A UNIQUENESS RESULT FOR ORTHOGONAL GROUPS AS 2-COMPACT GROUPS

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Abstract. Two connected compact Lie groups are isomorphic if and only if their maximal torus normalizers are isomorphic. It is conjectured that this result generalizes to p-compact groups. Here, we prove the generalization for orthogonal groups $O(n)$, the special orthogonal groups $SO(2k+1)$ and the spinor groups $Spin(2k+1)$ considered as 2-compact groups.

1. Introduction.

A p-compact group is the homotopy theoretic generalization of a compact Lie group. This concept was introduced by Dwyer and Wilkerson [8]. And it turned out that p-compact groups behave astonishingly similar as compact Lie groups; e.g. there exist maximal tori, Weyl groups and normalizer of maximal tori. Since their existence, the classification problem was one of the main questions. The classification scheme of connected p-compact groups is of the same form as for compact connected Lie groups. Every connected p-compact group has a finite covering which is a product of torus and a simply connected p-compact group [19], and every simply connected p-compact group splits into a product of finitely many simple simply connected p-compact groups [10] [23]. The classification boils down to the case of simple p-compact groups [15]. And in this situation, a case by case checking based on the Clark-Ewing list of rational pseudo reflection groups has solved the problem for odd primes ([2], [3], [4], [11], [17], [18], [20], [24]). For the prime 2, the known results in this direction concern unitary and special unitary groups [20], 2-compact groups with Weyl group isomorphic to an elementary abelian 2-group [11] and the case of the exceptional Lie group G_2 [26].

In this work we consider products of orthogonal groups and characterize them as 2-compact groups by their associated normalizer of the maximal torus. Before we state our main results, we first have to recall some basic notions of the theory of p-compact groups from [8]. For further notation and definition as well as the theory of p-compact groups see the survey articles [14] or [22] or the references given there. Since all the basic notions are well discussed in these and other papers we keep explanations short and refer the interested reader to the mentioned papers for more details.

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A loop space X is a pair (X, BX) of spaces, such that such that BX is pointed and such that the loop space ΩBX is equivalent to X. The loop space inherits properties from the space X, e.g. X is called connected if the space X is so and finite or $\mathbb{Z}$-finite if the integral cohomology $H^*(X;\mathbb{Z})$ vanishes in large degrees and is in any degree a finitely generated abelian group. The loop space X is called a p-compact group, if X is $\mathbb{F}_p$-finite and if X and BX are p-complete. Here, $\mathbb{F}_p$-finite means that $H^*(X;\mathbb{F}_p)$ is finite in each degree and vanishes in large degrees.

Every p-compact group X has a maximal torus $T_X \subset X$, a maximal torus normalizer $N_X \subset X$ which is a finite extension of T_X and a Weyl group W_X which acts on the p-adic lattice $L_X := \pi_2(BT_X)$. In this context, T_X denotes a completed torus and $\subset$ signifies maps $BT_X \to BX$ and $BN_X \to BX$ whose homotopy fibers are $\mathbb{F}_p$-finite. To ensure that $T_X \subset X$ is maximal torus we also have to put some maximality condition on the map $BT_X \to BX$ (e.g. see [8], [14] or [22]). The maximal torus normalizer fits into a fibration $BT_X \to BN_X \to BW_X$ which also determines the action of W_X on BT_X respectively on L_X. The map $BN_X \to BX$ is an extension of $BT_X \to BX$.

The composition $N_X \to X \to \pi_0(X)$ factors through a homomorphism $W_X \to \pi_0(X)$ and is always an epimorphism. The kernel of $N_X \to \pi_0(X) \cong \pi_1(BX)$ respectively the fiber of the map $BN_X \to B\pi_0(X)$ is a maximal torus normalizer of the component X_0 of the unit of X and the kernel of $W_X \to \pi_0(X)$ is the Weyl group of X_0. Examples of p-compact groups are given by the p-adic completion of compact Lie groups. If the group of components $\pi_0(G)$ of a compact Lie group G is a finite p-group, then the pair $(G_p^\wedge, BG_p^\wedge)$ gives rise to a p-compact group, also denoted by G. In particular, the orthogonal groups $O(n)$ give rise to 2-compact groups.

