type of BG. We will prove theorem 1.2 in these cases, i.e we have to prove the propositions 2.3 and 2.4.

By the propositions 11.1 and 12.1, there exists a triangle

$$(*) \qquad \begin{array}{ccc} & BN(T_G) & \\ {\scriptstyle Bi} \swarrow & & \searrow {\scriptstyle f_N} \\ BG^{\wedge}_p & \dashrightarrow & X \end{array}$$

commuting in mod–p cohomology. As in section 2 $\mathcal{R}_p(G)$ is the orbit category of p–stubborn subgroups of G contained in $N(T_G)$. For any p–stubborn subgroup $P \xrightarrow{i_p} N(T_G) \longrightarrow G$ of G, we defined in section 2 a map

$$f_P := f_N \circ Bi_p : BP \longrightarrow BN(T_G) \longrightarrow X\ .$$

The collection of these maps gives a diagram

$$\{BP\}_{\mathcal{R}_p(G)} \longrightarrow X$$

commuting in mod–p cohomology because of the diagram $(*)$

As first step we show that this diagram commutes up to homotopy, which is the statement of proposition 2.3.

Proof of proposition 2.3.

Let $c_g : G/P \longrightarrow G/P'$ be a map of $\mathcal{R}_p(G)$ given by a conjugation. We have to show that

$$\begin{array}{ccccc} BP & \xrightarrow{Bi_P} & BN(T_G) & \xrightarrow{f_N} & X \\ {\scriptstyle Bc_g}\downarrow & & & & \| \\ BP' & \xrightarrow{Bi_{P'}} & BN(T_G) & \xrightarrow{f_N} & X \end{array}$$

commutes up to homotopy. Without loss of generality we can assume that, according to a fixed chosen maximal torus and normalizer $T_G \longrightarrow N(T_G) \longrightarrow G$, the subgroup P is the special representative of the conjugacy class of P; i.e. P satisfies proposition 6.7. Moreover, we can think of $\alpha := i_{P'} c_g$ as being a homomorphism such that $f_N B\alpha$ and $f_N Bi_P$ induce the same map in mod–p cohomology, and i_P is conjugate to α in G. Under these assumptions, we will show that $f_N B\alpha \simeq f_N Bi_P$.

The intersection $P_T = P \cap T_G$ is an abelian group. The restrictions $\alpha|_{P_T}$ and $i_P|_{P_T}$ are conjugate in G, and hence, by theorem 10.1 (i) (see also remark 10.2 (i)), also conjugate in $N(T_G)$. That is that $f_N Bi_P|_{P_T}$ and $f_N B\alpha|_{P_T}$ are homotopic. The following proposition, the analogy of proposition 6.7 (iv), shows that the extensions of $f_N Bi_P|_{P_T}$ to maps $BP \longrightarrow X$ are classified by mod-p cohomology. This finishes the proof. $\square$

13.1 Proposition. *For any p–toral group P satisfying the hypothesis of proposition 6.7, the canonical map*

$$\pi_0(map(BP,X)_{g|_{BP_T}\simeq f_N Bi_P|_{BP_T}}) \longrightarrow Hom(H^*(X;\mathbb{Z}/p), H^*(BP;\mathbb{Z}/p))$$

is an injection.

Proof. We have to relate $map(BP,BG)$ to $map(BP,X)$.

The intersection $P_T = P \cap T_G$ is a normal subgroup of P, because $P \hookrightarrow N(T_G)$ normalizes T_G and P. The quotient $Q := P/P_T$ acts on $\widetilde{BP_T} \simeq BP_T$ and on $map(\widetilde{BP_T},-) \simeq map(BP_T,-)$. By remark 3.12 $map(BP,-) \simeq map(\widetilde{BP_T},-)^{hQ}$. We define $i_{P_T} := i_P|_{P_T}$. The diagram
(∗)

$$\begin{array}{ccc} & map(\widetilde{BP_T}, BN(T_G))_{Bi_{P_T}} & \\ \swarrow & & \searrow \\ map(\widetilde{BP_T}, BG_p^{\wedge})_{Bi_N Bi_{P_T}} & \dashrightarrow & map(\widetilde{BP_T}, X)_{f_N Bi_{P_T}} \end{array}$$

