

KERNELS OF MAPS BETWEEN CLASSIFYING SPACES

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ABSTRACT. For homomorphisms between groups, one can divide out the kernel to get an injection. Here, we develop a notion of kernels for maps between classifying spaces of compact Lie groups. We show that the kernel is a normal subgroup in a modified sense and prove a generalization of a theorem of Quillen, namely, a map $f: BG \rightarrow BH_p^\wedge$ is injective, iff the induced map in mod- p cohomology is finite. Moreover, for compact connected Lie groups, every map $f: BG \rightarrow BH_p^\wedge$ factors over a quotient of G in a modified sense and this factorisation is an injection.

1. Introduction.

For a homomorphism $\rho: G \rightarrow H$ between groups, we know that the kernel $\ker(\rho)$ of ρ is a normal subgroup of G , which gives rise to an exact sequence

$$\ker(\rho) \longrightarrow G \longrightarrow G/\ker(\rho) .$$

The induced homomorphism

$$\bar{\rho}: G/\ker(\rho) \longrightarrow H$$

is an injection. In this paper we will develop a analogous concept for maps between classifying spaces.

To investigate the topological situation we pass to the p -adic completion. We also allow a more general target. Let G be a compact connected Lie group. A space X is called **BG-local** if the evaluation induces an equivalence $\text{map}(BG, X) \simeq X$ between the mapping space and X . The space X is called **almost BG-local** if the evaluation induces an equivalence $\text{map}(BG, X)_{\text{const}} \simeq X$ between the component of the constant map $\text{const}: BG \rightarrow X$ and X . This is equivalent to the condition that the loop space ΩX is **BG-local**.

The situation, we are interested in, is the following

- (S) $f: BG \rightarrow X_p^\wedge$ is a map, where G is a compact Lie group, where X_p^\wedge is p -complete and almost $B\mathbb{Z}/p$ -local and where $H^*(X; \mathbb{F}_p)$ is of finite type.

When talking about the kernel of a map $f: BG \rightarrow X_p^\wedge$ as in (S), one has to look for elements $g \in G$ of order a power of p , such that $f|_{B\langle g \rangle}$ is homotopic to the constant map. Here, $\langle g \rangle$ denotes the subgroup generated by g . This leads to a definition of the prekernel, due to Ishiguro [7, 8], and of the kernel, which we explain now.

For every compact Lie group G there exists a maximal p -toral subgroup $S_p G$, unique up to conjugation, and every p -toral subgroup is subconjugated to $S_p G$ [9]. The group $S_p G$ is called the **p -toral Sylow group** of G . If G is finite, $S_p G$ is the usual p -Sylow group.

For a compact Lie group G , we denote by T_G the maximal torus, by NT_G the normalizer of T_G , and by W_G the Weyl group. Then, $S_p G$ is the counterimage of $S_p W_G$ in NT_G . We also denote $S_p G$ by $N_p T_G$ to indicate that $S_p G$ is the p -toral Sylow subgroup of NT_G , too. T_G is the component of the unit of $N_p T_G = S_p G$.

We define a subgroup $S_{p^\infty} G \subset S_p G$ by the commutative diagram

$$\begin{array}{ccccc} T_{p^\infty} & \longrightarrow & S_{p^\infty} G & \longrightarrow & \pi_0(S_p G) \\ \downarrow & & \downarrow & & \downarrow \\ T_G & \longrightarrow & S_p G & \longrightarrow & \pi_0(S_p G) , \end{array}$$

where $T_{p^\infty} \subset T_G$ denotes the subgroup generated by all elements of order a power of p . For a map $f: BG \rightarrow X_p^\wedge$, we define the prekernel of f as

$$\text{preker}(f) := \{g \in S_{p^\infty} G : f|_{B\langle g \rangle} \simeq \text{const}\} ,$$

and the kernel of f as

$$\text{ker}(f) := \text{cl}(\text{preker}(f)) ,$$

where $\text{cl}(\)$ denotes the closure in $S_p G$ or G .

1.1 Theorem. *Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S).*

- (1) *$\text{preker}(f)$ is a subgroup of $S_{p^\infty} G$.*
- (2) *$\text{ker}(f)$ is a p -toral subgroup of $S_p G$.*
- (3) *$f|_{B\text{ker}(f)}$ is nullhomotopic.*

As the proof in the next section shows, the theorem is true without the assumption that $H^*(X_p^\wedge; \mathbb{F}_p)$ is of finite type.

