

# FOR WHICH PSEUDO REFLECTION GROUPS ARE THE $p$ -ADIC POLYNOMIAL INVARIANTS AGAIN A POLYNOMIAL ALGEBRA?

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ABSTRACT. Let  $W$  be a finite group acting on a lattice  $L$  over the  $p$ -adic integers  $\mathbb{Z}_p^\wedge$ . The analysis of the ring of invariants of the associated  $W$ -action on the algebra  $\mathbb{Z}_p^\wedge[L]$  of polynomial functions on  $L$  is a classical question of invariant theory. If  $p$  is coprime to the order of  $W$ , classical results show that  $W$  is a pseudo reflection group, if and only if the ring of invariants is again polynomial. We analyze the situation for odd primes dividing the order of  $W$  and, in particular, determine those pseudo reflection groups for which the ring of invariants  $\mathbb{Z}_p^\wedge[L]^W$  is a polynomial algebra.

## 1. Introduction.

Let  $L$  be a  $p$ -adic lattice, i.e. a torsion free finitely generated  $\mathbb{Z}_p^\wedge$ -module, and let  $\mathbb{Z}_p^\wedge[L]$  be the graded polynomial algebra of polynomial functions on  $L$ . Every (faithful) representation  $W \rightarrow Gl(L)$  of a finite group  $W$  establishes a  $W$ -action on  $\mathbb{Z}_p^\wedge[L]$ . The analysis of the ring of invariants is a classical question of invariant theory. In particular, one might ask, whether  $\mathbb{Z}_p^\wedge[L]^W$  is again a polynomial algebra. If this is the case, we call the lattice  $L$  *polynomial* (with respect to the  $W$ -action).

If we work over a field of characteristic coprime to the order  $|W|$  of the group  $W$ , the Shephard–Todd–Chevalley–Theorem [16] [6] says that, if  $W$  is a pseudo reflection group, then the invariants are again a polynomial algebra. And the converse is also true (e.g. see [17; Theorem 7.4.1]).

For a vector space  $V$ , a representation  $W \rightarrow Gl(V)$  of a finite group is called a *pseudo reflection group* if it is faithful and if the image is generated by pseudo reflections; i.e. by elements of finite order which fix a hyperplane of  $V$  of codimension 1. For actions on  $p$ -adic lattices the same definition works. If we say that  $W$  is a pseudo reflection group, then we always have a particular representation in mind. Examples of  $p$ -adic pseudo reflection groups are given by the action of the Weyl group  $W_G$  of a connected compact Lie group  $G$  on a maximal torus  $T_G$  of  $G$ , actually on the  $p$ -adic lattice  $L_G := H_1(T_G; \mathbb{Z}_p^\wedge) \cong \pi_1(T_G) \otimes \mathbb{Z}_p^\wedge$ .

If  $p$  does not divide  $|W|$ , then, as an easy consequence, the Shephard–Todd–Chevalley Theorem as well as its converse also holds for  $p$ -adic integral representations; i.e. a representation  $W \rightarrow Gl(L)$  is a pseudo reflection group if and only if the lattice  $L$  is polynomial (see Corollary 2.6 and Proposition 2.7).

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In the modular case, that is  $p$  divides  $|W|$ , only the converse is true: if  $\mathbb{Z}_p^\wedge[L]^W$  is a polynomial algebra, then  $W \rightarrow Gl(L)$  is a pseudo reflection group (see Proposition 2.7).

In this work, we are mainly interested in the question, for which  $p$ -adic pseudo reflection groups the Shephard-Todd-Chevalley-Theorem holds; i.e. for which pseudo reflection groups  $W \rightarrow Gl(L)$  the  $W$ -lattice  $L$  is polynomial. For odd primes, we will give a complete answer to this question, in particular in the modular case.

Before we can state our results explicitly we first have to make some definitions and to fix notation. For the motivation of these definitions see [13].

### 1.1 Definitions and Remarks.

**1.1.1** Let  $L$  be a torsion-free  $\mathbb{Z}_p^\wedge$ -module. Every  $p$ -adic representation  $W \rightarrow Gl(L)$  gives rise to an representation  $W \rightarrow Gl(L_\mathbb{Q})$  over  $\mathbb{Q}_p^\wedge$ , where  $L_\mathbb{Q} := L \otimes_{\mathbb{Z}_p^\wedge} \mathbb{Q}$ . This  $p$ -adic rational representation might contain several  $W$ -lattices. From this point of view that are lattices of  $L_\mathbb{Q}$  which are stable under the action of  $W$ .

Several notions of  $L_\mathbb{Q}$  are inherited to  $L$ ; e.g. the representation  $W \rightarrow Gl(L)$  is called irreducible if  $W \rightarrow Gl(L_\mathbb{Q})$  is irreducible.

Two  $W$ -lattices  $L_1$  and  $L_2$  are called *isomorphic* if  $L_1$  and  $L_2$  are isomorphic as modules over the group ring  $\mathbb{Z}_p^\wedge[W]$ . They are called *weakly isomorphic* if there exists an automorphism  $\alpha : W \rightarrow W$  such that  $L_1$  and  $L_2^\alpha$  are isomorphic. Here, the action of  $W$  on  $L_2^\alpha$  is given by the composition of  $W \rightarrow Gl(L_2)$  and  $\alpha$ .

**1.1.2** Let  $U$  be a  $p$ -adic rational vector space. A pseudo reflection group  $W \rightarrow Gl(U)$  is called of *Lie type* if the representation is weakly isomorphic to an pseudo reflection group  $W_G \rightarrow Gl(L_G \otimes \mathbb{Q})$  for a suitable compact connected Lie group  $G$ . Otherwise it is called of *non Lie type*.

**1.1.3** Let  $L$  be  $p$ -adic lattice and let  $W \rightarrow Gl(L)$  be a pseudo reflection group. The *covariants*  $L_W := L/SL$  are defined as the quotient of  $L$  by  $SL$ , where  $SL \subset L$  is the sublattice generated by all elements of the form  $l - w(l)$  with  $l \in L$  and  $w \in W$ . The lattice  $L$  is called *simply connected* if  $L_W = 0$ . If  $p$  is odd, the lattice  $SL$  is always simply connected. [13; 3.2].

**1.1.4** For every  $W$ -lattice  $L$ , there exists a short exact sequence

$$0 \rightarrow L \rightarrow L_\mathbb{Q} \rightarrow L_\infty := L_\mathbb{Q}/L \rightarrow 0$$

of  $W$ -modules. The quotient  $L_\infty \cong (\mathbb{Z}/p^\infty)^n \subset (S^1)^n$  is called a  $p$ -discrete torus and can be considered as a subgroup of a torus whose dimension equals the rank of  $L$ .

For a  $p$ -discrete torus  $L_\infty$  with an action of  $W$ , we get an  $W$ -lattice by setting  $L := Hom(\mathbb{Z}/p^\infty, L_\infty)$ . In fact,  $L$  is a lattice because  $Hom(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \cong \mathbb{Z}_p^\wedge$ . For every  $W$ -lattice  $L$ , we always have  $L \cong Hom(\mathbb{Z}/p^\infty, L_\infty)$ .

**1.1.5** A  $W$ -lattice  $L$  is called centerfree if  $(L_\infty)^W = 0$ , i.e. the  $p$ -discrete torus  $L_\infty$  has no nontrivial fixed-point.

**1.1.6** Given a short exact sequence  $L \rightarrow M \rightarrow K$  of  $W$ -modules, such that  $L$  and  $M$  are lattices and such that  $K$  is finite, the two representations  $L_\mathbb{Q}$  and  $M_\mathbb{Q}$  of  $W$  are isomorphic. Thus, the above short exact sequence of 1.1.4 and the serpent lemma establish a short exact sequence  $K \rightarrow L_\infty \rightarrow M_\infty$ .