1.1 Definition.
(i) Two loop spaces or p-compact groups X and Y are called isomorphic if the associated classifying spaces BX and BY are homotopy equivalent.
(ii) Two p-compact groups X and Y are called N-isomorphic if the there exist isomorphisms $N_X \xrightarrow{\cong} N_Y$ and $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ such that the diagram

$$\begin{array}{ccc} BN_X & \longrightarrow & BN_Y \\ \downarrow & & \downarrow \\ B\pi_0(X) & \longrightarrow & B\pi_0(Y) \end{array}$$

commutes up to homotopy.

In analogy to compact Lie groups, the classification conjecture reads as

1.2 Conjecture. *Two p-compact groups X and Y are isomorphic if and only if they are N-isomorphic.*

As already mentioned, for odd primes, this conjecture is known to be true. In this case, one even can proof a weaker version, namely, X and Y are isomorphic if and only if N_X and N_Y are isomorphic, and, if in addition X is connected, then it suffices to have isomorphic Weyl group data; i.e. there exists an abstract isomorphism $W_X \cong W_Y$ such that L_X and L_Y are isomorphic as W_X-modules. (see the above mentioned references). But, as a comparison of $O(2)$ and $SO(3)$ and of $SO(2n+1)$ and $Sp(n)$ show, a weaker version cannot be true for the prime 2.

For $p = 2$, we want to prove the conjecture for finite products of orthogonal groups.

1.3 Theorem. *Let $G = \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group. Then, G and X are isomorphic if and only if they are N-isomorphic.*

The proof of this theorem is based on homotopy uniqueness results for products of orthogonal groups [21] in terms of mod-2 cohomology and

1.4 Theorem. *Let $G = \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group. If G and X are N-isomorphic, then there exists an isomorphism $H^*(BX) \cong H^*(BG)$ of algebras over the Steenrod algebra.*

Proof of Theorem 1.3. As shown in [21], every 2-complete space with the same mod-2 cohomology as BG is homotopy equivalent to the 2-adic completion $BG^\wedge_2$. □

As a corollary we will also see that the groups $Spin(2k+1)$ and $SO(2k+1)$ satisfy the above conjecture.

1.5 Corollary. *Let $G = Spin(2k+1)$ or $SO(2k+1)$ and let X be a 2-compact group. Then X and G are isomorphic if and only if they are N-isomorphic.*

Proof. Let X be a 2-compact group which is N-isomorphic to $SO(2k+1)$. Since $N_{O(2k+1)} \cong N_{SO(2k+1)} \times \mathbb{Z}/2$, the 2-compact group $X \times \mathbb{Z}/2$ is N-isomorphic to $O(2k+1)$ and therefore, by Theorem 1.3, isomorphic to $O(2k+1)$. The projection $BX \times B\mathbb{Z}/2 \to B\mathbb{Z}/2$ on the second factor is given by the classifying map of the non trivial 1-dimensional mod-2 cohomology class. The fiber of this map is BX. the same holds for $BO(2k+1)^\wedge_2$. This shows that BX and $BSO(2k+1)^\wedge_2$ are equivalent and that X and $SO(2k+1)$ are isomorphic as 2-compact groups.

Let X be a 2-compact group, which is N-isomorphic to $Spin(2k+1)$. The center of X can be read off from the maximal torus normalizer [9]. Hence, X and $Spin(2k+1)$ have the same center equal to $\mathbb{Z}/2$ and the quotients $\overline{X} := X/\mathbb{Z}/2$ and $SO(2k+1)$ are N-isomorphic and therefore isomorphic. Since X is the simply connected cover of $\overline{X}$ and $Spin(2k+1)$ the simply connected cover of $SO(2k+1)$, the 2-compact groups X and $Spin(2k+1)$ are isomorphic. □

Starting from Corollary 1.5, we plan in a further paper to prove uniqueness results for the 2-compact group $DI(4)$ constructed in [7] in terms of the maximal torus normalizer as well as in terms of the mod-2 cohomology of $BDI(4)$.

The paper is organized as follows. In Section 2 we provide some facts for orthogonal groups as 2-compact groups and fix some notation. Section 3, the last section, contains the tedious proof of Theorem 1.4. We also note that cohomology is always taken with $\mathbb{Z}/2$-coefficients; i.e. $H^*(-) = H^*(-;\mathbb{F}_2)$.

