P–ADIC LATTICES OF PSEUDO REFLECTION GROUPS

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Abstract. Let U be a vector space over the p–adic rationals, and let $W \longrightarrow Gl(U)$ be faithful representation of a finite group such that W is generated by pseudo reflections. For odd primes we study the p–adic W–lattices of this representation and achieve a complete classification. Examples of such situations are given by the Weyl group acting on the 1-dimesional homology of the maximal torus of a connected compact Lie group, or of the so called p–compact groups, a homotopy theoretic generalisation of compact Lie groups. The associated lattices are an important algebraic invariant in the study of these geometric object.

Introduction.

Let U be a finite dimensional vector space over the p–adic rationals $\mathbb{Q}_p^\wedge$. An elenment $1 \neq \sigma \in Gl(U)$ called a *pseudo reflection* if σ has finite order and if the kernel of $\sigma - id_U$ has codimension 1. The element σ is called a *honest reflection* or a *reflection* if σ has order 2. Because we are working in characteristic 0, the order of σ divides $p-1$ and the linear transformation $\rho(\sigma)$ is diagonalzable. i.e. U has a basis of eigenvectors with respect to σ.

A *pseudo reflection group* is a couple $W = (W, \rho)$, where W is a finite group and $\rho : W \longrightarrow Gl(U)$ a faithful representation such that the image $\rho(W)$ is generated by pseudo reflections. To emphasise the representation we also denote the pseudo reflection group by $W \longrightarrow Gl(U)$.

A *W–lattice* of $W \longrightarrow Gl(U)$ is a lattice $L \subset U$ of U of maximal rank fixed under the action of W, i.e. L is a $\mathbb{Z}_p^\wedge[W]$-module and $L \otimes \mathbb{Q} \cong U$ as vector spaces.

In this work we are concerned with the classification of all p–adic W–lattices $L \subset U$ of a given finite pseudo reflection group $W \longrightarrow Gl(U)$. For odd primes we will achieve a complete clasification. Our motivation to study this question comes from homotopy theory. The Weyl group W_G of a connected compact Lie group G acting on the tangent space of the maximal torus T of G or on the 1-dimensional homology $H_1(T; \mathbb{Z})$ provides an example of a honest reflection group and also of an integral lattice. This action is an important algebraic invariant in the study of connected compact Lie groups. In [5], Dwyer and Wilkerson gave the notion of p–compact groups, which is the homotopy theoretic generalisation of the notion of compact Lie groups. In their work, pseudo reflection groups occured in the same manner as honest reflection groups for connected compact Lie groups, namely as Weyl groups acting on a 'maximal torus'. These p–compact groups provide examples of pseudo reflection groups and associated p–adic lattices. Besides these geometric

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and homotopy theoretic aspects we believe that the study of p–adic W–lattices has interest from it's own.

We will use the following notation and definitions in this paper.

1.1 Notation, Definitions and Remarks

Let $W \to GL(U)$ be a pseudo reflection group.

1.1.1: The representation $W \longrightarrow Gl(U)$ is called *fixed–point free* if the fixed–point set $U^W = 0$ is trivial. A lattice $L \subset U$ is called *fixed–point free* if $L^W = 0$.

1.1.2 The pseudo reflection group $W \longrightarrow Gl(U)$ is called irreducible if the representation is irreducible. By [4], this is equivalent to the fact that the associated complex representation $U \otimes_{\mathbb{Q}_p^\wedge} \mathbb{C}$ is irreducible.

1.1.3: For every W–lattice L, we have a short exact sequence

$$0 \longrightarrow L \longrightarrow L_{\mathbb{Q}} \longrightarrow L_{\mathbb{Q}}/L =: T_{L,\infty} \longrightarrow 0$$

of W–modules. The quotient $T_{L,\infty} \cong (\mathbb{Z}/p^infty)^n \subset (S^1)^n$ is called a p–discrete torus and can be considered as a subgroup of a torus whose dimension equals the rank of L. Completing the classifying spaces of $T_{L,\infty}$ and passing to 2-dimensional homology establishes an isomorphism $H_2((BT_{L,\infty})^\wedge_p; \mathbb{Z}^\wedge_p) \cong L$ of W–modules.

1.1.4 For a W–lattice $L \subset U$, the fixed–point set $Z(L) := (T_{L,\infty})^W$ is called the *center* of L. There is an associated W–lattice PL given by the kernel of the composition

$$L \otimes \mathbb{Q} \longrightarrow T_{L,\infty} \longrightarrow T_{L,\infty}/Z(L) = T_{PL,\infty} \ .$$

and a W–equivariant map $L \longrightarrow PL$.

The lattice L is called *centerfree* if $Z(L) = 0$.

1.1.5 For a W–lattice $L \subset U$, the *covariants* L_W are given by the quotient L/SL, where $SL \subset L$ is the lattice generated by all elements of the form $l - w(l)$ with $l \in L$ and $w \in W$. The lattice L is called *simply connected* if $L_W = 0$.

1.1.6 A monomorphism $L \longrightarrow M$ of W–lattices is called a *W–trivial restriction* or *W–trivial extension* if W acts trivially on the quotient M/L and if M/L is finite.

1.1.7 Two W–lattices $L_1, L_2 \subset U$ are called isomorphic, if L_1 and L_2 are isomorphic as $\mathbb{Z}^\wedge_p[W]$–modules. That is that the two associated integral representations $\rho_1, \rho_2 : W \longrightarrow GL((\mathbb{Z}^\wedge_p)^n)$ are conjugated.

Most of these notions are motivated by an analogy to connected compact Lie groups. Let W_G denote the Weyl group and T_G the maximal torus of a connected compact Lie group G. Then, the action of W_G on $U_G := H_2(BT_G; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$ represents W as a finite reflection group, and $L_G := H_2(BT_G; \mathbb{Z}^\wedge_p) \subset U_G$ is a p–adic lattice. For odd primes, the center $Z(L_G)$ of L_G is a p–discrete approximation of the center of G [9]. Because the fundamental group $\pi_1(G)$ of G is isomorphic to the quotient of $\pi_1(T)$ by the translations of the extended Weyl group, one can also show that, for odd primes, $(L_G)_{W_G}$ is a p–discrete approximation of $\pi_1(G)$, i.e. $(L_G)_{W_G} \otimes \mathbb{Z}^\wedge_p \cong (L_G \otimes \mathbb{Z}^\wedge_p)_{W_G} \cong \pi_1(G) \otimes \mathbb{Z}^\wedge_p$.