commutes in mod–p cohomology (proposition 10.4). By the theorems 3.9 and 10.1

$$map(\widetilde{BP_T}, BG_p^{\wedge})_{Bi_N Bi_{P_T}} \simeq BT_{G_p}^{\wedge} \simeq map(\widetilde{BP_T}, X)_{f_N Bi_{P_T}} ,$$

and by proposition 3.8

$$map(\widetilde{BP_T}, BN(T_G))_{Bi_{P_T}} \simeq BT_G .$$

The diagonal maps in (∗) are Q–equivariant, because the components are fixed, and mod–p equivalences. Taking homotopy fixed–points we get
(∗∗)

$$\begin{array}{ccc} & map(\widetilde{BP_T}, BN(T_G))^{hQ}_{Bi_{P_T}} & \\ \swarrow & & \searrow \\ map(\widetilde{BP_T}, BG_p^{\wedge})^{hQ}_{Bi_N Bi_{P_T}} & & map(\widetilde{BP_T}, X)^{hQ}_{f_N Bi_{P_T}} \end{array} .$$

Both diagonal maps are also mod–p equivalences, because an equivariant mod–p equivalence between 1–connected spaces induces a mod–p equivalence between the homotopy fixed–point sets.

The components of $map(\widetilde{BP_T}, BG_p^{\wedge})^{hQ}_{Bi_N Bi_{P_T}}$ are distinguished by mod–p cohomology (proposition 6.7). The triangle

$$\begin{array}{ccc} & BN(T_G) & \\ \swarrow & & \searrow \\ BG & \dashrightarrow & X \end{array}$$

commutes in mod–p cohomology. Any map in $map(\widetilde{BP_T}, X)^{hQ}_{f_N Bi_{P_T}}$ has a lift to $BN(T_G)$. The obstruction groups which classify the extensions up to homotopy (see remark 3.12 and lemma 3.13)

$$\begin{aligned} H^2(Q;\pi_2(map(BP_T,X)_{f_N Bi_{P_T}})) &\cong H^2(Q;\pi_2(BT_G)) \\ &\cong H^2(Q;\pi_2(map(BP_T,BG^\wedge_p)_{Bi_N Bi_{P_T}}) \end{aligned}$$

are isomorphic. The isomorphisms of the obstruction groups follow from 10.1. All this together implies that the components of

$$map(BP_T,X)^{hQ}_{f_N Bi_{P_T}} \simeq map(BP,X)_{g|_{BP_T}\simeq f_N Bi_{P_T}}$$

are also distinguished by mod–p cohomology. □

Proposition 2.3 establishes a map from the 1–skeleton of the homotopy colimit $\operatorname{holim}_{\overrightarrow{\mathcal{R}_p(G)}} BP$ to X. In order to extend this map to the homotopy colimit, we have to calculate the obstruction groups

$$\lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^{i+1}\ \pi_i(map(BP,X)_{f_P}) \qquad i\geq 2$$

for extending this map on the 1–skeleton to the homotopy colimit. $\varprojlim^j$ is the j–th derived functor of the inverse limit functor. This we will do by comparing the two functors

$$\Pi_i(X), \Pi_i(G) : \mathcal{R}_p(G) \longrightarrow \mathcal{A}b\ ,$$

given by

$$\begin{aligned} \Pi_i(X)(G/P) &:= \pi_i(map(BP,X)_{f_P}) \\ \Pi_i(G)(G/P) &:= \pi_i(map(BP,BG^\wedge_p)_{Bi_P})\ . \end{aligned}$$

That is we have to contruct a natural transformation between the two functors which is an equivalence. The existence of this transformation is claimed in proposition 2.4.

Proof of proposition 2.4.

The triangle $(**)$ induces for every p–stubborn subgroup $P\hookrightarrow G$ homotopy equivalences

$$map(BP,BG^\wedge_p)_{Bi_P} \overset{\simeq}{\longleftarrow} (map(BP,BN(T_G))_{Bi_P})^\wedge_p \overset{\simeq}{\longrightarrow} map(BP,X)_{f_N B\alpha_N}\ ,$$

which depend on the chosen lift $i_P : BP\longrightarrow BN(T_G)$ of $i_P : BP\longrightarrow BG$. Two lifts only differ by a conjugation c_g (theorem 3.6). We already proved that $f_N Bi_P \simeq f_N c_g Bi_P$. Therefore the above equivalence induces well defined isomorphisms

$$\Pi_i(G)(G/P) \overset{\cong}{\longrightarrow} \Pi_i(X)(G/P)\ ,$$

which are compatible with the maps in $\mathcal{R}_p(G)$. □

The following lemma fills the last open gap of the program of section 2 and completes the proof of theorem 1.2 for pseudo simply connected Lie groups and products of unitary groups.

