

# FINITE LOOP SPACES ARE MANIFOLDS

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ABSTRACT. Using stable homotopy theory, the theory of  $p$ -compact groups, and surgery theory, we show that every loop space with finitely generated homology is homotopy equivalent to a smooth, parallelizable, compact manifold.

## 1. INTRODUCTION, STATEMENT OF RESULTS

One of the motivating questions for surgery theory was whether every finite  $H$ -space is homotopy equivalent to a Lie group. This question was answered in the negative by Hilton and Roitberg's discovery of some counterexamples [18]. However, the problem remained whether every finite  $H$ -space is homotopy equivalent to a closed, smooth manifold.

This question is still open, but in case the  $H$ -space admits a classifying space we have the following theorem.

**THEOREM.** *Let  $B$  be a CW complex and denote by  $X$  the loops on  $B$ ,  $\Omega B$ . If  $H_*(X) = \bigoplus_i H_i(X)$  is a finitely generated abelian group, then  $X$  is homotopy equivalent to a compact, smooth, parallelizable manifold.*

This condition on  $H_*(X)$  is often called *quasifiniteness*. We will briefly discuss the history of smoothing  $H$ -spaces in this introduction.

Suppose given a quasifinite space  $X = \Omega B$ ,  $B$  a CW-complex. It follows from [22] that  $X$  is finitely dominated, since it is a simple space with finitely generated homology. Finitely dominated means that up to homotopy it is a retract of a finite complex.

Recall [38] that an oriented,  $n$ -dimensional Poincaré duality space  $Y$  is a finitely dominated space  $Y$ , together with a class  $[Y] \in H_n(Y, \mathbf{Z})$  such that if  $[\tilde{Y}]$  is the transfer of  $[Y]$  to  $H_n^{1.f.}(\tilde{Y}, \mathbf{Z})$  then

$$[\tilde{Y}] \cap - : H_{c.s.}^*(\tilde{Y}, \mathbf{Z}) \rightarrow H_{n-*}(\tilde{Y}, \mathbf{Z})$$

is an isomorphism from cohomology with compact supports to homology of the universal cover. Obviously all oriented manifolds satisfy this kind of Poincaré duality, so a first step to prove that a loop space is a manifold, is to prove it is a Poincaré duality space.

Since  $X$  is finitely dominated,  $H^1(X; \mathbf{Z})$  is a free abelian group. Choose a classifying map  $X \rightarrow T^k$  representing a basis. This map has a section, using the  $H$ -space structure, given by the composite  $T^k \rightarrow X^k \rightarrow X$ , defining a basis of

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$\pi_1(X)$  modulo torsion. Denoting the homotopy fiber by  $X'$  it is easy to see that  $X' \times T^k \rightarrow X \times X \rightarrow X$  induces isomorphisms on homotopy groups, hence is a homotopy equivalence. Since  $X'$  inherits an  $H$ -space structure, it thus suffices to consider  $X$  with finite fundamental group for the question of Poincaré duality.

Assuming  $\pi_1(X)$  finite, we consider  $H^*(\tilde{X}; F)$  where  $F = \mathbf{F}_p$  or the rational numbers. The induced product map  $\tilde{X} \times \tilde{X} \rightarrow \tilde{X}$  induces a Hopf algebra structure on  $H^*(\tilde{X}; F)$ . It follows from the classification of finitely generated, connected, graded Hopf algebras over a field  $F$  [4, Theorem 6.1] and [19], that  $H^*(\tilde{X}; F)$  is a tensor product of exterior algebras and truncated polynomial algebras. The top dimension is generated by a product of the algebra generators to their maximal non-zero power. Denoting the top dimension by  $n_p$  for  $F = \mathbf{F}_p$  and by  $n_0$  for  $F = \mathbf{Q}$ , it follows that cap product with the homology dual of this top dimensional class induces an isomorphism

$$H^*(\tilde{X}; F) \rightarrow H_{n_p-*}(\tilde{X}; F).$$

Clearly  $n_0 \leq n_p$ . Since  $H_1(\tilde{X}; F)$  is 0 so is  $H^{n_p-1}(\tilde{X}; F)$ , hence the top class is the reduction mod  $p$  of a  $\mathbf{Z}$ -summand in integral cohomology, which means  $n_p \leq n_0$ , so all  $n_p$ 's are the same. Denoting this common dimension by  $n$ , there must be an integral class  $[\tilde{X}] \in H_n(\tilde{X}; \mathbf{Z})$ , so that cap product with the induced element in  $H_n(\tilde{X}; F)$  induces an isomorphism

$$H^*(\tilde{X}; F) \rightarrow H_{n-*}(\tilde{X}; F)$$

for all  $F$ , and hence an isomorphism from

$$H^*(\tilde{X}; \mathbf{Z}) \rightarrow H_{n-*}(\tilde{X}; \mathbf{Z})$$

Precisely the same arguments may be applied to  $H^*(X; F)$  to obtain dimensions  $n'_p$  for  $X$ . The spectral sequence of the fibration  $\tilde{X} \rightarrow X \rightarrow B\pi_1(X)$  is a spectral sequence of Hopf algebras, and analyzing this [5], it is shown that the  $n'_p = n_p$ , hence all  $n'_p$  are the same, but that is only possible if  $H_n(X; \mathbf{Z}) = \mathbf{Z}$ .

Denoting the generator by  $[X]$ , we now only need to see that the transfer of  $[X]$  is  $[\tilde{X}]$ . For this purpose we factor  $\tilde{X} \rightarrow X$  as a sequence of covering spaces  $\tilde{X} = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 = X$ , where each  $X_i \rightarrow X_{i-1}$  is a  $p$ -fold covering for some prime  $p$ . Each of the  $X_i$ 's are  $H$ -spaces, and the maps are maps of  $H$ -spaces so in mod  $p$  cohomology we get an induced map of Hopf algebras. Since in all cases there will be some class going to 0, and the top class is a product of all classes to maximal degree, it follows that the induced map is 0 on the top-dimensional class. In integral homology, transfer followed by the induced map is multiplication by  $p$ , but the induced map is 0 in mod  $p$  cohomology, so the transfer must send a generator to a generator in the top dimension.

What we have described here is a slight modification of Browder's argument [6, 5] that  $X$  is a Poincaré duality space.

The  $S$ -dual of a Poincaré duality space is the Thom space of the Spivak normal fibration. Being a Poincaré duality space,  $X$  can be written as an  $(n-1)$ -dimensional complex with one  $n$ -cell attached. Browder and Spanier [9] used the map  $X \times X \rightarrow X \rightarrow S^n$  to show that  $X$  is self-dual in the sense of  $S$ -duality, so stably the Thom space of the Spivak normal fibration is homotopy equivalent to  $S^k(X_+)$ . This means that the top class in  $S^k(X_+)$  is spherical, and a transversality

argument, making the map  $S^{n+k} \rightarrow S^k(X_+)$  transverse to  $X$  sets up a surgery problem

$$\begin{array}{ccc} \nu_M & \longrightarrow & \varepsilon \\ \downarrow & & \downarrow \\ M & \longrightarrow & X \end{array}$$

where  $\varepsilon$  denotes the trivial bundle. In the case  $\pi_1(X) = 0$ , Browder now proceeded to show that  $X$  is homotopy equivalent to a smooth manifold except possibly in dimensions  $4k+2$ . In odd dimensions this is because the surgery obstruction groups vanish, and in dimension  $4k$  the argument is that the rational cohomology of  $X$  is an exterior algebra hence the index of  $X$  is trivial. Hirzebruch's index formula shows that the index of  $M$  is trivial since stably, the normal bundle is trivial, and it follows that the surgery obstruction is trivial, being the difference. In the non-simply connected case these surgery obstruction groups can be very complicated even for finitely generated abelian groups.

NOTATION 1.1. We denote by  $\mathbf{Z}_p$  and  $\mathbf{Q}_p$  the  $p$ -adic integers and rationals, respectively. For a nilpotent space  $X$  and a commutative ring  $R$ , we denote by  $L_R X$  the localization of  $X$  with respect to  $H_*(-; R)$ . We abbreviate  $L_{\mathbf{Z}/p}$  by  $L_p$  and  $L_{\mathbf{Z}(p)}$  by  $L_{(p)}$ .

Denote the  $\mathbf{Q}_p$ -algebra  $H^*(X; \mathbf{Z}_p) \otimes \mathbf{Q}$  by  $H_{\mathbf{Q}_p}^*(X)$ . On finite nilpotent CW-complexes, this agrees with  $H^*(X; \mathbf{Q}_p)$ , but whereas the latter functor is not invariant under  $\mathbf{Z}/p$ -localization, the former one is. In algebra, this is mirrored by the fact that  $\text{hom}_{\mathbf{Z}}(\mathbf{Z}_p, \mathbf{Q}_p) \not\cong \mathbf{Q}_p$ , but  $\text{hom}_{\mathbf{Z}}(\mathbf{Z}_p, \mathbf{Z}_p) \cong \mathbf{Z}_p$ .

For a quasifinite loop space  $X$ , define the *dimension*  $\dim(X)$  to be the homological dimension, the *rank*  $\text{rk}(X)$  to be the number of exterior generators of  $H^*(X; \mathbf{Q})$ , and the *type* to be the multi-set of dimensions of those generators.

## 2. BASIC CONSIDERATIONS AND OUTLINE OF THE PROOF

Let  $X = \Omega B$  be a quasifinite connected loop space. Since  $H^2(B; \mathbf{Z})$  is free abelian, there is a map  $B \rightarrow K(\mathbf{Z}^r, 2) = BT^r$  inducing an isomorphism on  $H^2(-; \mathbf{Z})$ . Let  $B'$  be the homotopy fiber. We now have a fibration  $\Omega B' \rightarrow X \rightarrow T^r$ , and arguing as in the introduction, we have  $X \simeq \Omega B' \times T^r$ . Since  $T^r$  is a smooth parallelizable manifold, we may without loss of generality assume that  $X$  has finite fundamental group. We shall do so for the rest of this paper.

Surgery arguments are only valid in dimensions  $\geq 5$ . Clark's theorem [10] states that  $H^3(X; \mathbf{Q}) \neq 0$ , and assuming a finite fundamental group, the only instances of quasifinite loop spaces of dimension  $\leq 4$  are rational homology 3-spheres. By [7, Theorem 5.2], this means that  $X$  is homotopy equivalent to  $S^3$  or  $\text{SO}(3)$ , which are smooth parallelizable manifolds. We may thus assume the dimension of  $X$  is  $\geq 5$ .