By D_{2n} we denote the dihedral group of order $2n$.

1.2 Theorem. *Let p be an odd prime. Let $W \longrightarrow Gl(U)$ be a finite fixed–point free pseudo reflection group, and let $p \neq 3$ or $W \neq D_{12}$. Then the following holds:*

(1) *There exists a centerfree lattice $P \subset U$, unique up to isomorphism.*
(2) *There exists a simply connected lattice $S \subset U$, unique up to isomorphism.*

1.3 Theorem. *Let $p = 3$, $W = D_{12}$ and $W \to GL(U)$ be the irreducible representation of W as pseudo reflection group. Then, up to isomorphism, there exist two lattices $L_1, L_2 \subset U$, which are both centerfree and simply connected.*

A detailed description of these two lattices is given in Section 4. It also turns out that, for the two associated two dimensional integral representationa $\rho_1, \rho_2 : D_{12} \to GL(\mathbb{Z}_p^{\wedge 2})$, there exists an automorphism $\alpha : D_{12} \to D_{12}$ such that ρ_1 and $\rho_2\alpha$ are conjugate.

Next we describe centerfree and simply connected lattices of fixed–point free pseudo reflection groups. For a finite pseudo reflection group $W \to Gl(U)$, there exist splittings $U \cong U^W \oplus U_1 \oplus ... \oplus U_n$ of U as W–modules and $W \cong W_1 \times ... \times W_n$ of W such that W_i acts on U_i as an irreducible pseudo reflection group and trivially on every U_j for $j \neq i$ (e.g. see [10] or [4]).

1.4 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be a finite fixed–point free pseudo reflection group, and let $W \cong \prod_i W_i$ and $U \cong \bigoplus_i U_i$ be the associated splittings into irreducible pseudo reflection groups. Then the following holds:*

(1) *Every centerfree lattice $P \subset U$ splits into a direct sum $P \cong \bigoplus_i P_i$ of centerfree lattices $P_i \subset U_i$.*
(2) *Every simply connected lattice $S \subset U$ splits into a direct sum $S \cong \bigoplus_i S_i$ of simply connected lattices $S_i \subset U_i$.*

The last three theorems show that centerfree and simply connected lattices are "almost unique up to isomorphism". The only indeterminancy can come from the two different lattices of D_{12}.

1.5 Theorem. *Let $W \to GL(U)$ be a finite fixed–point free pseudo reflection group. Then the following holds:*

(1) *A lattice $P \subset U$ is centerfree if and only if every W–trivial restriction $L \to P$ factors over $L \to PL \to P$.*
(2) *A lattice $S \subset U$ is simply connected if and only if every W-trivial restriction $S \to L$ factors over $S \to SL \to L$.*

Finally we consider the general case of a pseudo reflection group $W \to Gl(U)$. Let $U \cong U^W \oplus U'$ be the splitting into the fixed–point set and the fixed–point free factor U'. Because W acts trivially on U^W, all lattices of U^W are isomorphic as W–modules. We choose a lattice of U^W and denote it by Z.

1.6 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be a pseudo reflection group, and let L be a W–lattice. Then the following holds:*

(1) *There exist a simply connected W–lattice S and a W–trivial restriction $Z \oplus S \to L$ with quotient $L/(Z \oplus S) \cong (L/L^W)_W$.*
(2) *There exist a centerfree W–lattice P and a W–trivial restriction $L \to Z \oplus P$.*

Lattices, which are centerfree and simply connected, do not allow non trivial W-restrictions. Also they only show up as direct summands as the next theorem shows.

1.7 Theorem. *Let $W \to Gl(U)$ be a finite pseudo reflection group. Let $L \to Z \oplus P$ be the W–trivial restriction of Theorem 1.6. Let $P \cong P_1 \oplus P_2$ be a splitting into* *centerfree lattices such that P_1 is also simply connected. Then, the lattice P_1 is a direct summand of L.*

The following corollary of all of the above results is obvious and classifies all lattices of finite pseudo reflection groups. This classification result is an analogue of the classification of connected compact Lie groups, where irreducible pseudo reflection groups play the same role as simple connected compact Lie groups and trivial representation the role of tori.

1.8 Corollary. *Let p be an odd prime. Then, every W–lattice is a W–trivial extension of a trivial W–lattice and a simply connected W–lattice, and every simply connected W–lattice is a direct sum of simply connected lattices of irreducible pseudo reflection groups.*

Our main theorems are statements about odd primes. This comes simply from the following lemma, which plays a little but important role in several proofs.

1.9 Lemma. *Let p be an odd prime, let W be a pseudo reflection group and let M be a $\mathbb{Z}_p^\wedge$–module with trivial W–action. Then, we have $H_1(W;M) = H^1(W;M) = 0$.*

Proof. As a pseudo reflection group for an odd prime, W is generated by elements of order coprime to p. By the Hurewicz theorem, the first homology group $H_1(W,\mathbb{Z})$ is isomorphic to the abelianization of W, which is a finite abelian group of order coprime to p. Universal coefficient theorems imply the statement. □

The paper is organized as follows: In Section 2, we discuss centerfree lattices and prove the first parts of Theorem 1.2, Theorem 1.4 and of Theorem 1.5. In Section 3 we study simply connected lattices and prove the second part of these theorems. Section 4 is devoted to the study of the dihedral group D_{12} at the prime 3. The last section contains an analysis of general finite pseudo reflection groups and the proof of Theorem 1.6 and Theorem 1.7.

Although the nature of this paper is mostly algebraic, sometimes we deal with completed topological spaces. Completion is always meant in the sense of Bousfield and Kan [2].