13.2 Lemma.

$$\lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^j \; \Pi_i(X)(G/P) = 0 \qquad i, j \geq 1 \; .$$

Proof. Because of proposition 2.4 we have to calculate $\lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^j \; \Pi_i(G)(G/P)$.

$G = G' \times G''$ splits into two factors; G'' is simply connected and G' a product of unitary groups. Since $\mathcal{R}_p(G' \times G'') \cong \mathcal{R}_p(G') \times \mathcal{R}_p(G'')$ (proposition 6.3), we have $\Pi_i(G) \cong \Pi_i(G') \times \Pi_i(G'')$. We have

$$\lim_{\overleftarrow{\mathcal{R}_p(G')}}{}^j \; \Pi_1(G') = 0$$

$$\lim_{\overleftarrow{\mathcal{R}_p(G'')}}{}^j \; \Pi_1(G'') = 0$$

$$\lim_{\overleftarrow{\mathcal{R}_p(G)}}{}^j \; \Pi_i(G) = 0 \qquad \text{for } i \geq 2 \; .$$

The last two equations follow from [J-M-O 1; theorem 4.8], and the first from the fact that centralizers in unitary groups are always products of unitary groups, and therefore $\Pi_1(G') \equiv 0$. □

14. The map $BG \longrightarrow X$ part II : The general case.

After having proved theorem 1.2 in the pseudo simply connected case we complete the proof of theorem 1.2 in this section. First we assume that G is a p–convenient simply connected Lie group. Then $G = G' \times G''$ splits into two factors, where $G' := SU(n_1) \times ... \times SU(n_l)$ is a product of special unitary groups and G'' a product of the other simply connected Lie groups. There exist exact sequences

$$K \longrightarrow G' \times (S^1)^l \longrightarrow U(n_1) \times ... \times U(n_l) \;\; =: \;\; \overline{G}'$$

and

$$K \longrightarrow G \times (S^1)^l \longrightarrow \overline{G}' \times G'' \;\; =: \;\; H \; ,$$

where $K := \mathbb{Z}/n_1 \times ... \times \mathbb{Z}/n_l$. Now let X be a space with the p–adic type of BG, and let $f_T : BT_G \longrightarrow X$ be the maximal torus of X.

We consider the composition

$$f_K := f_T \circ Bi : BK_p \longrightarrow BT_G \times (BS^1)^l \longrightarrow X \times (BS^1)^l \; ,$$

where K_p is the p–Sylow subgroup of the abelian group K. Because $K_p \subset G \times (S^1)^l$ is a central subgroup, the evaluation map

$$map(BK_p, X \times (BS^1{}^\wedge_p)^l)_{f_K} \longrightarrow X \times (BS^1)^{l\wedge}_p$$

is a mod–p equivalence (theorem 10.1). Moreover, the evaluation map is a homotopy equivalence, because both spaces are p–complete (theorem 10.1).

We can think of BK_p as a group which acts freely on

$$EBK_p \times map(BK_p, X \times (BS^1)^{l\wedge}_p) \simeq X \times (BS^1)^{l\wedge}_p \ .$$

We define the p–complete space Y by the fibration

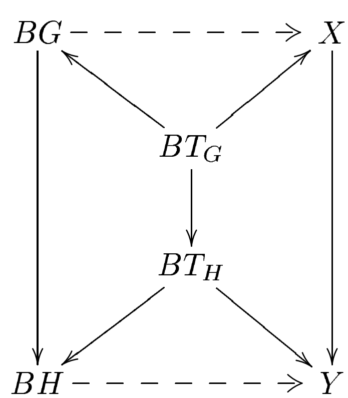
$$BK_p \longrightarrow X \times (BS^1)^{l\wedge}_p \longrightarrow (X \times (BS^1)^{l\wedge}_p)/BK_p =: Y \ .$$