We will use the language and theory of p-compact groups all over the places. For references, about p-compact groups we refer the reader to the survey articles [14] and [22] or the references mentioned there.

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2. Products of orthogonal type.

In this section we provide necessary facts about products of orthogonal groups and fix some notation.

2.1 Notation and Remarks. Let $n = 2k + \theta$ with $\theta \in \{0, 1\}$. We use Θ to denote the group $\mathbb{Z}/2^\theta$. The maximal torus normalizer of $O(n)$ is isomorphic to $O(2) \wr \Sigma_k \times \Theta$. The group $O(2)$ contains a maximal elementary abelian 2-subgroup $E_{O(2)}$ of rank 2. We denote by $(\mathbb{Z}/2)^k \cong t_{O(n)} \subset T_{O(n)}$ the maximal elementary abelian 2-subgroup of the maximal torus of $O(n)$. We have a sequence

$$t_{O(n)} \subset (\mathbb{Z}/2)^n \cong E_{O(n)} := E^k_{O(2)} \times \Theta \subset N_{O(n)} \subset O(n)$$

of inclusions.

Every elementary abelian 2-subgroup of $O(n)$ is contained in $E_{O(n)}$. The subgroup $E_{O(n)}$ is self centralizing, i.e. the inclusion $E_{O(n)} \subset C_{O(n)}(E_{O(n))}$ induces an isomorphism of $E_{O(n)}$ with the centralizer of $E_{O(n)}$. The Weyl group $\overline{W}_{O(n)} := W_{O(n)}(E_{O(n)})$ is isomorphic to Σ_n which acts by permutations on $E_{O(n)}$. Actually, the normalizer $N_{O(n)}(E_{O(n)})$ of $E_{O(n)}$ is isomorphic to $\mathbb{Z}/2 \wr \Sigma_n \subset O(n)$. All this follows from classical representation theory. Moreover, $H^*(BO(n); \mathbb{Z}/2) \cong H^*(BE_{O(n)}; \mathbb{Z}/2)^{\Sigma_n}$.

For $G = \prod_i O(n_i)$, taking products establishes the same type of groups which fit into the diagram of inclusions

$$\begin{array}{ccccccc} t_G & \longrightarrow & T_G & & & & \\ \downarrow & & \downarrow & & & & \\ E_G & \longrightarrow & N_G & \longrightarrow & N_G & \longrightarrow & G \end{array}$$

and satisfy the same properties. Among other things we collect these facts in the next proposition. And $\overline{W}_G := W_G(E_G)$ denotes the Weyl group of E_G in G.

By SG we denote the component of the unit of G. Then $SG = \prod_i SO(n_i)$ is a product of special orthogonal groups. We will also use the notation SX for the component of the unit of a p-compact group X.

2.2 Proposition. *Let $G \cong \prod_i O(n_i)$ be a finite product of orthogonal groups. Then, the following holds:*
(i) E_G is self centralizing.
(ii) The mod-2 cohomology of BN_G is detected by elementary abelian 2-subgroups and so is the mod-2 cohomology of BG.
(iii) For any monomorphism $\rho : E_G \to N_G$, the image is subconjugated to $E_G \subset N_G$.
(iv) Up to conjugation, every elementary abelian subgroup of G is contained in E_G.

(v) $\overline{W}_G := \prod_i \overline{W}_{O(n_i)} \cong \prod_i \Sigma_{n_i}$.
(vi) $H^*(BG;\mathbb{Z}/2) \cong H^*(BE_G;\mathbb{Z}/2)^{\overline{W}_G}$.
(vii) For any elementary abelian 2-group E*, the inclusion* $E_G \to G$ *induces a bijection* $[BE, BE_G]/\overline{W}_G \cong [BE, BG]$.
(viii) For any elementary abelian 2-subgroup $E \subset G$*, the centralizer* $C_G(E)$ *is again a finite product of orthogonal groups.*

Proof. : Most of the proof is obvious and/or a straight forward application of real representation theory. Only (ii) and (iii) need a more detailed argument.