Independently of us, Dwyer and Wilkerson got also proofs for some of the main results of this work [6].

Finally a warning: We are only dealing with odd primes, i.e. p always denotes an odd prime.

2. Centerfree lattices.

As mentioned in the introduction $W \to Gl(U)$ is a finite pseudo reflection group.

2.1 Lemma.

(1) *For a W–lattice L, the center $Z(L)$ is a finite abelian p–group if and only if L is fixed–point free.*
(2) *If L is fixed–point free, then we have $Z(L) \cong H^1(W;L)$.*

Proof. Taking fixed–points in the short exact sequence

$$0 \to L \to L_{\mathbb{Q}} \to T_{L,\infty} \to 0$$

gives rise to an exact sequence

$$0 \to L^W \to L_{\mathbb{Q}}^W \to (T_{L,\infty})^W \to H^1(W;L) \to H^1(W;L_{\mathbb{Q}}) = 0\ .$$

The first two terms vanishes if L is fixed–point free. Otherwise the quotient $L_{\mathbb{Q}}^W/L^W$ is a p–discrete torus. In particular, the quotient is not finite. Because $H^1(W;L)$ is always finite, both parts follow. $\square$

For a W–lattice L we denote by $L/p := L\otimes_{\mathbb{Z}_p^\wedge}\mathbb{F}_p$ the associated $\mathbb{F}_p[W]$–module.

2.2 Lemma. *A W–lattice P is centerfree if and only if $(P/p)^W = 0$.*

Proof. The multiplication $\mu_p : P \to P$ by p establishes a short exact sequence

$$0 \to P \xrightarrow{\mu_p} P \to P/p \to 0\ .$$

Passing to fixed–points gives an exact sequence

$$0 \to P^W \xrightarrow{\mu_p} P^W \to (P/p)^W \to H^1(W;P) \xrightarrow{\mu_p} H^1(W;P)\ .$$

If $(P/p)^W = 0$ then $\mu_p : P^W \to P^W$ and $\mu_p : H^1(W;P) \xrightarrow{\mu_p} H^1(W;P)$ are isomorphisms ($H^1(W;P)$ is a finite group). From the first isomorphism follows that P is fixed–point free. Because $H^1(W;P)$ is a finite abelian p–group, the second isomorphism implies that $H^1(W;P) = 0$. The other direction follows from Lemma 2.1. $\square$

The next statement shows that, for every W–lattice L, the associated W–lattice PL is centerfree.

2.3 Proposition. *Let L be a fixed–point free W lattice.*

(1) *There exists an exact sequence*

$$0 \to L \to PL \to Z(L) \to 0\ ,$$

and hence, L is a W–trivial restriction of PL.

(2) *The lattice PL is centerfree and $H^1(W;PL) = 0$.*

Proof. By definition of the lattice PL, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L\otimes\mathbb{Q} & \longrightarrow & T_{L,\infty} & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & PL & \longrightarrow & PL\otimes\mathbb{Q} & \longrightarrow & T_{PL,\infty} & \longrightarrow & 0\ , \end{array}$$

where the middle arrow is an isomorphism, where the left arrow is a monomorphism and where the right arrow is a epimorphism. Hence, the cokernel of the left arrow and the kernel of the right arrow, given by $Z(L)$ are isomorphic. This establishes the desired exact sequence of part (1).

Taking fixed–points in the exact sequence of (1) gives rise to the exact sequence

$$0 = PL^W \to Z(L)^W = Z(L) \to H^1(W;L) \to H^1(W;PL) \to H^1(W;Z(L))\ .$$

By Lemma 2.1, the second arrow is an isomorphism. By Lemma 1.4, the last term vanishes. Hence, we have $H^1(W;PL) = 0$. Again by Lemma 2.1, the lattice PL is centerfree. $\square$

The following technical proposition is the key for the proof of Theorem 1.2.

2.4 Proposition. *Let $p \neq 3$ or $W \neq D_{12}$. Let $W \longrightarrow Gl(U)$ be an irreducible pseudo reflection group, and let P be a centerfree lattice. If there exists an exact sequence*

$$0 \longrightarrow V_0 \longrightarrow P/p \longrightarrow V_1 \longrightarrow 0$$

of $\mathbb{F}_p[W]$–modules, such that $V_1^W = 0$, then either $V_0 = 0$ or $V_1 = 0$.

Proof. Let assume that V_0 and V_1 are nontrivial vector spaces. We choose a basis for V_0 and extend it to a basis of P/p. Then every element $w \in W$ can be represented by a matrix of the form

$$\begin{pmatrix} A_w & C_w \\ 0 & B_w \end{pmatrix}$$

where A_w decribes the action of w on V_0, B_w the action on V_1 and $C_w : V_1 \longrightarrow V_0$ the twisting, i.e. the failure to be a direct product. This description establishes a homomorphism $\phi : W \longrightarrow Gl(V_0) \times Gl(V_1)$ given by $\phi(w) := (A_w, B_w)$. Let W_i be the image of W in the factor $Gl(V_i)$. That is we have a homomorphism $\phi : W \longrightarrow W_0 \times W_1$. Because V_0 and V_1 have no non trivial fixed–point, both groups W_0 and W_1 are non trivial. For V_0 this follows from Lemma 2.2.

The kernel K of ϕ consists of those elements which are desribed by a matrix of the form $\begin{pmatrix} id & C \\ 0 & id \end{pmatrix}$. Therefore, every element of the kernel has order p and the kernel is an elementary abelian p–group and a normal subgroup of W.

Now let $\sigma \in W$ be a p–adic pseudo reflection. The matrix

$$\sigma - id = \begin{pmatrix} A_\sigma - id & C_\sigma \\ 0 & B_\sigma - id \end{pmatrix}$$

has rank 1. That is that all columns and all rows are multiple of one column or one row. We have $A_\sigma - id \neq 0$ if and only if $B_\sigma = id$. The equivalence follows from the fact that the order of σ is coprime to p. Therefore, W_0 and W_1 are generated by p-adic reflections. Let (w_0, w_1) be an element of $W_0 \times W_1$. We can assume that w_0 is the image of a product of p–adic reflections which are mapped onto the identity in W_1, and similiar for w_1. This shows that ϕ is an epimorphism.