These results lead to the following definitions: A map $f: BG \rightarrow X_p^\wedge$ as in (S) is called **injective** if $\text{ker}(f)$ is the trivial group, and, following [4] or [13], **monic** or **finite** if $H^*(BG; \mathbb{F}_p)$ is finitely generated over $H^*(X_p^\wedge; \mathbb{F}_p)$. In [13] is proved that the kernel of a homomorphism $\rho: G \rightarrow H$ between two compact Lie groups is finite and of order coprime to p if and only if the induced map $B\rho: BG \rightarrow BH_p^\wedge$ is monic. The following statement also generalizes a result of Dwyer and Wilkerson [4, Proposition 4.4]

1.2 Theorem. *Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S). Then, f is injective, if and only if f is monic.*

Let $\mathcal{O}(G)$ be the orbit category of G , and let $\mathcal{O}_p(G)$ be the full subcategory, whose objects are homogenous spaces G/P , where P is a p -toral subgroup of $S_p G$. Usually, $\mathcal{O}_p(G)$ is defined to be the full subcategory of the spaces G/P , where P is any p -toral subgroups of G . Our definition is more convenient for our purpose and gives a homotopy equivalent category. This follows because, up to conjugation, every p -toral subgroup is contained in $S_p G$ [9]. For a subgroup $K \subset S_p G$ and a p -toral subgroup $P \subset S_p G$, we define $K_P := K \cap P$.

1.3 Lemma. For a p -toral subgroup $\Gamma \subset S_p G$ the following conditions are equivalent:

- (1) For every $x \in \Gamma$ of order a power of p and for every $g \in G$, we have $g x g^{-1} \in S_p G$ if and only if $g x g^{-1} \in \Gamma$.
- (2) For every pair $P, P' \subset S_p G$ of p -toral subgroups and for every $g \in G$, such that $g P g^{-1} \subset P'$, we have $g \Gamma_P g^{-1} \subset \Gamma_{P'}$.

Proof. $S_{p^\infty} \Gamma \subset \Gamma$ is a dense subset and contains only elements of order a power of p . If Γ satisfies the first condition and if $g P g^{-1} \subset P'$, then $g(S_{p^\infty} \Gamma \cap P)g^{-1} \subset S_{p^\infty} \Gamma \cap P'$. Because P' is closed, this is also true for Γ , which is condition (2). Every element of order a power of p generates a finite p -group. Hence, (2) implies (1) obviously. \square

A p -toral subgroup $\Gamma \subset S_p G$ is called $\mathcal{O}_p(G)$ -**normal**, if Γ satisfies one of the conditions of Lemma 1.3.

1.4 Proposition. Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S). Then, $\ker(f) \subset S_p G$ is $\mathcal{O}_p(G)$ -normal.

Proof. In [7] is shown that for two subgroups $\Gamma, \Gamma' \subset S_p G$, which are conjugated in G , the restriction $f|_{B\Gamma}$ is nullhomotopic if and only if $f|_{B\Gamma'}$ is nullhomotopic. Hence, for every $x \in \ker(f)$, conjugated elements are also contained in $\ker(f)$.

For a map $f: BG \rightarrow X_p^\wedge$ as in (S), the kernel $\ker(f) \subset S_p G$ is not a normal subgroup of G in general. But, for G connected, $\ker(f)$ is the right invariant, which tells us, on which part the map $f: BG \rightarrow X_p^\wedge$ is trivial. To avoid discussions and arguments about homotopy colimits, we study $\mathcal{O}_p(G)$ -normal subgroups of $N_p T_G$. This investigation allows to prove the following theorem:

1.5 Theorem. Let G be a compact connected Lie group and let $f: BG \rightarrow X_p^\wedge$ be a map as in (S). Then, there exists a compact Lie group H and a commutative diagram

$$\begin{array}{ccc} BG & \xrightarrow{f} & X_p^\wedge \\ q \downarrow & & \parallel \\ BH_p^\wedge & \xrightarrow{\bar{f}} & X_p^\wedge \end{array}$$

such that $\bar{f}: BH_p^\wedge \rightarrow X_p^\wedge$ is injective.

Using a weak form of Theorem 1.1 (1), Ishiguro proved a similar result for simple Lie groups. [7, 8]. We remark that we have to take the completion of BH . Moreover, as the proof shows, the group H in the theorem is not connected in general. This statement says that, at least, we can divide out by the kernel to make the map injective. Moreover, the homotopy fiber of $BG_p^\wedge \xrightarrow{q} BH_p^\wedge$ is closely related to $\ker(f)$. One might think of this homotopy fiber as being the kernel of f and of q as a surjection. In general, the homotopy fiber of q might not be the completion of the classifying space of a compact Lie group.

The paper is organized as follows: In the next section we prove Theorem 1.1, the third section contains a proof of Theorem 1.2, and in the last section $\mathcal{O}_p(G)$ -normal subgroups are discussed to prove Theorem 1.5.

Completion are always meant in the sense of [2].

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2. Prekernels and kernels.