Our method of proof is to construct an orientable fibration  $S^1 \rightarrow X \rightarrow Y$  of quasifinite simple spaces. This suffices to prove that  $X$  is homotopy equivalent to a finite CW-complex using the theory of finiteness obstructions [37]. The finiteness obstruction is a generalized Euler characteristic defined by considering the chains of the universal cover of  $X$ , as a  $\mathbf{Z}[\pi]$ -module chain complex, which turns out to be chain homotopy equivalent to a finite length chain complex of finitely generated projective  $\mathbf{Z}[\pi]$ -modules. This allows the definition of an Euler characteristic in  $\tilde{K}_0(\mathbf{Z}[\pi])$ . The vanishing of this obstruction ensures that the space  $X$  is of the homotopy type of a finite complex.

To deal with the surgery obstructions, the fibration has to have some additional properties. We will discuss two slightly different concepts, *double 1-tori* and *special 1-tori*. In both cases the constructions rely on the theory of  $p$ -compact groups and on arithmetic square arguments. The construction of a double 1-torus is more elementary and needs less input from the theory of  $p$ -compact groups. Special 1-tori on the other hand reveal much more internal structure of finite loop spaces. For this reason, we have included both versions of the proof in this paper.

Our general arguments break down in some special cases, namely when the type of  $X$  is  $(3^k, 7^\epsilon)$ ,  $\epsilon = 0, 1$ . These cases have to be dealt with by special arguments.

### 3. THE SURGERY ARGUMENTS

The arguments in this section are modeled on the arguments in [29, 30]. In those papers the fourth author studied conditions on  $\mathbf{Z}_{(p)}$ -local  $S^1$ -fibrations making it possible to produce integral  $S^1$ -fibrations by gluing. The concept of spaces *admitting a 1-torus* and a *special 1-torus*, respectively, were used. In this paper, we propose a concept somewhere in between:

Let  $R$  be a commutative ring. Recall that a nilpotent space  $X$  is called  $R$ -finite if  $\bigoplus H_i(X; R)$  is finitely generated, and  $R$ -local if  $[Y, X] = 0$  for every  $HR$ -acyclic space  $Y$ . Note that  $\mathbf{Z}$ -finite is the same as quasifinite. For simple spaces  $\mathbf{Z}$ -locality is an empty condition and  $\mathbf{Z}/p$ -locality is the same as  $p$ -completeness.

DEFINITION 3.1. An  $R$ -finite  $R$ -local space  $X$  is called *stably reducible* if there is a stable map from an  $R$ -local sphere to  $X$  inducing an isomorphism in the top-dimensional homology.

DEFINITION 3.2. Let  $R$  be a ring and  $X$  be an  $R$ -finite,  $R$ -local, nilpotent, connected space. We call a fibration of nilpotent spaces  $X \xrightarrow{p} Y \rightarrow L_R B S^1$  an  *$R$ -local 1-torus* if it satisfies

- (1)  $Y$  is  $R$ -finite,  $R$ -local, and stably reducible and
- (2)  $\pi_1(p)$  is an isomorphism.

An  $R$ -local 1-torus is an  *$R$ -local double 1-torus* if this fibration is the pullback of a fibration of nilpotent spaces  $X \xrightarrow{p} Z \rightarrow L_R B S^1 \times L_R B \mathbf{Z}/2$  satisfying

- (1)  $Z$  is  $R$ -finite,  $R$ -local, and nilpotent and
- (2) The induced map  $\pi_1(Z) \rightarrow \mathbf{Z}/2$  is a split epimorphism.

We call the  $R$ -local 1-torus *rationally splitting* if the map  $p$  rationally has a retract of the form  $h : L_{\mathbf{Q}} L_R S^3 \rightarrow L_{\mathbf{Q}} L_R S^2$ , where  $h$  is  $L_{\mathbf{Q}} L_R$  applied to the Hopf map.

Notice that when  $\frac{1}{2} \in R$  there is no difference between a 1-torus and a double 1-torus. When  $\frac{1}{2} \notin R$  a double 1-torus leads to a diagram of fibrations

$$\begin{array}{ccccc}
 L_R S^1 & \longrightarrow & L_R S^1 & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 L_R S^1 \times \mathbf{Z}/2 & \longrightarrow & X & \longrightarrow & Z \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{Z}/2 & \longrightarrow & Y & \longrightarrow & Z
 \end{array}$$

where  $S^1 \rightarrow X \rightarrow Y$  is an orientable fibration of  $R$ -finite simple spaces and  $\pi_1(Y) \cong \pi_1(Z) \times \mathbf{Z}/2$ .

**PROPOSITION 3.3.** *Let  $X$  be a Poincaré duality space of dimension  $n \geq 5$  admitting an integral double 1-torus. Then  $X$  is homotopy equivalent to a compact, stably parallelizable, smooth manifold.*

*Proof.* Denote the double 1-torus by  $X \xrightarrow{p} Y \rightarrow Z$ . A quasifinite, simple space is finitely dominated by [22]. We first need to deal with the finiteness obstruction  $\sigma(X) \in \tilde{K}_0(\mathbf{Z}\pi_1(X))$ . The formula of [32] tells us that  $p_*(\sigma(X)) = \chi(S^1) \cdot \sigma(Y)$  where  $\chi(S^1)$  is the Euler characteristic, hence  $p_*(\sigma(X)) = 0$ , but  $p_*$  is an isomorphism so  $\sigma(X) = 0$  and  $X$  is thus homotopy equivalent to a finite complex. We now let  $E$  be the total space of the corresponding  $\mathbf{D}^2$ -fibration. It follows from [17] that  $Y$  is a Poincaré duality space, hence  $(E, X)$  is a Poincaré duality pair. We consider the classifying map of the Spivak normal fibration  $\nu_E : E \rightarrow BG$ . We have the equation

$$\nu_E = p^*(\nu_Y) \oplus p^*(p)^{-1}.$$

Now  $\nu_Y$  is trivial since  $Y$  was assumed to be stably reducible. Also  $p$  is an  $S^1$ -fibration classified by  $G(2)$ . But  $O(2) \subseteq G(2)$  is a homotopy equivalence so  $p$  is fiber homotopy equivalent to an  $O(2)$ -bundle, actually an  $S^1$ -bundle since the fibration was assumed orientable. We thus get a linear reduction  $\zeta$  of  $\nu_E$  and the reduction is trivial when restricted to  $X$  since the pullback of an  $S^1$ -bundle to its own total space is trivial. The procedure of surgery (see e. g. Browder [8, page 38]) sets up a degree 1 normal map

$$(M, \partial M) \xrightarrow{\phi} (E, X), \quad \hat{\phi} : \nu_M \rightarrow \zeta$$

with  $\pi = \pi_1(E) \cong \pi_1(Y) \cong \pi_1(X)$ . However,  $E$  is possibly not finite, only finitely dominated. Since  $Y \simeq E$ ,  $\sigma(E) = \sigma(Y)$ . This situation was studied in [31], where it was shown that the surgery obstruction of  $\partial M \rightarrow X$  is  $\delta([\sigma(E)])$  where  $\delta$  is the boundary in the Ranicki-Rothenberg exact sequence

$$\dots \rightarrow H^{n+1}(\mathbf{Z}/2; \tilde{K}_0(\mathbf{Z}\pi)) \xrightarrow{\delta} L_n^h(\mathbf{Z}\pi) \rightarrow L_n^p(\mathbf{Z}\pi) \rightarrow \dots$$

Since  $Z$  is an  $(n-1)$ -dimensional Poincaré duality space, the finiteness obstruction satisfies the formula  $\sigma(Z) = (-1)^{n-1}\sigma(Z)^*$ . Obviously,  $\sigma(Y)$  is just the restriction  $\text{Res}(\sigma(Z))$ . It now follows from [32] or just general covering space theory that  $p_{1*} \text{Res} \sigma(Z) = [(\mathbf{Z} \oplus \mathbf{Z}) \otimes P]$  where  $P$  is a projective module representing  $\sigma(Z)$ , and  $\pi_1(Z)$  acts on  $\mathbf{Z} \oplus \mathbf{Z}$  through its  $\mathbf{Z}/2$ -quotient by permuting the two factors. There is an exact sequence of  $\pi_1(Z)$ -modules  $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \oplus \mathbf{Z} \rightarrow \mathbf{Z}^- \rightarrow 0$  with trivial action on the first term, and nontrivial action on the last. This implies that  $p_* \text{Res} \sigma(Z) = [P] + [\mathbf{Z}^- \otimes P]$ . Let  $r : \pi_1(Z) \rightarrow \pi_1(Y)$  be a splitting. We then get  $\sigma(Y) = \text{Res} \sigma(Z) = r_*(p_1)_* \text{Res}(\sigma(Z)) = 2r_*(\sigma(Z)) = r_*(\sigma(Z)) + (-1)^{n-1}r_*(\sigma(Z)^*) = r_*(\sigma(Z)) + (-1)^{n-1}r_*(\sigma(Z))^*$

from which it follows that  $[\sigma(Y)] = 0$  in  $H^{n+1}(\mathbf{Z}/2; \tilde{K}_0(\mathbf{Z}\pi))$ .  $\square$

#### 4. THE REDUCTION TO A $\mathbf{Z}/p$ -LOCAL PROBLEM

**PROPOSITION 4.1.** *Let  $X$  be a quasifinite loop space such that for every  $p$ ,  $L_{(p)}X$  admits a rationally splitting double 1-torus. Then so does  $X$ .*

*Proof.* This was shown for ordinary rationally splitting 1-tori in [29, Prop. 3.2]. The extension to double 1-tori is immediate since  $B\mathbf{Z}/2$  is rationally trivial, so if  $X$  admits a 1-torus, and  $X_{(2)}$  admits a 2-local double 1-torus, then  $X$  admits a double 1-torus.  $\square$

To reduce the problem further to constructing double 1-tori in  $\mathbf{Z}/p$ -local loop spaces, we will make use of an easy fact about  $p$ -adic squares:

LEMMA 4.2. *Every  $p$ -adic rational number is the product of a rational number and the square of a  $p$ -adic integral unit.*

*Proof.* It is enough to show that every  $p$ -adic unit  $a \in \mathbf{Z}_p^\times$  can be written as a product of a rational integer and the square of a  $p$ -adic unit. Since the Legendre symbol is a group homomorphism, it suffices to exhibit an  $n \in \mathbf{Z}$  whose image in  $\mathbf{Z}_p$  is a unit with no square root. Any lift of a generator of  $(\mathbf{Z}/p)^\times$  (or  $(\mathbf{Z}/8)^\times$ ) to  $\mathbf{Z}$  will do.  $\square$

PROPOSITION 4.3. *Let  $X$  be a  $\mathbf{Z}_{(p)}$ -local,  $\mathbf{Z}_{(p)}$ -finite loop space such that  $L_p X$  admits a  $\mathbf{Z}/p$ -local rationally splitting double 1-torus. Then  $X$  admits a  $\mathbf{Z}_{(p)}$ -local double 1-torus.*

*Proof.* Let  $L_p X \rightarrow Y_p \rightarrow L_p BS^1$  be the rationally splitting 1-torus.