As a product of Θ and a wreath product of $O(2)$ with a symmetric group, the mod-2 cohomology of $BN_{O(n)}$ is detected by elementary abelian 2-subgroups [12] and so is the mod-2 cohomology of BN_G. This proves part (ii).

Part (iii) is a straightforward consequence of the lemma below, which contains the claim in a particular case. □

2.3 Lemma. *Let* $N := O(2) \wr \Sigma_n$*. The 2-rank* $rk_2(N)$*of* N *equals* $2n$ *and all elementary abelian 2-subgroups isomorphic to* $(\mathbb{Z}/2)^{2n}$ *are conjugated.*

Proof. Let $(\mathbb{Z}/2)^{2n} \cong \Delta_n \subset (O(2)^n$ be the elementary abelian 2-subgroup given by the product of all diagonal matrices. Since $\Delta_n \cong (\mathbb{Z}/2)^{2n}$, $rk_2(N) \geq 2n$.

The rest of the statement we prove by induction. For n=1 there is nothing to show. For abbreviation we set $N_k := O(2) \wr \Sigma_k$. Let $E \subset N_n$ be an elementary abelian 2-subgroup of dimension $2n$ and let $U \subset \Sigma_n$ be the image of E under the projection $N \to \Sigma_n$ and E' the kernel of $E \to U$. Then, $E' \subset (O(2)^n)^U$.

The group U acts on the set $\underline{n} := \{1, ..., n\}$. If this action has more than one orbit, U is subconjugated to the subgroup $\Sigma_k \times \Sigma_{n-k}$, where $k < n$, and E is subconjugated to $N_k \times N_{n-k}$. By induction hypothesis, $rk_2(N_k) = 2k$ and $rk_2(N_{n-k} = 2(n-k)$. Hence, the image of $E \cong E_k \times E_{n-k}$ where $E_K \subset N_K$ and $E_{n-k} \subset N_{n-k}$ are elementary abelian subgroups of maximal rank. Again by induction hypothesis, E_k is conjugated to Δ_k and E_{n-k} to Δ_{n-k}, which proves the statement in this case.

If the action of U on $\underline{n}$ produces exactly one orbit, then $E' \subset (O(2)^n)^U) \cong O(2)$ and E' has dimension ≤ 2. Since $rk_2(\Sigma_n) \leq n/2$, this implies that $2 + n/2 \geq 2n$ and hence that $n = 1$, which also proves the statement. □

3. Proof of Theorem 1.4.

Let $G \cong \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group which is N-isomorphic to G. In this section we want to prove a version of Proposition 2.2 for X. It will include Theorem 1.4.

Since X and G are N-isomorphic, we have the following diagram of inclusions

$$\begin{array}{ccccc} t_G & \longrightarrow & T_G & & \\ \downarrow & & \downarrow & & \\ E_G & \longrightarrow & N_G & \longrightarrow & X \end{array}$$

We denote by $i_X : E_G \to X$ the inclusion given by the bottom line. As already mentioned, SX denotes the component of the unit of X.

3.1 Proposition.
(i) The subgroup $E_G \subset X$ is self centralizing; i.e. $C_X(E_G) = E_G$.
(ii) The mod-2 cohomology of BX is detected by elementary abelian 2-subgroups.
(iii) Every elementary abelian 2-subgroup is subconjugated to E_G.
(iv) $\overline{W}_X = \overline{W}_G$.
(v) $H^*(BX) \cong H^*(BE_G)^{\overline{W}_G}$.

The proof of this proposition splits into several parts and is worked out in the rest of this section. In particular, the parts (iii) and (iv) are proved via inductions, which themselves split into several steps.

3.2 Remark. According to the splitting of $G = \prod_i O(n_i)$, the maximal torus normalizer $N_G = \prod N_{O(n_i)}$ splits into a product. By [23], this implies a splitting $X \cong \prod_i X_i$ such that $N_{O(n_i)} \subset X_i$ is a maximal torus normalizer and such that the diagram

$$\begin{array}{ccc} \prod_i N_{O(n_i)} & \xrightarrow{\cong} & N_G \\ \downarrow & & \downarrow \\ \prod_i X_i & \xrightarrow{\cong} & X \end{array}$$

commutes. In particular, X_i and $O(n_i)$ are N-isomorphic. Moreover, all statements of Proposition 3.1 also split in the same manner. Hence, on the way of proving the proposition, we always can assume that $G = O(n)$ is an orthogonal group. This even is possible for the inductions necessary for the proofs of (iii) and (iv).