We start with the following observation:

2.1 Lemma. *Let X_p^\wedge be a p -complete almost $B\mathbb{Z}/p$ -local space. Then, X_p^\wedge is almost BG -local for every compact Lie group G .*

Proof. A p -complete space X_p^\wedge is always $B\mathbb{Z}/p'$ -local for any prime $p' \neq p$. If X_p^\wedge is also $B\mathbb{Z}/p$ -local, then, by [11, §9], follows that X_p^\wedge is $B\pi$ -local for any locally finite group π . For every compact Lie group G , there exists a mod- p equivalence $B\pi \rightarrow BG$, where π is locally finite [6]. So, X_p^\wedge is BG -local. Because for a p -complete space X_p^\wedge , the loop space $\Omega(X_p^\wedge)$ is also p -complete, the same arguments apply to show that every p -complete almost $B\mathbb{Z}/p$ -local space is almost BG -local for every compact Lie group G . \square

Let π be a finite group. For $x \in \pi$, we define $\nu(x)$ to be the smallest subgroup of π , which is normal in π and contains x . This is welldefined because the intersection of two normal subgroups is also normal.

2.2 Lemma. *If π is a finite p -group and noncyclic, then, for every $x \in \pi$, $\nu(x) \subset \pi$ is a proper subgroup and $\nu(x) = \langle yxy^{-1} : y \in \pi \rangle$.*

For a set S of elements of π , we denote by $\langle S \rangle$ the subgroup generated by the elements of S .

Proof. The center of π is nontrivial and contains \mathbb{Z}/p as subgroup. Hence, there exists a central extension $\mathbb{Z}/p \rightarrow \pi \xrightarrow{q} \bar{\pi} := \pi/\mathbb{Z}/p$. If $\bar{\pi} \cong \mathbb{Z}/p^k$ is cyclic, all central group extensions are given by abelian groups. Thus, as a noncyclic group, $\pi \cong \mathbb{Z}/p \oplus \mathbb{Z}/p^k$ and obviously satisfies the statement.

If $\bar{\pi}$ is noncyclic, we can use an induction over the order of π . Let $\bar{x} := q(x)$. Then, $q(\nu(x)) \subset \nu(\bar{x}) \neq \bar{\pi}$ by induction hypothesis. This shows that $\nu(x) \neq \pi$.

To prove the second part of the statement, we observe that the group $\nu'(x) := \langle yxy^{-1} : y \in \pi \rangle$, generated by all the conjugates of x , is normal in π . Hence, $\nu(x) \subset \nu'(x)$. On the other hand, $yxy^{-1} \in \nu(x)$ for all $y \in \pi$, which shows that $\nu'(x) \subset \nu(x)$. \square

To prove Theorem 1.1 we need the following result, which may be found in [12], or [7].

2.3 Lemma. *Let $K \rightarrow G \xrightarrow{q} H$ be an exact sequence of groups, and let X be an almost BK -local space. Then,*

$$q^* : \text{map}(BH, X) \longrightarrow \bigcup_{f|_{BK} \simeq \text{const}} \text{map}(BG, X)_f$$

is an equivalence.

2.4 Proposition. *Let $f: B\pi \rightarrow X_p^\wedge$ be a map, where π is a finite p -group and X_p^\wedge a p -complete and almost $B\mathbb{Z}/p$ -local space. Let $\{x_1, \dots, x_r\}$ be a set of generators. If $f|_{B\langle x_i \rangle} \simeq \text{const}$ for all i , then f is homotopically trivial.*

Proof. We prove the statement by an induction over the order of π . If $\pi \cong \mathbb{Z}/p^k$ is cyclic, there is nothing to show because one of the elements must generate π . If π is noncyclic, by Lemma 2.1, there exists an exact sequence

$$\nu(x_1) \longrightarrow \pi \longrightarrow \bar{\pi} := \pi/\nu(x_1) .$$

$\nu(x_1)$ is generated by elements of the form yx_1y^{-1} , and $f|_{B\langle yx_1y^{-1} \rangle} \simeq \text{const}$. The order of $\nu(x_1)$ is smaller than the order of π . By induction hypothesis, $f|_{B\nu(x_1)} \simeq \text{const}$. By Lemma 2.1, X_p^\wedge is $B\nu(x_1)$ -local, and Lemma 2.2 establishes an equivalence $\text{map}(B\bar{\pi}, X_p^\wedge) \simeq \bigcup_{g|_{B\nu(x_1)} \simeq \text{const}} \text{map}(B\pi, X_p^\wedge)_g$. In particular, f factors over a map $\bar{f}: B\bar{\pi} \rightarrow X_p^\wedge$.

The quotient $\bar{\pi}$ is generated by the elements $\bar{x}_i := q(x_i)$. The exact sequence

$$\nu(x_1) \cap \langle x_i \rangle \longrightarrow \langle x_i \rangle \longrightarrow \langle \bar{x}_i \rangle$$

and another application of Lemma 2.2 show that $\bar{f}|_{B\langle \bar{x}_i \rangle} \simeq \text{const}$. Thus we can again apply the induction hypothesis, which shows that $\bar{f} \simeq \text{const}$. This finishes the proof. \square

Now, we are prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Let $x, y \in \text{preker}(f)$. We want to show that $xy \in \text{preker}(f)$ or, more generally, that $f|_{B\langle x, y \rangle} \simeq \text{const}$.