**1.1.7** A monomorphism  $L \rightarrow M$  of  $W$ -lattices is called a  $W$ -trivial extension if  $W$  acts trivially on the quotient  $M/L$  and if  $M/L$  is finite.

In [13] the following structure theorem is proved:

**1.2 Theorem.** *Let  $p$  be an odd prime. Let  $L$  be a  $p$ -adic lattice and let  $W \rightarrow Gl(L)$  be a pseudo reflection group.*

*i) The  $W$ -lattice  $L$  fits into two short exact sequences of  $W$ -modules, namely*

$$SL \rightarrow L \rightarrow L_W$$

and

$$SL \oplus L^W \rightarrow L \rightarrow K ,$$

where  $SL$  is simply connected and where  $W$  acts trivially on the finite  $W$ -module  $K \cong (L/L^W)_W \cong L_W/L^W$ .

*ii) Let  $S$  be a simply connected  $W$ -lattice. Then, there exist splittings  $W = \prod_i W_i$  and  $S \cong \bigoplus_i S_i$  in such a way that  $W_j$  acts trivially on  $S_i$  if  $i \neq j$  and  $S_i$  is a simply connected irreducible  $W_i$  lattice.*

*iii) If, in addition,  $L$  is an irreducible  $W$ -lattice, then, up to weak isomorphism, there exists a unique simply connected  $W$ -lattice  $S \subset L_{\mathbb{Q}}$ , given by  $SL$ .*

*Proof.* This follows from [13; theorems 1.2, 1.3, 1.4, 1.6]. The small differences in the statements come from the slightly different definitions of  $W$ -lattices.  $\square$

We can now state our main results.

**1.3 Theorem.** *Let  $p$  be an odd prime. Let  $L$  be a  $W$ -lattice. Then, the following statements are equivalent:*

- (1) *The  $W$ -lattice  $L$  is polynomial.*
- (2) *the  $W$ -lattice  $SL$  is polynomial and  $L_W$  is torsion free.*
- (3) *The  $W$ -lattice  $SL$  is polynomial and the composition  $K \rightarrow SL_{\infty} \oplus (L^W)_{\infty} \rightarrow (L^W)_{\infty}$  is a monomorphism.*

This result and the second part of Proposition 1.2 reduces the question of polynomial  $W$ -lattices to one about simply connected irreducible  $W$ -lattices.

**1.4 Theorem.** *Let  $p$  be an odd prime. Let  $W \rightarrow Gl(U)$  be an irreducible pseudo reflection group over  $\mathbb{Q}_p^{\wedge}$ . Let  $S \subset U$  be a simply connected  $W$ -lattice of  $U$ . Then,  $S$  is polynomial if and only if the pair  $(W, p)$  does not belong to one of the pairs  $(W_{F_4}, 3)$ ,  $(W_{E_6}, 3)$ ,  $(W_{E_7}, 3)$ ,  $(W_{E_8}, 3)$  and  $(W_{E_8}, 5)$ .*

The pairs excluded by the theorem are all given by the Weyl group action of exceptional connected compact Lie groups  $G$  on  $L_G$  at those odd primes which appear as torsion primes in the cohomology of  $G$ .

Clark and Ewing classified all irreducible  $p$ -adic rational pseudo reflection groups and wrote them down in a list [7]. In particular, Theorem 1.4 says that for all cases beside the listed exceptions the simply connected  $W$ -lattices are polynomial.

Starting from a simply connected polynomial  $W$ -lattice  $S$ , we can construct a polynomial lattice in the following way: We choose a lattice  $Z$  with trivial  $W$ -action, a finite subgroup  $K \subset (S_{\infty})^W$  and a monomorphism  $K \rightarrow Z_{\infty}$ . Then we define  $L_{\infty} := (S_{\infty} \oplus Z_{\infty})/K$  and  $L := Hom(\mathbb{Z}/p^{\infty}, L_{\infty})$ . By construction and because  $Ext(\mathbb{Z}/p^{\infty}, K) \cong K$ , we get a short exact sequence  $S \oplus Z \rightarrow L \rightarrow K$  of  $W$ -modules. By Theorem 1.3, the lattice  $L$  is polynomial and every polynomial  $W$ -lattice can be constructed this way.

The above construction allows the following description of polynomial  $W$ -lattices.

**Theorem 1.5.** *Let  $p$  be an odd prime. Let  $W \rightarrow Gl(L)$  be a pseudo reflection group such that  $L$  is polynomial lattice. Then, there exists a connected compact Lie group  $G$  and a finite number of irreducible pseudo reflection groups  $W_i \rightarrow Gl(L_i)$  such that  $W \rightarrow Gl(L)$  and  $W_G \times \prod_i W_i \rightarrow Gl(L_G \oplus \bigoplus_i L_i)$  are weakly isomorphic. The pseudo reflection groups  $W_i \rightarrow Gl(L_i)$  are rationally of non Lie type and, up to weak isomorphism, uniquely determined by the associated rational representation.*

The last claim of Theorem 1.5 is a consequence of the next proposition which describes lattices of irreducible  $p$ -adic rational pseudo reflection groups of non Lie type.

**1.6 Proposition.** *Let  $p$  be an odd prime. Let  $W \rightarrow Gl(U)$  be an irreducible  $p$ -adic rational pseudo reflection group of non Lie type. Then, up to isomorphism, there exists a unique  $W$ -lattice  $L \subset U$ . And this lattice is simply connected, centerfree and polynomial.*

To complete the picture we mention the following proposition.

**1.7 Proposition.** *Let  $p$  be an odd prime. Let  $G$  be a connected compact Lie group. Then, the integral cohomology  $H^*(G; \mathbb{Z})$  is  $p$ -torsion free if and only if the  $W_G$ -lattice  $L_G$  is polynomial.*

This proposition is proved in [15]. Actually, the main results of that paper are heavily based on the above statements. But Proposition 1.7 is never used in the proofs. The example of  $SO(3)$  shows that Proposition 1.7 is not true for the prime 2.

Similar results as in Theorem 1.4 are already proven by Demazure for Weyl groups of root systems, i.e. for pseudo reflection groups of Lie type [8]. Let  $L$  be the  $W$ -lattice defined by a root system. Demazure showed, that  $\mathbb{Z}_p^\wedge[L]^W \otimes \mathbb{F}_p$  is a polynomial algebra if the torsion index of the root system (for the definition see [8]) is coprime to  $p$ . Together with [8; Lemme 6] and the calculation of the torsion index for simple root systems [8, Proposition 8] this proves one half of Theorem 1.4 for pseudo reflection groups of Lie type.

There is also quite a lot of work done on the calculation of the mod- $p$  invariants  $\mathbb{F}_p[V]^W$  for pseudo reflection groups  $W \rightarrow Gl(V)$ , where  $V$  is  $\mathbb{F}_p$ -vector space. The most complete answer in this direction may be found in [12]. In fact, there is given a complete answer under the assumption that  $W \rightarrow Gl(V)$  is an irreducible pseudo reflection group. For further work calculating the mod- $p$  invariants, see the references in [12]. But unfortunately, if, for a  $W$ -lattice  $L$ , the reduction  $L/p$  is polynomial, it does not follow that  $L$  itself is polynomial. Knowing the mod- $p$  invariants may only allow to decide our question in the negative (see Lemma 2.3 and Section 6). The following example, demonstrating the first fact, was pointed out to us by A. Viruel.