Let  $\{e_1, \dots, e_k\}$  be a basis of the free part of  $\pi_3(X)$ , thus inducing a basis in  $\pi_3(L_{\mathbf{Q}}X)$  and in  $\pi_3(L_{\mathbf{Q}}L_p X)$ , and elements in  $\pi_3(L_p X)$ . We also denote the induced elements by  $\{e_1, \dots, e_k\}$ . Since the fibration is rationally splitting we may produce a diagram

$$\begin{array}{ccccc} L_{\mathbf{Q}}S^3 & \longrightarrow & L_p L_{\mathbf{Q}}S^3 & \longrightarrow & L_{\mathbf{Q}}L_p X \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathbf{Q}}S^2 & \longrightarrow & L_p L_{\mathbf{Q}}S^2 & \longrightarrow & L_{\mathbf{Q}}Y_p \end{array}$$

Let  $a$  be the image of a generator of  $\pi_3(L_{\mathbf{Q}}S^3)$  in  $\pi_3(L_{\mathbf{Q}}L_p X)$ . We have  $a = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_k e_k$  with  $\alpha_i \in \mathbf{Q}_p$ , and we choose to order the basis so that  $\alpha_1 \neq 0$ .

We first want to show that we can change the problem so that the  $\alpha_i$  are rational. The Hopf fibration  $h$  admits an automorphism of the form

$$\begin{array}{ccc} L_{\mathbf{Q}}L_p S^3 & \xrightarrow{u^2} & L_{\mathbf{Q}}L_p S^3 \\ \downarrow h & & \downarrow h \\ L_{\mathbf{Q}}L_p S^2 & \xrightarrow{u} & L_{\mathbf{Q}}L_p S^2 \end{array} \quad \text{for any } u \in \mathbf{Q}_p^\times$$

so by Lemma 4.2, this means we may assume that  $\alpha_1$  is rational.

We may however compose the splitting by any homotopy equivalence that can be lifted to a homotopy equivalence of  $L_p X$ . This still gives a rationally splitting one-torus, and it does not change  $X$  in its local genus, see Definition 4.5. We now show that we can find such a homotopy equivalence to change  $\alpha_2, \dots, \alpha_k$  to be rational.

Using the  $H$ -space structure on  $X$  we may produce a rational equivalence  $B \xrightarrow{b} X$  where  $B$  is a product of  $p$ -local odd dimensional spheres. The lifting problem

$$\begin{array}{ccc} & & B \\ & \nearrow c & \downarrow b \\ X & \xrightarrow{p^i} & X \end{array}$$

may be solved using obstruction theory for sufficiently large  $l$ : First try to lift the identity map, and use the fact that the homotopy groups of the fiber are finite  $p$ -groups. Whenever an obstruction is encountered, we may precompose with a map of degree  $p^i$ , to kill the obstruction. Given any map  $L_p S^3 \rightarrow L_p X$  sending  $e_j$  to  $\beta e_i$ , expressed in the basis chosen above, we now consider a map of the type

$$L_p X \xrightarrow{(L_p c, 1)} L_p B \times L_p X \rightarrow L_p S^3 \times L_p X \rightarrow L_p X \times L_p X \rightarrow L_p X$$

where  $L_p B \rightarrow L_p S^3$  is the projection on the  $i$ th 3-sphere. This map realizes the elementary matrix on  $\pi_3$ , where the  $(i, j)$  off-diagonal element is of type  $p^i \beta$ . To see we may choose all the  $\alpha_i$  to be rational, we observe that the element  $a = \alpha_1 e_1 + \dots + \alpha_k e_k$  already has  $\alpha_1$  rational, and denoting  $\alpha_1$  by  $q_1$ , the equations

$$\alpha_i + q_1 p^i \beta_i = q_i \in \mathbf{Q}$$

are solvable with  $\beta_i \in \mathbf{Z}_p$ , but the left hand side of the equation above is precisely the effect of applying an elementary operation.

We now extend the element  $a \in \pi_3(L_{\mathbf{Q}} X)$  to a basis  $\{a, a_2, \dots, a_k\}$  and denote the images of  $a_i$  in  $\pi_3(L_{\mathbf{Q}} Y_p)$  for  $i > 1$  by  $b_i$ . Choose a splitting

$$(L_{\mathbf{Q}} L_p X \rightarrow L_{\mathbf{Q}} Y_p) \rightarrow (L_{\mathbf{Q}} L_p S^3 \rightarrow L_{\mathbf{Q}} L_p S^2)$$

Since  $S^3 \rightarrow S^2 \rightarrow BS^1$  is a principal fibration, we see that we can vary this splitting by any compatible pair  $L_{\mathbf{Q}} L_p X \rightarrow L_{\mathbf{Q}} L_p S^3$  and  $L_{\mathbf{Q}} Y_p \rightarrow L_{\mathbf{Q}} L_p S^2$ , and after such a variation we may assume that the  $b_i$ 's map trivially on homotopy groups, without changing the image of  $a$ .

Using the basis  $\{a, a_2, \dots, a_k\}$  we may produce a diagram of fibrations with the horizontal maps homotopy equivalences

$$\begin{array}{ccc} L_{\mathbf{Q}} L_p S^1 & & L_{\mathbf{Q}} L_p S^1 \\ \downarrow & & \downarrow \\ L_{\mathbf{Q}} L_p X & \longrightarrow & L_{\mathbf{Q}} L_p S^3 \times \prod L_{\mathbf{Q}} L_p S^3 \times L_{\mathbf{Q}} L_p A \\ \downarrow & & \downarrow (h, 1, 1) \\ L_{\mathbf{Q}} Y_p & \longrightarrow & L_{\mathbf{Q}} L_p S^2 \times \prod L_{\mathbf{Q}} L_p S^3 \times L_{\mathbf{Q}} L_p A \end{array}$$

where  $h$  is the Hopf fibration and  $A$  is a product of  $S^{2n+1}$ ,  $n > 1$ .

We complete this diagram to the diagram

$$\begin{array}{ccccc}
L_{\mathbf{Q}}L_pX & \longleftarrow & & & L_{\mathbf{Q}}X \\
\downarrow & \searrow \simeq & & \swarrow & \downarrow \simeq \\
& & L_{\mathbf{Q}}L_pS^3 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA & & \\
& & \downarrow (h,1) & & \\
L_{\mathbf{Q}}Y_p & \searrow \simeq & & \swarrow & L_{\mathbf{Q}}S^3 \times \prod L_{\mathbf{Q}}S^3 \times L_{\mathbf{Q}}A \\
& & L_{\mathbf{Q}}L_pS^2 \times \prod L_{\mathbf{Q}}L_pS^3 \times L_{\mathbf{Q}}L_pA & & 
\end{array}$$

The desired fibration  $X \rightarrow Y$  is now obtained by presenting  $X$  as a pullback mapping to a pullback diagram defining  $Y$ .

$$\begin{array}{ccc}
\begin{array}{ccc} X & \longrightarrow & L_{\mathbf{Q}}X \\ \downarrow & & \downarrow \\ L_pX & \longrightarrow & L_pL_{\mathbf{Q}}X \end{array} & \longrightarrow & \begin{array}{ccc} Y & \longrightarrow & L_{\mathbf{Q}}S^2 \times \prod L_{\mathbf{Q}}S^3 \times L_{\mathbf{Q}}A \\ \downarrow & & \downarrow \\ Y_p & \longrightarrow & L_{\mathbf{Q}}Y_p \end{array}
\end{array}$$

The extension to a double 1-torus is obtained by just noting that the lifting of the map to  $B\mathbf{Z}/2$  is trivial rationally.  $\square$

The proof of the main theorem is divided up in some special cases and the general case. The special case is when the type of  $X$  is  $(3^k, 7^\epsilon)$ .

We first state the general case in the next theorem. Its proof will be given in the following section.

**THEOREM 4.4.** *Let  $X$  be a quasifinite loop space. Then for any  $p$ ,  $L_pX$  admits a rationally splitting 1-torus except possibly when  $p = 2$  and  $X$  is of type  $3^k$ . The 1-torus can be extended to a rationally splitting double 1-torus unless  $p = 2$  and  $X$  is of type  $(3^k, 7^\epsilon)$ ,  $\epsilon = 0, 1$ .*

We now turn to the special cases.

**DEFINITION 4.5.** Let  $X$  be a  $\mathbf{Z}/p$ -local space. The  $p$ -genus  $G_p(X)$  of  $X$  is the set of all  $\mathbf{Z}_{(p)}$ -local homotopy types  $Y$  such that  $L_pY \simeq X$ .

**LEMMA 4.6.** *If  $G$  is a center-free  $p$ -compact group which is a rational homology 3-sphere then  $G \simeq L_p\mathrm{SO}(3)$ . If  $X$  is a quasifinite rank 1 loop space, with  $L_pX$  center-free for all  $p$ , then  $X \simeq \mathrm{SO}(3)$ .*

*Proof.* Mixing with a rationalized sphere produces a  $\mathbf{Z}_{(p)}$ -local loop space. Mixing this with a sphere at the other primes produces an  $H$ -space which is a rational homology sphere. By Browder's theorem [7, Theorem 5.2] the only possibilities are  $S^3$ ,  $\mathrm{SO}(3)$ ,  $S^7$ , and  $\mathbf{R}P^7$ , but  $L_2S^3$  is not center-free, and  $L_2S^7$  is not a loop space. This at the same time proves the statement about  $X$  by [10].  $\square$

**LEMMA 4.7.**



- (1) *There is only one element in  $G_p(L_p \mathrm{SO}(3)^k)$ .*
- (2) *There is only one element in  $G_p(L_p \mathrm{SO}(5))$ .*
- (3) *There is only one element in  $G_p(L_p \mathrm{SO}(3)^k \times \mathrm{SO}(5))$ .*