We have $BT_H{}^\wedge_p \simeq B(T_G \times (S^1)^l/K)^\wedge_p \simeq BT_G{}^\wedge_p \times (BS^1)^{l\wedge}_p/BK_p$. Thus, the above fibration fits into the commutative diagram

$$\begin{array}{ccccc}
BK & \longrightarrow & BG \times (BS^1)^l & \longrightarrow & BH \\
\| & & \uparrow & & \uparrow \\
BK & \longrightarrow & BT_G \times (BS^1)^l & \longrightarrow & BT_H \\
\downarrow & & \downarrow & & \downarrow \\
BK_p & \longrightarrow & X \times (BS^1)^{l\wedge}_p & \longrightarrow & Y
\end{array} \ .$$

$BT_H \longrightarrow Y$ is a maximal torus. The Weyl group action of W_G on $BT_G \times (BS^1)^l$ commutes with the action of BK_p. Therefore $W_G = W_H$ acts as Weyl group on $BT_H \longrightarrow Y$. Analogously to the proof of proposition 4.2 follows that $H^*(Y;\mathbb{Z}^\wedge_p)$ is p–torsionfree, that $H^*(Y;\mathbb{Z}/p) \cong H^*(BT_H;\mathbb{Z}/p)^{W_H} \cong H^*(BH;\mathbb{Z}/p)$, and, by the lemma of Nakayama, that $H^*(Y;\mathbb{Z}^\wedge_p) \cong H^*(BT_G;\mathbb{Z}^\wedge_p)^{W_H} \cong H^*(BH;\mathbb{Z}^\wedge_p)$. That is to say that Y has the p–adic type of BH. We collect these considerations into the following proposition.

14.1 Proposition. *Let G be a p–convenient simply connected Lie group, and H the associated pseudo simply connected Lie group. Let X have the p–adic type of BG. Then there exists a p-complete space Y with the p–adic type of BH, and a diagram*



,

which commutes in p–adic cohomology.

Now we are ready to prove theorem 1.2. We do this in several steps, first for simply connected Lie groups, then for products of simply connected Lie groups and tori, and finally in the general case.

Proof of theorem 1.2 for simply connected Lie groups.

Let X be a space with the p–adic type of BG, and let Y be the space constructed above. We can think of X as the fibre of the map

$$Y = (X \times (BS^1)^{l\wedge}_p)/BK_p \longrightarrow (BS^1)^{l\wedge}_p/BK_p =: BT^\wedge_p$$

as well as of $BG^\wedge_p$ as the fibre of

$$BH^\wedge_p \longrightarrow BT^\wedge_p \ .$$

By section 13, there exists a homotopy equivalence $BH^\wedge_p \longrightarrow Y$, which can be extended to

$$\begin{array}{ccccc} BG^\wedge_p & \longrightarrow & BH^\wedge_p & \longrightarrow & BT^\wedge_p \\ \downarrow & & \simeq\downarrow & & \simeq\downarrow \\ X & \longrightarrow & Y & \longrightarrow & BT^\wedge_p \ . \end{array}$$

Obviously $BG^\wedge_p \longrightarrow X$ is a homotopy equivalence. $\square$

Proof of theorem 1.2 for products of simply connected Lie groups and tori.

Let X be a space with the p–adic type of $BG_s \times BT$, where G_s is a simply connected Lie group and T a torus. The projection $BG_{s}{}^\wedge_p \times BT^\wedge_p \longrightarrow BT^\wedge_p$ corresponds to a map $X \longrightarrow BT^\wedge_p$ which has an canonical left inverse $s : BT^\wedge_p \longrightarrow BT_{G_s}{}^\wedge_p \times BT^\wedge_p \longrightarrow X$. The fiber X_s of $X \longrightarrow BT^\wedge_p$ has the p–adic type of BG_s. Thus we have $BG_{s}{}^\wedge_p \simeq X_s$. By theorem 10.1 $map(BT^\wedge_p, X)_s \simeq X$. The adjoint

$$BG_{s}{}^\wedge_p \times BT^\wedge_p \longrightarrow X$$

of

$$BG_{s}{}^\wedge_p \simeq X_s \longrightarrow X \simeq map(BT^\wedge_p, X)_s$$

is a mod–p isomorphism and hence, a homotopy equivalence. $\square$

Proof of theorem 1.2 in the general case.