**1.9 Example.** Let  $p = 3$ . The symmetric group  $\Sigma_3$  can be considered as the Weyl group of  $SU(3)$  as well as of  $PU(3) := SU(3)/\mathbb{Z}/3$ . Then, the action of  $\Sigma_3$  on  $L_{SU(3)}$  and  $L_{PU(3)}$  make both to polynomial  $\Sigma_3$ -lattices. The ring of invariants  $\mathbb{F}_p[L_{PU(3)}/p]^{\Sigma_3}$  is generated by two elements, one of degree 1 and one of degree 6. For  $L_{PU(3)}$ , all this is stated in [5], but in that reference, the authors used the topological grading, i.e. all degrees are doubled. Because of the short exact sequence  $L_{SU(3)} \rightarrow L_{PU(3)} \rightarrow \mathbb{Z}/3$  of  $\Sigma_3$ -modules, we have  $L_{PU(3)} \otimes \mathbb{Q} \cong L_{SU(3)} \otimes \mathbb{Q} \cong \mathbb{Q}_p^\wedge^2$ , and the two pseudo reflection groups  $\Sigma_3 \rightarrow Gl(L_{PU(3)})$  and  $\Sigma_3 \rightarrow Gl(L_{SU(3)})$  are

rationality isomorphic. Therefore, the polynomial algebra  $(\mathbb{Z}_p^\wedge[L_{PU(3)}] \otimes \mathbb{Q})^{\Sigma_3}$  is generated by two elements of degree 2 and 3. Hence, reduction mod- $p$  does not establish an epimorphism  $\mathbb{Z}_p^\wedge[L_{PU(3)}]^{\Sigma_3} \rightarrow \mathbb{F}_p[L_G]^{\Sigma_3}$  between the invariants. By Lemma 2.3,  $L_{PU(3)}$  can't be a polynomial  $\Sigma_3$ -lattice.

The proof of Theorem 1.4 is very much based on the classification of  $W$ -lattices which is only known for odd primes [13]. It also uses the classification of irreducible  $p$ -adic rational pseudo reflection groups [7] and is done by a case by case checking. The restriction to odd primes comes also from the facts that, for odd primes, the kernel of  $Gl(n; \mathbb{Z}_p^\wedge) \rightarrow Gl(n; \mathbb{F}_p)$  is torsion free and that  $H^1(W; \mathbb{F}_p) = 0$  for a pseudo reflection group  $W$ .

The paper is organized as follows. In Section 2 we recall some material about  $W$ -lattices from [13]. Section 3 is devoted to a discussion of sufficient conditions for a  $W$ -lattice being polynomial. Section 4 contains the proof of Theorem 1.3, Sections 5 the proofs of half of Theorem 1.4, of Theorem 1.5 and of Proposition 1.6. In the last section we prove the other half Theorem 1.4.

## 2. Invariant theory for $p$ -adic pseudo reflection groups.

In this section we discuss sufficient conditions, that, for a given pseudo reflection group  $W \rightarrow Gl(L)$ , the  $W$ -lattice  $L$  or the vector space  $L/p := L \otimes \mathbb{F}_p$  are polynomial. We also notice that  $\mathbb{F}_p[L/p] \cong \mathbb{F}_p[L] \cong \mathbb{Z}_p^\wedge[L] \otimes \mathbb{F}_p$ .

The following well known result might be found in [17; 5.5.5].

**2.1 Proposition.** *Let  $V$  be an  $n$ -dimensional vector space over a field  $\mathbb{F}$ . Let  $W \rightarrow Gl(V)$  be a faithful representation of a finite group. For  $i = 1, \dots, n$ , let  $g_i \in \mathbb{F}[V]^W$ . Then the quotient  $\mathbb{F}[V]/(g_1, \dots, g_n)$  is a finite dimensional vector space over  $\mathbb{F}$  and  $\prod_i \deg(g_i) = |W|$  if and only if  $\mathbb{F}[V]^W$  is a polynomial algebra generated by  $g_1, \dots, g_n$ .*

We also need a  $p$ -adic version of this statement for lattices.

**2.2 Corollary.** *Let  $p$  be an odd prime. Let  $L \cong \mathbb{Z}_p^\wedge^n$  be a  $p$ -adic lattice carrying a faithful action of a finite group  $W$ . For  $i = 1, \dots, n$ , let  $g_i \in \mathbb{Z}_p^\wedge[L]^W$ . Then  $\prod_i \deg(g_i) = |W|$  and the quotient  $\mathbb{Z}_p^\wedge[L]/(g_1, \dots, g_n)$  is a finitely generated module over  $\mathbb{Z}_p^\wedge$  if and only if  $\mathbb{Z}_p^\wedge[L]^W$  is a polynomial algebra generated by  $g_1, \dots, g_n$ .*

*Proof.* Let  $A := \mathbb{Z}_p^\wedge[L]$ . Let  $\bar{\cdot} : A \rightarrow A/p \cong \mathbb{F}_p[L/p]$  denote the reduction mod  $p$ . Since  $p$  is odd and since  $W$  is a finite group the representation  $W \rightarrow Gl(L/p)$  is faithful and we can apply Proposition 2.1 for the  $W$ -action on  $L/p$ .

If  $A/(g_1, \dots, g_n)$  is a finitely generated  $\mathbb{Z}_p^\wedge$ -module, the quotient  $(A/p)/(\bar{g}_1, \dots, \bar{g}_n)$  is a finite dimensional  $\mathbb{F}_p$ -vector space. Since the degree equation is independent of the basic ring, this shows that  $(A/p)^W \cong \mathbb{F}_p[\bar{g}_1, \dots, \bar{g}_n] \cong A^W/p$  is a polynomial algebra generated by the elements  $\bar{g}_i$  (Proposition 2.1), and, by the Nakayama lemma, that  $A^W$  is also a polynomial algebra generated by the elements  $g_i$  for  $i = 1, \dots, n$ .

If  $A^W \cong \mathbb{Z}_p^\wedge[g_1, \dots, g_n]$  then  $A/(g_1, \dots, g_n)$  is a lattice in the finite dimensional  $\mathbb{Q}_p^\wedge$ -vector space  $A \otimes \mathbb{Q}/(g_1, \dots, g_n)$  and therefore a finitely generated  $\mathbb{Z}_p^\wedge$ -module. The degree equation also follows from Proposition 2.1 for  $\mathbb{F} = \mathbb{Q}_p^\wedge$ .  $\square$

**2.3 Lemma.** *Let  $p$  be an odd prime. Let  $L$  be a  $W$ -lattice. Then the following holds:*

*i) If  $L$  is polynomial, the map  $\mathbb{Z}_p^\wedge[L]^W \rightarrow \mathbb{F}_p[L/p]^W$  is an epimorphism and  $\mathbb{F}_p[L/p]^W$*

is a polynomial algebra.

ii) If  $\mathbb{F}_p[L/p]^W$  is polynomial and  $\mathbb{Z}_p^\wedge[L]^W \rightarrow \mathbb{F}_p[L/p]^W$  is an epimorphism, then  $L$  is polynomial.

(iii) If  $L$  is polynomial, then  $H^1(W; \mathbb{Z}_p^\wedge[L]) = 0$

*Proof.* Let  $A := \mathbb{Z}_p^\wedge[L]$ . Let  $g_1, \dots, g_n$  be polynomial generators of  $A^W$ . Then we have  $\prod_i \deg(g_i) = |W|$  (Corollary 2.2).