The above considerations show that W allows a short exact sequence

$$(*) \qquad 1 \longrightarrow K \longrightarrow W \longrightarrow W_0 \times W_1 \longrightarrow 1 \ .$$

where W_0 and W_1 are nontrivial groups, generated by elements coming from pseudo reflections in W, and where $K \subset W$ is an elementary abelian normal subgroup. For abbreviation, we say that W has the property $(*)$.

We want to show that either W_0 or W_1 is the trivial group. This would imply that either $V_0 = 0$ or $V_1 = 0$. The proof of this conclusion splits into two part, the nonmodular case, i.e. $(|W|, p) = 1$, and the modular case. For the modular case we use the classification of the irreducible pseudo reflection groups by Clark and Ewing [4]. We also use their numbering of the different cases.

First let $(|W|, p) = 1$. Then $K = 0$ and $W \cong W_0 \times W_1$ splits into a product of pseudo reflection groups. Because the representation $W \longrightarrow Gl(U)$ is irreducible, this implies that either W_0 or W_1 is the trivial group.

Now let p divide $|W|$. We are considering separately the case where K is central in W and the case in which it is not. If $K \subset W$ is a central subgroup, then every element of K establishes a W–equivariant automorphism $U \to U$. The W–representation U is irreducible if and only if $U \otimes_{\mathbb{Q}_p^\wedge} \mathbb{C}$ is irreducible [4]. Hence, the induced map $f \otimes \mathbb{C}$ is a multiple, in particular a p–adic multiple of the identity. That is to say that there exists a homomorphism $K \to \mathbb{Z}_p^{\wedge *} \cong \mathbb{Z}/p-1 \times \mathbb{Z}_p^\wedge$. Because W is finite and because W acts faithfully on U, this homomorphism is injective, and the kernel K is trivial. We can proceed as in the nonmodular case.

If $K \to W$ is not central, then there exists a pseudo reflection $\sigma \in W$ acting nontrivially on K. Because the order of σ is coprime to p, the representation K of the group $<\sigma>$, generated by σ, splits into 1-dimesional irreducible summands. Let $K' \subset K$ be one of the summands with a nontrivial action of σ and let $x \in K'$ be a generator. The subgroup $D := <\sigma, x\sigma x^{-1}> = <\sigma, x\sigma> = <\sigma, x>$ of W, generated by two pseudo reflections, fits into a short exact sequence

$$1 \to K' \to D \to <\sigma> \to 1\ .$$

The order $m = |\sigma|$ of σ is coprime to p. Therefore, the sequence splits and $D \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$ acts on U as a pseudo reflection group. As a $\mathbb{Z}/p$–module, $U \cong \bigoplus_i U_i$ splits into a direct sum of irreducible $\mathbb{Z}/p$–modules which are permuted by $\mathbb{Z}/m$. Each factor is either 1–dimensional with trivial $\mathbb{Z}/p$–action ($\mathbb{Q}_p^\wedge$ contains no p–th root of unity) or isomorphic to $U' \cong (\mathbb{Q}_p^\wedge)^{p-1}$ where we consider U' as the kernel of the map $(\mathbb{Q}_p^\wedge)^p \to \mathbb{Q}_p^\wedge$ given by summing up the coordinates and where $\mathbb{Z}/p$ acts via cyclic permutation on $(\mathbb{Q}_p^\wedge)^p$. The factors with trivial $\mathbb{Z}/p$–action does not lead to a faithful representation of D. Every factor isomorphic to U' is fixed under the action of $\mathbb{Z}/m$, and $\mathbb{Z}/m$ acts on U' via permutation associated to the action on $\mathbb{Z}/p$ considered as a set. Therefore, U' represents D as a pseudo reflection group if and only if $m = 2$. That is to say that $D \cong D_{2p}$ is a dihedral group. By the classification list of irreducible pseudo reflection groups [4] the only modular cases are given by D_6 and D_{12}. Hence, we have $p = 3$ and $D \cong D_6$.

By the above arguments it is only left to consider modular cases for $p = 3$. We will finish the proof by a case by case checking following the list of [4]. We only have to discuss the numbers 1 2a, 2b, 12, 28, 35, 36 and 37.

<u>Cases number 1 and 2a</u>. In this case $\Sigma_n \subset W \subset \mathbb{Z}/l \wr \Sigma_n$ where l divides $p-1$. In particular, the subgroup $K \subset W$ is a normal subgroup of Σ_n as well as of $A_n \subset \Sigma_n$. Here, A_n denotes the group of permutations of positive sign. For $n \geq 5$, the group A_n is simple. For $n = 4$, we have $A_4 \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/3$. Therefore, in these cases there exists no normal elementary abelian 3–subgroup, and we can proceed as in the nonmodular case.

Now let $n = 3$. If $W \neq \Sigma_3$, then there also exists no normal elementary abelian subgroup. If $W = \Sigma_3$, then the representation $W \to Gl(U)$ is desribed by the matrices $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$, which are reflections and generate Σ_3. For the obviuos associated lattice L, a straightforward calculation shows the existence of a short exact sequence $0 \to \mathbb{Z}/3 \to L/3 \to det/3 \to 0$ of W–modules. Here, W acts trivially on $\mathbb{Z}/3$ and det is the 1–dimensional representation given by the sign of the permutation. Every other lattice $L' \subset U$ has mod–3 the same composition factors. In particular, one of the modules V_0 or V_1 is isomorphic to the trivial representation and the other to $det/3$. This contradicts the fact that $V_0^{\Sigma_3} = 0 = V_1^{\Sigma_3}$.

For later purpose we note the following observation: For $n \geq 5$, the above argument shows that, if W has the property $(*)$, every pseudo reflection representation of W splits into two summands, where both factors carry a nontrivial W–action.