3.3 Proof of (i) and (ii): Part (ii) follows from Proposition 2.2(ii), since N_G is the maximal torus normalizer of X and since $H^*(BX) \to H^*(BN_G)$ is a monomorphism [8].

Since E_G is self centralizing in G, we have $C_{N_G}(E_G) = E_G$, too. By Proposition 2(iii), every monomorphism of $E_G \to N_G$ takes image in $E_G \subset N_G$. By [16], there exist a lift $E_G \to N_G$ of $i_X : E_G \to X$ such that $C_N(E_G) \subset C_X(E_G)$ is the maximal torus normalizer. Hence, the maximal torus normalizer of $C_X(E_G)$ is E_G. This implies that $C_X(E_G)$ is discrete, therefore its own maximal torus normalizer, and congruent to E_G. This proves the first part. □

Before we continue with the proof we digress for the following lemma:

3.4 Lemma. *Let $E \subset T_G$ be an elementary abelian 2-subgroup. Then $C_G(E)$ and $C_X(E)$ are N-isomorphic.*

Proof. By [9, 7.6], both $C_G := C_G(E)$ and $C_X := C_X(E)$ have the same Weyl groups and therefore the same maximal torus normalizer, namely $C_N := C_{N_G}(E)$. We only have to show that the components of the unit are N-isomorphic.

Since E is a toral subgroup and therefore $E \subset SX$ and since $SC_X(E)$ is contained in SX, we have $SC_X(E) = SC_{SX}(E)$. But the Weyl group of $SC_{SX}(E)$ is completely determined by the Weyl group action of W_{SX} on the maximal torus $T_G = T_X$ [9, 7.6]. Since SG and SX are N-isomorphic, and since the same holds for G, both centralizers, C_G and C_X, have N-isomorphic components of the unit. □

3.5 Claim. *Proposition 3.1 (iii) is true for $E \cong \mathbb{Z}/2$.*

Proof. We prove this via an induction over the order of the Weyl group of G. Because of Remark 3.2 we can assume that $G = O(n)$. If $G \cong O(2)$, then $X = N_G = G$. And for $O(2)$ all of Proposition 3.1 is true. This is the beginning of the induction.

Up to conjugation $E \subset N_G$. Since $N_G = O(2) \wr \Sigma_k \times \Theta$, the generator of E is given by an element $x = ((A_1, ..., A_k; \sigma), a) \in N_G$ with $A_i \in O(2)$, $\sigma \in \Sigma_k$ and $a \in \Theta$. Since $E \cong \mathbb{Z}/2$, the permutation σ is of order two. Without loss of generality we can assume that $\sigma = \prod_{j=1}^{r} \tau_{2j-1,2j}$ is a product of commuting transpositions. Therefore $x \in (O(2) \wr \mathbb{Z}/2)^r \times O(2)^{k-2r} \times \Theta$. If $r = 0$, then E obviously is subconjugated to E_G. If $r \geq 1$ then $T_G^E \neq T_G^{\Sigma_n}$ and there exists a $\mathbb{Z}/2 \cong E' \subset (T_G)^E$ which is neither central in N_G nor in G or in X. By Lemma 3.4, $C_G(E')$ and $C_X(E')$ are N-isomorphic. And by Proposition 2.8(viii), $C_G(E')$ is a product of orthogonal groups. Moreover, since $E' \subset T_G$ is not central in G, this centralizer has a smaller Weyl group than G. Since E_G and E are contained in $C_{N_G}(E') \subset X$, E is subconjugated to E_G by induction hypothesis. □

3.6 Claim. *Proposition 3.1 (iii) is true for any general elementary abelian 2-group E.*

Proof. This time we make an induction over the order of E. The starting point of the induction is given by Claim 3.5. Again, we can assume that $G = O(n)$. If the dimension of E is greater than 1, the composition $E \subset X \to \pi_0(X)$ has a non trivial kernel and we can split $E \cong E_0 \oplus E_1$ such that $\mathbb{Z}/2 \cong E_0 \subset SX$. Since E_0 is cyclic, it is subconjugated to T_G, in fact to t_G. The centralizers $C_X(E_0)$ and $C_G(E_0)$ are N-isomorphic (Lemma 3.4), and $E_G \subset C_X(E_0)$. By induction hypothesis, E_1 is subconjugated to E_G inside of $C_X(E_0)$, which shows that E is subconjugated to E_G inside X. □