$S_{p^\infty}G$ is a locally finite p group. In particular, there exists a sequence

$$\Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_r \subset \dots \subset S_{p^\infty}G$$

of finite p groups such that $S_{p^\infty}G = \bigcup_r \Gamma_r$ [5]. Therefore, $\langle x, y \rangle$ is a finite p -group, and, by the last proposition, $f|_{B\langle x, y \rangle} \simeq \text{const}$. This proves (1).

Let $\Gamma'_r := \Gamma_r \cap \text{preker}(f)$. Then, $\text{preker}(f) = \bigcup_r \Gamma'_r$. By Proposition 2.3, $f|_{B\Gamma'_r}$ is nullhomotopic. Because X_p^\wedge is almost $B\mathbb{Z}/p$ -local and hence almost $B\Gamma'_r$ -local (Lemma 2.1), $\varprojlim^1 \pi_1(\text{map}(B\Gamma'_r, X_p^\wedge)_{\text{const}}) \cong \varprojlim^1 \pi_1(X_p^\wedge)$ vanishes. The Milnor sequence for calculating the homotopy groups of inverse homotopy limits proves that $f|_{B\text{preker}(f)}$ is nullhomotopic.

Lemma 2.5 below shows that $B\text{preker}(f) \rightarrow B\ker(f)$ is a mod- p equivalence. For the p -complete space X_p^\wedge , the map $[B(\ker(f), X_p^\wedge) \rightarrow [B(\text{preker}(f), X_p^\wedge)]$ between homotopy classes of maps is a bijection [2]. This implies that $f|_{B\ker(f)}$ is nullhomotopic and proves part (3).

$\ker(f)$ is the closure of $\text{preker}(f)$ in S_pG . Thus, the group of the components of $\ker(f)$ is a finite p -group, and $\ker(f)$ is a p -toral group which is part (2). This finishes the proof. \square

2.5 Lemma. *Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S). Then, the map*

$$B(\text{preker}(f)) \longrightarrow B(\ker(f))$$

is a mod- p equivalence.

Proof. Let $T(f)$ denote the component of the unit of $\ker(f)$, $T_\infty(f)$ the intersection of $T(f)$ and $S_{p^\infty}G$, and let $\pi := \pi_0(\ker(f))$. These groups fit into the commutative diagram

$$\begin{array}{ccccc} T_\infty(f) & \longrightarrow & \text{preker}(f) & \longrightarrow & \pi \\ \downarrow & & \downarrow & & \parallel \\ T(f) & \longrightarrow & \ker(f) & \longrightarrow & \pi . \end{array}$$

Both rows are exact.

As a locally finite abelian p -group, $T_\infty(f) \cong (\mathbb{Z}/p^\infty)^r \times A$, where A is a finite abelian p -group. Because the closure of $T_\infty(f)$ is $T(f)$, A is trivial, and $T(f) \cong (S^1)^r$. So, $BT_\infty(f) \rightarrow BT(f)$ is a mod- p equivalence. The Serre spectral sequence for mod- p cohomology for the fibrations in the diagram

$$\begin{array}{ccccc} BT_\infty(f) & \longrightarrow & B\text{preker}(f) & \longrightarrow & B\pi \\ \downarrow & & \downarrow & & \parallel \\ BT(f) & \longrightarrow & B\ker(f) & \longrightarrow & B\pi \end{array}$$

proves the statement. \square

3. Injective and monic maps.

In this section we proof Theorem 1.2. Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S). Let $\mathcal{A}_p(G)$ denote the Quillen category. The objects are given by elementary abelian p -subgroups and the morphisms by conjugation in G [13]. To get a finite category, we take only one object for every isomorphism class of objects, i.e. for every conjugacy class of a group. The Quillen map

$$\phi: H^*(BG; \mathbb{F}_p) \longrightarrow \varprojlim_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p)$$

is $H^*(BG; \mathbb{F}_p)$ -linear and an F -isomorphism; i.e. kernel and cokernel are nilpotent [13, Theorem 7.2]. Let

$$B := \text{im}(\phi \circ f^*) \subset \varprojlim_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p) \subset \prod_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p)$$

be the image of $H^*(X_p^\wedge; \mathbb{F}_p)$.

Proof of Theorem 1.2. First, we assume that f is injective and show that f is monic. For any elementary abelian p -subgroup $V \subset G$, the restriction $f|_{BV}$ is also injective. By [4, Proposition 4.4], which is the analogous statement of Theorem 1.2 for elementary abelian p -groups, this implies that $H^*(BV; \mathbb{F}_p)$ is a finitely generated module over $H^*(X_p^\wedge; \mathbb{F}_p)$ and over B . Because $\mathcal{A}_p(G)$ is a finite category, $\prod_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p)$ is a finitely generated module over B and a finitely generated algebra over \mathbb{F}_p . Therefore, B is also a finitely generated algebra over \mathbb{F}_p [1, Proposition 7.8] and hence noetherian. This implies that $\varprojlim_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p)$, as

a submodule of $\prod_{V \in \mathcal{A}_p(G)} H^*(BV; \mathbb{F}_p)$, is a finitely generated module over B and over $H^*(X_p^\wedge; \mathbb{F}_p)$.