Case number 2b. In this case, we have $W = D_6$ or $W = D_{12}$ and U is 2–dimensional. The first case we already discussed and the second is excluded by the assumptions.

Case number 12. In this case we have $dim_{\mathbb{Q}_p^\wedge} U = 2$ and $W = Gl(2, \mathbb{F}_3)$. There exists a lattice $L \subset U$ such that the action of W on $L/3 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$ is isomorphic to the standard action, which gives an irreducible representation (details may be found in [1] or [10]). Hence, every other lattice $L' \subset U$ gives mod–p also an irreducible representation, which proves the statement in this case.

Case number 28. In this case, we have $W = W_{F_4} \cong ((\mathbb{Z}/2)^3 \rtimes \Sigma_4) \rtimes \Sigma_3$. The last isomorphism may be found in [7, p. 45]. A straight forward calculations shows that $K = 0$. We can proceed as in the nonmodular case.

Case number 35, 36, 37. In this case we have $W = W_{E_6}$, $W = W_{E_7}$ or $W = W_{E_8}$. We describe two maximal subgroups of maximal rank for each of these connected compact Lie groups.

G	H'	H''
E_6	$S^1 \times_{\mathbb{Z}/2} Spin(10)$	$SU(2) \times_{\mathbb{Z}/2} SU(6)$
E_7	$S^1 \times_{\mathbb{Z}/2} Spin(12)$	$S^1 \times_{\mathbb{Z}/3} E_6$
E_8	$SSpin(16)$	$SU(2) \times_{\mathbb{Z}/2} E_7$

A list of all maximal subgroups of maximal rank may be found in [8]. This establishes subgroups of W as follows:

W	W'	W''
W_{E_6}	$W_{H'} \cong (\mathbb{Z}/2)^5 \rtimes \Sigma_5$	$W_{SU(6)} \cong \Sigma_6$
W_{E_7}	$W_{H'} \cong (\mathbb{Z}/2)^5 \rtimes \Sigma_6$	W_{E_6}
W_{E_8}	$W_{H'} \cong (\mathbb{Z}/2)^7 \rtimes \Sigma_8$	W_{E_7}

In all cases, the two groups W' and W'' generate W. This follows because $H' \subset G$ is maximal of maximal rank. Moreover, the intersection $W' \cap W''$ is nonempty. We want to show that there exists no epimorphism $W \longrightarrow W_0 \times W_1$ as in $(*)$ with kernel given by an elementary abelian p–group.

Let us look at the case $W = W_{E_6}$. By the observation at the end of cases number 1 and 2a, if W' has the property $(*)$, the W'–module U splits into a direct sum of nontrivial W'–modules. The same is true for W''. But by the choice of the groups, both belong to case 2a with $n \geq 5$, we only can split of a trivial summand of U considered as a W' or W''–module. Therefore, W' as well as W'' have not the property $(*)$, and an epimorphism $W_{E_6} \longrightarrow W_0 \times W_1$ maps W' and W'' only into one factor. Because $W' \cap W''$ is nonempty, both are mapped into the same factor, let us say into W_0. Because W_{E_6} is generated by W' and W'', the group W_{E_6} is only mapped into W_0, too. Hence, W_1 is trivial. This proves the statement in this case. In particular, this argument also shows that there exists no epimorphism of the form $(*)$ with kernel given by an elementary abelian p–group.

For W_{E_7} and W_{E_8}, we can argue analogously using the result for W_{E_6} or W_{E_7}. This finishes the discusion of all possible cases and the proof of the statement. □

Remark. The last proposition as well as the proof originates in a discussion with C.Broto and J.Aguadé on a similar question.

2.5 Lemma. *Let $P \to L$ be a monomorphism between W–lattices of U. If P is centerfree, then we have $(L/P)^W = 0$.*

Proof. Because P is centerfree, it is also fixed–point free (lemma 2.1). Hence $U \cong P \otimes \mathbb{Q}$ as well as every lattice of U is fixed–point free. The short exact sequence $P \to L \to L/P$ gives rise to an exact sequence
$L^W = 0 \to (L/P)^W \to H^1(W;P) = 0$. Thus, the quotient L/P has no fixed–points. □

Proof of Theorem 1.2 (1). Let P and Q be two centerfree lattices of an irreducible pseudo reflection group $W \to Gl(U)$. Then, for r big enough, the lattice $p^rP := \{p^rv : v \in P\}$ is a sublattice of Q. Because p^rP and P are isomorphic W–lattices, there exists a W–equivariant monomorphism $\alpha : P \to Q$. Moreover, by choosing a minimal r, we can assume that $rk(Q/P) < rk(Q) = rk(P)$. Here, $rk(M)$ denotes the rank of a module, which we define to be the dimension of M/p over $\mathbb{F}_p$. Otherwise we have $P \subset pQ := \{px : x \in Q\}$ and $p^{r-1}P \subset Q$. Because P is centerfree we know that $(Q/P)^W = 0$ (Lemma 2.5). The monomorphism $P \xrightarrow{\alpha} Q$ is rationally an isomorphism, and the quotient Q/P is finite.

Applying the functor $\otimes\mathbb{F}_p$ yields an exact sequence

$$0 \to Tor(Q/P, \mathbb{F}_p) \to P/p \xrightarrow{\overline{\alpha}} Q/p \to Q/P \otimes \mathbb{F}_p \to 0$$

of W–modules. Let $V_0 := Tor(Q/P;\mathbb{F}_p)$ and let $V_1 := Im(\overline{\alpha})$ be the image of $\overline{\alpha}$ which is isomorphic to the kernel of $Q/p \to Q/P \otimes \mathbb{F}_p$. Because P and Q are centerfree we have $V_0^W = 0 = V_1^W$ (Lemma 2.2). Applying Proposition 2.4 (U is irreducible) shows that either V_0 or V_1 are trivial vector spaces. If $V_1 = 0$ then $rk(Q/P) = rk(Tor(Q/P;\mathbb{F}_p) = rk(P)$, which is a contradiction. Thus, $V_0 = 0$ and $Q/P = 0$. That is to say that $\alpha : P \to Q$ is an isomorphism. This proves the statement for irreducible pseudo reflection groups.