*Proof.* If  $Y \in G_p(X)$ , then  $Y$  is obtained as a pullback

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & L_{\mathbf{Q}}X \\ \downarrow & & \downarrow \\ L_p X & \xrightarrow{\quad} & L_{\mathbf{Q}}L_p X \xrightarrow{f} L_{\mathbf{Q}}L_p X. \end{array}$$

from a self-equivalence  $f \in \mathrm{Aut}(L_{\mathbf{Q}}L_p X)$ . Precomposing  $f$  with an element of  $\mathrm{Aut}(L_p X)$  leaves  $Y$  unchanged up to homotopy, as does postcomposing with an element of  $\mathrm{Aut}(L_{\mathbf{Q}}X)$ . Thus there is a bijection

$$G_p(L_p X) \cong \mathrm{Aut}(L_p X) \backslash \mathrm{Aut}(L_{\mathbf{Q}}L_p X) / \mathrm{Aut}(L_{\mathbf{Q}}X).$$

In the case  $X = \mathrm{SO}(3)^k$ ,  $\mathrm{Aut}(L_{\mathbf{Q}}L_p X) \cong \mathrm{GL}_k(\mathbf{Q}_p)$ , and since every  $p$ -adic integer can be realized as the degree of a self-map of  $L_p \mathrm{SO}(3)$ , we have that  $\mathrm{Aut}(L_p X) \cong \mathrm{GL}_k(\mathbf{Z}_p)$ . Thus,

$$G_p(L_p \mathrm{SO}(3)^k) \cong \mathrm{GL}_k(\mathbf{Z}_p) \backslash \mathrm{GL}_k(\mathbf{Q}_p) / \mathrm{GL}_k(\mathbf{Q}) \cong *.$$

For  $X = \mathrm{SO}(5)$  the gluing map is given by two  $p$ -adic rationals  $\alpha_3$  and  $\alpha_7$ , which describe the induced map on the homotopy groups in dimension 3 and 7. Looping down unstable Adams operations  $\psi^r : L_p B \mathrm{SO}(5) \rightarrow L_p B \mathrm{SO}(5)$  shows that for any  $p$ -adic unit  $r$ , the pair  $(r^2, r^4)$  can be realized by a self equivalence of  $L_p \mathrm{SO}(5)$ . We may also realize selfmaps of  $L_p \mathrm{SO}(5)$  of degree  $(s, s)$  for any  $p$ -adic unit  $s$ , and the proof is now completed by noting it is possible to choose  $r$  and  $s$  so that  $(r^2 s \alpha_3, r^4 s \alpha_7)$  is a pair of rational numbers. This shows that  $G_p(L_p \mathrm{SO}(5))$  contains only one element.

In the case of  $X = \mathrm{SO}(3)^k \times \mathrm{SO}(5)$ , the situation is slightly more complicated. We have to show that  $H_3(f) \in \mathrm{GL}_{k+1}(\mathbf{Q}_p)$  can be turned into the identity matrix by pre- and postcompositions as above. We argue similarly to the proof of Proposition 4.3. We have that  $\mathrm{Aut}(L_{\mathbf{Q}}X)$  surjects onto  $\mathrm{GL}_{k+1}(\mathbf{Q})$ .

As above, any  $p$ -adic integer can be realized as the  $H_3$ -degree of a map  $\mathrm{SO}(3) \rightarrow \mathrm{SO}(3)$ ,  $\mathrm{SO}(3) \rightarrow \mathrm{SO}(5)$ , or  $\mathrm{SO}(5) \rightarrow \mathrm{SO}(5)$ . Similarly, there is an integer  $N > 0$  and a map  $\mathrm{SO}(5) \rightarrow \mathrm{SO}(3)$  inducing multiplication by  $N$  on  $H_3$  ( $N = 48$  is possible, but that is irrelevant to the argument). To see this, represent the 3-dimensional generator of  $H^3(\mathrm{SO}(5))$  by a map  $g: \mathrm{SO}(5) \rightarrow K(\mathbf{Z}, 3)$  and consider the obstruction classes for lifting this map to  $\mathrm{SO}(3) \rightarrow K(\mathbf{Z}, 3)$ . They lie in finitely many torsion groups, so precomposing  $g$  with the product of their orders yields a map that lifts to  $\mathrm{SO}(3)$ .

This implies that in  $H_3$  any invertible matrix of the following form can be realized as an automorphism of  $L_p X$ :

$$\left( \begin{array}{c|c} & Nb_1 \\ * & \vdots \\ \hline * & Nb_k \\ * & * \end{array} \right)$$

where  $*$ ,  $b_i \in \mathbf{Z}_p$ . It remains to show that a matrix of the form  $\begin{pmatrix} I & * \\ 0 & 1 \end{pmatrix}$  can be written as a product of a matrix as above and a rational matrix. This follows from the easy fact that every  $p$ -adic rational can be written as a sum of a  $p$ -adic integer multiple of  $N$  and a rational.  $\square$

**THEOREM 4.8.** *Let  $X$  be a quasifinite loop space of type  $(3^k, 7^\epsilon)$ . Then  $X$  is homotopy equivalent to a compact, smooth, stably parallelizable manifold.*

*Proof.* In [24] it is proved  $L_p X$  is center-free for  $p$  large, and that the center is finite when  $\pi_1(X)$  is finite (our standing assumption). We may then construct a new space from  $L_p(X)/ZL_p(X)$  so that the original space is a finite covering space of this new space. Hence we may as well assume that  $L_p X$  is center-free for all  $p$ . By [16]  $L_p(X)$  can be written as a product of simple  $p$ -compact groups which by the classification of reflection groups will be of rank 1 in the case where there are only 3-dimensional generators. In this case it now follows from Lemma 4.6 that  $L_p(X)$  is a product of  $L_p SO(3)$ , and from Lemma 4.7(1) that  $L_{(p)}(X)$  is a product of  $L_{(p)} SO(3)$  and finally it follows from [30] that  $X$  is homotopy equivalent to a product of  $SO(3)$ . In case there is also a 7-dimensional generator we similarly get  $L_p X$  is a product of rank 1  $p$ -compact groups, and one of rank 2. It now follows from Theorem 6.1 that at the prime 2, the 2-compact group of rank 2 is  $L_2 Y = L_2 SO(5)$ , and by the classification at odd primes that the  $p$ -compact group of rank 2 is also  $L_p SO(5)$  (ignoring the loop structure). As above we now use Lemma 4.7(3) to show that  $X$  is in the Mislin genus of  $SO(3)^k \times SO(5)$ , hence by [30],  $X$  is homotopy equivalent to a stably parallelizable manifold.  $\square$

*Proof of the Main Theorem.* Theorems 4.4 and 4.8 together with the reduction steps in this section and the surgery arguments in section 3 imply the main theorem if we also show that the manifolds obtained are parallelizable, not only stably parallelizable.

To see this we use the criterion of Dupont [12, 34]. If the  $\dim(X)$  is even, the difference between parallelizability and stable parallelizability is determined by the Euler characteristic, which is obviously 0 for  $X$  being of the homotopy type of a loop space. In odd dimensions parallelizability is automatic in dimensions 1, 3, and 7, and in other dimensions it is determined by the mod 2 Kervaire semi-characteristic

$$\kappa(X; 2) = \sum \dim(H_{2i}(X; \mathbf{F}_2)) \in \mathbf{Z}/2.$$

But the cohomology of a loop space with  $\mathbf{F}_2$ -coefficients is a tensor product of truncated polynomial algebras  $\mathbf{F}_2[z]/(z^{2^k})$ , so this number is obviously zero.  $\square$

## 5. CONSTRUCTING 1-TORI IN $p$ -COMPACT GROUPS

The  $p$ -complete analog of a finite loop space is called a  $p$ -compact group. This is by definition a connected, pointed,  $\mathbf{Z}/p$ -local space  $BG$  such that  $G := \Omega BG$  is  $\mathbf{Z}/p$ -finite. We can (and will) choose a topological group model for  $G$  and call  $G$  itself a  $p$ -compact group. For a compact Lie group  $G$ , we will also write  $G$  for the associated  $p$ -compact group obtained by  $p$ -completion.

Recall [14] that every  $p$ -compact group  $G$  has a maximal torus  $T$ , a maximal torus normalizer  $N_G(T)$ , and a Weyl group  $W$  acting on  $T$ . These loop spaces fit

into a diagram

$$\begin{array}{ccccc} BT & \longrightarrow & BN_G(T) & \longrightarrow & BW \\ & \searrow & \downarrow & & \\ & & BG & & \end{array}$$

Here,  $BT \simeq K(\mathbf{Z}_p^n, 2)$  is homotopy equivalent to an Eilenberg-Mac Lane space of degree 2. We call  $n$  the rank of  $G$ . Let  $L = \pi_1(T) \cong \mathbf{Z}_p^n$  be the associated lattice. The top row of the diagram is a fibration and determines the action of  $W$  on  $T$ , or equivalently, on  $L$ . If  $G$  is connected, this representation is faithful and gives  $W$  the structure of a  $p$ -adic pseudo-reflection group.

We call a connected  $G$  semisimple if  $\pi_1(G)$  is finite, and simple if the associated representation  $W \rightarrow \mathrm{GL}(L \otimes \mathbf{Q})$  is irreducible.

The center of a  $p$ -compact group  $G$  is denoted by  $Z(G)$ . If  $G$  is either simply connected or center-free then it splits uniquely into a product of simple  $p$ -compact groups of the same sort.

For details and further notions we refer the reader to the survey articles [23] and [26] and the references mentioned there.

The main new ingredient in this section comes from the first author's thesis [3]. For any connected  $p$ -compact group, define  $S_G = (\Sigma_+^\infty G)^{hG}$  to be the homotopy fixed point spectrum of  $G$ , acting on its suspension spectrum by multiplication from the right. If  $G$  is the  $p$ -completion of a connected compact Lie group with Lie algebra  $\mathfrak{g}$  then  $S_G$ , equipped with the remaining left  $G$ -action, is equivariantly homotopy equivalent to the  $p$ -completion of  $\mathfrak{g} \cup \{\infty\}$  by results of Klein [21].

**THEOREM 5.1** ([3]). *For every  $p$ -compact group  $G$ ,  $S_G$  is a  $p$ -complete sphere. For an inclusion of  $p$ -compact groups  $H < G$ , we have a homotopy equivalence in the  $p$ -complete category*

$$G_+ \wedge_H S_H \simeq D(G/H)_+ \wedge S_G.$$

In particular, if  $T < G$  is a torus, the action of  $T$  on  $S_T$  is trivial, and we have

$$G/T_+ \simeq D(G/T)_+ \wedge S^{d-t}$$

where  $d = \dim G$  and  $t = \dim T$ . Hence  $G/T$  is a  $p$ -complete self-dual space of dimension  $d - t$ . In particular, it is finitely dominated and stably reducible.