The sequence  $A^W \rightarrow A \rightarrow A/A^W$  is short exact and splits as sequence of  $\mathbb{Z}_p^\wedge$ -modules, since  $A/A^W$  is torsion free and since  $A/A^W$  is a finitely generated  $\mathbb{Z}_p^\wedge$ -module (Corollary 2.2). Reducing mod  $p$  establishes a short exact sequence

$$A^W/p \rightarrow A/p \cong \mathbb{F}_p[L/p] \rightarrow (A/A^W)/p \cong (A/p)/(A^W/p).$$

The third term is a finite dimensional  $\mathbb{F}_p$ -vector space. Let  $\bar{g}_i$  be the image of  $g_i$  under the reduction. Then the elements  $\bar{g}_i \in A/p$  are invariants and generate  $A^W/p$ . Therefore,  $(A/p)/(\bar{g}_1, \dots, \bar{g}_n) \cong (A/p)/(A^W/p)$  is finite dimensional. As in the proof of Corollary 2.2 we can apply Proposition 2.1. Hence, because  $\prod_i \deg(\bar{g}_i) = \prod_i \deg(g_i) = |W|$ , the ring  $(A/p)^W$  of invariants is a polynomial algebra generated by  $\bar{g}_1, \dots, \bar{g}_n$  (Proposition 2.1) and the map  $\mathbb{Z}_p^\wedge[L]^W \rightarrow \mathbb{F}_p[L/p]^W$  is an epimorphism. This proves the first part.

Let  $\mathbb{F}_p[L/p]^W \cong \mathbb{F}_p[\bar{g}_1, \dots, \bar{g}_n]$  be a polynomial algebra. Let  $g_i \in \mathbb{Z}_p^\wedge[L]^W$  be a lift of  $\bar{g}_i$ . Then, the elements  $g_i$  are algebraically independent and, by the Nakayama lemma, generate  $\mathbb{Z}_p^\wedge[L]^W$  as a polynomial algebra, which proves the second part.

The third part is a consequence of (i) and the next lemma.  $\square$

**2.4 Lemma.** *Let  $G$  be a finite group and let  $M$  be a  $W$ -lattice. Then,  $M^G \rightarrow (M/p)^G$  is an epimorphism if and only if  $H^1(G; M) = 0$ .*

*Proof.* Multiplication by  $p$  gives rise to a short exact sequence  $M \rightarrow M \rightarrow M/p$ . Taking fixed-points establishes an exact sequence  $M^G \rightarrow M^G \rightarrow (M/p)^G \rightarrow H^1(G; M) \rightarrow H^1(G; M)$ . The first and last map are again given by multiplication by  $p$ . Since  $M$  is a  $\mathbb{Z}_p^\wedge$ -module and since  $G$  is finite, the group  $H^1(G; M)$  is a finite abelian  $p$  group. Therefore, the second map in the above sequence is an epimorphism iff  $H^1(G; M)$  vanishes.  $\square$

The following statement is a relative version of the theorem of Shephard and Todd [16] respectively of Chevalley [6].

**2.5 Proposition.** *Let  $V$  be vector space over a field  $\mathbb{F}$ . Let  $W \rightarrow Gl(V)$  be a pseudo reflection group. Let  $W_1 \subset W$  be a subgroup such that the index  $[W : W_1]$  is coprime to the characteristic  $\text{char}(\mathbb{F})$  of  $\mathbb{F}$ . If  $\mathbb{F}[V]^{W_1}$  is a polynomial algebra, then so is  $\mathbb{F}[V]^W$ .*

*Proof.* By [17; 6.4.4] it is sufficient to show that  $\mathbb{F}[V]^{W_1}$  is a free  $\mathbb{F}[V]^W$ -module, and by [17; 6.1.1] this follows if  $Tor_1^{\mathbb{F}[V]^W}(\mathbb{F}, \mathbb{F}[V]^{W_1})$  vanishes. Since  $([W : W_1], \text{char}(\mathbb{F})) = 1$ , there exists an averaging map

$$a : \mathbb{F}[V]^{W_1} \rightarrow \mathbb{F}[V]^W : f \mapsto (1/[W : W_1]) \sum_{w \in W/W_1} wf.$$

The vanishing of the  $Tor_1$ -term now follows analogously as in the proof of the Shephard-Todd theorem as given in [17; 7.4.1].  $\square$

Actually, for our purpose, we need a  $p$ -adic integral version of Proposition 2.5.

**2.6 Corollary.** *Let  $L$  be  $W$ -lattice. Let  $W \rightarrow Gl(L)$  be a pseudo reflection group. Let  $W_1 \rightarrow W$  be a subgroup of index coprime to  $p$ . If  $L$  is polynomial as  $W_1$ -lattice, then so is  $L$  as  $W$ -lattice.*

*Proof.* Since the index  $[W : W_1]$  is a unit in  $\mathbb{Z}_p^\wedge$ , there exists again the averaging map  $a : \mathbb{Z}_p^\wedge[L]^{W_1} \rightarrow \mathbb{Z}_p^\wedge[L]^W$  which, in particular, is an epimorphism. The same holds after reducing everything mod  $p$ . Therefore, in the diagram

$$\begin{array}{ccc} \mathbb{Z}_p^\wedge[L]^{W_1} & \xrightarrow{a} & \mathbb{Z}_p^\wedge[L]^W \\ \downarrow & & \downarrow \\ \mathbb{F}_p[L/p]^{W_1} & \xrightarrow{a} & \mathbb{F}_p[L/p]^W \end{array}$$

the bottom and the left vertical arrow are epimorphisms (Lemma 2.3) as well as the right vertical map. By Lemma 2.3, it follows that  $L$  is a polynomial  $W$ -lattice.  $\square$

If we choose the trivial group for  $W_1$  in the last two statements, we get the Shephard–Todd–Chevalley Theorem. We end this section with an  $p$ -adic integral version of the converse.

**2.7 Proposition.** *Let  $L$  be a finitely generated lattice and  $W \rightarrow Gl(L)$  be a faithful representation of a finite group  $W$ . If  $\mathbb{Z}_p^\wedge[L]^W$  is a polynomial algebra, then  $W \rightarrow Gl(L)$  is a pseudo reflection group.*

*Proof.* Because  $\mathbb{Q}_p^\wedge[L_\mathbb{Q}]^W \cong \mathbb{Z}_p^\wedge[L]^W \otimes \mathbb{Q}$ , both are polynomial algebras. Now, applying the Shephard–Todd–Chevalley Theorem in characteristic 0 shows that  $W \rightarrow Gl(L_\mathbb{Q})$  as well as  $W \rightarrow Gl(L)$  are pseudo reflection groups.

### 3. Proof of Theorem 1.3.

We start directly with the proof of Theorem 1.3.

#### Proof of Theorem 1.3.

Let us first assume that  $S := SL$  is polynomial and that  $Z := L_W$  is torsion free. For abbreviation we define  $A := \mathbb{Z}_p^\wedge[Z]$ ,  $B := \mathbb{Z}_p^\wedge[L]$  and  $C := \mathbb{Z}_p^\wedge[S]$ . Let  $i : A \rightarrow B$  be the inclusion and  $q : B \rightarrow C$  be the projection. The short exact sequence  $S \rightarrow L \rightarrow Z$  splits as a sequence of  $\mathbb{Z}_p^\wedge$ -modules. Hence, as a map of algebras, the homomorphism  $q$  has a right inverse  $s : C \rightarrow B$ . This section gives rise to an isomorphism  $f : A \otimes_{\mathbb{Z}_p^\wedge} C \xrightarrow{\cong} B$  of graded algebras. Let  $A^n \subset A$  denote the subset of  $A$  of all elements of degree  $n$ , and let  $I_n \subset A$  and  $J_n \subset B$  denote the ideals generated by  $A^n$  respectively by  $i(A^n)$ . This gives rise to filtrations

$$A = I_0 \supset I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$$

and

$$C = J_0 \supset J_1 \supset J_2 \supset \dots \supset J_n \supset \dots$$

The above isomorphism is compatible with these filtrations and maps  $I_n \otimes C$  isomorphically on  $J_n$ . Hence, we get isomorphisms

$$A^n \otimes C \xrightarrow{\cong} J_n/J_{n+1} : a \otimes c \mapsto a \cdot s(c)$$

between the filtration quotients. Moreover, this map is  $W$ -equivariant, since  $s(w(c)) - w(s(c))$  is contained in the kernel  $\ker(q) \cong J_1$  of  $q$  and since  $J_1 \cdot J_n \subset J_{n+1}$ .