Next we consider the case of a reducible fixed–point free pseudo reflection group W, i.e. $W \cong W_1 \times W_2$ splits into a nontrivial product of pseudo reflection groups. Moreover, $U \cong U_1 \times U_2$ also splits into a direct sum where $U_1 = U^{W_2}$ and $U_2 = U^{W_1}$. An apllication of the following proposition reduces the proof in this case to the case of irreducible pseudo reflection groups and finishes therefore the proof of Theorem 1.2 (1). □

2.6 Proposition. *Let $W \to Gl(U)$ be a reducible fixed–point free pseudo reflection group, and let P be a centerfree W–lattice of $U = U_1 \oplus U_2$. Then, the following holds:*

1. *The fixed–point set P^{W_1} is centerfree with respect to the W_2–action.*
2. *We have $P \cong P^{W_1} \oplus P^{W_2}$ as W–modules.*

Proof. The quotient P/P^{W_1} is torsion free. Hence, the sequence of W–modules

$$0 \to P^{W_1}/p \to P/p \to (P/p)/(P^{W_1}/p) \to 0$$

is short exact . Taking fixed–points yields an exact sequence

$$0 \to (P^{W_1}/p)^W \cong (P^{W_1}/p)^{W_2} \to (P/p)^W = 0 \ .$$

The last fixed–point set vanishes because P is centerfree and because of Lemma 2.2. Again by Lemma 2.2, the fixed–point set P^{W_1} is centerfree with respect to the W_2–action.

Applying the functor $\otimes\mathbb{Q}$ establishes an exact sequence

$$0 \to P^{W_1} \otimes \mathbb{Q} \to P \otimes \mathbb{Q} \to (P/P^{W_1}) \otimes \mathbb{Q} \to 0 \ .$$

Because $P^{W_1} \otimes \mathbb{Q} \cong (P \otimes \mathbb{Q})^{W_1}$, this sequence splits and shows that $(P/P^{W_1}) \otimes \mathbb{Q}$ as well as P/P^{W_1} are trivial W_2–module. Taking W_2–fixed–points establishes the exact sequence

$$0 = P^W = (P^{W_1})^{W_2} \to P^{W_2} \to (P/P^{W_1})^{W_2} = P/P^{W_1} \to H^1(W_2; P^{W_1}) = 0 \ .$$

The last identity follows from Lemma 2.1 since P^{W_1} is W_2–centerfree. This implies that the middle arrow is an isomorphism, and that $P^{W_1} \oplus P^{W_2} \to P$ is an isomorphism of W–modules. □

Proof of Theorem 1.4 (1). Let $W \to Gl(U)$ be a reducible pseudo reflection group. Using an induction over the number of irreducible summands of U, the statement follows from Proposition 2.6. □

Proof of Theorem 1.5 (1). Passing to associated centerfree lattices is a functor. If P is centerfree, every W–trivial restriction $L \to P$ establishes a W–trivial restriction $PL \to PP = P$. This establishes the desired factorisation.

To prove the other direction we consider the identity $id : P \to P$. By assunption this factors over $P \to PP \to P$. Hence, the second arrow is an epimoprphism and therefore, as a map of torsionfree $\mathbb{Z}_p^\wedge$–modules, an isomorphism. This shows that $PP \cong P$ and that P is centerfree. □

3. Simply connected lattices.

Again, $W \to Gl(U)$ denotes a finite pseudo reflection group. The situation for simply connected lattices is somehow dual to the case of centerfree lattices (see Proposition 5.1 and Corollary 5.2).

3.1 Lemma.

(1) *For a W–lattice L, the group L_W of covariants is finite if and only if L is fixed–point free.*
(2) *If L is fixed–point free, then we have $L_W \cong H_1(W, T_{L,\infty})$.*

Proof. Passing to covariants and using the fact that $L_W \cong H_0(W, L)$, the short exact sequence

$$0 \to L \to L \otimes \mathbb{Q} := L_{\mathbb{Q}} \to T_{L,\infty} \to 0$$

gives rise to the exact sequence

$$0 = H_1(W; L_{\mathbb{Q}}) \to H_1(W; T_{L,\infty}) \to L_W \to (L_{\mathbb{Q}})_W \to (T_{L,\infty})_W \to 0 \ .$$

We can split $L_{\mathbb{Q}} \cong U_1 \oplus U_2$ into a direct sum of a fixed–point free W–module U_1 and summand with trivial W–operation. Because every exact sequence of W–modules over $\mathbb{Q}_p^\wedge$ splits, we have $(U_1)_W = 0$ and $(L_{\mathbb{Q}})_W \cong L_{\mathbb{Q}}^W$. The homology group $H_1(W; T_{L,\infty})$ is finite. Thus, L_W is finite if and only if L is fixed–point free. The second part is obvious. □

In the introduction, for a W–lattice L, we defined SL to be the kernel of $L \to L_W$.

3.2 Proposition. *Let L be a fixed–point free W–lattice.*

(1) *There exists an exact sequence*

$$0 \to SL \to L \to L_W \to 0 \ ,$$

and L is a W–trivial extension of SL.

(2) *The lattice SL is simply connected.*

Proof. The first part follows from definition of L_W and SL and from Lemma 3.1. Passing to covariants, the short exact sequence of (1) establishes the exact sequence

$$H_1(W, L_W) \to SL_W \to L_W \to L_W \to 0 \ .$$

The first term vanishes (Lemma 1.9) and the second last arrow is an isomorphism. □

The next results connects simply connected and centerfree lattices.