**LEMMA 5.2.** *Let  $G$  be a nontrivial connected  $p$ -compact group with finite fundamental group, such that if  $p = 2$ ,  $G$  does not have type  $3^k$ . Then  $\mathrm{rk}_p(Z(G)) < \mathrm{rk}(G)$ . Moreover, if  $p = 2$  and  $\mathrm{rk}_2(Z(G)) = \mathrm{rk}(G) - 1$  then  $G$  has type  $(3^k, 7)$ .*

Here  $\mathrm{rk}_p$  is the  $p$ -rank of the abelian  $p$ -group  $Z(G)$ .

*Proof.* Let  $T < G$  be a maximal torus with Weyl group  $G$  and lattice  $L = \pi_1(T)$ . Dwyer and Wilkerson show in [15] that the  $p$ -discrete center of  $G$  is always contained in the center of  $\check{\mathbf{N}}(T)$ , the  $p$ -discrete normalizer of the maximal torus. This in turn is the same as the fixed points of  $\check{T}$  under the  $W$ -action. Thus the claim follows for  $p > 2$  from the classification of odd pseudo-reflection groups.

Now let  $p = 2$ . In terms of  $L$ , the order 2 elements of the center of  $G$  are contained in the invariants  $(L/2L)^W$ .

Assume that  $\mathrm{rk}_2(Z(G)) \geq \mathrm{rk}(G) - 1$ . This means that  $(L/2L)^W$  has codimension 1 or 0. We then know that the image of the representation  $W \rightarrow \mathrm{GL}(L/2L)$  consists

entirely of 2-torsion. Let  $K$  be the kernel and consider the diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & W & \longrightarrow & W/K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{Id} + 2 \text{End}(L) & \longrightarrow & \text{GL}(L) & \longrightarrow & \text{GL}(L/2L) & \longrightarrow & 1. \end{array}$$

Since  $\text{Id} + 2 \text{End}(L)$  is a 2-group, so is  $K$ , and since  $W/K$  is also an (elementary abelian) 2-group,  $W$  must be a 2-group. By inspection of the Clark-Ewing list of 2-adic pseudo-reflection groups, we see that the only possibilities are  $W = (Z/2)^k \times ((Z/2)^2 \wr Z/2)^l = W(\text{SU}(2)^k \times \text{Spin}(5)^l)$ . It follows from the classification of 2-compact groups up to rank 2 given in Section 6 that indeed,  $G/Z(G) \simeq L_2(\text{SO}(3)^k \times \text{SO}(5)^l)$ . Since  $Z \text{Spin}(5) = \mathbf{Z}/2$  and  $\text{rk}(\text{SO}(5)) = 2$ , only  $l = 0$  or  $1$  can occur, and  $l = 1$  if and only if  $\text{rk}_2(Z(G)) = \text{rk}(G) - 1$ . This concludes the proof since in this case the type is  $(3^k, 7)$ .  $\square$

**PROPOSITION 5.3.** *Let  $G$  be as in Lemma 5.2 and assume that  $H_{\mathbf{Q}_p}^3(G) \neq 0$ . Then  $G$  has a circle subgroup  $S$ , not meeting the center, such that  $H_{\mathbf{Q}_p}^*(G/S) \cong \bigwedge(t) \otimes R$  for some ring  $R$  and a 2-dimensional class  $t$ .*

*Proof.* Let  $T''$  be a subtorus of  $G$  which is minimal containing  $Z(G)$ . Extend to a maximal torus  $T = T' \times T''$ . Let  $r'$  be the dimension of  $T'$ .

By Lemma 5.2,  $r' \geq 1$ . Choose coordinates  $t_i : S^1 \rightarrow T$  in such a way that  $\{t_i \mid 1 \leq i \leq r'\}$  span  $T'$ . Let  $W$  be the Weyl group of  $G$  with respect to  $T$ . Rationally,

$$H_{\mathbf{Q}_p}^*(BG) \cong H_{\mathbf{Q}_p}^*(BT)^W \quad \text{and} \quad H_{\mathbf{Q}_p}^*(G/T) \cong H_{\mathbf{Q}_p}^*(BT) // H_{\mathbf{Q}_p}^*(BG).$$

Since  $H_{\mathbf{Q}_p}^2(BG) = 0$ , we know that  $H_{\mathbf{Q}_p}^2(BT) \xrightarrow{\sim} H_{\mathbf{Q}_p}^2(G/T) \cong \mathbf{Q}_p\{t_1, \dots, t_r\}$ , and since  $H_{\mathbf{Q}_p}^4(BG) \neq 0$ , we know that there is a nontrivial quadratic polynomial  $f(t_1, \dots, t_r) \in \mathbf{Q}_p[t_1, \dots, t_r]$  such that  $f \equiv 0$  in  $H_{\mathbf{Q}_p}^4(G/T)$ .

Define  $S \subseteq K(\mathbf{Z}_p^r, 1) = T$  by the coprime coordinates  $\alpha_1, \dots, \alpha_r \in \mathbf{Z}_p$ . Note that  $S$  will intersect  $Z(G)$  trivially if  $\alpha_1, \dots, \alpha_{r'}$  are also coprime. Let  $t \in H_{\mathbf{Q}_p}^2(G/S)$  be a generator. Under the map  $H_{\mathbf{Q}_p}^*(G/T) \rightarrow H_{\mathbf{Q}_p}^*(G/S)$ ,  $t_i$  is mapped to  $\alpha_i t$ . Therefore, the polynomial  $0 = f(t_1, \dots, t_r) \in H_{\mathbf{Q}_p}^4(G/T)$  is mapped to  $f(\alpha_1 t, \dots, \alpha_r t) = f(\alpha_1, \dots, \alpha_r) t^2$  which then must be 0. Since  $f$  is nonzero, we can choose  $\alpha_i$  such that  $f(\alpha_1, \dots, \alpha_r) \neq 0$  and  $\alpha_1, \dots, \alpha_{r'}$  coprime, whence  $t^2 = 0$ . Thus  $S$  satisfies the rational cohomology condition.  $\square$

*Proof of Theorem 4.4.* Let  $G = L_p X$ . Then  $G$  is a  $p$ -compact group, and by [10],  $H_{\mathbf{Q}_p}^3(G) \neq 0$ . By Prop. 5.3, there is a circle subgroup  $S^1 \rightarrow G$ , not meeting the center. By Theorem 5.1,  $G/S^1$  is stably parallelizable. The quasifiniteness condition on  $Y$  and the orientability of  $X \rightarrow Y$  are clearly satisfied.

The 1-torus is rationally splitting because the two dimensional class in  $H_{\mathbf{Q}_p}^*(G/S^1)$  produces a split map to  $L_{\mathbf{Q}} L_p S^2$  making the following diagram commute:

$$\begin{array}{ccc}
L_{\mathbf{Q}}G & \xleftrightarrow{\quad} & L_{\mathbf{Q}}L_pS^3 \\
\downarrow & & \downarrow \\
L_{\mathbf{Q}}G/S^1 & \xleftrightarrow{\quad} & L_{\mathbf{Q}}L_pS^2 \\
& \searrow & \swarrow \\
& K(\mathbf{Q}_p, 2) & 
\end{array}$$

Now let  $p = 2$ . By Lemma 5.2, the dimension of a maximal torus  $T'$  not meeting the center is at least 2. We may choose  $T'$  in such a way that the constructed  $S^1$  is contained in  $T'$ , this can be extended to an  $S^1 \times \mathbf{Z}/2 < T'$ , thus giving a rationally splitting double 1-torus  $\square$

## 6. 2-COMPACT GROUPS OF RANK 2

In this section we will classify all simple 2-compact groups of rank 2. For the purpose of this paper, we really only need the uniqueness of  $\text{Spin}(5)$ , but little extra work will also deal with the general case.

**THEOREM 6.1.** *Any simple 2-compact group  $G$  of rank 2 is isomorphic to the 2-adic completion of  $\text{SU}(3)$ ,  $\text{Spin}(5) = \text{Sp}(2)$ ,  $\text{SO}(5)$ , or  $G_2$ .*

The rest of this section is devoted to the proof of this statement.

Let  $U$  be a finite dimensional  $\mathbf{Q}_2$ -representation of a finite group  $W$ . A  $W$ -lattice  $L$  of  $U$  is a  $\mathbf{Z}_2$ -lattice  $L \subset U$  of maximal rank fixed under the action of  $W$ ; i.e.  $L$  is a  $\mathbf{Z}_2[W]$ -module and  $L \otimes \mathbf{Q} \cong U$ . Two  $W$ -lattices  $L$  and  $L'$  of  $U$  are called *isomorphic* if  $L \cong L'$  as  $\mathbf{Z}_2[W]$ -modules. A  $W_1$ -lattice  $L_1$  and a  $W_2$ -lattice  $L_2$  are called *weakly isomorphic* if there exists an isomorphism  $W_1 \xrightarrow{\sim} W_2$  such that  $L_1$  and  $L_2$  are isomorphic as  $W_1$  lattices.

We say that two  $p$ -compact groups  $G$  and  $G'$  have the same Weyl group data if the representations  $W_G \rightarrow \text{GL}(L_G)$  and  $W_{G'} \rightarrow \text{GL}(L_{G'})$  are weakly isomorphic. Renaming the elements of  $W_{G'}$ , we can always identify  $W_{G'}$  with  $W_G$  and assume that the two lattices are actually isomorphic.

From the Clark-Ewing list [11] we get a complete list of all irreducible reflection groups of rank 2 defined over  $\mathbf{Q}_2$ . They are the dihedral groups  $D_6$ ,  $D_8$  and  $D_{12}$  with their standard representation as reflection groups. In fact, these are the only dihedral groups which can be represented as reflection groups over  $\mathbf{Q}_2$ . They correspond to the rational Weyl group representations of  $\text{SU}(3)$ ,  $\text{Spin}(5)$  or  $\text{SO}(5)$ , and  $G_2$ , respectively. The classification of Clark and Ewing only works up to weak equivalence.

The Lie groups  $\text{Spin}(5)$  and  $\text{Sp}(2)$  are isomorphic, hence they have the same Weyl group data. In the following, we will always use the one of these two which seems more natural.

If a  $p$ -compact group  $G$  has finite fundamental group then the universal cover  $\tilde{G}$  is again a  $p$ -compact group, and  $G$  and  $\tilde{G}$  have the same rational Weyl group data. In that case,  $G \cong \tilde{G}/Z$ , where  $Z \subset \tilde{G}$  is a central subgroup [24]. Simple  $p$ -compact groups have finite fundamental groups [24]. Therefore, Theorem 6.1 is a consequence of the following classification result for simply connected simple 2-compact groups.