If we restrict the filtrations to elements of a fixed degree  $n$ , then  $J_{n+1}^n = 0 = I_{n+1}^n$ , the filtrations become finite, and, for  $r \leq n$ , we have isomorphisms  $J_n^r/J_n^{r+1} \cong A^r \otimes C^{n-r}$ . We use these considerations to show that  $B^W \rightarrow C^W$  is an epimorphism. Since  $J_1 = \ker(q)$ , we only have to show that  $H^1(W; J_1^n) = 0$  for every  $n$ . Because  $S$  is polynomial, we have

$$H^1(W; C^{n-r}) = 0 = H^1(W; A^r \otimes C^{n-r}) \cong H^1(W; J_n^r/J_n^{r+1})$$

for each  $0 \leq r \leq n$ . By induction based on the above filtrations, this shows that  $H^1(W; J_1)$  vanishes and that  $B^W \rightarrow C^W$  is an epimorphism.

Let  $g_1, \dots, g_n$  be polynomial generators of  $C^W$ . Since  $W$  acts trivially on  $A$ , choosing lifts of these polynomials in  $B^W$  establishes an isomorphism  $A \otimes C^W \xrightarrow{\cong} B^W$ , which shows that  $B^W$  is a polynomial algebra and that  $L$  is polynomial.

Now let us assume that  $L$  is polynomial. We first want to show that  $L_W$  is torsion free. The lattice  $L$  fits into a  $W$ -trivial extension  $S \oplus Z \rightarrow L \rightarrow K$  with  $Z \cong L^W$  (Theorem 1.2). We dualize this sequence, i.e. we apply the functor  $\text{Hom}(\cdot, \mathbb{Z}_p^\wedge)$ . Let  $L^\sharp$  denote the dual. We get a short exact sequence

$$L^\sharp \rightarrow S^\sharp \oplus Z^\sharp \rightarrow \text{Ext}(K, \mathbb{Z}_p^\wedge) \cong K .$$

Taking fixed-points yields the exact sequence given by the top row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{\sharp W} & \longrightarrow & S^{\sharp W} \oplus Z^\sharp \cong Z^\sharp & \longrightarrow & K & \longrightarrow & H^1(W; L^\sharp) \\ & & \downarrow & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & L^\sharp & \longrightarrow & S^\sharp \oplus Z^\sharp & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^\sharp/L^{\sharp W} & \xrightarrow{\cong} & S^\sharp & \longrightarrow & 0 & & . \end{array}$$

All rows and columns are exact. The isomorphism in the middle of the top row follows from the fact that  $S$  as well as  $S^\sharp$  are fixed-point free. And  $H^1(W; L^\sharp) = 0$  by Lemma 2.3 (iii).

Having in mind that, for a lattice  $M$ , we have  $(M^\sharp)^\sharp \cong M$  and dualizing the first column establishes the short exact sequence

$$S \rightarrow L \rightarrow ((L^\sharp)^W)^\sharp \cong L_W .$$

Therefore, the module  $L_W$  is torsion free.

Now we want to show that  $S$  is polynomial. Again, let  $B := \mathbb{Z}_p^\wedge[L]$  and  $C := \mathbb{Z}_p^\wedge[S]$ . By assumption we have  $B^W \cong \mathbb{Z}_p^\wedge[f_1, \dots, f_r, g_1, \dots, g_s]$  for suitable elements  $f_i$  and  $g_j$ , where  $\deg(f_i) = 1$  and  $\deg(g_j) > 1$ . The number of generators are determined by the dimension of the lattices. That is that  $s + r = \dim(L)$ . Since  $S^\sharp$  is fixed-point free, the elements  $f_i$  are images of elements of  $Z^\sharp$  and  $r \leq \dim(Z)$ . Since  $Z^\sharp \subset L^{\sharp W}$  we have  $r \geq \dim(Z)$  and therefore  $r = \dim(Z)$  and  $s = \dim(S)$ . Now let  $g'_j \in C^W$  be the images of  $g_j \in B^W$ . Then



$\prod_j \deg(g'_j) = \prod_j \deg(g_j) \prod_i \deg(f_i) = |W|$  (Corollary 2.2). Moreover, the map  $B/(f_i, g_j) \rightarrow C/(g'_j)$  is an epimorphism and therefore, the target is also finitely generated. Applying Corollary 2.2 again yields  $C^W \cong \mathbb{Z}_p^\wedge[g_j]$ . In particular, the lattice  $S$  is polynomial. This finishes the proof of the first equivalence of statements. The second follows from the lemma below.  $\square$

**3.1 Lemma.** *Let  $S \oplus Z \rightarrow L \rightarrow K$  be a  $W$ -trivial extension, such that  $W$  acts trivially on  $Z$  and such that  $S$  is simply connected. Then  $L_W$  is torsion free if and only if the composition  $K \rightarrow S_\infty \oplus Z_\infty \rightarrow Z_\infty$  is injective.*

*Proof.* Since  $S$  is simply connected and since  $H_1(W, K) = 0$  ( $p$  is odd), passing to covariants establishes a short exact sequence  $Z \rightarrow L_W \rightarrow K$ . If  $L_W$  is torsion free, this is a  $W$ -trivial extension and establishes the short exact sequence  $K \rightarrow Z_\infty \rightarrow (L_W)_\infty$ . In particular,  $K \rightarrow Z_\infty$  is a monomorphism.

If  $K \rightarrow Z_\infty$  is a monomorphism the quotient  $Z'_\infty := Z_\infty/K$  defines a lattice  $Z' := \text{Hom}(\mathbb{Z}/p^\infty, Z'_\infty)$  with trivial  $W$ -action and fits into the diagram

$$\begin{array}{ccccc}
 & & S_\infty & \xlongequal{\quad} & S_\infty \\
 & & \downarrow & & \downarrow \\
 K & \longrightarrow & S_\infty \oplus Z_\infty & \longrightarrow & L_\infty \\
 \parallel & & \downarrow & & \downarrow \\
 K & \longrightarrow & Z_\infty & \longrightarrow & Z'_\infty .
 \end{array}$$

Switching to  $W$ -lattices, i.e. applying the functor  $\text{Hom}(\mathbb{Z}/p^\infty, \ )$  shows that  $L_W \cong Z'$  is torsion free.  $\square$

#### 4. $W$ -lattices of irreducible pseudo reflection groups.

We start with the proof of the first half of Theorem 1.4. That is that, for a simply connected  $W$ -lattice  $L$  of an irreducible pseudo reflection group  $W \rightarrow \text{Gl}(n; \mathbb{Q}_p^\wedge)$ , the ring  $\mathbb{Z}_p^\wedge[L]^W$  of invariants is a polynomial algebra for all but the exceptional cases listed in Theorem 1.4. The exceptional cases are discussed in the next section.