If f is not monic, i.e the finitely generated \mathbb{F}_p -algebra $H^*(BG; \mathbb{F}_p)$ is not a finitely generated module over $H^*(X_p^\wedge; \mathbb{F}_p)$, there exists an element $y \in H^*(BG; \mathbb{F}_p)$ such that $\{y^i: i \in \mathbb{N}\}$ is a set of linearly independent elements over $H^*(X; \mathbb{F}_p)$. By the above considerations, for $r \in \mathbb{N}$ big enough, there exists a relation $\phi(y^r) = \sum_{i=0}^{r-1} x_i \phi(y^i)$ with $x_i \in H^*(X_p^\wedge; \mathbb{F}_p)$. That is that $y^r - \sum_{i=0}^{r-1} x_i y^i$ is in the kernel of ϕ and hence nilpotent. Thus, for $s \in \mathbb{N}$ big enough,

$$0 = (y^r - \sum_{i=0}^{r-1} x_i y^i)^s = y^{rs} - \sum_{j=0}^{r-1} x'_j y^j$$

for suitable $x'_i \in H^*(X_p^\wedge; \mathbb{F}_p)$. This is a contradiction and proves that f is monic.

Now, we assume that f is monic. Let $\mathbb{Z}/p \subset G$ be a subgroup of G . Up to conjugation, \mathbb{Z}/p is contained in $S_p G$. By [13], $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ is finitely generated over $H^*(BG; \mathbb{F}_p)$, and therefore, also over $H^*(X_p^\wedge; \mathbb{F}_p)$. That is that the map $B\mathbb{Z}/p \rightarrow BG \xrightarrow{f} X_p^\wedge$ is homotopically nontrivial. This implies that $\ker(f) = \{1\}$ and that f is injective. \square

4. $\mathcal{O}_p(G)$ -normal subgroups.

For every compact connected Lie group G , there exists a finite covering

$$K \longrightarrow \tilde{G} \xrightarrow{\alpha} G,$$

where $\tilde{G} \cong G_s \times T$ is a product of a simply connected Lie group G_s and a torus T . $G_s \cong \prod G_i$ is a product of simply connected simple Lie groups. K is a finite central subgroup of \tilde{G} . The group \tilde{G} is unique up to isomorphisms. Such coverings we call universal finite.

4.1 Lemma. *Let $K \rightarrow \tilde{G} \xrightarrow{\alpha} G$ be an exact sequence of compact Lie groups, K finite and \tilde{G} and G connected. Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S).*

(1) *The sequence*

$$S_p K \rightarrow \text{preker}(f \circ B\alpha) \rightarrow \text{preker}(f)$$

is exact.

(2) *$\ker(f \circ B\alpha) \rightarrow \ker(f)$ is an epimorphism, and $\ker(f \circ B\alpha) = S_p \alpha^{-1} \ker(f)$.*

Proof. Obviously, $\alpha^{-1}(\text{preker}(f)) \cap S_p \tilde{G}$ is contained in $\text{preker}(f \circ B\alpha)$. In particular, $S_p K \subset \text{preker}(f \circ B\alpha)$. Let $\Gamma := \text{preker}(f \circ B\alpha) / S_p K \supset \text{preker}(f)$. Then, by Lemma 2.3, $\text{map}(B\Gamma, X_p^\wedge) \simeq \bigcup_{g|_{BS_p K} \simeq \text{const}} \text{map}(B\text{preker}(f \circ B\alpha), X_p^\wedge)_g$. This implies that $f|_{B\Gamma}$ is homotopically trivial, and hence, that $\Gamma \subset \text{preker}(f)$, which establishes the desired sequence of (1).

To prove the second statement we first observe that in this case epimorphisms are maintained under taking closures. The second part in (2) follows from the facts that, as a p -toral group, $\ker(f \circ B\alpha) \subset S_p \alpha^{-1} \ker(f)$, and that $f|_{BS_p \alpha^{-1} \ker(f)}$ is homotopically trivial. \square

4.2 Lemma. *Let $K \rightarrow \tilde{G} \xrightarrow{\alpha} G$ be an exact sequence of compact Lie groups, K finite and \tilde{G} and G connected. Let $P \subset S_p G$ be a p -toral subgroup. Then, P is $\mathcal{O}_p(G)$ -normal if and only if $S_p \alpha^{-1}(P) \subset S_p \tilde{G}$ is $\mathcal{O}_p(\tilde{G})$ -normal.*

Proof. Let $\tilde{Q} := \alpha^{-1}(P)$ and $\tilde{P} := S_p \tilde{Q}$. The composition $\tilde{P} \rightarrow \tilde{Q} \rightarrow P$ is an epimorphism. This follows, because \tilde{P} and \tilde{Q} have identical components of the unit, and because, passing to the components, the composition $\pi_0(\tilde{P}) = S_p \pi_0(\tilde{Q}) \rightarrow \pi_0(P)$ is an epimorphism of finite groups.