**THEOREM 6.2.** *Let  $H = \mathrm{SU}(3)$ ,  $\mathrm{Sp}(2)$  or  $G_2$ . A simply connected 2-compact group  $G$  has the same rational Weyl group data as  $H$  if and only if  $G$  and  $H$  are isomorphic as 2-compact groups.*

For the proof of this theorem we first have to classify all 2-adic lattices of the representation  $W_H \rightarrow \mathrm{GL}(L_H \otimes \mathbf{Q})$ .

**LEMMA 6.3.** *Let  $W \rightarrow \mathrm{GL}(U)$  be a reflection group, where  $U = (\mathbf{Q}_2)^2$ .*

- (1) *If  $W = D_6$  or  $W = D_{12}$  then, up to isomorphism, there exists exactly one  $W$ -lattice of  $U$ .*
- (2) *If  $W = D_8$  each  $W$ -lattice of  $U$  is isomorphic either to  $L_{\mathrm{SO}(5)}$  or to  $L_{\mathrm{Spin}(5)}$ . Both lattices are weakly isomorphic.*

*Proof.* Case (1) follows from [2, Proposition 4.3 and Theorem 6.2].

Now let  $W = D_8$ , and let  $L$  be a  $W$ -lattice of  $U$ . For  $r$  big enough, the lattice  $2^r L$ , the submodule of all elements divisible by  $2^r$ , is a submodule of  $L_{\mathrm{SO}(5)} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ . We choose  $r$  minimal with this property, i.e.  $2^r L \subset L_{\mathrm{SO}(5)}$  but  $2^r L \not\subset 2L_{\mathrm{SO}(5)}$ . Since  $L \otimes \mathbf{Q} \cong L_{\mathrm{SO}(5)} \otimes \mathbf{Q}$ , we get a short exact sequence of  $\mathbf{Z}_2[W]$ -modules

$$0 \rightarrow 2^r L \rightarrow L_{\mathrm{SO}(5)} \xrightarrow{\rho} Q \rightarrow 0.$$

The minimality of  $r$  implies that  $Q$  is a finite cyclic group; i.e.  $Q \cong \mathbf{Z}/(2^s)$  generated by  $\rho(1, 0)$  or  $\rho(0, 1)$ . The dihedral group  $D_8$  is generated by the three elements  $\sigma_1, \sigma_2, \tau$ , where  $\sigma_i$  multiplies the  $i$ -th coordinate by  $-1$  and  $\tau$  exchanges the two coordinates. Since the automorphism group of  $Q$  is abelian, the action of  $W$  on  $Q$  factors through the abelianization of  $W$ . It follows that the element  $\sigma_1 \sigma_2 = \sigma_1 \tau \sigma_1 \tau$  acts trivially on  $Q$ . Hence the elements  $(1, 0), (0, 1) \in L_{\mathrm{SO}(5)}$  are mapped onto elements of order 2 in  $Q$ . Thus, either  $Q = 0$  or  $Q = \mathbf{Z}/2$ . In the first case, we have  $L \cong L_{\mathrm{SO}(5)}$ . In the second case,  $D_8$  acts trivially on  $Q$  with  $\rho(1, 0) = \rho(0, 1) \neq 0$  in  $\mathbf{Z}/2$  and consequently  $L \cong L_{\mathrm{Spin}(5)}$ . This proves the first part of (2).

The second part follows from the facts that  $L_{\mathrm{Sp}(2)}$  and  $L_{\mathrm{Spin}(5)}$  are weakly isomorphic and that  $L_{\mathrm{Sp}(2)}$  and  $L_{\mathrm{SO}(5)}$  are isomorphic.  $\square$

*Proof of Theorem 6.2:* If  $G$  and  $H$  have the same rational Weyl group data, the above lemma shows that they also have the same 2-adic Weyl group data. We can assume that  $W := W_G = W_H$  and that  $L := L_G = L_H$ . We can also identify the maximal tori  $T := T_G \cong T_H$ .

This implies  $H \cong G$  for  $H = \mathrm{SU}(3)$  [25] and for  $H = G_2$  [36].

For  $H = \mathrm{Sp}(2)$ , uniqueness result are only known in terms of the maximal torus normalizer [28, 35]. We have to show that  $BN_G \simeq BN_{\mathrm{Sp}(2)}$ .

Since  $G$  and  $\mathrm{Sp}(2)$  have the same rational Weyl group data, they have isomorphic rational cohomology. Hence,  $H_{\mathbf{Q}_2}^*(X)$  is an exterior algebra with generators in dimensions 3 and 7. If  $H^*(G; \mathbf{Z}_2)$  has 2-torsion, then  $G$  and  $G_2$  have isomorphic mod-2 cohomology [20]. The Bockstein spectral sequence then shows that  $G$  does not have the correct rational cohomology. Therefore,  $G$  has no 2-torsion, and  $H^*(G; \mathbf{Z}_2)$  is an exterior algebra with generators in dimensions 3 and 7. Hence  $H^*(BG; \mathbf{F}_2) \cong \mathbf{F}_2[x_4, x_8]$ . Since  $H^*(BG; \mathbf{F}_2)$  is a finitely generated module over  $H^*(BT; \mathbf{F}_2)$ , the composition

$$\begin{aligned} H^*(BG; \mathbf{F}_2) &\cong H^*(BG; \mathbf{Z}_2) \otimes \mathbf{F}_2 \rightarrow H^*(BT; \mathbf{Z}_2)^W \otimes \mathbf{F}_2 \\ &\cong H^*(B\mathrm{Sp}(2); \mathbf{F}_2) \rightarrow H^*(BT; \mathbf{F}_2) \end{aligned}$$

is a monomorphism.

The isomorphism  $H^*(BT; \mathbf{Z}_2)^W \otimes \mathbf{F}_2 \cong H^*(B\mathrm{Sp}(2); \mathbf{F}_2)$  follows from the fact that  $G$  and  $\mathrm{Sp}(2)$  have the same 2-adic Weyl group data (Lemma 6.3). Since the first and third term are both polynomial algebras of the same type,

$$H^*(BG; \mathbf{F}_2) \rightarrow H^*(BT; \mathbf{Z}_2)^W \otimes \mathbf{F}_2 \cong H^*(B\mathrm{Sp}(2); \mathbf{F}_2)$$

is an isomorphism.

Let  $t \subset T$  denote the elements of order 2 and  $K := \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$  the subgroup of diagonal quaternionic matrices. We have a chain of inclusion  $t \subset T \subset K \subset \mathrm{Sp}(2)$  and  $K = C_{\mathrm{Sp}(2)}(t)$ . The action of  $D_8$  on  $t$  factors through the  $\mathbf{Z}/2$ -action on  $t$  given by switching the coordinates.

Now we use Lannes' T-functor theory (e.g. see [33]). We get a map  $f : Bt \rightarrow BG$  which looks in mod 2 cohomology like the map  $Bt \rightarrow B\mathrm{Sp}(2)$ . This map is  $\mathbf{Z}/2$ -equivariant up to homotopy. The mod 2 cohomology of the classifying space  $BC_G(t) := \mathrm{map}(Bt, BG)_f$  of the centralizer  $C_G(t)$  can be calculated with the help of Lannes' T-functor and  $H^*(BC_H(t); \mathbf{F}_2) \cong H^*(BC_{\mathrm{Sp}(2)}(t); \mathbf{F}_2) \cong H^*(BG; \mathbf{F}_2)$ . Moreover, the Weyl group of  $C_G(t)$  is given by the elements of  $D_8$  acting trivially on  $t$ . Hence  $W_{C_G(t)} \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ . By [13][Theorem 0.5B], this implies that  $BC_G(t) \simeq BK$ . We will identify  $C_G(t)$  with  $K$ . The  $\mathbf{Z}/2$ -action on  $t$  induces a  $\mathbf{Z}/2$ -action on  $K$ . Since  $Bt \rightarrow BG$  was  $\mathbf{Z}/2$ -equivariant up to homotopy, the inclusion  $BC_G(t) \rightarrow BG$  extends to a map  $BL := BK_{h\mathbf{Z}/2} \rightarrow BG$ . In this case, the homotopy orbit space  $BL$  happens to be a 2-compact group and has the same Weyl group as  $G$ . That is,  $N_L = N_G$ . Moreover, the space  $BL$  is part of a fibration

$$BK \rightarrow BL \rightarrow B\mathbf{Z}/2,$$

which is classified by obstructions in  $H^*(B\mathbf{Z}/2; \pi_*(B\mathrm{aut}_1(BK)))$ .

Here,  $\mathrm{aut}_1(BK)$  is the monoid of self-equivalences of  $BK$  homotopic to the identity. Since  $\mathrm{aut}_1(BK) \simeq (B\mathbf{Z}/2)^2$  [15] and since  $\mathbf{Z}/2$  acts on  $\pi_2(B^2(\mathbf{Z}/2)^2) \cong (\mathbf{Z}/2)^2$  by switching the coordinates, all obstruction groups vanish and the above fibration splits. This shows that  $BL \simeq B(K \rtimes \mathbf{Z}/2) := BK'$  and that  $BN_G = BN_L \simeq BN_{K'} = BN_{\mathrm{Sp}(2)}$ . That is,  $G$  and  $\mathrm{Sp}(2)$  have isomorphic maximal torus normalizer, and hence  $G \cong \mathrm{Sp}(2)$ .  $\square$

REMARK. The only simply connected 2-compact group of rank 1 is  $S^3$ . Hence we get the following complete list (up to isomorphism) of connected 2-compact groups of rank 2:

$S^1 \times S^1$ ,  $S^1 \times S^3$ ,  $U(2)$ ,  $S^1 \times \mathrm{SO}(3)$ ,  $S^3 \times S^3$ ,  $S^3 \times \mathrm{SO}(3)$ ,  $\mathrm{SO}(3) \times \mathrm{SO}(3)$ ,  $\mathrm{SO}(4)$ ,  $\mathrm{SU}(3)$ ,  $\mathrm{Sp}(2)$ ,  $\mathrm{SO}(5)$ , and  $G_2$ .