*Proof of the first half of Theorem 1.4.* Let  $W \rightarrow \text{Gl}(n; \mathbb{Q}_p^\wedge)$  be an irreducible pseudo reflection group over the  $p$ -adic rationals, and let  $(W, p)$  be a pair which is not listed in the statement of Theorem 1.4.

If  $(p, |W|) = 1$ , an application of Corollary 2.7 shows that every  $W$ -lattice  $L \subset U$  is polynomial. Moreover, we have  $H^1(W; L) = 0 = H_1(W; L_\infty)$ . Since  $L$  has no fixed-points, the vanishing of the cohomology and the homology group implies that the  $W$ -lattice  $L$  is simply connected and centerfree [13; Lemma 3.1 and Lemma 2.1].

Checking the complete list of irreducible pseudo reflection groups over  $\mathbb{Q}_p^\wedge$  given by Clark and Ewing [7], there are only a few cases left. For all these cases, there exists a lattice  $L \subset U$  and a connected compact Lie group  $G$  such that the composition  $W_G \subset W \rightarrow \text{Gl}(L)$  makes  $L$  to  $W_G$ -lattice isomorphic to  $L_G := H_1(T_G; \mathbb{Z}_p^\wedge)$  and such that  $L_G$  is polynomial. The particular information about the Lie groups, the subgroups and the primes are given in the following table. The numbering refers to the Clark-Ewing list [7].

No	$G$	$W_1$	$W$	prime
1	$SU(n)$	$\Sigma_n$	$\Sigma_n$	$p \leq n$
2a	$U(n)$	$\Sigma_n$	$W \subset \mathbb{Z}/r \wr \Sigma_n$	$r (p-1), p \leq n$
2b	$SU(3)$	$\Sigma_3$	$D_{12}, D_6$	$p = 3$
12	$SU(3)$	$\Sigma_3$	$Gl(2; \mathbb{F}_3)$	$p = 3$
29	$SU(5)$	$\Sigma_5$	$W$	$p = 5$
31	$SU(5)$	$\Sigma_5$	$W$	$p = 5$
34	$SU(7)$	$\Sigma_7$	$W$	$p = 7$

For all other cases, the pair  $(W, p)$  belongs to the excluded list of Theorem 1.4 or  $(p, |W|) = 1$ . The information given in this table is obvious for No. 1 and for No. 2b for  $W = D_6 \cong \Sigma_3$ . It may be found in [14] for No. 2a, in [13] for No. 2b for  $W = D_{12}$  and in [2] for all other cases. For all cases besides No. 2a,  $L_G$  is a simply connected  $W_1 = W_G$ -lattice and therefore,  $L$  is simply connected as  $W$ -lattice. For No. 2a, we have  $(L^\sharp/p)^W = 0$  [14; Proposition 1.4]. Hence, by [13; Lemma 2.2] the  $W$ -lattice  $L^\sharp$  is centerfree and by [13; Proposition 5.1] the  $W$ -lattice  $L$  simply connected.

Since  $L_G$  is a polynomial  $W_1 = W_G$ -lattice in all cases, we can apply Corollary 2.7. to show that  $L$  is a polynomial  $W$ -lattice. Since the simply connected lattice of  $U$  is unique up to weak isomorphism (Theorem 1.2) this finishes the proof of one half of Theorem 1.4.

□

From the mentioned references [2], [13] and [14], for the lattices of the above table, one can get a little extra piece of information about the subgroup  $W_1 \subset W$ . Actually, if  $\Delta \cong \mathbb{Z}/p \subset L/p$  describes the diagonal of  $L/p$ , then  $W_1 \cong Iso_W(\Delta)$  is given as the isotropy subgroup of  $\Delta$ .

For  $(p, |W|) = 1$ , we define  $\Delta := L/p$ . Then,  $W_1 := Iso_W(\Delta)$  is the trivial group ( $p$  is odd). Taking the torus as  $G$  we get the same picture as described in the above table. This shows that the following proposition is true at least for simply connected irreducible  $W$ -lattices.

**4.1 Proposition.** *Let  $p$  be an odd prime. Let  $L$  be a polynomial  $W$ -lattices. Then, there exists a compact connected Lie group  $G$  and a subspace  $\Delta \subset L/p$  such that the following holds:*

- i)  $W_G \subset W \rightarrow Gl(L)$  is isomorphic to the  $W_G$ -lattice  $L_G$ .*
- ii)  $L$  is a polynomial  $W_G$ -lattice and  $\mathbb{Z}_p^\wedge[L]^{W_G} \cong H^*(BG; \mathbb{Z}_p^\wedge)$ .*
- iii)  $W_G = Iso_W(\Delta)$  and the index  $[W : W_G]$  is coprime to  $p$ .*

*Proof.* The statement is true for simply connected  $W$ -lattices, since these split into a product of irreducible simply connected lattices.

Every polynomial lattice  $L$  fits into an exact sequence  $SL \times Z \rightarrow L \rightarrow K$ , where  $W$  acts trivially on  $Z$  and  $K$ , such that  $SL$  is simply connected and polynomial and

such that the associated homomorphism  $K \rightarrow Z_\infty$  is a monomorphism (Theorem 1.3). Now, we choose  $\Delta \subset SL/p$  and a connected compact Lie group  $G$  which satisfy the statement for  $SL$ . The inclusion  $SL_\infty \subset T_G$  is  $W_G$ -equivariant, and  $K \subset T_G^{W_G}$ . By the lemma below, the index of the center  $Z(G) \subset T_G^{W_G}$  is a power of 2. Hence,  $K \subset T_G \subset G$  is a central subgroup.

We also can realize  $Z$  as the  $p$ -adic lattice of an integral torus  $T$  with trivial  $W_G$ -action. Let  $H := (G \times T)/K$ . Then,  $W_H \cong W_G \subset W$  has index coprime to  $p$  and, by construction, the composition  $W_H \subset W \rightarrow Gl(L)$  is isomorphic to the  $W_H$ -lattice  $L_H$ . Moreover,  $\Delta \subset L/p$  and  $W_H \cong Iso_W(\Delta)$ . This shows that the statement is true for general polynomial  $W$ -lattices.  $\square$

The following lemma is known, but we couldn't find a reference for it stating the result in terms of compact Lie groups.

**4.2 Lemma.** *For any connected compact Lie group  $G$ , the index  $[T_G^{W_G} : Z(G)]$  is a power of 2.*

*Proof.* We only have to show that, for odd primes every  $p$ -toral subgroup  $P \subset T_G^{W_G}$  is already contained in  $Z(G)$ . Let  $C \subset G$  be the centralizer of  $P$ . Then,  $C$  is of maximal rank, the Weyl groups  $W_C = W_G$  are identical and  $\pi_0(C)$  is a finite  $p$ -group. The first two claims are obvious and the third follows from [11; A. 4]. Since the normalizer  $N_C(T_G)$  of  $T_G$  taken in  $C$  maps onto  $\pi_0(C)$  and since  $W_G$  is generated by elements of order 2, the group  $C$  is connected. Because  $C$  and  $G$  have identical Weyl groups, they are isomorphic. This shows that  $P \subset Z(G)$ .  $\square$

**4.3 Remark.** The only simply connected  $W$ -lattices, which are not polynomial, come from the Weyl group action of some exceptional connected compact Lie groups. Hence, for odd primes, the first and the third part of Proposition 4.1 are true for general  $W$ -lattices of a given pseudo reflection group  $W$ .

Similar arguments as in the proof of Proposition 4.1 allow to show the following statement.