$K \subset \tilde{G}$ is a central subgroup. The multiplication $\mu: (K \cap \tilde{Q}) \times \tilde{P} \rightarrow \tilde{Q}$ fits into the pull back diagram

$$\begin{array}{ccc} (K \cap \tilde{Q}) \times \tilde{P} & \xrightarrow{\mu} & \tilde{Q} \\ \downarrow & & \downarrow \alpha \\ \tilde{P} & \xrightarrow{\alpha} & P. \end{array}$$

Thus, μ is an epimorphism, and $\tilde{P} \subseteq \tilde{Q}$ is a normal subgroup. That is that \tilde{P} is the only p -toral Sylow subgroup of \tilde{Q} .

Every element $x \in P$ of order a power of p has a lift $\tilde{x} \in \tilde{P}$, also of order a power of p . Let $\tilde{g} \in \tilde{G}$ and $g := \alpha(\tilde{g})$. Then, $\tilde{g}\tilde{x}\tilde{g}^{-1} \in S_p \tilde{G}$ if and only if $g x g^{-1} \in S_p G$, and, because $\tilde{P} \subset \tilde{Q}$ is the only p -toral Sylow subgroup, $\tilde{g}\tilde{x}\tilde{g}^{-1} \in \tilde{P}$ if and only if $g x g^{-1} \in P$. This proves the statement. \square

Lemma 4.1 and Lemma 4.2 reduce the calculation of kernels and $\mathcal{O}_p(G)$ -normal subgroups, G connected, to the case of products of simply connected Lie groups and tori. Let $\tilde{G} \cong G_s \times T$ be such a product. In order to describe $\mathcal{O}_p(\tilde{G})$ -normal subgroups, we associate for every prime p a subgroup $H(G, p)$ to each simply connected simple Lie group G . We define

$$H(G, p) := \begin{cases} NT_G & \text{if } (p, |W_G|) = 1 \\ SU(2) \rtimes \mathbb{Z}/2 & \text{if } G = G_2 \text{ and } p = 3 \\ G & \text{else.} \end{cases}$$

We define $H(G_s, p) := \prod H(G_i, p)$ for a product $G_s = \prod G_i$ of simply connected simple Lie groups. Then, $BH(G_s, p) \hookrightarrow BG$ is a mod- p equivalence. If $p = 3$ and $G = G_2$, this follows from the isomorphism $H^*(BG_2; \mathbb{Z}/p) \cong H^*(BSU(2); \mathbb{Z}/p)^{\mathbb{Z}/2}$, and if $(p, |W_{G_s}|) = 1$ from the isomorphism $H^*(BG_s; \mathbb{Z}/p) \cong H^*(BT_{G_s}; \mathbb{Z}/p)^{W_{G_s}}$.

4.3 Proposition. *Let $\tilde{G} = \prod G_i \times T$ be a product of simply connected simple Lie groups G_i and a torus T . Let $\Gamma \subset N_p T_{\tilde{G}}$ be a $\mathcal{O}_p(\tilde{G})$ -normal subgroup. Then, we can split $G_s = G' \times G''$ such that $\Gamma \cong N_p T_{G'} \times \hat{\Gamma}$ and $\hat{\Gamma} \subset T_{G''} \times T$. Moreover, $\hat{\Gamma}$ is normal in $H(G'', p) \times T$ and the image of $\hat{\Gamma}$ in G'' is finite.*

We postpone the proof. This result enables us to prove Theorem 1.5.

Proof of Theorem 1.5. Let $f: BG \rightarrow X_p^\wedge$ be a map as in (S), and let

$$K \longrightarrow \tilde{G} \longrightarrow G$$

be a universal finite covering, where $\tilde{G} \cong G_s \times T$. By the last proposition and Proposition 1.3, $G_s \cong G' \times G''$ and $\ker(f \circ B\alpha) \cong N_p T_{G'} \times \hat{\Gamma}$. Now, we define $H =$

$(H(G'', p) \times T)/\hat{\Gamma}$. The classical kernel of the projection $G' \times H(G'', p) \times T \longrightarrow H$ is given by $G' \times \hat{\Gamma}$, which contains K by construction. We get a commutative diagram

$$\begin{array}{ccccc}
B(G' \times H(G'', p) \times T) & \xrightarrow{Bi} & B\tilde{G} & \xrightarrow{f \circ B\alpha} & X_p^\wedge \\
B\hat{\alpha} \downarrow & & B\alpha \downarrow & & \parallel \\
B(G' \times H(G'', p) \times T)/K & \xrightarrow{B\bar{i}} & BG & \xrightarrow{f} & X_p^\wedge \\
\hat{q} \downarrow & & q \downarrow & & \parallel \\
BH_p^\wedge & \xlongequal{\quad} & BH_p^\wedge & \xrightarrow{\bar{f}} & X_p^\wedge .
\end{array}$$