3.3 Proposition. *Let S be a simply connected W–lattice. Let $P := PS$ be the associated centerfree lattice. Then, we have $SP \cong S$ and $Z(S) \cong P_W$.*

Proof. By construction there exists a short exact sequence

$$0 \to S \to P \xrightarrow{q_S} Z(S) \to 0 \ .$$

Because $Z(S)$ is a trivial W–module, the map q_S factors over the covariants P_W. This establishes a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SP & \longrightarrow & P & \longrightarrow & P_W & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & P & \longrightarrow & Z(S) & \longrightarrow & 0 \end{array}$$

where the cokernel S/SP of the monomorphism $SP \to S$ is a W–submodule of P_W. Therefore, the quotient S/SP is a module with trivial W–action, and the epimorphism $S \to S/SP$ factors over $S_W = 0$. This shows that all vertical arrows are isomorphisms. □

We finish this section with proofs of Part (2) of theorems 1.2, 1.4 and 1.5.

Proof of Theorem 1.2 (2). The existence of a simply connected lattice follows from Proposition 3.2. Let S and S' be two simply connected lattices. Let P and P' be the associated centerfree lattices. By Theorem 1.2 (1), we know that $P \cong P'$, and by Proposition 3.3 follows that $S \cong SP \cong SP' \cong S'$. □

Proof of Theorem 1.4 (2). Let $S \subset U$ be a simply connected lattice and let $P := PS \subset U$ be the associated centerfree lattice. By Theorem 1.4 (1), we have a splitting $P \cong \bigoplus_i P_i$ of P into centerfree lattices $P_i \subset U_i$ of the irreducible pseudo reflection groups $W_i \to Gl(U_i)$. The lattices $S_i := SP_i \subset U_i$ are simply connected. The sequence $S \cong SP \cong \bigoplus_i SP_i = \bigoplus_i S_i$ proves the statement. The first isomorphism follows from Proposition 3.3, and the second from Proposition 2.6 and Proposition 3.2. □

Proof of Theorem 1.5 (2). Passing to the associated simply connected lattices is a functor. If S is simply connected, every W–trivial restriction $S \to L$ establishes a W–trivial restriction $S = SS \to SL$. This is desired factorisation.

To prove the other direction we consider the identity $id : S \to S$. By assumption, it factors over $S \to SS \to S$. The second arrow is an isomorphism, because it is an epimorphism and because all modules are torsion free. This shows that S is simply connected. □

4. The case $p = 3$ and $W = D_{12}$.

The 2–dimensional homolgy $S := H_2(BT_{SU(3)}; \mathbb{Z}_p^\wedge)$ of the classifying space of the maximal torus of $SU(3)$ gives a representation of Σ_3 as pseudo reflection group. The action can be represented as in case no. 2a of the proof of Proposition 2.4. A straightforward calculation shows that S is simply connected and that $Z(S) \cong \mathbb{Z}/3$. Hence, by Proposition 3.2 and Theorem 1.2, there exist two Σ_3–lattices of $U := S \otimes \mathbb{Q}$, namely S and $P := PS$. They fit into a W–trivial restriction

$$0 \to S \to P \to \mathbb{Z}/3 \to 0 \ .$$

The action of Σ_3 on U can be extended to an action of $D_{12} \cong \Sigma_3 \times \mathbb{Z}/2$ by saying that the subgroup $\mathbb{Z}/2$ acts via multiplication by -1 or trivially. Because the centralizer of $\Sigma_3 \subset Gl(U)$ is given by $\{id, -id\}$, these are the only possible extensions. The first represents D_{12} as a pseudo reflection group, the second does not. Let $D_{12} \to Gl(U)$ be the representation of D_{12} as pseudo reflection group and let S and P also denote the D_{12}–lattices of U.

4.1 Proposition. *The D_{12}–lattices S and P are both simply connected and centerfree, but not isomorphic.*

Proof. Because the representation $D_{12} \to GL(U)$ contains multiplication by -1, we have $S_{D_{12}} = 0 = P_{D_{12}}$ and $Z(S) = 0 = Z(P)$. The two lattices are not isomorphic because they are nonisomorphic as Σ_3–modules.

Proof of Theorem 1.3. We already constructed two non isomorphic lattices. Let $L \subset U$ another lattice. Then, as Σ_3–lattice L has to be isomorphic to either S or P. Because for both lattices there exists a unique extension to a D_{12}–lattice, representing D_{12} as a pseudo reflection group, we also know that L is isomorphic to S or P as D_{12}–lattice. □

Remark. Let $\rho_S, \rho_P : D_{12} \to Gl(\mathbb{Z}_p^{\wedge 2})$ denote the representations associated with S and P. Here we identify both lattices with $\mathbb{Z}_p^{\wedge 2}$. There exists an automorphism $\alpha : D_{12} \to D_{12}$ which maps every reflection $s \in D_{12}$ on $-id \circ s$. Using α we can construct new representations given by $\rho_S\alpha, \rho_P\alpha : D_{12} \to Gl(\mathbb{Z}_p^{\wedge 2})$ with associated lattices S_α and P_α. Rationally, the representations $\rho_S\alpha$ and ρ_s are isomorphic, because restriction represents Σ_3 in both cases as pseudo reflection group, because there is only one representation with this property and because there exists a unique extension to D_{12} as a pseudo reflection group. Hence, S_α and P_α also describe lattices of $U := S \otimes \mathbb{Q}$.

The lattices P_α and P are not isomorphic. Otherwise, we have $S_\alpha \cong S$, because $\alpha\alpha = id : D_{12} \to D_{12}$. Moreover, there exists a short exact sequence $0 \to S \to P \to (\mathbb{Z}/3)_\alpha \to 0$. Taking invariants with respect to the action of Σ_3 shows that

$H^1(\Sigma_3; S) \cong H^1(\Sigma_3; P) = 0$. The last equation follows from Lemma 2.2. The same lemma would also show that S is a centerfree Σ_3–lattice.

Because there exist only two lattices of U, these considerations show that S and P_α are isomorphic as well as P and S_α.