COROLLARY 6.4. *For any simple connected 2-compact group  $G$  of rank 2, there exists a homomorphism  $S^3 \rightarrow G$  such that the composition  $S^3 \rightarrow G \rightarrow \overline{G}$  is a monomorphism and such that  $H_{\mathbf{Q}_2}^4(BG) \cong H_{\mathbf{Q}_2}^4(BS^3)$ .*

*Proof.* Because of Theorem 6.1 we only have to check this for the compact connected Lie groups  $SU(3)$ ,  $Sp(2)$ ,  $SO(5)$  and  $G_2$ . There exists a chain of monomorphisms  $S^3 = SU(2) \subset SU(3) \subset G_2$ . Both groups,  $SU(3)$  and  $G_2$ , are 2-adically center-free. This proves the claim in these two cases. Let  $S^3 \subset Sp(2)$  denote the inclusion into the first coordinate. Since the intersection of  $S^3$  and the center of  $Sp(2)$  is trivial, the composition  $S^3 \subset Sp(2) \rightarrow SO(5)$  is also a monomorphism. This proves the claim in the other cases. The condition on the rational cohomology is obvious.  $\square$

## 7. GEOMETRIC PROPERTIES OF LOOP SPACES

In the final sections we describe a different proof of our main theorem, which is based on the concept of special 1-tori. This concept was exploited by the fourth author to prove that finite loop spaces in the genus of a compact connected Lie group are homotopy equivalent to stably parallelizable manifolds [29, 30].

DEFINITION 7.1. For a subring of the rationals  $R$ , a nilpotent  $R$ -local space  $X$  admits an  $R$ -local *special 1-torus* if, up to homotopy, there exists a diagram of orientable fibrations of nilpotent spaces

$$\begin{array}{ccccc}
 L_R S^1 & \longrightarrow & L_R S^3 & \longrightarrow & L_R S^2 \\
 \parallel & & \downarrow & & \downarrow \\
 L_R S^1 & \longrightarrow & X & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & Z & \xlongequal{\quad} & Z
 \end{array}$$

such that

- (1)  $Z$  is  $R$ -finite.
- (2)  $Y$  is  $R$ -finite and stably reducible.
- (3) Localized at 0, the diagram is homotopy equivalent to

$$\begin{array}{ccccc}
 L_{\mathbf{Q}} S^1 & \longrightarrow & L_{\mathbf{Q}} S^3 & \longrightarrow & L_{\mathbf{Q}} S^2 \\
 \parallel & & \downarrow & & \downarrow \\
 L_{\mathbf{Q}} S^1 & \longrightarrow & L_{\mathbf{Q}} Z \times L_{\mathbf{Q}} S^3 & \longrightarrow & L_{\mathbf{Q}} Z \times S_0^2 \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & L_{\mathbf{Q}} Z & \xlongequal{\quad} & L_{\mathbf{Q}} Z
 \end{array}$$

where all vertical fibrations are trivial.

In [30] the fourth author showed that for a quasi-finite Poincaré complex  $X$ , the existence of rationally splitting  $\mathbf{Z}_{(p)}$ -local special 1-tori implies the existence of a global special 1-torus and that, as a consequence,  $X$  is homotopy equivalent to a compact, smooth, stably parallelizable manifold. The next proposition, which will be proved in Section 8, allows to establish  $\mathbf{Z}_{(p)}$ -local special 1-tori.

PROPOSITION 7.2. *Let  $X$  be a connected quasifinite loop space which is not of type  $3^k$ . Then there exists a loop space  $Y$  and a fibration  $A \rightarrow L_{(p)}BS^3 \xrightarrow{f} L_{(p)}BY$  such that  $A$  is simple,  $\mathbf{Z}_{(p)}$ -finite and such that  $Y$  and  $X$  are homotopy equivalent spaces. Moreover, localized at 0, there exists a left inverse  $s : L_{\mathbf{Q}}BY \rightarrow L_{\mathbf{Q}}BS^3$  of  $f$ , i. e.  $sf_0 = id_{L_{\mathbf{Q}}BS^3}$ .*

The proof of this proposition will be given in Section 9.

COROLLARY 7.3. *Under the above assumption, the localization  $L_{(p)}X$  admits a  $\mathbf{Z}_{(p)}$ -local special 1-torus.*

*Proof.* Since the loop space  $Y$  of the last proposition is equivalent to  $X$  we only have to prove the claim for  $Y$  or equivalently, we may assume that there exist a fibration  $L_{(p)}BS^3 \rightarrow L_{(p)}BX$  with the desired properties.





8. PARTICULAR SUBGROUPS OF  $p$ -COMPACT GROUPS.

In this section we will construct particular subgroups of  $p$ -compact groups whose center-free quotient contains one simple factor of rank at least 2.

PROPOSITION 8.1. *Let  $G$  be a semisimple connected  $p$ -compact group such that  $r = \dim H_{\mathbf{Q}_p}^4(BX) \neq 0$ . If  $p = 2$ , assume that  $G$  is not of type  $3^k$ . Then there exists a compact Lie group  $H$  and a map  $f : BH \rightarrow BG$  such that the following hold:*

- (1)  $H \cong S^3 \times H'$  with  $H'$  semisimple and its universal cover  $\tilde{H}'$  isomorphic to  $(S^3)^{r-1}$ . If  $p$  is odd, we can choose  $H = (S^3)^r$ .
- (2) The induced map  $H_{\mathbf{Q}_p}^4(BG) \rightarrow H_{\mathbf{Q}_p}^4(BH)$  is an isomorphism.
- (3) The homotopy fiber  $G/H$  of  $f$  is simple and  $\mathbf{Z}/p$ -finite.

For the proof we need the following lemma.

LEMMA 8.2. *Let  $G$  be a simple, simply connected 2-compact group satisfying that  $H_{\mathbf{Q}_2}^4(BG) \neq 0$ . Then there exists a map  $BS^3 \rightarrow BG$  such that*

- (1)  $H_{\mathbf{Q}_2}^4(BG) \rightarrow H_{\mathbf{Q}_2}^4(BS^3)$  is an isomorphism;
- (2) if  $\text{rk}(G) \geq 2$ ,  $BS^3 \rightarrow BG \rightarrow B\overline{G}$  is a monomorphism.

*Proof.* Since  $G$  is simple and since  $H_{\mathbf{Q}_2}^4(BG) \neq 0$ , the Weyl group  $W_G$  is an honest reflection group and already defined over  $\mathbf{Z}$ . This follows from the classification of irreducible pseudo-reflection groups. Actually, the Weyl group is isomorphic to the Weyl group of a compact Lie group.

If  $W_G$  is abelian then  $BG$  is either  $BSO(3)$  or  $BS^3$  and the first part is obvious. Hence we can assume that  $W_G$  is nonabelian. Let  $W' \subset W_G$  be a subgroup of the Weyl group of  $G$  generated by two non-commuting reflections. Let  $T \subset T_G^{W'} \subset T_G$  denote the connected component of the fixed-point set of the  $W_G$ -action on  $T_G$ , which has codimension 2. The centralizer  $C = C_G(T)$  is a connected 2-compact group whose Weyl group  $W_C$  contains  $W'$  [24]. There exists a finite covering of  $C$  which splits into a product  $K \times T$  where  $K$  is a simply connected 2-compact group of rank 2 with Weyl group isomorphic to  $W_C$ . The action of  $W'$  on the maximal torus  $T_K$  of  $K$  gives rise to an irreducible representation over  $\mathbf{Q}_2$ . Otherwise,  $W'$  would split into a product and the two chosen reflections would commute. Hence, the 2-compact group  $K$  is simple and of rank 2.

Let  $G'$  be the simple simple simply connected compact Lie group with the same Weyl group. The above construction is only based on the Weyl group action of  $W_G$  on  $T_G$ , Hence, applying the construction to  $G'$ , establishes a map  $BK' \rightarrow BG'$  which as a map between classifying spaces of compact Lie groups, is defined globally and which in rational cohomology induces the same map as the composition  $BK \rightarrow BC \rightarrow BG$  only defined  $\mathbf{Z}/p$ -locally. By [1]  $H_{\mathbf{Q}_p}^4(BG') \rightarrow H_{\mathbf{Q}}^4(BK')$  is nontrivial, in fact an isomorphism. And the same holds for  $BK \rightarrow BG$ .

Let  $S^3 \subset K$  be the subgroup constructed in Proposition 6.4

The subgroup  $S^3 \subset G$  is constructed via the composition  $S^3 \rightarrow K \rightarrow G$ . where  $K$  is a 2-compact group of rank 2. we get a diagram

$$\begin{array}{ccccc}
 S^3 & \longrightarrow & K & \longrightarrow & X \\
 \downarrow = & & \downarrow & & \downarrow \\
 S^3 & \longrightarrow & K/A & \longrightarrow & \overline{X} \\
 \downarrow = & & \downarrow & & \\
 S^3 & \longrightarrow & K/A & & 
 \end{array}$$

where  $A$  denotes the kernel of the composition  $K \rightarrow G \rightarrow \overline{G}$ . The bottom right arrow is a monomorphism. And since  $S^3 \rightarrow K \rightarrow \overline{K}$  is a monomorphism, the same holds for the bottom left arrow as well as for the composition in the bottom row. This proves the second part. Part (1) also follows from Proposition 6.4.  $\square$

REMARK. The same statement holds for odd primes. In this case, the proof for the second claim even simplifies. Since  $S^3$  is center-free at odd primes, we do not need to construct a subgroup of rank 2. Choosing a subgroup of  $W_G$  generated by one reflection will produce a monomorphism  $S^3 \rightarrow G$  with all the desired properties. This clarifies a detail overseen in [27]. The argument for the first part given there is not complete.

*Proof of Proposition 8.1.* We can compare the statement with [27, Prop. 3.1]. For odd primes there is no difference. Hence we have to prove the statement only for  $p = 2$ . In this case we have an extra assumption on the rational type and the additional output is that  $H$  contains a factor  $S^3$ .

Since  $H_{\mathbf{Q}_p}^*(G)$  has a generator of degree greater than 3, the Weyl group  $W_G$  is nonabelian [13, Theorem 0.5B]. The universal cover  $\tilde{G}$  of  $G$  splits into a direct product  $\tilde{G} \cong \prod G_i$  of simple, simply connected pieces [16]. Since  $G$  and  $\tilde{G}$  have isomorphic Weyl groups, we can assume that  $G_1$  has a nonabelian Weyl group  $W_1$ . If  $H_{\mathbf{Q}_p}^4(BG_1) \neq 0$ , Lemma 8.2 will produce a monomorphism  $f : BS^3 \rightarrow BG_1$  such that  $H_{\mathbf{Q}_p}^4(f)$  is an isomorphism and such that  $BS^3 \rightarrow BG_1 \rightarrow \overline{BG_1}$  is a monomorphism.