Bi and $B\bar{i}$ are mod- p equivalences. BH_p^\wedge is p -complete, because $\pi_1(BH)$ is a finite group [2]. This establishes the map $q: BG \longrightarrow BH_p^\wedge$. The quotient $(G' \times \hat{\Gamma})/K$ is a normal subgroup of $(G' \times H(G'', p) \times T)/K$, and X_p^\wedge is almost $B(G' \times \hat{\Gamma})/K$ -local (Lemma 2.1). Therefore, the map $f \circ B\bar{i}$ factors over $\bar{f}: BH_p^\wedge \longrightarrow X_p^\wedge$ (Lemma 2.3). Moreover, $\bar{f} \circ q \simeq f$, because $\bar{f} \circ \hat{q} \simeq f \circ B\bar{i}$. This proves the first half of Theorem 1.5.

By Lemma 4.1, $\ker(f \circ B\alpha \circ Bi) \xrightarrow{\beta \circ \hat{\alpha}} \ker(\bar{f})$ is an epimorphism. This shows that \bar{f} is injective. \square

In the rest of this section, we prove Proposition 4.3.

Proof of Proposition 4.3. The subgroup $\Gamma_T = \Gamma \cap T_{\tilde{G}}$ is invariant under the Weyl group action. In particular, $W_{\tilde{G}}$ acts on the component Γ_e of the unit of Γ , and $H^2(B\Gamma_e; \mathbb{Q})$ is a $W_{\tilde{G}}$ -submodule of

$$H^2(BT_{\tilde{G}}; \mathbb{Q}) \cong \bigoplus H^2(BG_i; \mathbb{Q}) \oplus H^2(BT; \mathbb{Q}) .$$

The first summands are irreducible. Thus, $\Gamma_T \cap T_{G_i} = T_{G_i}$ or the intersection is trivial.

Let G' be the product of all factors G_i with $T_{G_i} \subset \Gamma$, and G'' the product of the other factors of \tilde{G} . Let $x \in N_p T_{G'}$ but $x \notin T_{G'}$. Then x is conjugated to an element in $T_{G'}$ and therefore, $x \in \Gamma$. This implies that $N_p T_{G'} \subset \Gamma$. Moreover, $N_p T_{G'}$ is a normal subgroup of Γ . In the commutative diagram

$$\begin{array}{ccccc}
N_p T_{G'} & \xrightarrow{i} & \Gamma & \longrightarrow & \hat{\Gamma} := \Gamma/N_p T_{G'} \\
\parallel & & \downarrow & & \downarrow \\
N_p T_{G'} & \xrightarrow{j} & N_p T_{G'} \times N_p T_{G''} \times T & \longrightarrow & N_p T_{G''} \times T
\end{array}$$

both rows are central extensions. The map j has a section, which establishes also a section for i . Therefore, the upper sequence is the trivial extension, $\Gamma \cong N_p T_{G'} \times \hat{\Gamma}$, and $\hat{\Gamma} \subset N_p T_{G''} \times T$. This is the first part of the statement.

Because $\Gamma_e \subset N_p T_{G'} \times T$, the image of $\hat{\Gamma}$ in G'' is a finite group and $\mathcal{O}_p(G'')$ -normal. We have to investigate finite $\mathcal{O}_p(G)$ -normal subgroups of simply connected simple Lie groups. The following proposition finishes the proof. \square

4.4 Proposition. *Let G be a simply connected simple Lie group, and $\Gamma \subset N_p T_G$ a $\mathcal{O}_p(G)$ -normal finite p -subgroup.*

- (1) *If p divides $|W_G|$, and if $G \neq G_2$ or $p \neq 3$, then Γ is central in G .*
- (2) *If $G = G_2$ and $p = 3$, then Γ is central in $SU(3)$ and hence normal in $H(G_2, 3)$.*
- (3) *If $(p, |W_G|) = 1$, then Γ is normal in $H(G, p)$.*

Proof. If $x \in \Gamma$ then $txt^{-1}x^{-1} \in \Gamma$ for all $t \in T_G$. Because Γ is finite, all the commutators are trivial. Thus, x centralizes T_G . Because $C_G(T_G) = T_G$, $\Gamma = \Gamma_T \subset T_G$.

If $(p, |W_G|) = 1$, Γ is normal in $NT_G = H(G, p)$. If p divides $|W_G|$, we have to prove that Γ is central in G or, for $G = G_2$ and $p = 3$, central in $SU(3)$. Let $x \in \Gamma \setminus Z(G)$ and $x^{p^k} = 1$. Using this element, we construct an element $\Gamma \setminus T_G$, which gives a contradiction. This is done by a case by case checking and very much in the flavour of the proof of [7, Theorem 2].