**4.4 Proposition.** *Let  $p$  be an odd prime. Let  $W \rightarrow GL(U)$  be a  $p$ -adic rational pseudo reflection group of Lie type. Let  $L \subset U$  be a  $W$ -lattice. Then, there exists a compact connected Lie group  $G$  such that the  $W$ -lattice  $L$  and the  $W_G$ -lattice  $L_G$  are weakly isomorphic.*

For the proof of Proposition 4.4, we need one more lemma. For every connected compact Lie group  $G$  there exists a finite covering  $K \rightarrow H \times T \rightarrow G$  where  $H$  is a simply connected compact Lie group,  $T$  a torus and  $K$  a finite central subgroup of  $H \times T$ . We have associated short exact sequences  $K \rightarrow T_H \times T \rightarrow T_G$  for the maximal tori  $K_p \rightarrow (L_H)_\infty \oplus (L_T)_\infty \rightarrow (L_G)_\infty$  for the  $p$ -discrete tori and  $L_H \oplus L_T \rightarrow L_G \rightarrow K_p$  for the the associated lattices. If we identify  $W_H$  with  $W_G$  then the last sequence is a sequence of  $W_G$  modules. Here,  $K_p$  denoted the  $p$ -torsion of  $K$ . Moreover, both  $W_G$  lattices are rationally weakly isomorphic. The long exact sequence of the homotopy groups for the fibration  $K \rightarrow H \times T \rightarrow G$  shows that  $\pi_1(G) \cong K$ .

**4.5 Lemma.** *Let  $p$  be an odd prime. Let  $G$  be a connected compact Lie group. Then, the fundamental group  $\pi_1(G)$  has no  $p$ -torsion if and only if  $L_G$  is a simply connected  $W_G$ -lattice.*

As the example of  $SU(2)$  shows this statement is not true for the prime 2.

*Proof.* The inclusion  $T_G \rightarrow G$  of the maximal torus induces an epimorphism  $\pi_1(T_G) \rightarrow \pi_1(G)$  [4] between the fundamental groups. This homomorphism factors over the covariants and, tensoring with  $\mathbb{Z}_p^\wedge$ , establishes an epimorphism  $(L_G)_{W_G} \rightarrow \pi_1(G) \otimes \mathbb{Z}_p^\wedge$ . Hence, if  $L_G$  is simply connected, the fundamental group  $\pi_1(G)$  is  $p$ -torsion free.

Now let us assume that  $\pi_1(G)$  has no  $p$ -torsion. Let  $K \rightarrow H \times T \rightarrow G$  be the finite covering described above. Then, the abelian group  $K$  has no  $p$ -torsion and the lattices  $L_G$  and  $L_H \oplus L_T$  are isomorphic as  $W_G$ -modules. Since every simply connected compact Lie group splits into a product of simple simply connected compact Lie groups, we have therefore to prove the assertion only for simple simply connected compact Lie groups.

Since  $Spin(n) \rightarrow SO(n)$  is a 2-fold covering, both have isomorphic lattices for odd primes as well as  $Sp(n)$  and  $SO(2n+1)$ . For  $SU(n)$  and  $SO(n)$  the description of the Weyl group on the lattice may be found in [4]. A straight forward and easy calculation proves the claim for these cases.

For the exceptional Lie groups we can find the following subgroups of maximal rank:  $SU(3) \subset G_2$ ,  $Spin(9) \subset F_4$ ,  $SU(2) \times_{\mathbb{Z}/2} SU(6) \subset E_6$ ,  $SU(8)/(\mathbb{Z}/4) \subset E_7$  and  $SU(2) \times \mathbb{Z}/2E_7 \subset E_8$ . In all cases, for odd primes, the lattices are already simply connected considered as lattice with respect to the Weyl group of the subgroup of maximal rank. This finishes the proof of the statement.  $\square$

*Proof of Proposition 4.4.* Let  $W \rightarrow Gl(U)$  be a  $p$ -adic rational pseudo reflection group, weakly isomorphic to  $W_{H'} \rightarrow Gl(L_{H'} \otimes \mathbb{Q})$  for a suitable connected compact Lie group  $H'$ . By the above considerations, needed for the proof of Lemma 4.5, we can assume that  $H' = H \times T$  is a product of a simply connected compact Lie group  $H$  and a torus  $T$ . Since  $p$  is odd, the  $W_H$ -lattice  $L_H \subset U/U^W$  is simply connected. Therefore, by Theorem 1.2, there exists a  $W$ -trivial extension  $L_H \oplus L_T \rightarrow L \rightarrow K$ . Now we can proceed as in the proof of Proposition 4.1. The composition

$$K \rightarrow (L_H)_\infty \oplus (L_T)_\infty \rightarrow T_H \times T$$

makes  $K$  to a central subgroup of  $H \times T$  and the quotient  $G := H \times T/K$  establishes a pseudo reflection group weakly isomorphic to  $W \rightarrow Gl(L)$ . This proves the statement.  $\square$

We finish this section with the proof of Theorem 1.5 and of Proposition 1.6.

*Proof of proposition 1.6.* Let  $W \rightarrow Gl(U)$  be an irreducible  $p$ -adic rational pseudo reflection group. In [3] is analyzed, for which of these representations there exists, up to weak isomorphism, a unique  $W$ -lattice. It turns out that, if this is not true, then  $p = 2$  or  $W$  is of Lie type. Hence, for all pseudo reflection groups under consideration, there exists a unique  $W$ -lattice  $L \subset U$ . Since, for odd primes, every  $p$ -adic rational pseudo reflection group contains a simply connected lattice (Theorem 1.2) as well as a centerfree lattice [13], the lattice  $L \subset U$  is simply connected and centerfree. Moreover, because the excluded cases in Theorem 1.4 are of Lie type, the  $W$ -lattice  $L$  is also polynomial. This finishes the proof.  $\square$

*Proof of Theorem 1.5.*

Let  $W \rightarrow Gl(U)$  be a  $p$ -adic rational pseudo reflection group. Let  $L \subset U$  be a  $W$ -lattice. Then, by Theorem 1.2, there exists a finite number of pseudo reflection groups  $W_i \rightarrow Gl(S_i)$  such that  $S_i$  is a simply connected  $W_i$ -lattice, such that the

two  $p$ -adic pseudo reflection groups  $W \rightarrow Gl(SL)$  and  $\prod_i W_i \rightarrow Gl(\bigoplus_i S_i)$  are weakly isomorphic and such that there exists a  $W$  trivial extension  $\bigoplus_i S_i \oplus Z \rightarrow L \rightarrow K$ . We split  $W \cong W' \times W''$  and  $SL \cong S' \oplus S''$  into two factors, the first contains all factors which are rationally of Lie type and the second all of non Lie type. Passing to  $p$ -adic discrete tori, the above  $W$ -trivial extension transforms into the short exact sequence

$$K \rightarrow S'_\infty \oplus Z_\infty \oplus S''_\infty \rightarrow L_\infty$$

of  $W$ -modules. By Proposition 1.6, the  $W''$ -lattice  $S''$  is centerfree. That is that  $(S''_\infty)^W = 0$ , and therefore, the composition  $K \rightarrow S'_\infty \oplus Z_\infty \oplus S''_\infty \rightarrow S''_\infty$  is trivial and  $L_\infty \cong (S'_\infty \oplus Z_\infty)/K \oplus S''_\infty$ . The first summand  $L' := (S'_\infty \oplus Z_\infty)/K$  gives a  $W'$ -lattice which is rationally of Lie type. By Lemma 4.4, there exists a connected compact Lie group  $G$  such that the  $W'$ -lattice  $L'$  is weakly isomorphic to the  $W_G$ -lattice  $L_G$ . This finishes the proof of Theorem 1.5.  $\square$

## 5. Proof of the second half of Theorem 1.4.

In this section we prove that for the pairs  $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 3)$ , and  $(E_8, 5)$  the ring  $\mathbb{Z}_p^\wedge[L_G/p]^{W_G}$  is not polynomial. For abbreviation we denote the set of listed pairs by  $\mathcal{E}$ .