5. General lattices.

Before we discuss the case of general finite pseudo reflection groups, i.e. before we prove the theorems 1.6 and 1.7, we need some informations about the dual representations. Let $W \to Gl(U)$ be a pseudo reflection group. We consider U as a left $\mathbb{Q}_p^\wedge[W]$–module. The set $Hom_{\mathbb{Q}_p^\wedge}(U, \mathbb{Q}_p^\wedge)$ becomes a left $\mathbb{Q}_p^\wedge[W]$–module by defining $w(x^*) := x^* w^{-1}$ for $x^* \in U^*$ and $w \in W$. The vector space U^* again represents W as a pseudo reflection group. For a W–lattice $L \subset U$ we define $L^* := Hom_{\mathbb{Z}_p^\wedge}(L; \mathbb{Z}_p^\wedge)$ which becomes analogously as above a left $\mathbb{Z}_p^\wedge[W]$–module. Because $L^* \otimes \mathbb{Q} \cong (L \otimes \mathbb{Q})^*$ as $\mathbb{Q}_p^\wedge[W]$–modules, the lattice L^* is a lattice of U^*.

5.1 Proposition. *Let $W \to Gl(U)$ be a fixed–point free finite pseudo reflection group. Then, for every lattice $L \subset U$ we have $(SL)^* \cong P(L^*)$ and $(PL)^* \cong S(L^*)$.*

Proof. Dualizing a W–trivial restriction

$$0 \to L \to M \to M/L =: Q \to 0$$

gives a W–trivial restriction

$$0 \to M^* \to L^* \to Ext_{\mathbb{Z}_p^\wedge}(Q, \mathbb{Z}_p^\wedge) \to 0 \ .$$

Because Q carries the trivial W–action, we have $Ext_{\mathbb{Z}_p^\wedge}(Q, \mathbb{Z}_p^\wedge) \cong Q$ as W–modules.

Let $L \subset U$ be a lattice. Dualizing the W–trivial restriction

$$0 \to (SL)^* \to P((SL)^*) \to P((SL)^*)/(SL)^* \cong ZS((SL)^*) \to 0$$

gives

$$0 \to (P((SL)^*))^* \to SL \to Z((SL)^*) \to 0 \ .$$

The equivalence in the top row follows from Proposition 2.3. Because SL is simply connected, taking covariants show that $Z((SL)^*)_W = 0$. But, as a center of a lattice, $Z((SL)^*)$ is a module with trivial W–action. This shows that $Z((SL)^*) = 0$ and that $(SL)^*$ is centerfree. Dualizing the W–trivial restriction $SL \to L$ gives the W–trivial restriction $L^* \to (SL)^*$. Applying the construction P establishes the W–trivial restriction $\alpha : P(L^*) \to (SL)^*$. Since $P(L^*)$ is centerfree, the map α is a W–trivial restriction and since $H^1(W, P(L^*)) = 0$ (Proposition 2.3 (2)), applying fixed–points, the cokernel of α is trivial. Hence, α is an isomorphism.

The other equation is proved analogously, but dual. □

The proof of the following is obvious.

5.2 Corollary. *Let $W \to Gl(U)$ be a finite fixed–point free pseudo reflection group.*

1. *A lattice $P \subset U$ is centerfree if and only if $P^* \subset U^*$ is simply connected.*
2. *A lattice $S \subset U$ is simply connected if and only if $S^* \subset U^*$ is centerfree.*

Proof of Theorem 1.6. Let $L \subset U$ be a W–lattice. The quotient $L/L^W =: \overline{L}$ is a fixed–point free W–lattice of U', where $U \cong U^W \oplus U'$ splits into the direct sum of the fixed–points U^W and a fixed–point free part U'. Let $S := S(\overline{L}) \subset U'$ be the associated simply connected lattice. Using pullbacks the W–trivial restriction $S \to \overline{L}$ establishes a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^W \cong Z & \longrightarrow & L' & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^W \cong Z & \longrightarrow & L & \longrightarrow & \overline{L} & \longrightarrow & 0 \,. \end{array}$$

The top row describes an element of the group $Ext_{\mathbb{Z}_p^\wedge[W]}(S, Z)$ of extensions. We have the following sequence of isomorphisms:

$$\begin{aligned} Ext_{\mathbb{Z}_p^\wedge[W]}(S, Z) &\cong H^1(W; Hom_{\mathbb{Z}_p^\wedge}(S, Z)) \\ &\cong H^1(W, S^* \otimes Z) \\ &\cong H^1(W; S^*) \otimes Z \\ &= 0 \,. \end{aligned}$$

The first identity follows, because S and Z are free modules over $\mathbb{Z}_p^\wedge$ [3, III; 2.2], the second from the isomorphism between the coefficients, the third because W acts trivially on Z and because of Lemma 1.9, and the last because S^* is centerfree (Corollary 5.2 and Lemma 2.1). That is to say that $L' \cong Z \oplus S$. Moreover, we have an isomorphism $L/(Z \oplus S) \cong \overline{L}/S \cong \overline{L}_W$ which shows that $Z \oplus S \to L$ is a W–trivial restriction. This proves part (1).

For the second statement we dualize the above argument. There exists a W–trivial restriction $Z^* \oplus P^* \to L^*$ where $P^* \subset U^*$ is a simply connected lattice (Corollary 5.2 and part (1)). Dualizing again gives a short exact sequence

$$0 \to L \to Z \oplus P \to Ext(L^*/(Z^* \oplus P^*); \mathbb{Z}_p^\wedge) \to 0 \,,$$

which shows that the first arrow is a W–trivial restriction. □

Remark. Using Proposition 5.1 and Corollary 5.2 one can easily prove the second parts of the theorems 1.2, 1.4 and 1.5 as a consequence of the first parts. The idea is the same as in the proof of Theorem 1.6.

Proof of Theorem 1.7. Let $0 \to L \to P \oplus Z \to P \oplus Z/L =: Q \to 0$ be the W–trivial restriction of Theorem 1.6. Let $W \cong W_1 \times W_2$ and $P \cong P_1 \oplus P_2$ be a splitting into centerfree W_i–lattices. We also assume that P_1 is simply connected. The composition $P_1 \to P \oplus Z \to Q$ factors over $(P_1)_{W_1}$ and is therefore trivial. Hence, the inclusion $P_1 \to P \oplus Z$ lifts to L which shows that P_1 is a direct summand of L. □

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