Let G be a p–convenient compact connected Lie group and X a space with the p–adic type of BG. The finite covering

$$K \longrightarrow G_s \times T \longrightarrow G \ ,$$

G_s simply connected and T a torus, establishes a fibration

$$BK_p \longrightarrow BG_{s}{}^\wedge_p \times BT^\wedge_p \longrightarrow BG^\wedge_p \ ,$$

where K_p is the p–Sylow subgroup of K. This fibration is classified by a map $BG_p^\wedge \longrightarrow BBK_p$. Since $H^*(BG;\mathbb{Z}/p^r) \cong H^*(X;\mathbb{Z}/p^r)$ for all r, there is a corresponding map $X \longrightarrow BBK_p$ classifying the fibration

$$BK_p \longrightarrow Y \longrightarrow X \ .$$

This fibration fits into

$$\begin{array}{ccccc} BK_p & \longrightarrow & BG_{s}{}_p^\wedge \times BT_p^\wedge & \longrightarrow & BG_p^\wedge \\ \| & & \uparrow & & \uparrow \\ BK_p & \longrightarrow & BT_{G_s}{}_p^\wedge \times BT_p^\wedge & \longrightarrow & BT_{G}{}_p^\wedge \\ \| & & \downarrow & & \downarrow \\ BK_p & \longrightarrow & Y & \longrightarrow & X \ . \end{array}$$

The Weyl group $W_G = W_{G_s}$ acts as Weyl group on $BTG_{s}{}_p^\wedge \times BT_p^\wedge \longrightarrow Y$. The space Y is p–complete and of the p–adic type of $BG_{s}{}_p^\wedge \times BT_p^\wedge$. This follows from a comparison of the Serre spectral sequences for p–adic cohomology of all three horizontal fibrations and implies that $BG_s \times BT$ and Y are homotopy equivalent. Moreover, this equivalence fits in a diagram

$$(*) \qquad \begin{array}{ccccc} BK_p & \longrightarrow & BG_{s}{}_p^\wedge \times BT_p^\wedge & \longrightarrow & BG_p^\wedge \\ \| & & \downarrow \simeq & & \\ BK_p & \longrightarrow & Y & \longrightarrow & X \ . \end{array}$$

We have to fill in the arrow in the right collumn.

The canonical G–equivariant homotopy equivalence

$$X \longrightarrow map(\widetilde{BK}_p, X)_c \ ,$$

$c : \widetilde{BK}_p \longrightarrow X$ the constant map, induces equivalences

$$map(BG, X) = X^{hG} \simeq map(\widetilde{BK}_p, X)_c^{hG} \simeq map(BG_{s}{}_p^\wedge \times BT_p^\wedge, X)_{g|_{BK_p} \simeq c}$$

(see remark 3.12), where we take in the last mapping space only the components of those maps whose restriction to BK_p is trivial up to homotopy. This implies that there exists a map $BG_p^\wedge \longrightarrow X$ making the diagram $(*)$ commutative. Moreover, this map is an equivalence. □

15. The integral homotopy type.

In this chapter we prove theorem 1.6.

Proof of theorem 1.6.
Let X be a 1–connected CW–complex of finite type. If $H^*(X;\mathbb{Z})$ is torsionfree the λ–ring structure of $K(X)$ completely determines the algebra $H^*(X;\mathbb{Z}/p)$ as well as the Steenrod algebra action on $H^*(X;\mathbb{Z}/p)$ for all primes p [At]. Therefore, if $H^*(X;\mathbb{Z})$ and $H^*(BG;\mathbb{Z})$ are torsion free,

$$H^*(X;\mathbb{Z}/p) \cong H^*(BG;\mathbb{Z}/p)$$

as algebras over the Steenrod algebra. If in addition G is 2–convenient or G is a product of unitary groups we can construct maximal tori

$$f_{T_p^\wedge} : BT_{X_p^\wedge} \longrightarrow X_p^\wedge$$

and Weyl group actions of W_G on $BTX_p^\wedge$ for all p. Moreover, $H^*(X;\mathbb{Z}_p^\wedge) \cong H^*(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge)^{W_G}$. All this follows from proposition 7.2.