If for all factors with nonabelian Weyl group this rational cohomology group vanishes, there exist a factor  $G_2$  of rank 1 with  $H_{\mathbf{Q}_p}^*(BG_2) \cong H_{\mathbf{Q}_p}^*(BS^3)$ . This implies that  $G_2 \cong S^3$ . We define  $G'_1 := G_1 \times G_2$ . Since the Weyl group  $W_{G_1}$  is defined over  $\mathbf{Q}_2$  it is honest reflection group and the arguments of Lemma 8.2 show that the second part does hold in this case. Hence, we can then define a map  $BS^3 \rightarrow BG'_1 = BG_1 \times BG_2$  which is the identity on the second factor and satisfies the claims of Lemma 8.2.

Now we can proceed similarly as in [27]. For all other pieces with  $H_{\mathbf{Q}_p}^4(BG_i) \neq 0$  there exist monomorphisms  $BH_i \rightarrow BG_i$  inducing an isomorphism on  $H^4(-; \mathbf{Z}_p) \otimes \mathbf{Q}$  such that  $H_i$  is isomorphic to  $S^3$  or to  $SO(3)$  (see [27]). This produces a homomorphism  $\prod H_i \rightarrow \prod G_i \cong \tilde{X} \rightarrow X$  of  $p$ -compact groups. The kernel  $K$  of this homomorphism, which might be nontrivial, is a central subgroup of  $H_1 \times \prod_{i>1} H_i$ . Since the center-free quotient  $\overline{G}$  is isomorphic to  $\prod_i \overline{G}_i$  we have a homomorphism  $G \rightarrow \overline{G}_1$ . By construction the composition  $S^3 \rightarrow G_1 \rightarrow \overline{G}_1$  is a monomorphism.

We get a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & S^3 \times \prod_{i>1} H_i & \longrightarrow & G \\ \downarrow = & & \downarrow & & \downarrow \\ K & \longrightarrow & S^3 & \longrightarrow & \overline{G}_1 \end{array}$$

where the right arrow in the bottom row is a monomorphism. Since  $\overline{G}_1$  is center-free the composition  $K \rightarrow S^3 \times \prod_{i>1} H_i \rightarrow S^3$  is trivial. Therefore,  $K$  is a subgroup of  $\prod_{i>1} H_i$  and the map  $S^3 \times \prod_{i>1} H_i \rightarrow G$  factors through a monomorphism  $H := S^3 \times ((\prod_{i>1} H_i)/K) \rightarrow G$  with all the desired properties.  $\square$

## 9. PROOF OF PROPOSITION 7.2

The proof of Proposition 7.2 is based on an arithmetic square argument. First we need a statement about the existence of a particular sub-loop space. Actually, a  $\mathbf{Z}_{(p)}$ -local version of the next proposition would be sufficient for our purpose, but with no extra effort we can prove a global result.

**PROPOSITION 9.1.** *Let  $X$  be a semisimple  $\mathbf{Z}$ -finite loop space not of type  $3^k$ . Then there exist a semisimple compact Lie group  $H \cong S^3 \times H'$ , loop spaces  $U$  and  $Y$ , and a fibration*

$$A \rightarrow BU \rightarrow BY$$

such that the following hold:

- (1) *The universal cover of  $H$  is isomorphic to a product of  $S^3$ 's.*
- (2)  *$H^4(BY; \mathbf{Q}) \rightarrow H^4(BU; \mathbf{Q})$  is an isomorphism.*
- (3)  *$A$  is simple and  $\mathbf{Z}$ -finite.*
- (4) *The spaces  $H$  and  $U$  as well as  $X$  and  $Y$  are homotopy equivalent.*
- (5) *For each prime  $p$  there exists a commutative diagram*

$$\begin{array}{ccc} L_p BU & \longrightarrow & L_p BY \\ \downarrow & & \downarrow \\ L_p BH & \longrightarrow & L_p BX \end{array}$$

where the vertical maps are equivalences. The same holds for the rationalizations of the classifying spaces.

*Proof.* This statement is a refinement of [27, Prop. 1.4]. The proof of that statement is an arithmetic square argument which uses its  $p$ -completed version as input [27, Prop. 3.1]. The proof carries over word for word. We only have to replace that proposition by a  $p$ -completed version of the above claim, namely by Proposition 8.1. Claim (5), which is not part of [27, Prop. 1.4], follows from the arithmetic square argument and Proposition 8.1.  $\square$

**REMARK.** The above proposition establishes an oriented fibration  $H \rightarrow X \rightarrow A$ . The existence of such an oriented fibration is sufficient to show that the finiteness obstruction vanishes and that every quasifinite loop space is actually finite (see [27]). The existence of a special 1-tori is needed for the vanishing of the appropriate surgery obstruction.

For the proof of Proposition 7.2 we need a higher dimensional version of Lemma 4.2.

LEMMA 9.2. *Let  $A \in \mathrm{GL}(n, \mathbf{Z}_p)$ . Then, there exists a vector  $v = (v_1, \dots, v_n) \in (\mathbf{Z}_p)^n$  such that  $v_i$  is a square of a  $p$ -adic unit for all  $i$  and such that  $Av$  is a vector whose components are given by elements of  $\mathbf{Z}_{(p)}$ .*

*Proof.* Let  $B = A^{-1}$ . We have to solve the following problem: Find a vector  $w \in \mathbf{Z}_{(p)}^n$  such that  $Bw \neq 0$  has as components squares of  $p$ -adic units. The question whether a  $p$ -adic unit is a square can be decided by reducing to  $\mathbf{Z}/p$  for  $p$  odd or to  $\mathbf{Z}/8$  for  $p = 2$ . In both cases the reduction  $\bar{B}$  of  $B$  is an invertible matrix and therefore induces an epimorphism on  $V = (\mathbf{Z}/p)^n$ . In particular, if  $\bar{v} \in V$  is a vector with components given by squares mod  $p$  such that all entries are units in  $\mathbf{Z}/p$ , there exists a vector  $w \in (\mathbf{Z})^n$  such that  $\bar{B}w = \bar{v}$ . Hence,  $Bw$  is a vector whose components are squares of nontrivial  $p$ -adic units. For  $p = 2$ , the same argument works, we only have to replace  $\mathbf{Z}/p$  by  $\mathbf{Z}/8$ .  $\square$

*Proof of Proposition 7.2.* Let  $U$  and  $Y$  denote the loop spaces and  $H$  the Lie group constructed in Proposition 9.1 Since  $L_p BU \simeq L_p BH$  and  $L_{\mathbf{Q}} BU \simeq L_{\mathbf{Q}} BH$  we have a pullback diagram

$$\begin{array}{ccc} L_{(p)}BU & \xrightarrow{\quad} & L_pBH \\ \downarrow & & \downarrow \\ L_{\mathbf{Q}}BH & \xrightarrow{\quad} & L_{\mathbf{Q}}L_pBH \xrightarrow{\alpha} L_{\mathbf{Q}}L_pBH \end{array}$$

The map  $\alpha$  is an equivalence and induces a continuous map in homotopy. The homotopy groups  $\pi_*(L_{\mathbf{Q}}L_pBH)$  carry a natural topology since  $\pi_*(L_pBH) \cong \pi_*(BH) \otimes \mathbf{Z}_p$  (details may be found in [39]).  $L_{\mathbf{Q}}L_pBH \simeq K(\mathbf{Q}_p^r, 4)$  is a rational Eilenberg-Mac Lane space. Since self-maps of rational Eilenberg-Mac Lane spaces are determined by the induced maps in homotopy, and since  $\alpha$  induces a continuous map in homotopy, we can think of  $\alpha$  as a matrix in  $\mathrm{GL}(n, \mathbf{Q}_p)$  inducing a continuous self-equivalence of  $(\mathbf{Q}_p)^n$ . Such matrices can be written as a product  $\gamma\rho$  where  $\gamma \in \mathrm{GL}(n, \mathbf{Z}_p)$  and  $\rho \in \mathrm{GL}(n; \mathbf{Q})$ .

Since  $\rho$  can be realized as a self-equivalence of  $L_{\mathbf{Q}}HG$ , replacing  $\alpha$  by  $\gamma$  does not change the homotopy type of the pullback. Hence we may assume that  $\alpha \in \mathrm{GL}(n, \mathbf{Z}_p)$ .

Every square unit of  $\mathbf{Z}_p$ , considered as a self-map of  $\pi_4(L_pBS^3)$  can be realized by self-equivalence  $L_pBS^3 \rightarrow L_pBS^3$ . Since  $H \cong S^3 \times H'$  and since the universal cover of  $H'$  is a product of  $S^3$ 's, Lemma 9.2 shows that there exists a map  $BS^3 \rightarrow L_pBS^3 \times L_pB\tilde{H}'$  such that the composition  $L_pBS^3 \rightarrow L_pBS^3 \times L_pB\tilde{H}' \rightarrow L_pBH$  is a monomorphism and such that the composition

$$\begin{aligned} L_{(p)}BS^3 &\rightarrow L_pBS^3 \rightarrow L_p(BS^3 \times B\tilde{H}') \\ &\rightarrow L_{\mathbf{Q}}L_p(BS^3 \times B\tilde{H}') \simeq L_{\mathbf{Q}}L_pBH \xleftarrow{\alpha^{-1}} L_{\mathbf{Q}}L_pBH \end{aligned}$$

lifts to a map  $L_{(p)}BS^3 \rightarrow L_{\mathbf{Q}}BH \simeq L_{\mathbf{Q}}(BS^3 \times BH')$ . Moreover, localized at 0, composition with the projection on the first factor is an equivalence. This establishes a map  $L_{(p)}BS^3 \rightarrow L_{(p)}BU$  such that the completion of  $L_{(p)}BS^3 \rightarrow L_{(p)}BU$  is induced by the monomorphism  $L_pS^3 \rightarrow L_pS^3 \times L_pH' \cong L_pH$  of  $p$ -compact groups. This shows that the homotopy fiber of  $L_{(p)}BS^3 \rightarrow L_{(p)}BU$  is simple and  $\mathbf{Z}_{(p)}$ -finite

as is the homotopy fiber of the composite  $f : L_{(p)}BS^3 \rightarrow L_{(p)}BU \xleftarrow{g} L_{(p)}BY$ . Since  $H^4(BY; \mathbf{Q}) \cong H^4(BU; \mathbf{Q})$ , there exists a left inverse  $s : L_{\mathbf{Q}}BY \rightarrow L_{\mathbf{Q}}BU$  for  $L_{\mathbf{Q}}g$ . Projection onto the first factor gives a left inverse of  $L_{\mathbf{Q}}BS^3 \rightarrow L_{\mathbf{Q}}BU \simeq L_{\mathbf{Q}}BS^3 \times L_{\mathbf{Q}}BH'$ . This shows that, localized at 0, the map  $f : L_{(p)}BS^3 \rightarrow L_{(p)}BY$  has a left inverse and finishes the proof of the proposition.  $\square$

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