We make the map $BE_G \to BX$ into a fibration and denote the fiber by X/E_G. The inclusion $i_X : E_G \to X$ establishes a proxy action of E_G on X/E_G (see [8]) and the homotopy fixed point set $(X/E_G)^{E_G}$ fits into a fibration

$$(X/E_G)^{E_G} \to map(BE_G, BE_G)_{\text{lifts}} \to map(BE_G, BX)_{Bi_X} .$$

The total space is given by all self maps of BE_G which are lifts of Bi_X. Since $E_G \subset X$ is self centralizing (Proposition 3.1 (i)) the fiber is homotopically discrete and the set of components $\overline{W}_X := \pi_0((X/E_G)^{E_G})$ is a group, the Weyl group $W_X(E_G)$ of $E_G \subset X$. Since $\pi_0(map(BE_G, BE_G) = Hom(E_G, E_G)$, every element of $\overline{W}_X$ is represented by an automorphism of E_G. This establishes an action of $\overline{W}_X$ on E_G as well as on BE_G.

3.7 Claim. *For $G = O(n)$ the following holds:*
(i) $H^1(BX) \cong \mathbb{Z}/2$.
(ii) Let $0 \neq d \in H^1(BX)$ represented by the map $d : BX \to B\mathbb{Z}/2$. Then, the composition $BN_G \to BX \to B\mathbb{Z}/2$ is homotopic to the map induced by the determinant $N_G \subset O(n) \xrightarrow{det} \mathbb{Z}/2$ where det denotes the determinat.

Proof. Since X and $O(n)$ are N-isomorphic, there exists a commutative diagram

$$\begin{array}{ccc} & \pi_1(BN_G) & \\ \swarrow & & \searrow \\ \pi_1(BX) & \longrightarrow & \pi_1(BO(n)) \end{array} \qquad .$$

The Hurewicz map and universal coefficient theorems establish a commutative diagram

$$\begin{array}{ccc} & H^1(BN_G) & \\ \swarrow & & \searrow \\ H^1(BX) & \xrightarrow{\cong} & H^1(BO(n)) \end{array} \qquad .$$

This proves part (i). And part (ii) is a straightforward consequence of this diagram. □

3.8 Claim.
$\overline{W}_G \subset \overline{W}_X$. Moreover there exist a chain of inclusions $H^*(BX) \subset H^*(BE_G)^{\overline{W}_X} \subset H^*(BE_G)^{\overline{W}_G}$

Proof. Again we can assume that $G = O(n)$. That is that $\overline{W}_G = \Sigma_n$ For $n \geq 3$ the two subgroups $\Sigma_2 \times \Sigma_{n-2}$ and $\Sigma_{n-2} \times \Sigma_2$ generate Σ_n. The inclusions of both subgroups are induced by the Weyl group inclusion $\overline{W}_{C_G(E)} \subset \overline{W}_G$ for an elementary abelian 2-subgroup $\mathbb{Z}/2 \cong E \subset t_G$. By Lemma 3.4, $C_G(E)$ and $C_X(E)$ are N-isomorphic. Now, the first part of the claim follows by an induction over the order of the Weyl group. And the last is then a consequence of part (ii) and (iii) of Proposition 3.1. □

The next claim is nothing but part (iv) of Proposition 3.1. The argument in the proof was pointed out to us by the referee. It simplifies and shortens our original proof.