Before we start, we make two observations. First, if (1) is true for two groups G and H , then it is obviously true for the product $G \times H$ and, Second, if $G \xrightarrow{q} \overline{G}$ is a covering of connected Lie groups, then (1) is true for G if and only if \overline{G} satisfies condition (1). To see this, we consider a finite $\mathcal{O}_p(G)$ -normal p -subgroup $\Gamma \subset N_p T_G$. Then, $\Gamma' := \langle \Gamma, S_p Z(G) \rangle$ is also a finite $\mathcal{O}_p(G)$ -normal p -subgroup. Because $\Gamma' = S_p(q^{-1}(q(\Gamma')))$, the group $q(\Gamma')$ is $\mathcal{O}_p(\overline{G})$ -normal if and only if Γ' is $\mathcal{O}_p(G)$ -normal (Lemma 4.2), and $q(\gamma') \subset \overline{G}$ is central if and only if $\Gamma' \subset G$ is central.

Let $G = SU(n)$, $n \geq p$. Up to conjugation, x can be represented by a diagonal matrix $D = D(a_1, \dots, a_n)$, where a_i is a p^k -th root of unity, and $a_1 \neq a_2$. The element $y = D(a_2, a_1, a_3, \dots, a_n)$ is conjugate to x and thus, $xy^{-1} = D(a_1 a_2^{-1}, a_1^{-1} a_2, 1, \dots, 1) \in \Gamma$. Because $a_1 a_2^{-1} \neq 1$, conjugates of xy^{-1} generate a subgroup of T_G , which contains the maximal elementary abelian p -subgroup V_G of T_G . Every element of order p is conjugate to an element in V_G . This gives a contradiction. For $G = U(n)$, $n \geq p$, the same proof works.

Let $G = Sp(n)$, $n \geq p$. Then, $U(n)$ and $SU(2)^n$ are subgroups of maximal rank. Therefore, $\Gamma \subset T_{Sp(n)}$ is central in $U(n)$ and $SU(2)^n$. But $Z(U(n)) \cap Z(SU(2)^n) = \mathbb{Z}/2 = Z(Sp(n))$.

By the above observation, the case of $G = Spin(n)$, $n \geq 2p$ and $n \geq 5$, can be reduced to the case of $SO(n)$. Let $n = 2k$ or $n = 2k + 1$. Then, $U(k) \subset SO(2k) \subset SO(2k + 1)$ is a subgroup of maximal rank, and Γ is central in $U(k)$. For $n \geq 5$, the only $W_{SO(2k)}$ or $W_{SO(2k+1)}$ -invariant subgroup of $S^1 = Z(U(k))$ is $\mathbb{Z}/2$. For $SO(2k)$, $\mathbb{Z}/2 \subset U(k) \subset SO(2k)$ is central, and for $SO(2k + 1)$, $\mathbb{Z}/2 \subset U(k) \subset SO(2k + 1)$ is conjugate to a subgroup in $NT_{SO(2k+1)} \setminus T_{SO(2k+1)}$, which proves the statement for $SO(n)$.

Let G be an exceptional Lie group and p a divisor of $|W_G|$. In this case, we choose subgroups of maximal rank, given by the following list:

| | | | | |
|---------|---------------|---|---------|---|
| $G_2 :$ | $p = 2, 3$ | $H = SU(3)$ | | |
| $F_4 :$ | $p = 2$ | $H = SU(3) \times_{\mathbb{Z}/3} SU(3)$ | $p = 3$ | $Spin(9)$ |
| $E_6 :$ | $p = 2, 5$ | $H = SU(3)^3 / \mathbb{Z}/3$ | $p = 3$ | $H = SU(2) \times_{\mathbb{Z}/2} SU(6)$ |
| $E_7 :$ | $p = 3, 5, 7$ | $H = SU(8) / \mathbb{Z}/2$ | $p = 2$ | $H = SU(3) \times_{\mathbb{Z}/3} SU(6)$ |
| $E_8 :$ | $p = 3, 5, 7$ | $H = SU(2) \times_{\mathbb{Z}/2} E_7$ | $p = 2$ | $H = SU(9) / \mathbb{Z}/3$ |

Beside the case $G = G_2$ and $p = 3$, the inclusion $H \subset G$ always induces an isomorphism $S_p Z(H) \cong S_p Z(G)$ between the p -Sylow subgroups of the centers. The data may be obtained from [9], where one can find a complete list of maximal subgroups of maximal rank of the exceptional Lie groups, and from [14].

Now, we can argue as follows: Let $\Gamma \subset N_p T_G$ be an $\mathcal{O}_p(G)$ -normal subgroup. Then, by induction over the rank, by the above observations, and by the already calculated cases, Γ is central in H and hence, central in G .

For $G = G_2$ and $p = 3$, the argument only shows that Γ is central in $SU(3)$. That is $\Gamma = \mathbb{Z}/3$ or Γ is the trivial group. In both cases, Γ is normal in $SU(3) \rtimes \mathbb{Z}/2 = H(G_2, 3)$. This finishes the last open case and the proof of the proposition. \square

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