**5.1 Remark.** Let  $V$  be a finite dimensional vector space with a fixed basis over  $\mathbb{F}_p$  and let  $V^\sharp := Hom_{\mathbb{F}_p}(V, \mathbb{F}_p) \cong \mathbb{F}_p[V]^1$  denote the dual respectively the linear forms on  $V$ . The Frobenius induces a linear map  $\phi : V^\sharp \rightarrow (\mathbb{F}_p[V])^p$  from the linear forms to the polynomial of degree  $p$ .

A symmetric quadratic  $n \times n$  matrix  $A$  defines a quadratic form  $q \in \mathbb{F}_p[V]^2$  by  $q := xAx^{tr}$ . Here,  $x \in V^\sharp$  is interpreted as a row. This correspondence is one to one. Using the Frobenius, for every quadratic form  $q$ , we can define a polynomial  $B(q) := xA\phi(x)^{tr} \in \mathbb{F}_p[V]^{p+1}$ . By construction, for every map  $f : V \rightarrow V$ , we have  $f^*(B(q)) = B(f^*(q))$ . In particular, if  $q$  is an invariant quadratic form with respect to an action of a group on  $V$ , then  $B(q)$  is an invariant polynomial, too.

**5.2 Remark.** If  $G$  is a simple connected compact Lie group, the representation  $W_G \rightarrow Gl(I)$  is fixed-point free and fixes an integral quadratic form  $q \in \mathbb{Z}[I]$ , where  $I := \pi_1(T_G)$ . Moreover, if  $q$  is not divisible by an integer, it generates the invariants  $(\mathbb{Z}[I]^2)^{W_G} \cong \mathbb{Z}$  of degree 2. By  $q'_G$ , we denote one of the two generators, by  $q_G$  the image in  $\mathbb{Z}_p^\wedge[L_G]^{W_G}$  and by  $\bar{q}_G$  the image in  $\mathbb{F}_p[L_G]^{W_G}$ .

The following lemma is proved in [10; Theorem 1.2 and Theorem 3.4].

**5.3 Lemma.** *Let  $(G, p)$  be one of the pairs contained in  $\mathcal{E}$ . Then the two polynomials  $\bar{q}_G, B(\bar{q}_G) \in \mathbb{F}_p[L_G]$  are algebraically independent. In particular,  $(\bar{q}_G)^{(p+1)/2}$  and  $B(\bar{q}_G)$  are linearly independent.*

Now, we can complete the proof of Theorem 1.4.

*Proof of the second half of Theorem 1.4.* Let  $(G, p) \in \mathcal{E}$ . And let us assume that  $L_G$  is a  $W_G$ -polynomial lattice. Then, by Lemma 2.3,  $\mathbb{Z}_p^\wedge[L_G]^{W_G} \rightarrow \mathbb{F}_p[L_G/p]^{W_G}$  is an epimorphism onto a polynomial algebra. In particular, polynomial generators of  $\mathbb{F}_p[L_G/p]^{W_G}$  have the same degrees as generators of  $\mathbb{Q}_p^\wedge[L_G \otimes \mathbb{Q}]^{W_G}$ . These degrees, described in [7], are given in the following table:

$G$	degree
$F_4$	2, 6, 8, 12
$E_6$	2, 5, , 6, 8, 9, 12
$E_7$	2, 6, 8, 10, 12, 14, 18
$E_8$	2, 8, 12, 14, 18, 20, 24, 30

This table shows, that, in all cases, the vector space of invariants of degree  $p + 1$  is 1-dimensional. On the other hand, by Lemma 5.3, the invariants  $B(\bar{q}_G)$  and  $(\bar{q}_G)^{(p+1)/2}$  are linearly independent. This gives a contradiction and proves the second half of Theorem 1.4.  $\square$

**Remark.** Actually, for  $(G, p) = (F_4, 3)$ ,  $(E_7, 3)$ ,  $(E_8, 3)$  and  $(E_8, 5)$ , the mod- $p$  invariants  $\mathbb{F}_p[L_G]^{W_G}$  are already shown to be a non polynomial algebra [12; Theorem 7.2]. Together with Lemma 2.3 (ii) this proves Theorem 1.4 in the above cases. But, since the case  $(E_6, 3)$  is missing we had to find another argument, which happens to work in all cases under consideration.

We can draw the following corollary:

**5.4 Corollary.** *For  $G = F_4, E_6, E_7$ , or  $E_8$ , the integral ring  $H^*(BT_G; \mathbb{Z})^{W_G}$  of polynomial invariants is not a polynomial algebra.*

*Proof.* Otherwise, for  $p = 3$ , the  $p$ -adic integral ring of invariants  $\mathbb{Z}_p^\wedge[L_G]^{W_G}$  were a polynomial algebra.  $\square$

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**ERRATA TO:  
FOR WHICH PSEUDO REFLECTION GROUPS  
ARE THE POLYNOMIAL INVARIANTS  
AGAIN A POLYNOMIAL ALGEBRA ?**

D. NOTBOHM

Unfortunately, it turned out that my above mentioned paper (for the reference see below) contains a closable gap. The proof of the first half of Theorem 1.4 in Section 4 is not complete. First of all in the table given in the proof some cases are missing. Beside the mentioned cases of irreducible pseudo reflection groups over the  $p$ -adic rationals there are three more for which  $p$  divides the order of the group and which are not excluded by the assumptions, namely

No	$G$	$W_1$	$W$	prime
35	$SU(2) \times SU(6)$	$\Sigma_2 \times \Sigma_7$	$W_{E_6}$	$p = 5,$
36	$SU(7)$	$\Sigma_8$	$W_{E_7}$	$p = 5, 7$
37	$SU(8)$	$\Sigma_9$	$W_{E_8}$	$p = 7$

For all cases the information may be found in [11]. For No. 36 and 37 the information is also contained in [2]. We use the same notation as in the original manuscript. The numbering refers to the Clark-Ewing list [7]. For quotations we refer to the list of references in the original manuscript. All three cases come from exceptional simply connected compact Lie groups ( $E_6$ ,  $E_7$  and  $E_8$ ). and the associated lattices are therefore simply connected (they are already simply connected as  $W_G$ -lattice).

Now we want to apply Corollary 2.6 to conclude that the lattices are polynomial as  $W$ -lattices. But this corollary is based on Proposition 2.5 whose proof has a gap. A more detailed analysis shows that we cannot argue analogously as in the proof of the Shephard-Todd theorem as given in [17; 7.4.1]. But the argument given in the proof of Corollary 2.6 is sufficient to prove a weaker version:

**Lemma.** *Let  $L$  be a  $p$ -adic lattice and  $W \rightarrow Gl(L)$  a pseudo reflection group. Let  $W_1 \subset W$  be a subgroup of index coprime to  $p$  such that  $L$  is polynomial as  $W_1$  lattice. Then, if  $\mathbb{F}_p[L/p]^W$  is a polynomial algebra,  $L$  is polynomial as  $W$ -lattice.*

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To complete the proof of Theorem 1.4 we only have to show that in all cases given in the tables the ring of invariants  $\mathbb{F}_p[L/p]^W$  is always a polynomial algebra. This makes the above lemma applicable. For the cases No. 1 and 2b this is obvious, for No. 12 we get the Dickson algebra and for No. 2a this may be found in [14] and in all other cases in [12].

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