3.9 Claim. $\overline{W}_X = \overline{W}_G$.

Proof. As usual we can assume that $G = O(n)$. To prove the statement we only have to compare the orders of both groups. We consider the element $x := (-1, -1, 1, ..., 1) \in (\mathbb{Z}/2)^n \cong E_G$. Actually, this element is contained in T_G and generates the subgroup E in the proof of Claim 3.8. By induction, $C_X(E) \cong O(2) \times O(n-2)$. On the other hand we can calculate the mod-2 cohomology of $BC_X(E)$ with the help of the Lannes' T-functor [8]. Let α denote any of the compositions

$$H^*(BX) \to H^*(BE_G)^{\overline{W}_X} \to H^*(BE_G)^{\overline{W}_X} \to H^*(BE_G) \to H^*(BE)$$

with target $H^*(BE)$. Let $T^E_\alpha(\ )$ denote the componet of the T-functor associated to the morphism α [13]. And let I_X respectively I_G denote the isotropy group of E

with respect to the action of $\overline{W}_X$ or $\overline{W}_G$ on E_G. Since the T-functor is exact and commutes, for finite groups, with taking invariants (e.g. see [25]) we get a sequence of monomorphisms

$$H^*(BC_X(E)) \cong T^E_\alpha(H^*(BX)) \to H^*(BE_G)^{I_X} \to H^*(BE_G)^{I_G}.$$

By induction hypothesis, $C_G(E) \cong O(2) \times O(n-2)$ as 2-compact groups. Since $I_G = \Sigma_2 \times \Sigma_{n-2} = W_{O(2)\times O(n-2)}$, all monomorphism are isomorphism and $\Sigma_2 \times \Sigma_{n-2} = I_G = I_X$.

Now we look at the orbit of x under $\overline{W}_X$. Let x' be an element in the orbit which has k cordinates equal to -1. Since x and x' are conjugate in X both have the same determinant. This implies that k is even. Hence, since $\Sigma_n \subset \overline{W}_X$, x' is conjugate to an element in T_G with the same number of -1's. We can assume that $x' \in T_G$. Since both are conjugate in X they are already conjugate in N_G. Therefore, there exist an element in $W_X = W_G \subset \Sigma_n$ which transports x into x'. this implies that $k = 2$, that the $\overline{W}_X$-orbit of x is the same as the Σ_n-orbit, that both groups have the same order and that they are equal. □

Finally we have to prove part (v) of Proposition 3.1. We denote by $\mathcal{A}_p$ the Steenrod algebra and by $\mathcal{K}$ and $\mathcal{U}$ the categories of unstable algebras and unstable modules over $\mathcal{A}_p$. For an object A^* in $\mathcal{K}$ we denote by $F(A^*)$ the field of fractions. The action of $\mathcal{A}_p$ on A^* can be extended to an action on $F(A^*)$ which, in general, is not unstable. We denote $Un(F(A^*))$ the unstable part of $F(A^*)$. This is again an object of $\mathcal{K}$.

3.10 Claim.
$H^*(BX) \cong H^*(BG)$.

Proof. By Proposition 3.1 (ii) and (iii), $H^*(BX) \to H^*(BE_G)$ is the Adams-Wilkerson embedding [1]. Let $R^* := H^*(BX)$, $A^* := H^*(BE_G)$ and $D^* \subset A^*$ be the smallest Hopf-subalgebra containing R^* which is also an algebra over the Steenrod algebra. By [6, 3.6], $D^* \cong T^{E_G}_i(R^*) \cong H^*(BC_X(E_G)$. Since $E_G \cong C_X(E_G)$ (part (i)), we have $D^* = A^*$.

Since A^* is a finitely generated as an R^*-module $F(R^*) \subset F(A^*)$ is an algebraic extension. We can apply [27, Theorem II]. In our situation, it tells us that there exists a "Galois group" $W \subset Gl(E_G)$ such that $R^* \subset Un(F(R^*) = (A^*)^W$. Since the homotopy classes of maps $BE_G \to BX$ are in a one to one relation with the algebraic maps of algebras over the Steenrod algebra $H^*(BX) \to H^*(BE_G)$, the Galois group W can be identified with $\overline{W}_X = \overline{W}_G$. This establishes a chain of monomorphism

$$H^*(BX) \cong R^* \xrightarrow{j} H^*(BG) \cong (A^*)^{W_G} \to H^*(BN_G) \; .$$

Since the homotopy fiber of $BN_G \to BX$ is $\mathbb{F}_p$-finite and has Euler characteristique equal to 1 [8], there exists a transfer $tr : H^*(BN_G) \to H^*(BX)$ which is a $\mathcal{U}$-map $H^*(BX)$-linear. Moreover, $tr\, j = id$ [5]. Let $x \in H^*(BG)$. Then, there exist classes $a, b \in H^*(BX)$ such that $j(a) = x\, j(b)$. Applying the transfer gives

$tr(x)\, b = a$ and applying j yields $j(tr(x))\, j(b) = j(a)$. Since $H^*(BG)$ is a polynomial ring, this shows that $j\, tr(x) = x$ and that j is an isomorphism. This finishes the proof of the claim and of Proposition 3.1. □

References.