Using the results of [At] a Chern character argument shows that

$$K(X;\mathbb{Z}_p^\wedge) \cong K(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge)^{W_G} \ .$$

$K(-;\mathbb{Z}_p^\wedge)$ denotes complex K–theory with p-adic coefficients. By [Wil] there exist λ–ring maps

$$\begin{aligned} \phi &: K(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge) \longrightarrow K(BT_{G_p^\wedge};\mathbb{Z}_p^\wedge) \\ \psi &: K(BT_{G_p^\wedge};\mathbb{Z}_p^\wedge) \longrightarrow K(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge) \end{aligned}$$

such that

$$\begin{array}{ccccc} K(X_p^\wedge;\mathbb{Z}_p^\wedge) & \cong & K(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge)^{W_G} & \longrightarrow & K(BT_{X_p^\wedge};\mathbb{Z}_p^\wedge) \\ \cong\downarrow & & \cong\downarrow & & \phi\downarrow\ \uparrow\psi \\ K(BG_p^\wedge;\mathbb{Z}_p^\wedge) & \cong & K(BT_{G_p^\wedge};\mathbb{Z}_p^\wedge)^{W_G} & \longrightarrow & K(BT_{G_p^\wedge};\mathbb{Z}_p^\wedge) \ . \end{array}$$

By an argument analogous to [A-M; 1.7] $\phi\psi$ and $\psi\phi$ differ from the identity only by an element of the Weyl group. Thus, ϕ and ψ are isomorphisms. Because K–theory classifies maps $BT_{G_p^\wedge} \longrightarrow BG_p^\wedge$ up to homotopy [No 1] [N-S 1], the representations of W_G, given by the action on BT_G and $BT_{X_p^\wedge}$, are conjugate over $\mathbb{Z}_p^\wedge$. That is to say that $X_p^\wedge$ has the p–adic type of BG. By theorem 1.2 we get $X_p^\wedge \simeq BG_p^\wedge$.

The rationalizations $X_\mathbb{Q}$ and $BG_\mathbb{Q}$ are homotopy equivalent, because $BG_\mathbb{Q}$ is a product of rational Eilenberg–McLane spaces and because

$$H^*(X;\mathbb{Q}) \cong K(X)\otimes\mathbb{Q} \cong K(BG)\otimes\mathbb{Q} \cong H^*(BG;\mathbb{Q}) \ .$$

Hence X is in the p–adic genus of BG. The finite universal cover $G_s \times T \longrightarrow G$ does not contain any $Sp(n)$ or $Spin(n)$, $n \geq 3$, as factor. Therefore, in this case complex K–theory classifies the adic genus [No 2; remark 5.4, theorem 1.8 and the proof]. Since $K(X) \cong K(BG)$ this implies that $BG \simeq X$. $\square$

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List of notation.

$\mathbb{Z}/p$	group of order p
$\mathbb{Z}_{(p)}$	integers localised at p
$\mathbb{Z}_p^\wedge$	p–adic integers
$X_p^\wedge$	mod–p completion in the sense of Bousfield and Kan
X_p°	fiberwise completion
$map(X,Y)$	mapping space
$map(X,Y)_f$	the component of $map(X,Y)$ associated to the map f
$Fib(B.F)$	set of equivalence classes of fibrations $F\longrightarrow E\longrightarrow B$
BG	classifying space of the group G
$im(\rho),\ ker(\rho)$	image and kernel of a homomorphism between groups
$Rep(G,H)$	set of representations from G to H,i.e. conjugacy classes of homomorphisms $G\longrightarrow H$
S_pG	p–toral Sylow subgroup of a compact Lie group G
$Z(G)$	center of the group G
$C_G(H),\ C_G(\rho)$	centralizer of a subgroup H or of the image of a homorphism ρ in G
$N_G(H)$	normalizer of H in G
$GrExt(G,H)$	set of equivalence classes of exact sequences $H\longrightarrow K\longrightarrow G$ of groups
L/p	$L\otimes\mathbb{Z}/p$
$L_p^\wedge$	p–adic completion of a module L, or $L\otimes\mathbb{Z}_p^\wedge$
$X \dashrightarrow Y$	map which exists only in cohomology as $H^*(Y)\longrightarrow H^*(X)$ and which is waiting for a topological realization

For a compact connected Lie group G, we denote by

T_G	the maximal torus of G
W_G	the Weyl group of G
$N(T_G)$	the normalizer of T_G
V_G	the maximal elementary abelian p–subgroup of T_G
LT_G	$H_2(BT_G;\mathbb{Z})$
L^*T_G	$H^2(BT_G;\mathbb{Z})$