[1] J.F. Adams & C.W. Wilkerson, *Finite H–spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95-143.

[2] K. Anderson, J. Grødal, J.M. Møller, and A. Viruel, *The classification of p-compact groups, p odd*, in preparation.

[3] C. Broto & A. Viruel, *Homotopy uniqueness of $BPU(3)$*, Group representations: cohomology, group actions, and topology (Seattle, WA, 1996), Amer. Math. Soc., Providence, RI, (1998), 85-93.

[4] C. Broto & A. Viruel, *Projective unitary groups are totally N-determined p-compact groups*, preprint.

[5] W.G. Dwyer, *Transfer maps for fibrations*, Math. Proc. Camb. Phil. Soc. **120** (1996), 221-235.

[6] W.G. Dwyer, H.Miller, & C.W. Wilkerson, *The homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29-45.

[7] W.G. Dwyer & C.W. Wilkerson, *A new finite loop space at the prime two*, J. AMS **6** (1993), 37-63.

[8] W.G. Dwyer & C.W. Wilkerson, *Homotopy fixed–point methods for Lie groups and finite loop spaces*, Ann. Math. (2) **139** (1994), 395-442.

[9] W.G. Dwyer & C.W. Wilkerson, *The center of a p-compact group*, The Cech Contennial. Contemporary Math. **181** (1995), 119-157.

[10] W.G. Dwyer & C.W. Wilkerson, *Product splittings of p–compact groups*, Fund. Math. **147** (1995), 279-300.

[11] W.G. Dwyer & C.W. Wilkerson, *p-compact groups with abelian Weyl groups*, Preprint.

[12] J.H. Gunawardena, J. Lannes, and S. Zarati, *Cohomlogie des groupes symmétrique et application de Quillen*, L.M.S Lectures Notes Math. **139**.

[13] J. Lannes, *Sur les espaces fonctionelles dont la source est la classifiant d'un p–groupe abélien élemementaires*, IHES **75** (1992), 135-244.

[14] J.M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), 413-428.

[15] J.M. Møller, *Deterministic p-compact groups*, Stable and unstable homotopy theory, (Toronto, ON, 1996), Amer. math. Soc., Providence, RI (1998), 255-278.

[16] J.M. Møller, *Normalizers of maximal tori*, Math. Z. **231** (1999), 51-74.

[17] J.M. Møller, *N-determined p-compact groups*, Preprint.

[18] J.M. Møller, *N-determinism of the p-compact groups of the A-family*, Preprint.

[19] J.M. Møller & D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. reine u. angew. Math. **456** (1994), 99-133.

[20] J.M. Møller & D. Notbohm, *Connected finite loop spaces with maximal tori*, Trans. AMS **350** (1998), 3483-3504.

[21] H. Morgenroth, *Homotopieeindeutigkeit von Produkten orthogonaler Gruppen*, Thesis, Göttingen 1996 or ftp://hopf.math.purdue.edu/pub/morgenroth/BOn.dvi.

[22] D. Notbohm, *Classifying spaces of compact lie groups and finite loop spaces*, in: Handbook of Algebraic topology, Ed.: I. James, Elsevier (1995), 1049-1094.

[23] D. Notbohm, *Unstable splittings of classifying spaces of p–compact groups*, to appear Quart. J. Math. Oxford.

[24] D. Notbohm, *Spaces with polynomial cohomology*, Math. Proc. Camb. Phil. Soc. **9** (1999), 55-99.

[25] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed-point set conjecture, University of Chicago Press* (1994).

[26] A. Viruel, *Homotopy uniqueness of BG_2*, Manuscr. Math. **95** (1998), 471-497.

[27] C.W. Wilkerson, *Rings of invariants and insepaprable forms of algebras over the Steenrod algebra*, Preprint.

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